Computation of Corner Singularities in Linear Elasticity

MARTIN COSTABEL
IRMAR, Campus de Beaulieu, Université de Rennes 1, F–35042 RENNES Cedex

MONIQUE DAUGE
Département de Mathématiques, Université de Nantes, F–44072 NANTES Cedex 03

1 INTRODUCTION

Elliptic boundary value problems for scalar operators or systems admit non-regular solutions when the domain has corners on its boundary. Near a conical point $\mathcal{O}$, these corner singularities have the form $r^\nu\varphi(\theta)$ where (r,θ) denote the polar coordinates centered in $\mathcal{O}$ and ν spans a discrete set in $\mathbb{C}$. The ν are the *exponents of singularities.*

In this note, we are interested in the problem of computing the exponents ν in two-dimensional domains for ADN-elliptic systems $\boldsymbol{M}(\partial_{x_1},\partial_{x_2})$.

The classical Kondratev method [3] consists in the separation of variables in polar coordinates. The exponents ν appear as "eigenvalues" of a boundary value problem for a (system of) ordinary differential equations $\mathscr{M}(\nu;\,\theta,\partial_\theta)$ on an interval $(0,\omega)$. If one knows a solution basis for this system of differential equations with right hand side 0 and *without boundary conditions*, one can compute the exponents as the complex roots of explicitly given analytic functions.

Classical situations where this is possible are the well-known cases of the Laplace operator, the biharmonic operator, the Stokes system, the system of the isotropic elasticity (Lamé). In all these cases, the corresponding system of ordinary differential equations $\mathscr{M}$ has *constant coefficients*. More generally, we are in this situation if the system is *rotationally invariant*, which implies that the complex characteristics of the initial two-dimensional problem $\boldsymbol{M}$ are $\xi_1=\pm i\xi_2$.

If the characteristics are different from $\xi_1=\pm i\xi_2$, the system of differential equations

$\mathscr{M}$ has variable coefficients in θ. In the scalar case, it is possible to write the operator $\boldsymbol{M}$ in the two-dimensional domain as the product of first order terms of the form $\partial_{x_1} - a\,\partial_{x_2}$ and with the help of such a factorization, one can find an expression of a solution basis for the ordinary differential equation $\mathscr{M}(\nu)\varphi = 0$ on the interval.

Our result is an explicit construction for a solution basis in the most general case of ADN-elliptic systems. In this note, relying on our results in [1], we describe this construction for 2×2 systems of order 2. We give the solution basis and the resulting explicit method for the computation of the exponents of the singularities for the case of orthotropic elasticity (a field where the absence of explicit formulas has been observed (and regretted) repeatedly in the literature). We present some of the numerical results so obtained.

2 A SOLUTION BASIS FOR 2×2 SYSTEMS

As an illustrative example, we consider here the case of the Dirichlet problem for a 2×2 elliptic system of partial differential operators of order 2:

$$\begin{cases} \boldsymbol{M}\boldsymbol{u} &= \boldsymbol{f} \quad \text{in} \quad \Omega \\ \boldsymbol{u} &= 0 \quad \text{on} \quad \partial\Omega\,, \end{cases} \tag{2.1}$$

in the neighborhood of a vertex $\mathcal{O}$ of a polygonal domain Ω. Let ω be the opening of the tangent sector to $\partial\Omega$ in $\mathcal{O}$. Since the exponents of singularities in $\mathcal{O}$ depend only on the principal part of the operator frozen in $\mathcal{O}$, we can assume without restriction that $\boldsymbol{M}$ is homogeneous with constant coefficients.

