On representation formulas and radiation conditions

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Dedicated to Professor G. C. Hsiao on the occasion of his 60th anniversary

Abstract. *Solutions of linear elliptic partial differential equations in unbounded domains can be represented by boundary potentials if they satisfy certain conditions at infinity. These radiation conditions depend on the fundamental solution chosen for the integral representation. We prove some basic results about radiation conditions in a rather general framework.*

Fundamental solutions G *are considered that are defined only on the complement of a compact set. It turns out, however, and we present examples for this, that the more interesting results only hold if* G *is defined on all of* $\mathbb{R}^n$ *or if it is a Green function for an exterior boundary value problem.*

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1 INTRODUCTION

An essential step in the boundary element method for an elliptic boundary value problem is the representation of the unknown solution by boundary potentials. As is well known, even if the domain under consideration is bounded, any analysis of the integral equation method involves the complementary domain. Therefore one always has to study boundary potential representations on unbounded domains.

Such integral representations make use of a fundamental solution of the elliptic differential operator, and their validity is strongly linked with a certain behavior at infinity: a function will have such a representation only if, in a certain sense, it behaves at infinity like the fundamental solution chosen to represent it.

There are several classical approaches to the question of characterizing the behavior at infinity of solutions of elliptic partial differential equations:

- In the widely studied cases of the Laplace or Helmholtz equations, conditions guaranteeing existence and uniqueness of exterior boundary value problems are known. These conditions allow the construction of Green functions and representation formulas (see [5, Chap.II]).
- For strongly elliptic homogeneous operators, weighted Sobolev spaces have been studied by Nedelec [11, 12] and Giroire [6], which allow variational formulations of exterior problems, and as a consequence, representation formulas by boundary potentials.
- In [10], Nazarov and Plamenevskii study spaces with asymptotics at infinity and discuss the validity of Green formulas and variational formulations. Radiation conditions appear as conditions for obtaining well-posed problems.

Here we concentrate on the question of validity of the representation by boundary potentials in exterior domains. We can consider rather general elliptic operators having a fundamental solution, and we can also consider general fundamental solutions, including Green functions for exterior domains. This gives a lot of flexibility in the choice of the behavior at infinity (see the examples in section 7).

For the behavior at infinity guaranteeing the validity of the representation formulas in the exterior domain, we choose the expression "radiation condition". The study of radiation conditions in this generalized sense was started in [3, 4].

In this paper, we prove some general results on representation formulas and radiation conditions. In particular, we prove for fundamental solutions in $\mathbb{R}^n$:

- Every fundamental solution satisfies its own radiation condition;

- If a solution can be represented by any combination of boundary or volume potentials, then it can be represented by its own Cauchy data on a given surface;

- There is a one-to-one correspondence between radiation conditions and fundamental solutions for any given elliptic differential operator.

We generalize these results to the case of Green functions of an exterior domain.

Next, we consider boundary conditions covering the elliptic differential operator. It is well known that such an elliptic boundary value problem induces a Fredholm operator in the interior bounded domain. With the help of the representation formulas, we prove that we also obtain a Fredholm operator in the exterior unbounded domain on spaces of functions satisfying the radiation condition at infinity.

We discuss many examples, some of them well known, some less. We see in particular that the general results above are no longer true in general if the fundamental solution is not defined on all of $\mathbb{R}^n$.

2 FUNDAMENTAL SOLUTIONS AND GREEN FORMULAS

Throughout the paper, we consider the following geometric situation:

$$K \text{ a compact set in } \mathbb{R}^n, \quad D \text{ its complement } \mathbb{R}^n \setminus K.$$

We assume that D is *connected* and D will be the underlying domain of everything we want to consider. In fact the case when $D = \mathbb{R}^n$ is the most important case for us, but the possibility of considering other underlying domains allows to treat for example the Green function related to the domain D.

The basic boundary on which we will work is denoted Γ. We assume that Γ is a bounded $\mathscr{C}^\infty$ manifold of dimension $n-1$ and that Γ is the boundary

$$\begin{cases} \text{of a bounded open set } \Omega^- \supset K \\ \text{and of an unbounded connected domain } \Omega^+ = \mathbb{R}^n \setminus \overline{\Omega^-}. \end{cases}$$

We refer to Ω^- as the *interior* and to Ω^+ as the *exterior* domain.

Our purpose is to study representation formulas in Ω^+ by boundary potentials on Γ. Let

$$L(x, \partial_x) = \sum_{|\alpha| \leq 2m} a_\alpha(x)\, \partial_x^\alpha$$

be a properly elliptic differential operator with scalar coefficients

$$a_\alpha \in \mathscr{C}^\infty(\overline{D}).$$

We denote by

$$L'(x,\partial_x) = \sum_{|\alpha|\,\leq\, 2m} (-1)^{|\alpha|} \partial_x^\alpha\, a_\alpha(x)$$

the formal transpose of L.

The two main assumptions concern the availability of a fundamental solution and of a Green formula.

A *fundamental solution* of L in D is a distribution $G \in \mathscr{D}'(D\times D)$ satisfying

$$\begin{cases} \forall y\in D: & L(x,\partial_x)\, G(x,y) &=& \delta_y(x) \\ \forall x\in D: & L'(y,\partial_y)\, G(x,y) &=& \delta_x(y). \end{cases} \tag{2.1}$$

It is well known that fundamental solutions always exist if L has constant coefficients. At the opposite, there are elliptic operators without fundamental solutions [7]. Furthermore the following properties are well known.

Lemma 2.1 (i) $G \in \mathscr{C}^\infty\Big(\{(x,y)\in D\times D \mid x\neq y\}\Big)$.
(ii) $G(x,y) = \mathscr{O}\left(|x-y|^{2m-n}\log|x-y|\right)$ *as* $|x-y|\to 0$. *Hence*

$$\forall x\in D: \quad G(x,\cdot)\in L^1_{\mathrm{loc}}(D).$$

(iii) *Let* G_1 *be a fundamental solution for* L *in* D. *Then* $G_2\in\mathscr{D}'(D\times D)$ *is another fundamental solution for* L *in* D *if and only if*

$$G_2 = G_1 + H$$

where $H\in\mathscr{C}^\infty(D\times D)$ *satisfies*

$$\forall (x,y)\in D\times D: \quad L(x,\partial_x)\, H(x,y) = L'(y,\partial_y)\, H(x,y) = 0.$$

(iv) *Let* G *denote the integral operator with kernel* $G(x,y)$:

$$Gf(x) = \int_D G(x,y)\, f(y)\, dy.$$

Then G *as a continuous operator* $\mathscr{C}_0^\infty(D)\to\mathscr{C}^\infty(D)$ *satisfies*

$$\begin{aligned} \forall f\in\mathscr{C}_0^\infty(D):&\ LGf = f \\ \forall u\in\mathscr{C}_0^\infty(D):&\ GLu = u. \end{aligned}$$

Moreover, G *has a continuous extension on the space* $\mathscr{E}'(D)$ *of compactly supported distributions* $G:\mathscr{E}'(D)\to\mathscr{D}'(D)$ *and satisfies*

$$\forall f\in\mathscr{E}'(D):\ LGf = f \tag{2.2a}$$

$$\forall u\in\mathscr{E}'(D):\ GLu = u. \tag{2.2b}$$

Note that (**??**) means in particular that there are no non-trivial solutions of $Lu = 0$ in D with compact support.

