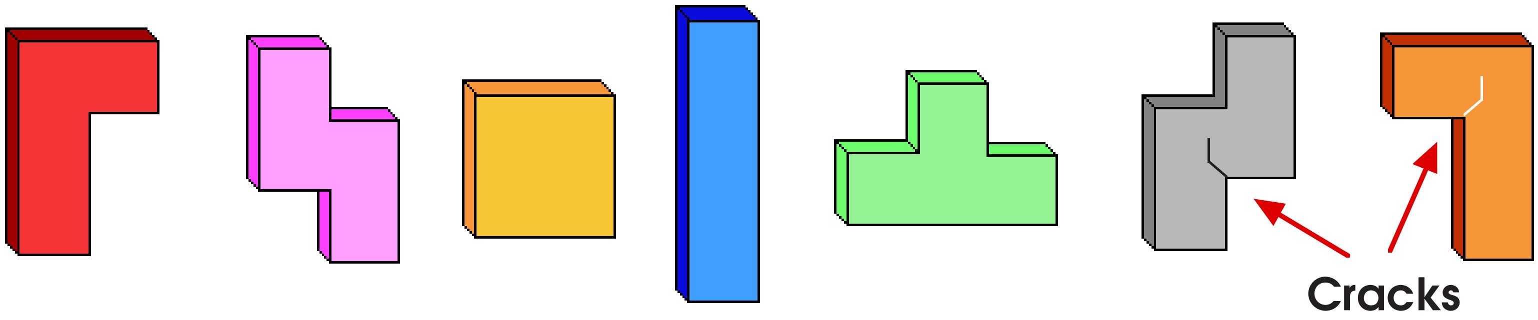
Stuttgart'98

Singularities of Corner Problems

Monique DAUGE

Institut de Recherche MAthématique de Rennes

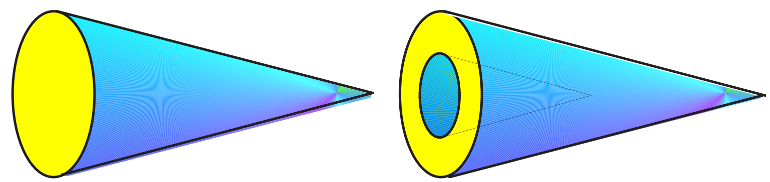
Corners



2D Vertex of a polygon

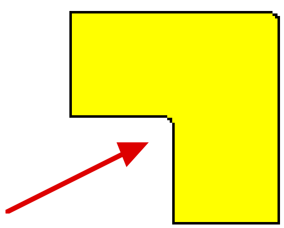
3D Edge or vertex of a polyhedron

Vertex of a cone



Curved faces or cracks possible (but ~~cusps~~ - different theory)

Related to: Smooth domains with large curvatures
(closer to physics)



Corners: World-wide-spread, mainly by industry...

Genealogy

	Smooth elliptic problems				
57	Agmon - Douglis - Nirenberg			Lions - Magenes	
	Corner problems				
67	Maz'ya	Kondrate'v		Grisvard	
		Kondrat'ev - Oleinik		Moussaoui	
78	Maz'ya - Plamenevskii			Lemrabet	
		Nikishkin	Wendland		
	Nazarov		Costabel - Stephan		
	Maz'ya - Nazarov - Plamenevskii				Rempel - Schulze
88			Dauge		
			Nicaise		
	Kozlov		Dauge - Nicaise		
	Maz'ya - Rossmann		Costabel - Dauge		
	Kozlov - Maz'ya		Nicaise - Sändig		Schulze - Schrohe
98	Kozlov - Maz'ya - Rossmann		Costabel - Dauge - Nicaise		

Time arrow

Domains and Equations

Domain Ω **:** 3D polyhedral *(or with curved faces, conical points, in higher dimensions)*

Inner Operator L **:** second order strongly elliptic, scalar or matrix-valued *(or ord.* $2m$ *)*, defined via a first order bilinear form a (usual ingredients: grad, div, curl, ε).