The exponents of singularities are the complex numbers ν such that there exists a non-zero couple $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ solution of the problem

$$\begin{cases} \boldsymbol{M}\Big(r^\nu(\varphi_1(\theta), \varphi_2(\theta))\Big) &= 0 \quad \text{in} \quad (0,\omega) \\ \varphi_1(0) = \varphi_2(0) &= 0 \\ \varphi_1(\omega) = \varphi_2(\omega) &= 0\,. \end{cases} \tag{2.2}$$

In polar coordinates, $\boldsymbol{M}(\partial_{x_1}, \partial_{x_2})$ can be written as $r^{-2}\mathscr{M}(r\partial_r;\, \theta, \partial_\theta)$ and (2.2) is equivalent to the system of ordinary differential equations

$$\begin{cases} \mathscr{M}(\nu;\, \theta, \partial_\theta)\Big(\varphi_1(\theta), \varphi_2(\theta)\Big) &= 0 \quad \text{in} \quad (0,\omega) \\ \varphi_1(0) = \varphi_2(0) &= 0 \\ \varphi_1(\omega) = \varphi_2(\omega) &= 0\,. \end{cases} \tag{2.3}$$

As a consequence of [1, Th. 2.1], there holds:

LEMMA 2.1 *For all $\nu \in \mathbb{C}$, the space of solutions to*

$$\mathscr{M}(\nu;\, \theta, \partial_\theta)\Big(\varphi_1(\theta), \varphi_2(\theta)\Big) = 0 \tag{2.4}$$

has the dimension 4.

Let us denote by $\boldsymbol{\varphi}_\ell(\nu) = (\varphi_{1,\ell}(\nu,\cdot), \varphi_{2,\ell}(\nu,\cdot))$ for $\ell = 1,2,3,4$ a solution basis of (2.4). We are going to give explicit formulas for such $\boldsymbol{\varphi}_\ell(\nu)$. The functions $r^\nu \boldsymbol{\varphi}_\ell(\nu)$ for $\ell = 1,2,3,4$ are a basis of the space

$$\mathfrak{W}(\nu) := \Big\{ r^\nu \boldsymbol{\varphi}(\theta) \ \ | \quad \boldsymbol{M}(r^\nu \boldsymbol{\varphi}(\theta)) = 0 \Big\}.$$

From $\boldsymbol{M}$, we introduce the 2 matrices:

$$\boldsymbol{M}_+(\alpha) = \boldsymbol{M}(\alpha+1, i(\alpha-1)) \quad \text{and} \quad \boldsymbol{M}_-(\alpha) = \boldsymbol{M}(1+\alpha, i(1-\alpha))$$

and, with $\zeta = r\,e^{i\theta}$, the 2 spaces

$$\mathfrak{W}^+(\nu) := \Big\{ \int_{|\alpha|=1} \Big(\mathrm{diag}(\alpha\zeta + \overline{\zeta})^\nu\Big) \boldsymbol{M}_+^{-1}(\alpha)\, \boldsymbol{f}(\alpha)\, d\alpha \ \ | \ \ \boldsymbol{f} \text{ holomorphic} \Big\}$$

and

$$\mathfrak{W}^-(\nu) := \Big\{ \int_{|\alpha|=1} \Big(\mathrm{diag}(\zeta + \alpha\overline{\zeta})^\nu\Big) \boldsymbol{M}_-^{-1}(\alpha)\, \boldsymbol{f}(\alpha)\, d\alpha \ \ | \ \ \boldsymbol{f} \text{ holomorphic} \Big\}.$$

The inclusions $\mathfrak{W}^\pm(\nu) \subset \mathfrak{W}(\nu)$ are straightforward. We deduce from [1, Th. 2.1]

LEMMA 2.2 *For any non-integral* $\nu \in \mathbb{C}$, $\dim \mathfrak{W}^\pm(\nu) = 2$ *and*

$$\mathfrak{W}(\nu) = \mathfrak{W}^+(\nu) \oplus \mathfrak{W}^-(\nu).$$

As a consequence of the ellipticity, each of the 2 determinants

$$\det \boldsymbol{M}_+(\alpha) \quad \text{and} \quad \det \boldsymbol{M}_-(\alpha)$$

has two roots inside the unit disk $|\alpha| < 1$, denoted:

$$\alpha_1^+, \quad \alpha_2^+ \quad \text{and} \quad \alpha_1^-, \quad \alpha_2^-.$$