A *Green formula* for the operator L in a bounded domain Ω such that $\overline{\Omega} \subset D$ and with $\mathscr{C}^\infty$ boundary $\partial\Omega$ consists of two Dirichlet systems [9] of order $2m$ on $\partial\Omega$:

$$(B_j)_{j=0,\dots,2m-1} \quad \text{and} \quad (Q_{2m-1-j})_{j=0,\dots,2m-1}$$

such that

$$\forall u, v \in \mathscr{C}^\infty(\overline{\Omega}) : \qquad \int_\Omega \Big(Lu\, v - u\, L'v \Big)\, dx = \sum_{j=0}^{2m-1} \int_{\partial\Omega} B_j u\, Q_j v\, ds. \tag{2.3}$$

Thus B_j and Q_{2m-1-j} are differential operators on $\partial\Omega$ with $\mathscr{C}^\infty$ coefficients whose total order is j and whose order with respect to the normal derivative is also j. It is known [9] that to any choice of a Dirichlet system $(B_j)_{j=0,\dots,2m-1}$ there exists a complementary Dirichlet system $(Q_{2m-1-j})_{j=0,\dots,2m-1}$ such that (2.3) holds.

One standard consequence of Green's formula is the representation formula in a *bounded domain*:

Theorem 2.2 *Under the above assumptions, one has*

$$\forall u, f \in \mathscr{C}^\infty(\overline{\Omega}) \;\; \textit{with} \;\; f = Lu \;\; \textit{in} \;\; \Omega, \quad \forall x \in \Omega :$$
$$u(x) = \int_\Omega G(x,y)\, f(y)\, dy + \sum_{j=0}^{2m-1} \int_{\partial\Omega} B_j u(y)\, Q_j(y, \partial_y) G(x,y)\, ds(y). \tag{2.4}$$

Proof. For the proof, one uses

$$\big\langle Lu, v \big\rangle - \big\langle u, L'v \big\rangle = 0 \tag{2.5}$$

which is true, by the definition of L', for any $u \in \mathscr{C}_0^\infty(D)$ and for any $v \in \mathscr{D}'(D)$, together with (2.3): By combining (2.5) and (2.3), we see that (2.3) remains true for any $u \in \mathscr{C}^\infty(\overline{\Omega})$ and any distribution $v \in \mathscr{D}'(D)$ whose singular support does not intersect $\partial\Omega$. For any $x \notin \partial\Omega$, such a distribution is given by

$$v(y) = G(x,y).$$

Using $L'v = \delta_x$, one obtains (2.4). ■

Remark 2.3 The above proof shows also that under the hypotheses of the theorem, one has $\forall x \in D \setminus \overline{\Omega}$:

$$\int_\Omega G(x,y)\, f(y)\, dy + \sum_{j=0}^{2m-1} \int_{\partial\Omega} B_j u(y)\, Q_j(y, \partial_y) G(x,y)\, ds(y) = 0.$$
■

3 RADIATION CONDITIONS

Let us now study the representation formula in the unbounded domain Ω^+. That is, we look for conditions on the function u such that

$$u(x) = \int_{\Omega^+} G(x,y)\, f(y)\, dy + \sum_{j=0}^{2m-1} \int_\Gamma B_j u(y)\, Q_j(y,\partial_y) G(x,y)\, ds(y) \tag{3.1}$$

holds for $x \in \Omega^+$. As above, we write $f = Lu$ in Ω^+. First of all, it is clear that (3.1) holds if $u \in \mathscr{C}^\infty(\overline{\Omega^+})$ has compact support in Ω^+.

Secondly, let for ρ sufficiently large,

$$\Omega_\rho := \Omega^+ \cap \{x \in \mathbb{R}^n \mid \ |x| < \rho\}, \qquad \Gamma_\rho := \{x \in \mathbb{R}^n \mid \ |x| = \rho\}.$$

Then (2.4) holds for the bounded domain Ω_ρ:

$$\begin{aligned} u(x) = \int_{\Omega_\rho} G(x,y)\, f(y)\, dy \ &+ \sum_{j=0}^{2m-1} \int_\Gamma B_j u(y)\, Q_j(y,\partial_y) G(x,y)\, ds(y) \\ &- \sum_{j=0}^{2m-1} \int_{\Gamma_\rho} B_j u(y)\, Q_j(y,\partial_y) G(x,y)\, ds(y). \end{aligned} \tag{3.2}$$

Here we used $\partial\Omega_\rho = \Gamma \cup \Gamma_\rho$ and we chose the signs for the boundary integrals such as to remember the importance of the orientation of the boundary: If $\{B_j, Q_j\}$ are complementary Dirichlet systems for L on a part Γ_0 of a domain Ω, then $\{-B_j, Q_j\}$ are complementary Dirichlet systems for L on Γ_0 if one considers Green's formula in a domain whose boundary contains Γ_0, but lies on the other side of Γ_0 as Ω. But, strictly speaking, this sign has no meaning, since the operators B_j and Q_j on Γ_ρ are a priori different from those on Γ, their only relation being the the validity of Green's formula in Ω_ρ:

$$\begin{aligned} \forall u,\, v \in \mathscr{C}^\infty(\overline{\Omega}_\rho):\qquad & \int_\Omega \Big(Lu\, v - u\, L'v \Big)\, dx = \\ & \sum_{j=0}^{2m-1} \left(\int_\Gamma B_j u\, Q_j v\, ds - \int_{\Gamma_\rho} B_j u\, Q_j v\, ds \right). \end{aligned} \tag{3.3}$$

The Green formula (2.3) shows in particular that the quantity

$$\mathscr{J}(u,v;\partial\Omega) := \sum_{j=0}^{2m-1} \int_{\partial\Omega} B_j u\, Q_j v\, ds \tag{3.4}$$

does not depend on the choice of the complementary Dirichlet systems $\{B_j, Q_j\}$ on $\partial\Omega$, if u and v are $\mathscr{C}^\infty$ in some one-sided neighborhood of $\partial\Omega$ in $\overline{\Omega}$. It depends, however, on the orientation of $\partial\Omega$.

Likewise, (3.3) shows that $\mathscr{J}(u,v;\Gamma_\rho)$ does not depend on the choice of the complementary Dirichlet systems $\{B_j, Q_j\}$ on Γ_ρ provided the Green formula (3.3) is satisfied.

The following definition is therefore independent on the choice of $\{B_j, Q_j\}$ on Γ_ρ for ρ sufficiently large.

Definition 3.1 Let $u \in \mathscr{C}^\infty(\Omega^+)$.
We say that u satisfies the *radiation condition* $(\mathscr{R}_G)$ if

$$(\mathscr{R}_G) \qquad \forall x \in \Omega^+: \quad \lim_{\rho\to\infty} \sum_{j=0}^{2m-1} \int_{\Gamma_\rho} B_j u(y)\, Q_j(y,\partial_y) G(x,y)\, ds(y) = 0. \qquad \blacksquare$$

The reason of this definition is immediately clear if one compares (3.1) and (3.2).