Coefficients of a **:** constant, or piecewise-constant in a polyhedral domain decomposition Ω_j of Ω *(or piecewise-smooth).*

Boundary conditions B **:** Dirichlet or Neumann on each face (of Ω or of Ω_j [a]) defining a subspace V of H^1 (except for Maxwell)

$$(\mathcal{P}) \qquad u \in V, \quad \forall v \in V, \quad \int_\Omega a(u,v)\,dx = \int_\Omega f\,v\,dx$$

or

$$\begin{cases} Lu = f & \text{in } \Omega_j, \\ Bu = 0 & \text{on } \partial\Omega \cap \partial\Omega_j, \qquad [u] = [Nu] = 0 \ \text{ on } \partial\Omega_j \setminus \partial\Omega \end{cases}$$

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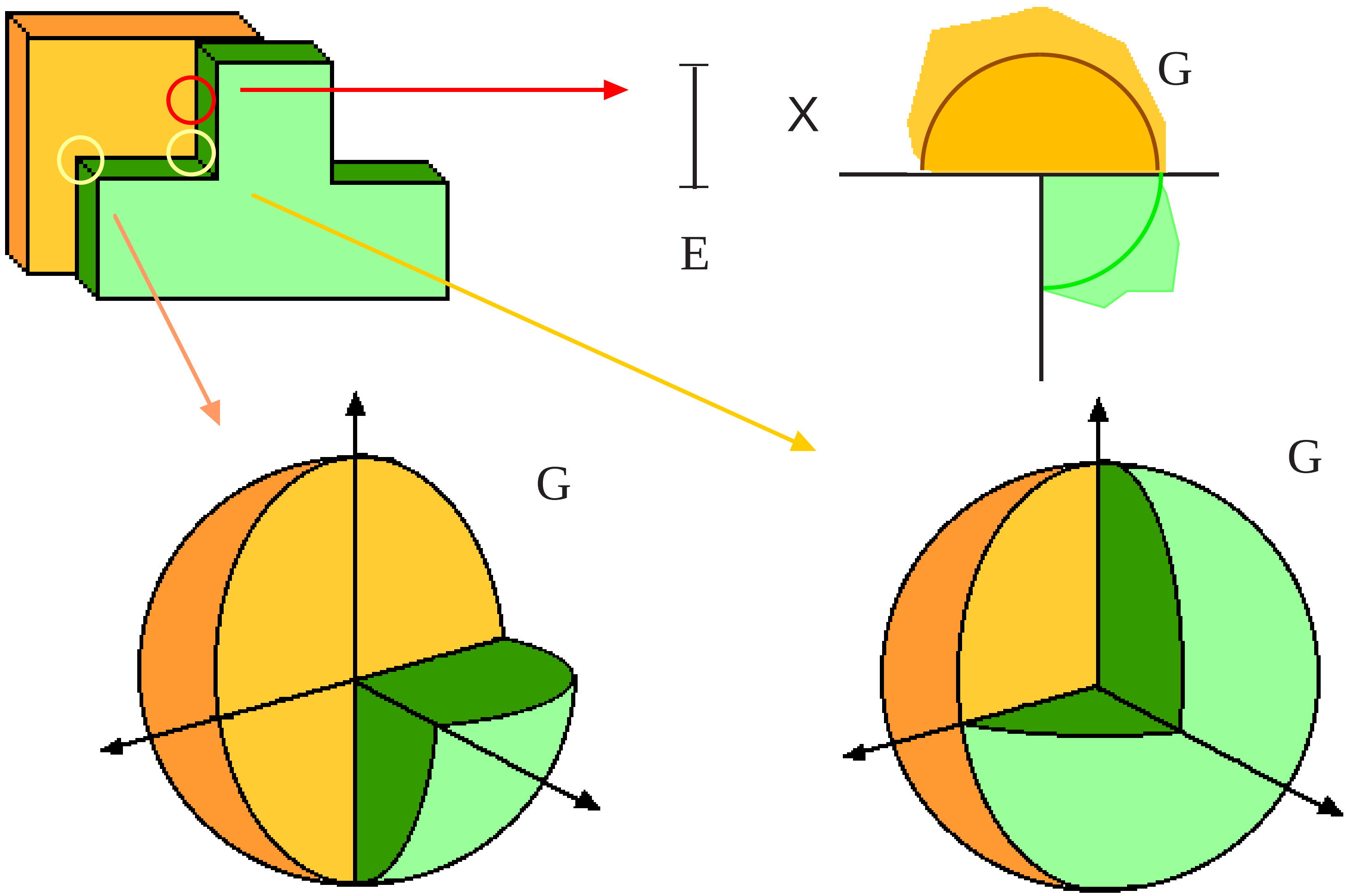
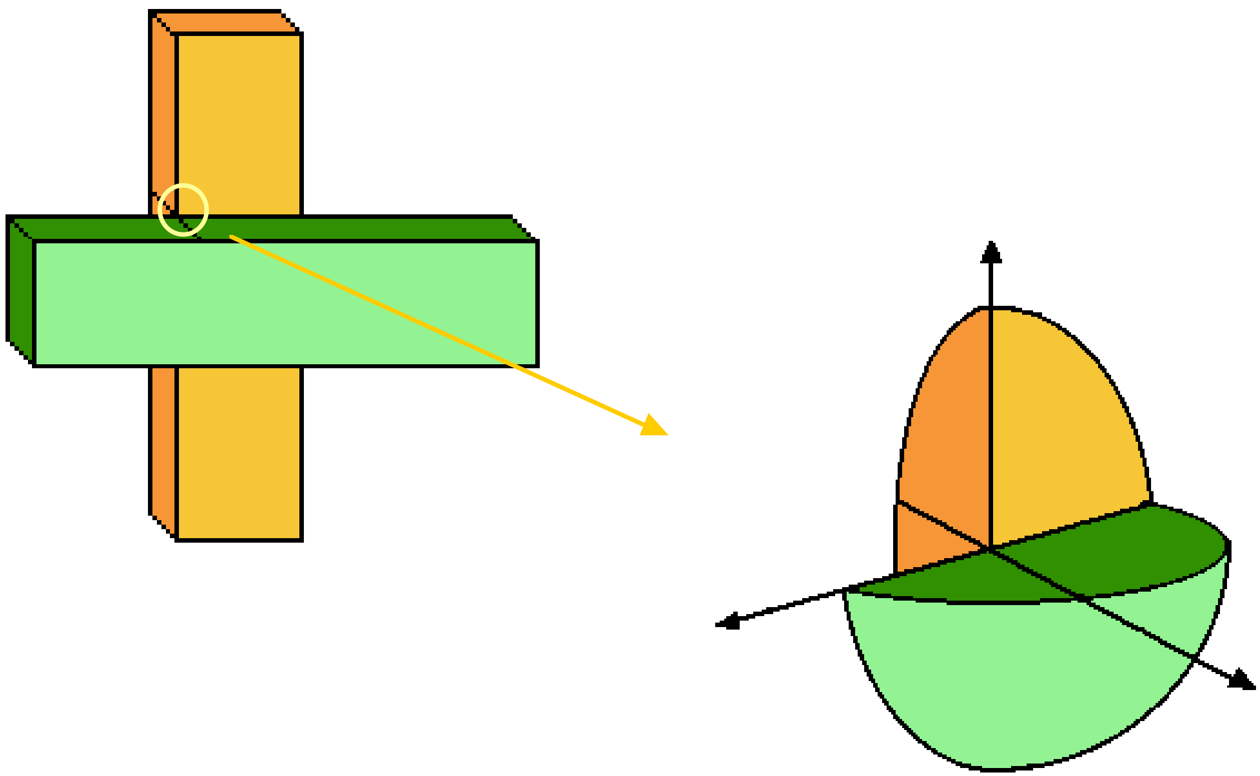
[a] *Careful! Mixed boundary conditions on a smooth domain are a trap!*

Vertices and Edges



A non-Lipschitz polyhedron



Vertex Mellin Symbols

Vertex v (of Ω or Ω_j). In polar coordinates $(\rho,\vartheta)\in\mathbb{R}_+\times G$ of center v,

$$L=\rho^{-2}\mathcal{L}(\vartheta;\rho\partial_\rho,\partial_\vartheta), \qquad \text{and} \qquad B=\rho^{-\deg B}\mathcal{B}(\vartheta;\rho\partial_\rho,\partial_\vartheta).$$