It is clear that, with $\boldsymbol{M}_{\mathrm{ad}}^\pm$ denoting the matrix of cofactors of $\boldsymbol{M}_\pm$,

$$\begin{aligned}
\mathfrak{W}^+(\nu) &= \Big\{ \int_{|\alpha|=1} \Big(\mathrm{diag}(\alpha\zeta + \overline{\zeta})^\nu\Big) \frac{\boldsymbol{M}_{\mathrm{ad}}^+(\alpha)}{(\alpha-\alpha_1^+)(\alpha-\alpha_2^+)}\, (\boldsymbol{q} + \alpha\boldsymbol{q}')\, d\alpha \ \ | \ \ \boldsymbol{q},\ \boldsymbol{q}' \in \mathbb{C}^2 \Big\} \\
\mathfrak{W}^-(\nu) &= \Big\{ \int_{|\alpha|=1} \Big(\mathrm{diag}(\zeta + \alpha\overline{\zeta})^\nu\Big) \frac{\boldsymbol{M}_{\mathrm{ad}}^-(\alpha)}{(\alpha-\alpha_1^-)(\alpha-\alpha_2^-)}\, (\boldsymbol{q} + \alpha\boldsymbol{q}')\, d\alpha \ \ | \ \ \boldsymbol{q},\ \boldsymbol{q}' \in \mathbb{C}^2 \Big\}.
\end{aligned} \tag{2.5}$$

If $\alpha_1^+ \neq \alpha_2^+$, the range of $\boldsymbol{M}_{\mathrm{ad}}^+(\alpha_j^+)$ is of dimension 1 for $j = 1,2$ and is generated by one of the columns of $\boldsymbol{M}_{\mathrm{ad}}^+(\alpha_j^+)$. From Lemma 2.2 and (2.5) we deduce

THEOREM 2.3 *If* $\alpha_1^+ \neq \alpha_2^+$ *and* $\alpha_1^- \neq \alpha_2^-$*, let* $\boldsymbol{q}_1^+$, $\boldsymbol{q}_2^+$, $\boldsymbol{q}_1^-$ *and* $\boldsymbol{q}_2^-$ *be non-zero columns of* $\boldsymbol{M}_{\mathrm{ad}}^+(\alpha_1^+)$, $\boldsymbol{M}_{\mathrm{ad}}^+(\alpha_2^+)$, $\boldsymbol{M}_{\mathrm{ad}}^-(\alpha_1^-)$ *and* $\boldsymbol{M}_{\mathrm{ad}}^-(\alpha_2^-)$ *respectively. Then, setting for* $j = 1,2$

$$\boldsymbol{\varphi}_j(\theta) = e^{-i\theta\nu}(\alpha_j^+ e^{2i\theta} + 1)^\nu \boldsymbol{q}_j^+ \quad \text{and} \quad \boldsymbol{\varphi}_{2+j}(\theta) = e^{i\theta\nu}(\alpha_j^- e^{-2i\theta} + 1)^\nu \boldsymbol{q}_j^-$$

we obtain a solution basis of (2.4).

REMARK 2.4 When we have multiple roots $\alpha_1^+ = \alpha_2^+$, a residue calculus in (2.5) still yields explicit formulas, involving also $\zeta\,(\alpha\zeta + \overline{\zeta})^{\nu-1}$. ■

3 THE CASE OF ORTHOTROPIC ELASTICITY

First, we recall the situation for the Lamé system (for isotropic materials). Next we apply our Theorem 2.3 to the case of orthotropic elasticity.

3.1 Isotropic elasticity

The system

$$\boldsymbol{M} = \mu\,\Delta + (\lambda+\mu)\nabla(\mathrm{div}),$$

where λ and μ are the Lamé constants, is "equivariant": if we introduce the radial and tangential components of $\boldsymbol{u}$, $u_r = r^\nu \varphi_r(\theta)$ and $u_\theta = r^\nu \varphi_\theta(\theta)$, the equation (2.4) is equivalent to

$$\left\{\begin{array}{lcl} \mu\,\partial_\theta^2\varphi_r + (\lambda+2\mu)(\nu^2-1)\,\varphi_r + \Big((\lambda+\mu)\nu - (\lambda+3\mu)\Big)\partial_\theta\varphi_\theta & = & 0 \\ (\lambda+2\mu)\,\partial_\theta^2\varphi_\theta + \mu(\nu^2-1)\,\varphi_\theta + \Big((\lambda+\mu)\nu - (\lambda+3\mu)\Big)\partial_\theta\varphi_r & = & 0\,. \end{array}\right. \tag{3.1}$$