Proposition 3.2 *Let* $u \in \mathscr{C}^\infty(\overline{\Omega^+})$ *with* $Lu = f \in \mathscr{C}_0^\infty(\overline{\Omega^+})$. *Then* u *is represented by formula* (3.1) *if and only if* u *satisfies the radiation condition* $(\mathscr{R}_G)$.

A simple consequence of (3.2) is the following: If $Lu(x) = 0$ for $|x| > \rho_0$, then the function

$$p_u^G(x) := \mathscr{J}(u, G(x,\cdot);\Gamma_\rho) \quad \text{for} \quad \rho > \rho_0 \text{ and } \rho > |x| \tag{3.5}$$

does not depend on ρ. It is therefore well defined for any $x \in D$:

$$p_u^G \in \mathscr{C}^\infty(D) \quad \text{and} \quad \forall x \in D, \quad Lp_u^G(x) = 0.$$

So condition $(\mathscr{R}_G)$ is equivalent to the vanishing of p_u^G in Ω^+.

4 FUNDAMENTAL SOLUTIONS ON THE WHOLE SPACE

Here we consider the case

$$K = \emptyset, \quad \text{that is} \quad D = \mathbb{R}^n.$$

In this case, the class of functions satisfying $(\mathscr{R}_G)$ allows several useful equivalent characterizations.

The following theorem shows that the class of functions u such that $Lu = 0$ outside a bounded set and satisfying the radiation condition $(\mathscr{R}_G)$ is precisely the class of functions given either as volume potentials or as boundary potentials using the fundamental solution G. We introduce the following multilayer potentials for densities defined on a surface Γ: For $\varphi \in L^1(\Gamma)$:

$$\forall x \in \mathbb{R}^n \setminus \Gamma, \quad \mathscr{Q}_j\varphi(x) := \int_\Gamma \varphi(y)\, Q_j(y, \partial_y)G(x, y)\; ds(y).$$

The natural extension to $\varphi \in \mathscr{D}'(\Gamma)$ can be written as

$$\mathscr{Q}_j\varphi(x) := \big\langle \varphi, Q_jG(x, \cdot)\big\rangle.$$

Theorem 4.1 *Let G a fundamental solution for L on $D = \mathbb{R}^n$. For a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$, the following assertions are equivalent:*

(i) *There is a $\rho_0 > 0$ such that $Lu = 0$ for $|x| > \rho_0$ and u satisfies condition $(\mathscr{R}_G)$.*

(ii) *There is a $\rho_1 > 0$ and $f \in \mathscr{E}'(\mathbb{R}^n)$ such that $u = Gf$ in $|x| > \rho_1$.*

(iii) *There is $f \in \mathscr{E}'(\mathbb{R}^n)$ such that $u = Gf$ in $\mathbb{R}^n$.*

(iv) *There is a smooth closed surface Γ with exterior Ω^+ such that $\forall x \in \Omega^+$:*

$$u(x) = \sum_{j=0}^{2m-1} \int_\Gamma B_ju(y)\, Q_j(y, \partial_y)G(x, y)\; ds(y). \tag{4.1}$$

(v) *There is a bounded set such that for any smooth closed surface Γ containing it in its interior,* (4.1) *holds.*

(vi) *There is a smooth closed surface Γ with exterior Ω^+ and $2m$ functions $\varphi_j \in \mathscr{C}^\infty(\Gamma)$ such that:*

$$\forall x \in \Omega^+: \quad u(x) = \sum_{j=0}^{2m-1} \int_\Gamma \varphi_j(y)\, Q_j(y, \partial_y)G(x, y)\; ds(y). \tag{4.2}$$

(vii) *There is a smooth closed surface Γ with exterior Ω^+ and $2m$ distributions $\varphi_j \in \mathscr{D}'(\Gamma)$ such that:*

$$u = \sum_{j=0}^{2m-1} \mathscr{Q}_j\varphi_j \quad \textit{in} \quad \Omega^+. \tag{4.3}$$

Proof. We show the chain of implications *(ii)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(i)* $\Rightarrow$ *(v)* $\Rightarrow$ *(iv)* $\Rightarrow$ *(vi)* $\Rightarrow$ *(vii)* $\Rightarrow$ *(ii)*. Some of these, namely *(v)* $\Rightarrow$ *(iv)* $\Rightarrow$ *(vi)* $\Rightarrow$ *(vii)*, are trivial. The equivalence of *(i)* and *(v)* was seen above (Proposition 3.2).

(ii) $\Rightarrow$ *(iii)*. Suppose u and f as in *(ii)*. Let $u_0 = u - Gf$. Then u_0 has compact support and therefore $u_0 = GLu_0$. Let $f_1 = f + Lu_0 \in \mathscr{E}'(\mathbb{R}^n)$. It follows $u = u_0 + Gf = Gf_1$.

(iii) $\Rightarrow$ *(i)*. If $u = Gf$ with $f \in \mathscr{E}'(\mathbb{R}^n)$, we have $Lu = 0$ outside $\operatorname{supp} f$, thus for $|x| > \rho_0$ for some ρ_0. Let us first assume that $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$. For $\rho > \rho_0$,

the representation formula (2.4) holds in the ball B_ρ with radius ρ. This can be written as

$$u(x) = Gf(x) - \mathscr{J}(u, G(x,\cdot); \Gamma_\rho).$$

Since $u = Gf$, we find $\mathscr{J}(u, G(x,\cdot); \Gamma_\rho) = 0$.
Let now $f \in \mathscr{E}'(\mathbb{R}^n)$ with $\operatorname{supp} f \subset B_{\rho_0}$. We can approximate f in $\mathscr{E}'$ by $(f_k)_{k \in \mathbb{N}}$ with $f_k \in \mathscr{C}_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} f_k \subset B_{\rho_0}$. For $\rho > \rho_0$, Gf_k converges to $u = Gf$ uniformly with all its derivatives on Γ_ρ. Hence, once again we find

$$\mathscr{J}(u, G(x,\cdot); \Gamma_\rho) = 0.$$

Thus u satisfies $(\mathscr{R}_G)$.