Mellin Symbol :

$$\mathbb{C}\ni\lambda \longmapsto \mathcal{L}(\vartheta;\lambda,\partial_\vartheta)=:\mathcal{L}(\lambda) \qquad \text{and} \qquad \mathbb{C}\ni\lambda \longmapsto \mathcal{B}(\vartheta;\lambda,\partial_\vartheta)=:\mathcal{B}(\lambda).$$

The coerciveness of problem $(\mathcal{P})$ implies the solvability of

$$\mathcal{P}[\mathrm{v},\lambda] \quad \begin{cases} \mathcal{L}(\lambda)\,\mathrm{U} = \mathrm{F} & \text{in } G, \\ \mathcal{B}(\lambda)\,\mathrm{U} = 0 & \text{on } \partial G, \qquad [\mathrm{U}]=[\mathcal{N}\mathrm{U}]=0 \quad \text{on } \partial G_j\setminus\partial G \end{cases}$$

in $H^1(G)$ except for λ in a discrete set $\mathcal{E}[\mathrm{v}]$: for $\lambda\in\mathcal{E}[\mathrm{v}]$, the kernel of $\mathcal{P}[\mathrm{v},\lambda]$ is *“generically”* one-dimensional, generated by $\mathrm{U}_{\mathrm{v},\lambda}$.

The *singularity exponents* λ and the *singular functions* $U_{\mathrm{v},\lambda}$ associated with v are

$$\lambda\in\mathcal{E}[\mathrm{v}], \quad \text{with} \quad \operatorname{Re}\lambda > -\tfrac{1}{2} \qquad \text{and} \qquad U_{\mathrm{v},\lambda}=\rho^{\lambda}\,\mathrm{U}_{\mathrm{v},\lambda}(\vartheta).$$

Edge Mellin Symbols

Edge $E \ni \mathrm{e}$ (of Ω or of Ω_j). In cylindrical coordinates $(r, \theta, z) \in \mathbb{R}_+ \times (0, \omega) \times T_{\mathrm{e}}A$ of center e,

$$L = r^{-2}\mathcal{L}(\theta; r\partial_r, \partial_\theta) + r^{-1}\mathcal{L}_1(\theta; r\partial_r, \partial_\theta, \partial_z) + \mathcal{L}_2(\theta; r\partial_r, \partial_\theta, \partial_z).$$

Mellin Symbol :

$$\mathbb{C} \ni \lambda \longmapsto \mathcal{L}(\theta; \lambda, \partial_\theta) =: \mathcal{L}(\lambda) \quad \text{and} \quad \mathbb{C} \ni \lambda \longmapsto \mathcal{B}(\theta; \lambda, \partial_\theta) =: \mathcal{B}(\lambda).$$

The corresponding problem $\mathcal{P}[\mathrm{e}, \lambda]$ is uniquely solvable in $H^1(0, \omega)$ except for λ in a discret set $\mathcal{E}[\mathrm{e}]$: for $\lambda \in \mathcal{E}[\mathrm{e}]$, the kernel of $\mathcal{P}[\mathrm{e}, \lambda]$ is "generically" one-dimensional, generated by $u_{\mathrm{e},\lambda}$.

The *singularity exponents* λ and the *singular functions* $U_{\mathrm{e},\lambda}$ associated with e are

$$\lambda \in \mathcal{E}[\mathrm{e}], \quad \text{with} \quad \mathrm{Re}\,\lambda > 0 \quad \text{and} \quad U_{\mathrm{e},\lambda} = r^\lambda u_{\mathrm{e},\lambda}(\theta).$$

Regularity Theorem

There holds (use piecewise – H^s for transmission problems)

$$\boxed{f \in H^{\sigma-1}(\Omega) \quad \Longrightarrow \quad u \in H^{\sigma+1}(\Omega)}$$

if and only if $\sigma < \sigma[\Omega, L, B]$

$$\boxed{\sigma[\Omega, L, B] = \min\left\{ \min_{\mathrm{v\ vertex}} \xi_{\mathrm{v}} + \tfrac{1}{2} \ , \min_{\mathrm{e\ in\ edges}} \xi_{\mathrm{e}} \right\}}$$

with

ξ_{v} the least real part $> -\frac{1}{2}$ of the exponents $\lambda \in \mathcal{E}[\mathrm{v}]$.

and

ξ_{e} the least real part > 0 of the exponents $\lambda \in \mathcal{E}[\mathrm{e}]$.