3.2 Orthotropic elasticity

In *2D*, the system $\boldsymbol{M}$ for the orthotropic elasticity takes the form $(u_1,u_2) \mapsto (f_1,f_2)$ with:

$$\left\{\begin{array}{lclcl} (c_{11}\,\partial_1^2 + c_{66}\,\partial_2^2)\,u_1 \;+\; (c_{12}+c_{66})\,\partial_1\partial_2 u_2 & = & f_1 & \text{in} & \Omega \\ (c_{66}\,\partial_1^2 + c_{22}\,\partial_2^2)\,u_2 \;+\; (c_{12}+c_{66})\,\partial_1\partial_2 u_1 & = & f_2 & \text{in} & \Omega, \end{array}\right. \tag{3.2}$$

satisfying the ellipticity conditions $c_{11} > |c_{12}|$ and $c_{66} > 0$.

For the *3D* equations of an orthotropic material, the computation of the exponents of singularities along an edge is reduced to the *2D* system above uncoupled from a Laplace type equation:

$$(c_{55}\,\partial_1^2 + c_{44}\,\partial_2^2)\,u_3 = f_3.$$

For an isotropic material, we have the relations

$$c_{12} = \lambda \qquad c_{44} = c_{55} = c_{66} = \mu \qquad c_{11} = c_{22} = \lambda + 2\mu\,.$$

For general coefficients, there is no way to obtain a system $\mathscr{M}(\nu)$ with constant coefficients as in (3.1).

Our method yields the following solution basis

$$\boldsymbol{\varphi}_j = e^{-i\theta\nu}(\alpha_j^+ e^{2i\theta}+1)^\nu \cdot \begin{pmatrix} (c_{66}-c_{22})(\alpha_j^{+2}+1) + 2(c_{66}+c_{22})\alpha_j^+ \\ -i(c_{66}+c_{12})(\alpha_j^{+2}-1) \end{pmatrix}$$

and

$$\boldsymbol{\varphi}_{2+j} = e^{i\theta\nu}(\alpha_j^- e^{-2i\theta}+1)^\nu \cdot \begin{pmatrix} (c_{66}-c_{22})(\alpha_j^{-2}+1) + 2(c_{66}+c_{22})\alpha_j^- \\ i(c_{66}+c_{12})(\alpha_j^{-2}-1) \end{pmatrix}$$

for $j = 1, 2$.

Note that it is also possible to define

$$\boldsymbol{\varphi}_{2+j} = e^{i\theta\nu}(\alpha_j^- e^{-2i\theta}+1)^\nu \cdot \begin{pmatrix} -i(c_{66}+c_{12})(\alpha_j^{-2}-1) \\ (c_{11}-c_{66})(\alpha_j^{-2}+1) + 2(c_{11}+c_{66})\alpha_j^- \end{pmatrix}.$$

4 CHARACTERISTIC DETERMINANTS FOR BOUNDARY VALUE AND TRANSMISSION PROBLEMS

4.1 Dirichlet problem

The Dirichlet conditions are

$$\left\{\begin{array}{lllll} u_1 & = & 0 & \text{on} & \partial\Omega \\ u_2 & = & 0 & \text{on} & \partial\Omega\,. \end{array}\right. \tag{4.1}$$