(vii) $\Rightarrow$ *(ii)*. For $\varphi_j \in \mathscr{D}'(\Gamma)$, let $f_j \in \mathscr{E}'(\mathbb{R}^n)$ be the distribution with support on Γ defined by

$$\forall \chi \in \mathscr{C}^\infty(\mathbb{R}^n): \quad \langle f_j, \chi \rangle = \langle \varphi_j, Q_j \chi \big|_\Gamma \rangle.$$

Then $\mathscr{Q}_j \varphi_j = G f_j$, and by assumption we have

$$u = G\Big(\sum_j f_j\Big) \quad \text{in} \quad \Omega^+.$$

Thus *(ii)* is satisfied. ■

Remark 4.2 The *Calderón operator* $\mathscr{C}^+$ for Ω^+ corresponding to the Dirichlet systems $\{B_j\}$ and $\{Q_j\}$ is classically defined by the mapping

$$\mathscr{C}^\infty(\Gamma)^{2m} \ni \Big(\varphi_0, \dots, \varphi_{2m-1}\Big) \longmapsto \Big(B_0 u, \dots, B_{2m-1} u\Big)\Big|_\Gamma \in \mathscr{C}^\infty(\Gamma)^{2m}$$
$$\text{where} \quad u = \sum_{k=0}^{2m-1} \mathscr{Q}_k \varphi_k \quad \text{in} \quad \Omega^+.$$

It is well known [1], [4], that this is a matrix of pseudodifferential operators of orders $(j-k)_{j,k=0,\dots,2m-1}$. From Theorem 4.1 — especially the implication *(vi)* $\Rightarrow$ *(v)*, we see that $(\mathscr{C}^+)^2 = \mathscr{C}^+$ and that $\mathscr{C}^+$ is a *projector* onto the space of traces $(B_j u)_j$ of solutions u of $Lu = 0$ in Ω^+ satisfying $(\mathscr{R}_G)$. ■

Remark 4.3 For fixed y_0, if we consider the right hand side $f = \delta_{y_0}$, one has $G(\cdot, y_0) = Gf$. Therefore the fundamental solution $G(\cdot, y_0)$ satisfies the radiation condition $(\mathscr{R}_G)$. It can therefore be represented by (4.1):

$$G(x_0, y_0) = \mathscr{J}(G(\cdot, y_0), G(x_0, \cdot); \Gamma)$$

if y_0 is in the interior Ω^- and x_0 in the exterior Ω^+ of the closed surface Γ. The radiation condition $(\mathscr{R}_G)$ gives directly

$$\mathscr{J}(G(\cdot, y_0), G(x_0, \cdot); \Gamma) = 0 \tag{4.4}$$

if x_0 and y_0 are both in the interior domain Ω^-. Of course, the representation formula (2.4) in the interior domain Ω^- yields (4.4) again if x_0 and y_0 are both in the exterior domain Ω^+. ∎

Here are a few important consequences of Theorem 4.1.

Corollary 4.4 *For a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ with $Lu \in \mathscr{E}'(\mathbb{R}^n)$ there holds*

$$u \text{ satisfies } (\mathscr{R}_G) \quad \Longleftrightarrow \quad u = GLu.$$

Corollary 4.5 *The problem*

$$Lu = 0 \text{ in all } \mathbb{R}^n \quad \text{and} \quad u \text{ satisfies } (\mathscr{R}_G)$$

has only the trivial solution $u = 0$.

Corollary 4.6 *Let G and $\tilde{G}$ be two fundamental solutions for L in $\mathbb{R}^n$. If $(\mathscr{R}_G)$ implies $(\mathscr{R}_{\tilde{G}})$, then $G = \tilde{G}$.*

PROOF. The difference $u = G(\cdot, y_0) - \tilde{G}(\cdot, y_0)$ satisfies $Lu = 0$ in $\mathbb{R}^n$ and the condition $(\mathscr{R}_{\tilde{G}})$. Hence $u = 0$. ∎

5 GREEN FUNCTIONS

For $D \neq \mathbb{R}^n$, the results of the previous section are not true, in general. In particular, functions having a representation as a multilayer potential cannot always be represented by their own Cauchy data, and the corresponding Calderón operator is not a projection operator, in general. But these results do remain true if G is a *Green function* for L in D. To define this notion in our general setting, we introduce the following assumptions:

α) The compact K is the closure of an open set with smooth boundary $\partial K = \partial D$.

β) There are two closed subspaces V and V' of the space of functions $\mathscr{C}^\infty$ in a neighborhood of ∂D in $\overline{D}$ which are orthogonal with respect to the bilinear form $\mathscr{J}$ (3.4):

$$\forall u \in V, \quad \forall v \in V' : \qquad \mathscr{J}(u, v; \partial D) = 0. \tag{5.1}$$

The fundamental solution G satisfies the boundary conditions corresponding to V and V' in both variables:

$$\forall y \in D : \ G(\cdot, y) \in V; \qquad \forall x \in D : \ G(x, \cdot) \in V'. \tag{5.2}$$

$\gamma)$ There holds the following trace lifting property: For any $u \in \mathscr{C}^\infty(\overline{D})$, there exists $u_0 \in \mathscr{C}_0^\infty(\overline{D})$ such that $u - u_0 \in V$.

Typical examples of such spaces V and V' are defined by m boundary conditions: The set $\{0, \dots, 2m-1\}$ is partitioned into 2 sets:

$$\{0, \dots, 2m-1\} = M_1 \cup M_2, \quad M_1 \cap M_2 = \emptyset$$

and the spaces V and V' are defined as:

$$V = \{u \mid \ \forall j \in M_1 : \ B_j u = 0 \ \text{ on } \ \partial D\}, \quad V' = \{u \mid \forall j \in M_2 : \ Q_j u = 0 \ \text{ on } \ \partial D\}.$$

For example $V = V'$ can correspond to Dirichlet conditions, and if L is selfadjoint, Neumann conditions can also correspond to some $V = V'$.

If Γ is a smooth closed surface containing K in its interior Ω^-, the conditions (5.2) imply that in the representation formula (2.4) for $u \in V$ in the domain $\Omega = \Omega^- \cap D$ whose boundary is $\partial\Omega = \partial D \cup \Gamma$, no boundary integral over ∂D appears. In particular, if u has a compact support, one can choose a surface Γ surrounding the support of u and one obtains the following lemma

Lemma 5.1 *Let* $u \in \mathscr{C}_0^\infty(\overline{D})$, $u \in V$. *Then*

$$u = GLu.$$

Let us note that, because of (5.2), $G(x, \cdot) \in \mathscr{C}^\infty(\overline{D} \setminus \{x\})$; therefore the operator $f \to Gf$ has a natural extension to $f \in \mathscr{E}'(\mathbb{R}^n)$ with $\operatorname{supp} f \subset \overline{D}$.

The statement corresponding to Theorem 4.1 is

Theorem 5.2 *Let* $\alpha)$ *and* $\beta)$ *above be satisfied. For a distribution* $u \in \mathscr{D}'(D)$ *satisfying* $u \in V$, *the following assertions are equivalent:*

(i) Lu *has compact support in* $\overline{D}$ *and* u *satisfies condition* $(\mathscr{R}_G)$.

(ii) *There is a* $\rho_1 > 0$ *and* $f \in \mathscr{E}'(D)$ *such that* $u = Gf$ *in* $|x| > \rho_1$.

(iii) *There is* $f \in \mathscr{E}'(\mathbb{R}^n)$ *with* $\operatorname{sing\,supp} f \subset D$ *such that* $u = Gf$ *in* D.