Remark : for L^{p} *Sobolev spaces, we have*

$$\sigma^{(\mathrm{p})} = \min\left\{ \min_{\mathrm{v\ vertex}} \xi_{\mathrm{v}} + \tfrac{3}{\mathrm{p}} - 1 \ , \min_{\mathrm{e\ in\ edges}} \xi_{\mathrm{e}} + \tfrac{2}{\mathrm{p}} - 1 \right\}$$

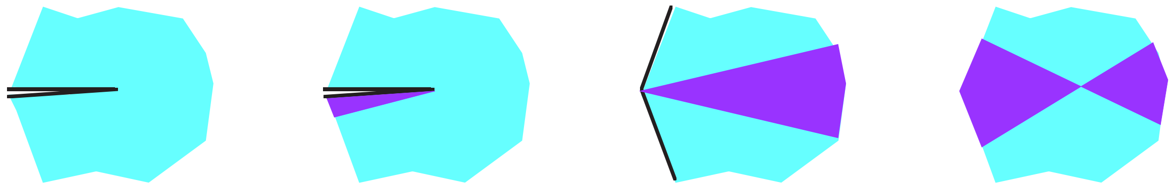
First Example : Electrostatic Potential

Associated with the bilinear form (ε_j is the electric permittivity of material Ω_j)

$$a(u,v) = \sum_j \int_{\Omega_j} \varepsilon_j \,\mathrm{grad}\, u \cdot \mathrm{grad}\, v \; dx, \qquad \text{for} \quad u, v \in \overset{\circ}{H}{}^1(\Omega).$$

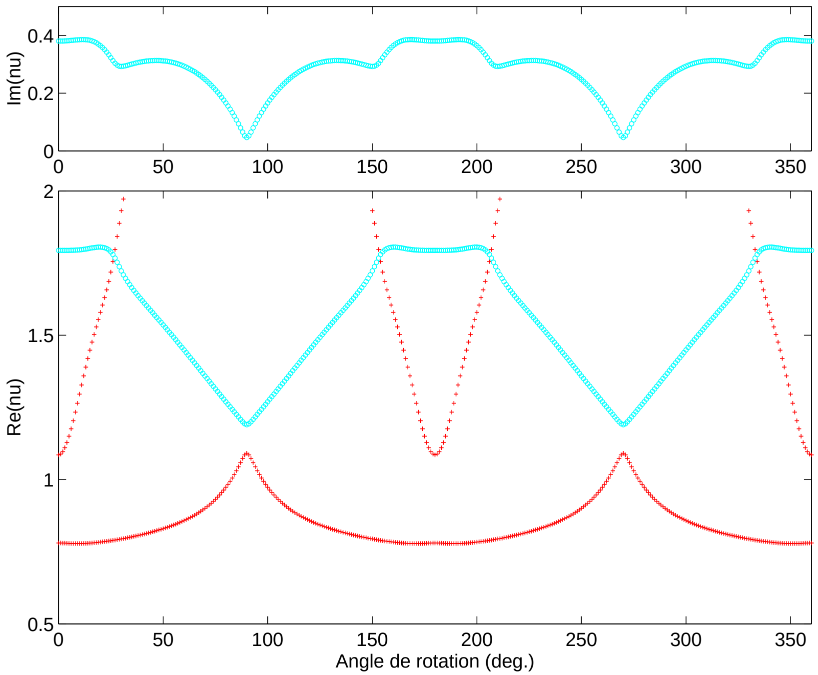
Optimal Minima for ξ_{e} :

Exterior Angle	1 material	2 materials	3 materials	4 materials
$\leq \frac{\pi}{2}$	2	1	0	0
Convex	1	$\frac{1}{2}$	0	0
Any	$\frac{1}{2}$	$\frac{1}{4}$	0	0
None	∞	$\frac{1}{2}$	$\frac{1}{4}$	0