The exponents of singularities are the roots of the equation $\det \mathcal{A}(\nu) = 0$ where $\mathcal{A}(\nu)$ is the matrix of Dirichlet conditions in $\theta = 0$ and $\theta = \omega$ of the solution basis $\boldsymbol{\varphi}_1(\nu)$, $\boldsymbol{\varphi}_2(\nu)$, $\boldsymbol{\varphi}_3(\nu)$ and $\boldsymbol{\varphi}_4(\nu)$:

$$\mathcal{A}(\nu) = \begin{pmatrix} \varphi_{1,1}(\nu,0) & \varphi_{1,2}(\nu,0) & \varphi_{1,3}(\nu,0) & \varphi_{1,4}(\nu,0) \\ \varphi_{2,1}(\nu,0) & \varphi_{2,2}(\nu,0) & \varphi_{2,3}(\nu,0) & \varphi_{2,4}(\nu,0) \\ \varphi_{1,1}(\nu,\omega) & \varphi_{1,2}(\nu,\omega) & \varphi_{1,3}(\nu,\omega) & \varphi_{1,4}(\nu,\omega) \\ \varphi_{2,1}(\nu,\omega) & \varphi_{2,2}(\nu,\omega) & \varphi_{2,3}(\nu,\omega) & \varphi_{2,4}(\nu,\omega) \end{pmatrix}.$$

4.2 Neumann and mixed problems

The Neumann boundary conditions of the orthotropic elasticity are

$$\left\{\begin{array}{lllll} (c_{11}\, n_1\, \partial_1 + c_{66}\, n_2\, \partial_2)\, u_1 \;+\; (c_{12}\, n_1\, \partial_2 + c_{66}\, n_2\, \partial_1)\, u_2 & = & 0 & \text{on} & \partial\Omega \\ (c_{66}\, n_1\, \partial_1 + c_{22}\, n_2\, \partial_2)\, u_2 \;+\; (c_{12}\, n_2\, \partial_1 + c_{66}\, n_1\, \partial_2)\, u_1 & = & 0 & \text{on} & \partial\Omega\,. \end{array}\right. \tag{4.2}$$

The exponents of singularities associated to Neumann boundary conditions or other boundary conditions (for example mixed Dirichlet-Neumann) are obtained in the same way as Dirichlet with the help of the same solution basis. The corresponding 4×4 matrix $\mathcal{A}(\nu)$ is formed by the 4 boundary conditions applied to $\boldsymbol{\varphi}_1(\nu)$, $\boldsymbol{\varphi}_2(\nu)$, $\boldsymbol{\varphi}_3(\nu)$ and $\boldsymbol{\varphi}_4(\nu)$.

4.3 Bi-material transmission problems

We suppose that the two domains Ω_1 and Ω_2 are tangent with plane sectors Γ_1 and Γ_2 in $\mathcal{O}$, with

$$\Gamma_1 = \{(x_1, , x_2) \;\mid\; \omega_0 < \theta < \omega_1\} \qquad \text{and} \qquad \Gamma_2 = \{(x_1, , x_2) \;\mid\; \omega_1 < \theta < \omega_2\},$$

for certain angles ω_0, ω_1 and ω_2 with $0 \le \omega_0 < \omega_1 < \omega_2 \le 2\pi$. We have a system $\boldsymbol{M}_1$ on Ω_1 and a system $\boldsymbol{M}_2$ on Ω_2. The transmission conditions on the interface $\theta = \omega_1$ are the continuity of the Dirichlet and Neumann traces.

Let us denote by $\boldsymbol{\Phi}^{(k)}(\nu)$ the quadruple $\boldsymbol{\varphi}_1(\nu)$, $\boldsymbol{\varphi}_2(\nu)$, $\boldsymbol{\varphi}_3(\nu)$ and $\boldsymbol{\varphi}_4(\nu)$ corresponding to $\boldsymbol{M}_k$ for $k = 1, 2$. The Dirichlet traces on the ray $\theta = \theta_0$ are denoted D_{θ_0} and the "conormal" Neumann traces (4.2) associated to $\boldsymbol{M}_k$ are denoted $N^{(k)}_{\theta_0}$. The exponents of singularities are the roots of the equation $\det \mathcal{A}(\nu) = 0$ where $\mathcal{A}(\nu)$ is the 8×8 matrix of boundary and jump conditions of $\boldsymbol{\Phi}^{(1)}(\nu)$ and $\boldsymbol{\Phi}^{(2)}(\nu)$.