(iv) *There is a smooth closed surface* Γ *whose exterior* Ω^+ *is a subdomain of* D *such that:*

$$\forall x \in \Omega^+ : \quad u(x) = \mathscr{J}(u, G(x, \cdot); \Gamma). \tag{5.3}$$

(v) *There is a bounded set such that for any smooth closed surface* Γ *containing it in its interior,* (5.3) *holds.*

(vi) *There is a smooth closed surface* Γ *with exterior* Ω^+ *and* $2m$ *functions* $\varphi_j \in \mathscr{C}^\infty(\Gamma)$ *such that:*

$$u = \sum_{j=0}^{2m-1} \mathscr{Q}_j \varphi_j \quad \textit{in} \quad \Omega^+. \tag{5.4}$$

(vii) *There is a smooth closed surface* Γ *with exterior* Ω^+ *and* $2m$ *distributions* $\varphi_j \in \mathscr{D}'(\Gamma)$ *such that* (5.4) *holds.*

Proof. In order to see that the proof of Theorem 4.1 is working here, we have to note the following points:

Because of (5.2), any distribution represented either as Gf with $f \in \mathscr{E}'(D)$ or as a multilayer surface potential as in *(vi)* or *(vii)*, satisfies also $u \in V$.

In the implication *(ii)* $\Rightarrow$ *(iii)*, the distribution u_0 will now have compact support in $\overline{D}$ — but not in D in general. But since it satisfies $u_0 \in V$, the relation $u_0 = GLu_0$ still holds (Lemma 5.1).

In the implication *(iii)* $\Rightarrow$ *(i)*, one has for $Lu = f \in \mathscr{C}^\infty(\overline{D} \cap \overline{B}_\rho)$ the representation formula in $D \cap B_\rho$:

$$u(x) = Gf(x) - \mathscr{J}(u, G(x,\cdot); \Gamma_\rho).$$

Again, no boundary term from ∂D appears.

All other arguments remain the same as above. ∎

Corollary 5.3 *For any fixed* $y_0 \in D$, $G(\cdot, y_0)$ *satisfies* $(\mathscr{R}_G)$.

Corollary 5.4 *The problem* "$Lu = 0$ in D, u satisfies $u \in V$ and $(\mathscr{R}_G)$" *has only the trivial solution* $u = 0$.

Corollary 5.5 *Let* G *and* $\tilde{G}$ *be two fundamental solutions for* L *in* D. *If both* G *and* $\tilde{G}$ *satisfy* (5.2) *with the same spaces* V *and* V', *and if* $(\mathscr{R}_G)$ *implies* $(\mathscr{R}_{\tilde{G}})$, *then* $G = \tilde{G}$.

So far we have considered representation formulas for u in the exterior Ω^+ of a closed surface Γ which is disjoint from the boundary ∂D where the boundary conditions (spaces V and V') are considered. It is obvious that for the validity of the radiation condition $(\mathscr{R}_G)$ for u, the condition $u \in V$ has no importance. Thus we have

Theorem 5.6 *Let* $u \in \mathscr{C}^\infty(\overline{D})$ *satisfy* $Lu = f \in \mathscr{C}_0^\infty(\overline{D})$ *and condition* $(\mathscr{R}_G)$. *Then*

$$\forall x \in D: \quad u(x) = Gf(x) + \mathscr{J}(u, G(x,\cdot); \partial D). \tag{5.5}$$

PROOF. Choose ρ sufficiently large. Then the representation formula between ∂D and Γ_ρ is

$$u(x) = Gf(x) + \mathscr{J}(u, G(x,\cdot); \partial D) - \mathscr{J}(u, G(x,\cdot); \Gamma_\rho).$$

But $\mathscr{J}(u, G(x,\cdot); \Gamma_\rho) = 0$ for $\rho > |x|$ if u satisfies $(\mathscr{R}_G)$. ∎

If we make the supplementary assumption $\gamma)$ about a lifting of traces, for the equivalence of the assertions of Theorem 5.2 with the exception of *(iii)*, the condition $u \in V$ can be dropped.

Corollary 5.7 *Let u_0 and $f \in \mathscr{C}_0^\infty(\overline{D})$ be given. Then there exists a unique solution of the exterior boundary value problem*

$$Lu = f \text{ in } D; \quad u - u_0 \in V; \quad u \text{ satisfies } (\mathscr{R}_G). \tag{5.6}$$

Moreover $u \in \mathscr{C}^\infty(\overline{D})$ and there holds the Poisson-type representation

$$u(x) = Gf(x) + \mathscr{J}(u_0, G(x,\cdot); \partial D).$$

PROOF. The uniqueness follows from Corollary 5.4. Existence follows from the explicit solution formula for u:

$$u = u_0 + G(f - Lu_0).$$

The Poisson-type representation is then a consequence of Theorem 5.6. ∎

6 ELLIPTICITY OF THE EXTERIOR PROBLEM

In this section, we show that the radiation condition $(\mathscr{R}_G)$ is an elliptic boundary condition in a certain sense. The classical notion of elliptic boundary conditions (see [9], for example) means in a bounded domain that the resulting operators are Fredholm operators between suitable Sobolev spaces.

In an unbounded domain, in general, conditions at infinity will be required so that the resulting operator is Fredholm, and we can call such conditions *elliptic*. Consider, for instance, the exterior Dirichlet problem: It is well known and easy to see that the operator

$$u \mapsto ((-\Delta + 1)u, u\big|_\Gamma) \;:\; H^2(\Omega^+) \to L^2(\Omega^+) \times H^{3/2}(\Gamma)$$

is an isomorphism, whereas the operator

$$u \mapsto (-\Delta u, u\big|_\Gamma) \;:\; H^2(\Omega^+) \to L^2(\Omega^+) \times H^{3/2}(\Gamma)$$

is not a Fredholm operator. In the latter case, one has to consider conditons at infinity that are different from those implied by the standard Sobolev spaces $H^s(\Omega^+)$. For the example of the Laplace operator, this can be done by using weighted Sobolev spaces, see [11, 12]. This approach is, however, restricted to homogeneous strongly elliptic operators, and is not applicable to the Helmholtz equation, for example. On the other hand, our radiation condition $(\mathscr{R}_G)$ requires that the homogeneous differential equation $Lu = 0$ is satisfied in a neighborhood of infinity. Thus we shall consider only the homogeneous differential equation. The corresponding operator which we want to be Fredholm is then given by traces of u on Γ.

We will show now that the radiation condition $(\mathscr{R}_G)$ is an elliptic boundary condition in this sense. To this purpose, we assume that we are given on our $\mathscr{C}^\infty$ manifold Γ, whose exterior is the domain Ω^+, a set of boundary operators $R = (R_1, \dots, R_m)$ that defines elliptic boundary conditions on Γ. That is, R satisfies the usual [9] normality and covering conditions of Shapiro-Lopatinski.

We show that the exterior boundary value problem

$$Lu = 0 \text{ in } \Omega^+; \quad Ru = g; \quad u \text{ satisfies } (\mathscr{R}_G). \tag{6.1}$$

is elliptic. Let

$$\mathcal{L}_+ = \{u \in \mathscr{C}^\infty(\overline{\Omega^+}) \mid \quad Lu = 0 \text{ in } \Omega^+; \quad u \text{ satisfies } (\mathscr{R}_G)\}$$

and $\mathcal{H}(\Gamma) = \mathscr{C}^\infty(\Gamma)^m$.