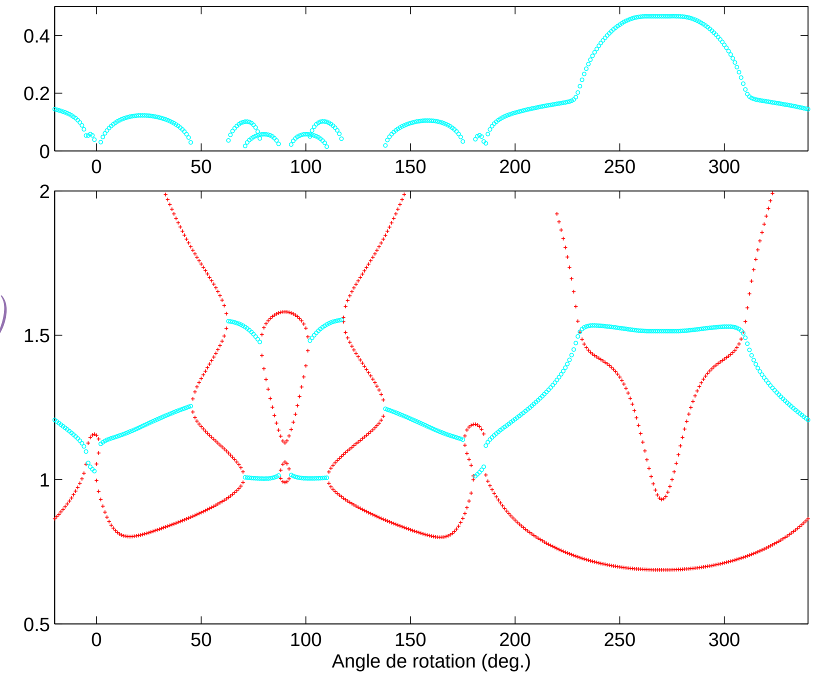


Second Example : bi-material Elasticity

(i)



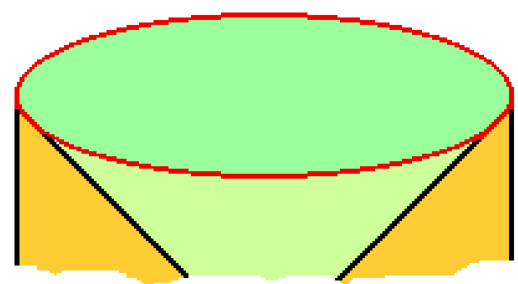
(ii)



The domain is a straight cylinder formed by two materials.

(i) **Isotropic - Orthotropic** (ii) **Orthotropic - Anisotropic**

Bound. Cond. (D) on the flat bound., (N) on the circular bound.



Computed with `ExSiEl`

Expansion Theorem

Let $u_{\mathrm{v},\lambda}$ be a localization of $U_{\mathrm{v},\lambda}$ and $u_{\mathrm{e},\lambda}$ be a localization of $U_{\mathrm{e},\lambda}$.

For $\sigma[\Omega, L, B] < \sigma < 1 + \sigma[\Omega, L, B]$, there holds in a "sub-generic" way

$$u - \Bigg(\sum_{\mathrm{v\ vertex},} \sum_{\substack{\lambda \in \mathcal{E}[\mathrm{v}] \\ -1/2 < \mathrm{Re}\,\lambda < \sigma - 1/2}} c_{\mathrm{v},\lambda}\, u_{\mathrm{v},\lambda} + \sum_{\mathrm{e\ in\ edges},} \sum_{\substack{\lambda \in \mathcal{E}[\mathrm{e}] \\ 0 < \mathrm{Re}\,\lambda < \sigma}} \mathcal{K}\{c_{\mathrm{e},\lambda}\}\, u_{\mathrm{e},\lambda} \Bigg) \in H^{\sigma+1}(\Omega),$$