Example of exterior Neumann conditions (the blocks are 2×4 matrices):

$$\mathcal{A}(\nu) = \begin{pmatrix} \boxed{N^{(1)}_{\omega_0}\,\boldsymbol{\Phi}^{(1)}(\nu)} & 0 \\ \boxed{D_{\omega_1}\,\boldsymbol{\Phi}^{(1)}(\nu)} & -\boxed{D_{\omega_1}\,\boldsymbol{\Phi}^{(2)}(\nu)} \\ \boxed{N^{(1)}_{\omega_1}\,\boldsymbol{\Phi}^{(1)}(\nu)} & -\boxed{N^{(2)}_{\omega_1}\,\boldsymbol{\Phi}^{(2)}(\nu)} \\ 0 & -\boxed{N^{(2)}_{\omega_2}\,\boldsymbol{\Phi}^{(1)}(\nu)} \end{pmatrix}.$$

Example when $\omega_0 = 0$, $\omega_2 = 2\pi$ and transmission conditions are imposed on both interfaces $\theta = 0$ and $\theta = \omega$:

$$\mathcal{A}(\nu) = \begin{pmatrix} \boxed{D_0\,\boldsymbol{\Phi}^{(1)}(\nu)} & -\boxed{D_{2\pi}\,\boldsymbol{\Phi}^{(2)}(\nu)} \\ \boxed{N^{(1)}_0\,\boldsymbol{\Phi}^{(1)}(\nu)} & -\boxed{N^{(2)}_{2\pi}\,\boldsymbol{\Phi}^{(2)}(\nu)} \\ \boxed{D_\omega\,\boldsymbol{\Phi}^{(1)}(\nu)} & -\boxed{D_\omega\,\boldsymbol{\Phi}^{(2)}(\nu)} \\ \boxed{N^{(1)}_\omega\,\boldsymbol{\Phi}^{(1)}(\nu)} & -\boxed{N^{(2)}_\omega\,\boldsymbol{\Phi}^{(2)}(\nu)} \end{pmatrix}.$$

4.4 General transmission problems

We can treat the situation of a transmission problem between K materials in domains Ω_k whose interfaces meet in $\mathcal{O}$ and whose tangents in $\mathcal{O}$ are all distinct from each other. For each material we have an operator $\boldsymbol{M}_k$ and a solution basis $\boldsymbol{\Phi}^{(k)}$. The matrix $\mathcal{A}(\nu)$ is the $4K \times 4K$ matrix obtained as the product

$$\mathcal{A}(\nu) = \mathscr{L} \times \mathscr{C}(\nu),$$

where $\mathscr{C}(\nu)$ is the $8K \times 4K$ matrix of the Cauchy data of the $\boldsymbol{\Phi}^{(k)}(\nu)$ on the two tangent segments to the sides limiting Ω_k and $\mathscr{L}$ is the $4K \times 8K$ matrix of the boundary and interface conditions. For instance, for the first example of bi-material transmission problem above, the matrix $\mathscr{L}$ can be written in 2×2 blocks as:

$$\mathscr{L} = \begin{pmatrix} \mathbb{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{I} & 0 & -\mathbb{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I} & 0 & -\mathbb{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\mathbb{I} & 0 \end{pmatrix}.$$

5 SOME NUMERICAL RESULTS

We have made some numerical experimentations, to compare our results with other methods for the computation of exponents of singularities.

Concerning the isotropic case (Lamé system), let us quote Grisvard [2] and Sändig & al. [7] for one material, and Nicaise & Sändig for transmission problems [5].

Concerning the general (anisotropic) case, we can quote Leguillon & Sanchez-Palencia [4], whose method is numeric (*determinant method*): this consists in computing for "each" ν the determinant of the matrix of a variational problem on the interval $(0,\omega)$, discretized by FEM and to find values of ν for which this determinant is zero.

We also quote Papadakis [6] who uses a *shooting method* to compute a solution basis of problem (2.4): this consists in the numerical solution of a Cauchy problem.