We will first give a formulation for $\mathscr{C}^\infty$ regularity and afterwards, as an easy consequence, the corresponding result for finite regularity measured by a Sobolev index $s \in \mathbb{R}$.

Theorem 6.1 *Let either $D = \mathbb{R}^n$ or G be a Green function for D in the sense of §5, where $\Omega^+ \subset D$. Then the mapping*

$$\gamma R : \mathcal{L}_+ \longrightarrow \mathcal{H}(\Gamma), \quad \gamma R u := Ru\Big|_\Gamma$$

has finite-dimensional kernel and cokernel.

Proof. We compare the problem on the unbounded domain Ω^+ with one on a bounded domain $\Omega^\# = \Omega^+ \cap \Omega_0^\#$, where $\Omega_0^\#$ is some large ball with boundary $\Gamma^\#$. On $\Gamma^\#$, we choose Dirichlet conditions $R^\# = (1, \partial_n, \dots, \partial_n^{m-1})$ and we denote by $\gamma^\#$ the restriction operator on $\Gamma^\#$.

We also introduce *non-local* boundary conditions on $\Gamma^\#$:

$$\gamma^\# R_\Gamma^\# u = \gamma^\# R^\# \Big(\sum_{j=0}^{2m-1} \int_\Gamma B_j u(y)\, Q_j(y, \partial_y) G(\cdot, y)\, ds(y) \Big).$$

Since $\Gamma \cap \Gamma^\# = \emptyset$, the traces $\gamma^\# R^\#_\Gamma$ are defined by integral operators with $\mathscr{C}^\infty(\Gamma^\# \times \Gamma)$ kernels. Let

$$\mathcal{L}_\# = \{u \in \mathscr{C}^\infty(\overline{\Omega^\#}) \mid \quad Lu = 0 \text{ in } \Omega^\#; \quad \gamma^\#(R^\# - R^\#_\Gamma)u = 0\}.$$

Since the boundary conditions in $\mathcal{L}_\#$ are given by a regularizing perturbation of the elliptic conditions $\gamma^\# R^\#$, the standard elliptic theory shows that the mapping

$$\gamma R : \mathcal{L}_\# \longrightarrow \mathcal{H}(\Gamma)$$

has a finite-dimensional kernel $\mathcal{N}^\#_R$ and a closed range $\gamma R \mathcal{L}_\#$ with finite codimension in $\mathcal{H}(\Gamma)$.

We will complete the proof by showing that the kernel $\mathcal{N}^+_R$ of the mapping

$$\gamma R : \mathcal{L}_+ \longrightarrow \mathcal{H}(\Gamma)$$

and its range $\gamma R \mathcal{L}_+$ satisfy

$$\mathcal{N}^+_R \subset \mathcal{N}^\#_R \tag{6.2}$$

and

$$\gamma R \mathcal{L}_+ + \mathcal{K}_\# \supset \gamma R \mathcal{L}_\#, \tag{6.3}$$

where $\mathcal{K}_\# \subset \mathcal{H}(\Gamma)$ is some finite-dimensional space.

Let u be in $\mathcal{L}_+$. Then u satisfies $Lu = 0$ in Ω^+ and the condition $(\mathscr{R}_G)$. It can therefore be represented by its Cauchy data on Γ, and therefore

$$\gamma^\# R^\# u = \gamma^\# R^\#_\Gamma u,$$

hence $u\Big|_{\Omega^\#} \in \mathcal{L}_\#$. Thus $\mathcal{N}^+_R \subset \mathcal{N}^\#_R$.

For (6.3), we give the proof in the case of $D = \mathbb{R}^n$, and we leave the case of a Green function of $D \neq \mathbb{R}^n$ to the reader (one uses Corollary 5.7). Let $g \in \gamma R \mathcal{L}_\#$: $g = \gamma R u^\#$ with $u^\# \in \mathcal{L}_\#$. We define u on Ω^+ by

$$u(x) = \mathscr{J}(u^\#, G(x,\cdot); \Gamma)$$

and $u_0 = u^\# - u$. Since u satisfies $Lu = 0$ and $(\mathscr{R}_G)$, we have also

$$u(x) = \mathscr{J}(u, G(x,\cdot); \Gamma) \quad \text{in} \quad \Omega^+$$

and therefore

$$\mathscr{J}(u_0, G(x,\cdot); \Gamma) = 0 \quad \text{in} \quad \Omega^+.$$

The representation formula for u_0 in $\Omega^\#$ is thus

$$\begin{aligned} u_0(x) &= \mathscr{J}(u_0, G(x,\cdot); \Gamma) - \mathscr{J}(u_0, G(x,\cdot); \Gamma^\#) \\ &= -\mathscr{J}(u_0, G(x,\cdot); \Gamma^\#). \end{aligned}$$

This formula shows that u_0 has a continuation as a solution of $Lu = 0$ in all of $\Omega_0^\#$. In addition, $\gamma^\# R^\# u_0 = 0$, because $\gamma^\# R_\Gamma^\# u_0 = \gamma^\# R^\# \mathscr{J}(u_0, G; \Gamma) = 0$ and $u_0 = u^\# - u\big|_{\Omega^\#} \in \mathcal{L}_\#$. Altogether, we have $g = \gamma R u + \gamma R u_0$ where $u \in \mathcal{L}_+$, and u_0 belongs to the finite-dimensional space of solutions of the homogeneous Dirichlet problem in $\Omega_0^\#$:

$$Lu_0 = 0 \ \text{ in } \ \Omega_0^\#; \quad \gamma^\# R^\# u_0 = 0.$$

Thus (6.3) is shown. ■

For the analogous result in Sobolev spaces, we introduce the space:

$$\mathcal{L}_+^s = \{u \in H^s_{\mathrm{loc}}(\overline{\Omega^+}) \mid \quad Lu = 0 \ \text{ in } \ \Omega^+; \quad u \ \text{ satisfies } \ (\mathscr{R}_G)\}.$$

Let $\mu_1, \ldots, \mu_m \in \{0, \ldots, 2m-1\}$ be the orders of $R_1, \ldots, R_m$. The trace space on Γ is then

$$\mathcal{H}^s(\Gamma) = \prod_{j=1}^{m} H^{s-\mu_j-1/2}(\Gamma).$$

It is well known [1] that

$$\gamma R : \mathcal{L}_+^s \longrightarrow \mathcal{H}^s(\Gamma), \quad u \longmapsto Ru\big|_\Gamma$$

is well defined and continuous for any $s \in \mathbb{R}$.

Theorem 6.2 *Let either $D = \mathbb{R}^n$ or G be a Green function for D in the sense of §5, where $\Omega^+ \subset D$. Then the mapping*

$$\gamma R : \mathcal{L}_+^s \longrightarrow \mathcal{H}^s(\Gamma)$$

has finite-dimensional kernel and cokernel.

The proof is the same as for Theorem 6.1, and it shows in fact that this kernel and cokernel are independent of s and the same as in the $\mathscr{C}^\infty$ case.