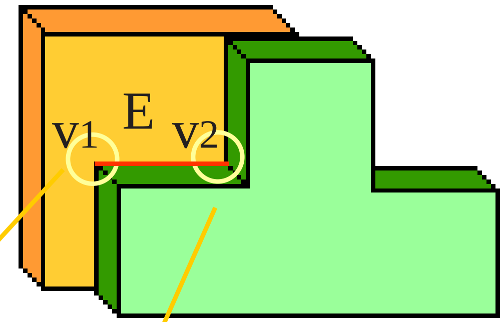
where

- $c_{\mathrm{v},\lambda} \in \mathbb{C}$ are the vertex coefficients
- $E \ni \mathrm{e} \mapsto c_{\mathrm{e},\lambda}$ are the edge coefficients on E
- $\mathcal{K}$ is a smoothing operator (inside Ω).

"Sub-generic" assumption: $\forall \mathrm{e}$, *the* λ *are neither integer nor multiple.*

Vertex – Edge interaction: the geometry

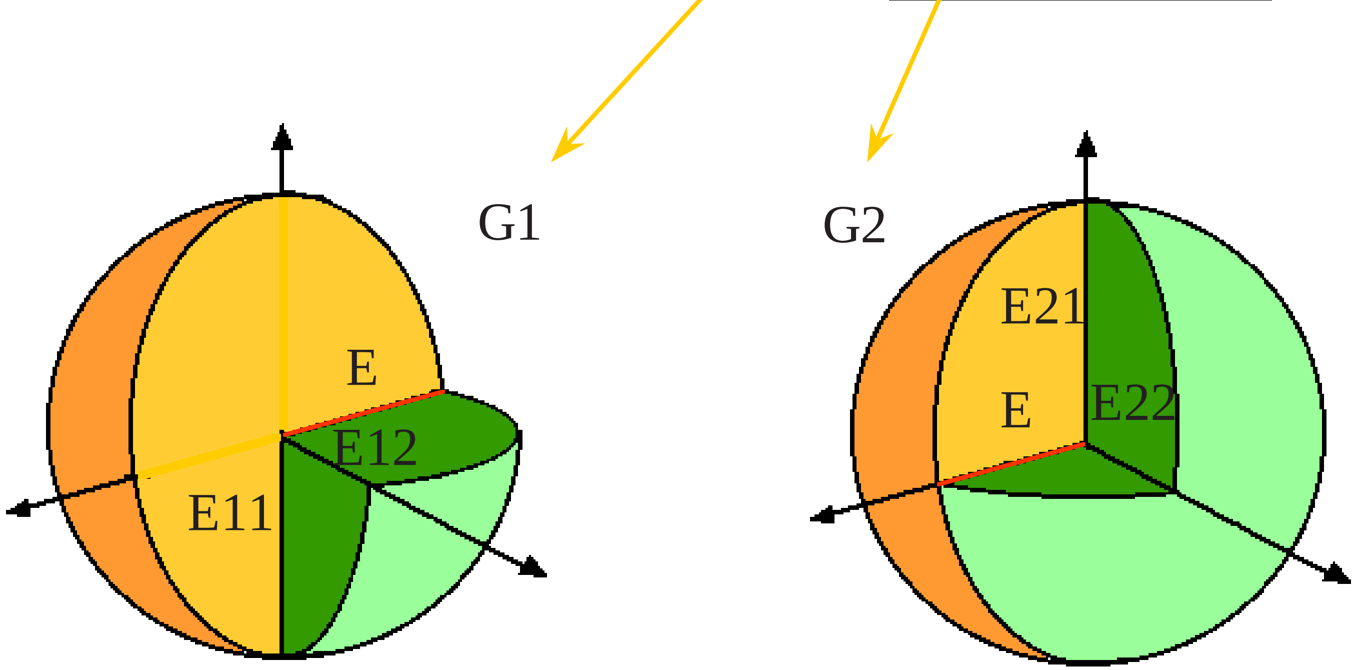
The edge E has the vertices $\mathrm{v}1$ and $\mathrm{v}2$ as end points.



Vertices $\mathrm{v}1$ – $\mathrm{v}2$ $\longleftrightarrow$ spherical polygons $G1$ – $G2$.

Corners of $G1$ $\longleftrightarrow$ edges E, $E11$ and $E12$.