Note that our numerical method, due to its simplicity, is by several orders of magnitude faster than both the determinant method and the shooting method and this with arbitrary precision.

5.1 Bi-material Neumann transmission problem

We take exterior Neumann conditions and symmetric materials ($\omega_0 = 0$, $\omega_1 = \omega$ and $\omega_2 = 2\omega$). Both materials are isotropic, with Poisson coefficient σ equal to 0.35 for both and with Young moduli $E_1 = 1$, $E_2 = 10$. Let us recall the relations with the Lamé coefficients

$$\lambda = \frac{E\,\sigma}{1-\sigma-2\sigma^2}\,; \qquad \mu = \frac{E}{2+2\sigma}\,.$$

In the table below, we give the values $0 < \nu_1 \le \nu_2$ of the two exponents ν with least positive real part (they are real in these cases). We compare the results of [4] **[LeSa]** with ours **[CoDa]**.

	[LeSa]		**[CoDa]**	
ω	ν_1	ν_2	ν_1	ν_2
90	0.771	1.00	0.77415	1.00000
85	0.821	1.00	0.82236	1.00000
80	0.876	1.00	0.87819	1.00000
75	0.937	0.989	0.94287	1.00000
70	0.987	1.03	1.00000	1.01800
65	0.998	1.11	1.00000	1.10567
60	1.00	1.21	1.00000	1.20866

We observe a crossing between $\nu = 1$, corresponding to the rigid displacement

$$\begin{cases} u_1(x_1,x_2) &= -x_2 \\ u_2(x_1,x_2) &= \quad x_1\,, \end{cases}$$

and another exponent which takes the values ... 0.87819, 0.94287, 1.01800, 1.10567 ... and depends smoothly on the angle ω.

5.2 Bi-material Dirichlet transmission problem

Here $\omega_0 = 0$, $\omega_1 = \omega < \pi$ and $\omega_2 = \pi$. Both materials are orthotropic. The constants for the first material are (cf. [6, §6.3]

$$c^{(1)}_{11} = c^{(1)}_{22} = 20.41337, \quad c^{(1)}_{12} = 0.91860168, \quad c^{(1)}_{66} \text{ according to the table.}$$

The constants for the second material are as in [6, §6.3]

$$c_{11}^{(2)} = c_{22}^{(2)} = 1.5384615, \quad c_{12}^{(2)} = 0.461538 \quad c_{66}^{(2)} = 0.7$$

Here are numerical results for different values of ω and $c_{66}^{(1)}$.

ω	80°		90°		100°	
c_{66}	ν_1	ν_2	ν_1	ν_2	ν_1	ν_2
1.5	1.00272	1.26291	0.95676	1.07738	0.79954	0.99665
1.1	1.00160	1.27132	0.96249	1.05602	0.77691	0.99821
0.7	1.00000	1.28411	0.99793	1.00000	0.75054	1.00000
0.3	0.997224	1.30497	0.98007 ± 0.06681 i		0.72076	1.00248

We observe the appearance of the exponent $\nu = 1$ for the case $c_{66}^{(1)} = c_{66}^{(2)}$. This corresponds to the solution

$$\begin{cases} u_1(x_1, x_2) = x_2 \\ u_2(x_1, x_2) = 0\,. \end{cases}$$

5.3 Bi-material crack problem

Here $\omega_0 = 0$, $\omega_1 = \pi$ and $\omega_2 = 2\pi$. We investigate the behaviour of the first 6 exponents for two materials at the limit when the rigidity of one of them tends to $+\infty$. We choose as starting values

$$c_{11}^{(1)} = c_{22}^{(1)} = 20, \quad c_{12}^{(1)} = 1, \quad c_{66}^{(1)} = 2, \qquad c_{11}^{(2)} = c_{22}^{(2)} = 5, \quad c_{12}^{(2)} = 2, \quad c_{66}^{(2)} = 1.$$

We indicate in the table the ratio 10^κ by which we multiply the coefficients of the first material, the coefficients of the second material remaining always the same. We consider the cases of Dirichlet and Neumann conditions on the crack respectively. In all cases, we find a double root $\nu = 1$, corresponding to piecewise linear solutions.