7 EXAMPLES

7.a Operators with invertible symbol

We assume here that $D = \mathbb{R}^n$ and that L has constant coefficients. If the symbol $\sigma(L)$ does not vanish on $\mathbb{R}^n$, then there is a unique tempered fundamental solution given by Fourier transformation:

$$G(x,y) = \mathscr{F}^{-1}\Big(\frac{1}{\sigma(L)(\xi)}\Big)(x-y).$$

The corresponding radiation condition can be expressed simply by requiring that

$$u \in \mathscr{S}'(\mathbb{R}^n). \tag{7.1}$$

In fact, if $u \in \mathscr{S}'(\mathbb{R}^n)$ with $Lu = f \in \mathscr{E}'(\mathbb{R}^n)$, one has $\sigma(L)(\xi)\,\mathscr{F}u(\xi) = \mathscr{F}f(\xi)$, hence $u = Gf$. On the other hand any distribution $u = Gf$ with $f \in \mathscr{E}'(\mathbb{R}^n)$ decays exponentially at infinity. Thus an ***equivalent*** formulation of $(\mathscr{R}_G)$ is in this case:

$$\exists\,\varepsilon > 0 \;\text{ such that }\; u(x) = \mathscr{O}\left(e^{-\varepsilon|x|}\right) \;\text{ as }\; |x| \to \infty. \tag{7.2}$$

Of course there are many other equivalent formulations between (7.1) and (7.2).

As the well-known example of the operator

$$L = -\Delta + 1 \quad \text{in} \quad \mathbb{R}^3$$

shows, besides the tempered fundamental solution

$$G_0(x,y) = \frac{e^{-|x-y|}}{4\pi|x-y|}\,,$$

there exist exponentially growing fundamental solutions

$$G_t(x,y) = \frac{1}{4\pi|x-y|}\Big(t\,e^{+|x-y|} + (1-t)\,e^{-|x-y|}\Big)$$

that give rise to different conditions $(\mathscr{R}_{G_t})$ for each $t \in \mathbb{C}$. These seem to have no simple characterization in terms of asymptotics at infinity.

7.b Homogeneous operators

We assume again $D = \mathbb{R}^n$ and L has constant coefficients, so we can choose G as a convolution kernel in $\mathscr{S}'(\mathbb{R}^n)$. For any u with $Lu = 0$ in Ω^+ there holds the representation formula, cf (3.5):

$$u(x) = u_0(x) - p_u^G(x) \quad \text{where} \quad \begin{cases} u_0(x) = \mathscr{J}(u, G(x,\cdot);\Gamma) \;\text{ satisfies }\; (\mathscr{R}_G) \\ p_u^G \;\text{ satisfies }\; Lp_u^G = 0 \;\text{ on }\; \mathbb{R}^n. \end{cases}$$

If L is homogeneous and $u \in \mathscr{S}'(\mathbb{R}^n)$, this implies that p_u^G is a polynomial. Its vanishing is equivalent to $(\mathscr{R}_G)$, and this is therefore a condition on the expansion of u at infinity (whose terms behave as products of $\log|x|$ and powers of $|x|$).

7.c The Laplace operator

(i) THE STANDARD FUNDAMENTAL SOLUTION IN $\mathbb{R}^2$. For $D = \mathbb{R}^2$ and $L = \Delta$:

$$G_0(x, y) = -\frac{1}{2\pi} \log |x - y|.$$

The corresponding condition $(\mathscr{R}_{G_0})$ is

$$\exists\, c \in \mathbb{C} : \quad u(x) = c \,\log |x| + \mathscr{O}\,(1) \quad \text{as} \quad |x| \to \infty.$$

In connection with integral equations methods, this condition has been used for a long time instead of the more classical condition of boundedness at infinity [8].

Corollary 4.6 shows that there exists no fundamental solution $\tilde{G}$ on all of $\mathbb{R}^2$ such that its radiation condition $(\mathscr{R}_{\tilde{G}})$ characterizes the class of harmonic functions vanishing at infinity.

(ii) FUNDAMENTAL SOLUTIONS ON $\mathbb{R}^2 \setminus \{0\}$. We examine three families of fundamental solutions on $\mathbb{R}^2 \setminus \{0\}$. Of course they cannot be considered as Green functions. Thus we can expect that the self-representation will not always occur.

Example 1. Let

$$G_t(x, y) = G_0(x, y) - t\, G_0(x, 0) = G_0(x, y) + \frac{t}{2\pi} \log |x|.$$

For all $t \in \mathbb{C}$, this is a fundamental solution on $\mathbb{R}^2 \setminus \{0\}$. By using Remark 4.3 for G_0, it is easy to verify that for $|x_0|$, $|y_0| < \rho$ one has

$$\mathscr{J}\,(G_t(\cdot, y_0), G_t(x_0, \cdot); \Gamma_\rho) = -\frac{t^2 - t}{2\pi} \log |x_0|.$$

This means that
- for $t \neq 0$, the condition $(\mathscr{R}_{G_t})$ is equivalent to

$$u(x) = \mathscr{O}\,(1) \quad \text{as} \quad |x| \to \infty;$$

- the fundamental solution G_t is representable in Ω^+ by its own Cauchy data — and therefore the conclusions of Theorem 4.1 are true with $K = \{0\}$ — if and only if $t = 0$ or $t = 1$.

Thus

$$G_1(x, y) = -\frac{1}{2\pi} \log \frac{|x - y|}{|x|}$$

is suitable for the representation of harmonic functions vanishing at infinity if $0 \notin \Omega^+$.

Example 2. Let $\tilde{G}_t(x,y) = G_t(y,x)$. Then one has

$$\mathscr{J}(\tilde{G}_t(\cdot, y_0), \tilde{G}_t(x_0, \cdot); \Gamma_\rho) = -\frac{t-t^2}{2\pi} \log |y_0|.$$

Thus, once again, the conclusions of Theorem 4.1 are valid in $D = \mathbb{R}^2 \setminus \{0\}$ if and only if $t = 0$ or $t = 1$.

From the relation

$$p_u^{G_t}(x) = p_u^{G_0}(x) - t\, p_u^{G_0}(0)$$

one obtains that $(\mathscr{R}_{G_0})$ implies $(\mathscr{R}_{\tilde{G}_t})$ for any t. In fact, $(\mathscr{R}_{G_0})$ is equivalent to $(\mathscr{R}_{\tilde{G}_t})$ for $t \neq 1$. For $t = 1$, $(\mathscr{R}_{\tilde{G}_1})$ is equivalent to

$$\exists\, c, d \in \mathbb{C}: \quad u(x) = c \log|x| + d + \mathscr{o}(1) \quad \text{as} \quad |x| \to \infty.$$

Example 3. By symmetrisation of G_1, one obtains a fundamental solution

$$G(x,y) = -\frac{1}{2\pi} \log \frac{|x-y|}{|x|\,|y|}$$

for $-\Delta$ on $\mathbb{R}^2 \setminus \{0\}$ whose condition $(\mathscr{R}_G)$ is equivalent to the boundedness of the harmonic function at infinity.