Corners of $G2$ $\longleftrightarrow$ edges E, $E21$ and $E22$.



Vertex – Edge interaction: the singular functions

The edge coefficients $c_{\mathrm{e},\mu}$ belong to the weighted Sobolev space $\mathcal{H}^{\sigma-\mathrm{Re}\,\mu}_{-\sigma}(E)$ with

$$\mathcal{H}^s_\gamma(E) = \{c \in \mathcal{D}'(E)\ ;\ \delta^{\gamma+\alpha}\, d^\alpha c \in L^2(E),\ \alpha \le s\}$$

where δ is the distance to the endpoints of E (if vertex $\mathrm{v} \in \overline{E}$ then $\delta \simeq \rho$ near v).

Splitting only along the edges yields new edge coefficients $\tilde{c}_{\mathrm{e},\mu} \in \mathcal{H}^{\sigma-\mathrm{Re}\,\mu}_0(E)$.

The vertex singular functions are $\rho^\lambda\, U_{\mathrm{v},\lambda}(\vartheta)$ and each $U_{\mathrm{v},\lambda}$ has singularities at the corners of G (corresponding to an $\mathrm{e} \in E$ for any edge $E \ni \mathrm{v}$):

$$U_{\mathrm{v},\lambda} \;-\; \sum_{E \ni \mathrm{v}} \;\sum_{\substack{\mu \in \mathcal{E}[\mathrm{e}] \\ 0 < \mathrm{Re}\,\mu < \sigma}} \boxed{a^{\mathrm{v},\lambda}_{\mathrm{e},\mu}}\, u_{\mathrm{e},\mu} \;\in\; H^{\sigma+1}(G).$$

Expansion of the edge coefficients

$$\tilde{c}_{\mathrm{e},\mu} \;-\; \sum_{\substack{\lambda \in \mathcal{E}[\mathrm{v}] \\ -1/2 < \mathrm{Re}\,\lambda < \sigma - 1/2}} \boxed{a^{\mathrm{v},\lambda}_{\mathrm{e},\mu}}\, \rho^\lambda \;=\; c_{\mathrm{e},\mu}$$

Stable Singularities

In presence of curved edges or variable coefficients, $U_{\mathrm{e},\lambda}$ depends on $\mathrm{e} \in E$.

If for $\mathrm{e}_0 \in E$ and $\lambda_0 \in \mathcal{E}[\mathrm{e}]$, the inverse $\lambda \longmapsto \mathcal{P}[\mathrm{e},\lambda]^{-1}$ has a *double* pole (branching), there exists *two* associated singularities

$$r^{\lambda_0}\, u_{\mathrm{e}_0,\lambda_0}(\theta) \qquad \text{and} \qquad r^{\lambda_0}\Big(\log r\; u_{\mathrm{e}_0,\lambda_0}(\theta) + v_{\mathrm{e}_0,\lambda_0}(\theta)\Big).$$

For $\mathrm{e} \neq \mathrm{e}_0$ there exists *two* singularities $U_1(\mathrm{e}) = U_{\mathrm{e},\lambda_1}$ and $U_2(\mathrm{e}) = U_{\mathrm{e},\lambda_2}$ for which $\lambda_1(\mathrm{e}) \longrightarrow \lambda_0$ and $\lambda_2(\mathrm{e}) \longrightarrow \lambda_0$ as $\mathrm{e} \longrightarrow \mathrm{e}_0$.

Similar if $\lambda_0 \in \mathbb{N}$: logarithmic singularity in e_0 and $U_1(\mathrm{e})$ close to a polynomial $U_2(\mathrm{e})$ if $\mathrm{e} \neq \mathrm{e}_0$ (crossing). We set $\lambda_2(\mathrm{e}) \equiv \lambda_0$.

Stable Singularities: by sum and divided difference

$$\widetilde{U}_1(\mathrm{e}) = U_1(\mathrm{e}) + U_2(\mathrm{e}) \qquad \text{and} \qquad \widetilde{U}_2(\mathrm{e}) = \frac{U_1(\mathrm{e}) - U_2(\mathrm{e})}{\lambda_1(\mathrm{e}) - \lambda_2(\mathrm{e})}.$$