	Dirichlet	
κ	ν_1, ν_2	ν_5, ν_6
0	0.50000000 ± 0.04798949 i	1.50000000 ± 0.04798949 i
1	0.50000000 ± 0.08459395 i	1.50000000 ± 0.08459395 i
3	0.50000000 ± 0.09002542 i	1.50000000 ± 0.09002542 i
6	0.50000000 ± 0.09008219 i	1.50000000 ± 0.09008219 i
10	0.50000000 ± 0.09008225 i	1.50000000 ± 0.09008225 i
∞	0.50000000 ± 0.09008225 i	1.50000000 ± 0.09008225 i

The line $\kappa = \infty$ gives the computations for the mixed Dirichlet-Neumann problem for the second material.

	Neumann	
κ	ν_1, ν_2	ν_5, ν_6
0	0.50000000 $\pm$ 0.04199760 i	1.50000000 $\pm$ 0.04199761 i
1	0.50000000 $\pm$ 0.07860206 i	1.50000000 $\pm$ 0.07860206 i
3	0.50000000 $\pm$ 0.08403353 i	1.50000000 $\pm$ 0.08403354 i
6	0.50000000 $\pm$ 0.08409030 i	1.50000000 $\pm$ 0.08409030 i
10	0.50000000 $\pm$ 0.08409036 i	1.50000000 $\pm$ 0.08409036 i
∞	0.50000000 $\pm$ 0.08409036 i	1.50000000 $\pm$ 0.08409037 i

The line $\kappa = \infty$ gives the computations for the mixed Dirichlet-Neumann problem for the first material.

5.4 Mixed problem at the limit of ellipticity

The domain is the half space, with Dirichlet conditions in $\theta = 0$ and Neumann conditions in $\theta = \pi$. In the following computations, we take $c_{66} = 1$ and we have $c_{11} \to |c_{12}| = 10$ in two different situations: $c_{12} = -10$ and $c_{12} = 10$. We observe very different behaviors at the limit.

c_{11}	c_{66}	c_{12}	ν_1, ν_2
20	1	-10	0.5 $\pm$ 0.11761517
11	1	-10	0.5 $\pm$ 0.34769686
10.1	1	-10	0.5 $\pm$ 0.68591184
10.001	1	-10	0.5 $\pm$ 1.41522496
10.000001	1	-10	0.5 $\pm$ 2.51459115
10.000000001	1	-10	0.5 $\pm$ 3.613994
20	1	10	0.5 $\pm$ 0.03999793
11	1	10	0.5 $\pm$ 0.02003839
10.1	1	10	0.5 $\pm$ 0.00673842
10.001	1	10	0.5 $\pm$ 0.00067859
10.00001	1	10	0.5 $\pm$ 0.00006786
10.0000001	1	10	0.5 $\pm$ 0.00000679
10.000000001	1	10	0.5 $\pm$ 0.00000068

REFERENCES

1. M. Costabel, M. Dauge. Construction of corner singularities for Agmon-Douglis-Nirenberg elliptic systems. *Math. Nachr.* **162** (1993) 209–237.

2. P. Grisvard. Singularités en élasticité. *Arch. Rational Mech. Anal.* **107 (2)** (1989) 157–180.

3. V. A. Kondrat'ev. Boundary-value problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.* **16** (1967) 227–313.

4. D. Leguillon, E. Sanchez-Palencia. *Computation of singular solutions in elliptic problems and elasticity.* RMA 5. Masson, Paris 1987.

5. S. Nicaise, A. M. Sändig. General interface problems I. Pub. IRMA, Vol. 30, No. V, Lille University 1992.

6. P. Papadakis. Computational aspects of the determination of the stress intensity factors for two-dimensional elasticity. PhD Thesis, University of Maryland 1989.

7. A. M. Sändig, U. Richter, R. Sändig. The regularity of boundary value problems for the Lamé equations in a polygonal domain. *Rostock. Math. Kolloq.* **36** (1989) 21–50.