(*iii*) Green functions for the exterior of a circle. A Green function for the Dirichlet problem in the exterior D_ρ of a circle of radius ρ is obtained by the classical reflexion method. We set

$$x^* = \frac{\rho^2}{|x|^2}\, x. \tag{7.3}$$

Then a Green function is given by

$$G_{\rho,0}(x,y) = -\frac{1}{2\pi} \log \frac{\rho}{|x|} \frac{|x-y|}{|x^*-y|}.$$

$G_{\rho,0}$ is symmetric as can be seen by the complex writing

$$G_{\rho,0}(x,y) = -\frac{1}{2\pi} \log \frac{\rho|x-y|}{|\rho^2 - \bar{x}y|}.$$

The corresponding radiation condition is the boundedness at infinity.

For any $t \in \mathbb{C}$, the following is also a Green function for D_ρ:

$$G_{\rho,t}(x,y) = G_{\rho,0}(x,y) + t\,\log\frac{|x|}{\rho}\,\log\frac{|y|}{\rho}.$$

For every t, the results of §5 hold. In particular, conditions $(\mathscr{R}_{G_{\rho,t}})$ are different for different t, and for each t the exterior Dirichlet problem

$$\Delta u = 0 \ \text{ in } \ D_\rho, \quad u = g \ \text{ on } \ \Gamma_\rho$$

has a unique solution satisfying $(\mathscr{R}_{G_{\rho,t}})$.

(iv) The standard fundamental solution in $\mathbb{R}^n$. For $L = -\Delta$ and $n \geq 3$:

$$G_0(x,y) = c_n\,|x-y|^{2-n}.$$

It corresponds to harmonic functions vanishing at infinity. By Corollary 4.6, there does not exists any fundamental solution on all of $\mathbb{R}^n$ corresponding to the class of harmonic functions bounded at infinity. However, the latter class corresponds to $(\mathscr{R}_{G_1})$ for

$$G_1(x,y) = c_n\Big(|x-y|^{2-n} - |y|^{2-n}\Big) \ \text{ on } \ \mathbb{R}^n \setminus \{0\}.$$

(v) Green functions for the exterior of a sphere. In $\mathbb{R}^3$, the following is a Green function for the Dirichlet problem in the exterior of a sphere of radius ρ:

$$G_{\rho,0}(x,y) = \frac{1}{4\pi}\left(\frac{1}{|x-y|} - \frac{\rho}{|x|}\,\frac{1}{|x^*-y|}\right)$$

where x^* is defined as in (7.3). Condition $(\mathscr{R}_{G_{\rho,0}})$ is equivalent to the vanishing at infinity. For every t

$$G_{\rho,t}(x,y) = G_{\rho,0}(x,y) + t\left(1 - \frac{\rho}{|x|}\right)\left(1 - \frac{\rho}{|y|}\right)$$

is also a Green function.

7.d Helmholtz equation

For the operator $L = \Delta + k^2$, $k > 0$, in $\mathbb{R}^n$, the classical outgoing Sommerfeld radiation condition

$$(\partial_{|x|} - ik)u(x) = \mathscr{O}\left(|x|^{-(n-1)/2}\right) \quad \text{as} \quad |x| \to \infty \tag{7.4}$$

is equivalent to the condition $(\mathscr{R}_{G_+})$ for the fundamental solution

$$G_+(x,y) = \frac{1}{4i}\left(\frac{k}{2\pi|x-y|}\right)^{\frac{n}{2}-1} H^{(1)}_{\frac{n}{2}-1}(k|x-y|). \tag{7.5}$$

To see this, one first has to show that (7.4) implies $\int_{\Gamma_\rho} |u|^2\, ds = \mathscr{O}(1)$ as $\rho \to \infty$, see [2].

Thus in $\mathbb{R}^3$, the standard outgoing and incoming fundamental solutions

$$G_{\pm}(x,y) = \frac{1}{4\pi|x-y|}\, e^{\pm\, ik|x-y|}$$

correspond to the outgoing and incoming Sommerfeld conditions

$$(\partial_{|x|} \mp ik)u(x) = \mathscr{o}\left(|x|^{-1}\right).$$

On the other hand, the fundamental solution

$$G(x,y) = \frac{1}{2}\Big(G_+(x,y) + G_-(x,y)\Big) = \frac{\cos k|x-y|}{4\pi|x-y|}$$

for $\Delta + k^2$ in $\mathbb{R}^3$ does not allow a simple description of the asymptotic behavior of the functions u satisfying $(\mathscr{R}_G)$, or so it appears.

7.e Bilaplacian

For the operator $L = \Delta^2$ in $\mathbb{R}^2$, we look at the fundamental solution

$$G(x,y) = \frac{1}{8\pi}|x-y|^2 \log|x-y|.$$

The condition $(\mathscr{R}_G)$ is given by

$$\begin{aligned}
&\exists\, a, b, b', c, d, d' \in \mathbb{C},\\
&\qquad u(x) = a|x|^2 \log|x| + (bx + b'\bar{x})(2\log|x| + 1)\\
&\qquad\qquad + c(\log|x| + 1) + \frac{1}{|x|^2}(dx^2 + d'\bar{x}^2) + \mathscr{o}(1), \qquad \text{as } |x| \to \infty.
\end{aligned}$$

Here we used the complex variable $x \in \mathbb{C} \equiv \mathbb{R}^2$.

For the operator $L = \Delta^2 - k^4$, $k > 0$, in $\mathbb{R}^n$, a fundamental solution is given by

$$G = \frac{1}{2k^2}(G_1 - G_2),$$

where G_1 is a fundamental solution for $\Delta - k^2$ and G_2 is a fundamental solution for $\Delta + k^2$. This follows from $\Delta^2 - k^4 = (\Delta - k^2)(\Delta + k^2)$. If we choose G_1 exponentially decaying and G_2 as the standard outgoing fundamental solution G_+ (7.5), then the radiation condition is the *same as for the Helmholtz equation*:

$$(\partial_{|x|} - ik)u(x) = \mathscr{o}\left(|x|^{-(n-1)/2}\right) \quad \text{as} \quad |x| \to \infty. \tag{7.6}$$

To see that (7.6) is sufficient, suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ satisfies $Lu = f \in \mathscr{E}'(\mathbb{R}^n)$ and (7.6). From (7.6) it follows that $u \in \mathscr{S}'(\mathbb{R}^n)$, hence $u_1 = (\Delta + k^2)u$ and $u_2 = (\Delta - k^2)u = u_1 - 2k^2 u$ satisfy

$$u_1 \in \mathscr{S}'(\mathbb{R}^n) \quad \text{and} \quad (\Delta - k^2)u_1 = f,$$

hence u_1 satisfies $(\mathscr{R}_{G_1})$, is representable by G_1 and is therefore exponentially decreasing (see §7.a). Hence u_2 satisfies $(\Delta + k^2)u_2 = f$ and (7.6) and is therefore representable by G_2. Thus $u_1 = G_1 f$ and $u_2 = G_2 f$, hence $u = Gf$.

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