EDGE ASYMPTOTICS ON A SKEW CYLINDER

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1. POSITION OF THE PROBLEM

1.a Origin. This problem was posed as a question to the first author by I. Babuška (Maryland).

It is well known that singularities of the domain give rise to a loss of regularity for the solutions of any elliptic boundary value problem. The situation is rather well understood when the singularities are isolated points of the boundary and are of conical type (see [4], [8]).

When a conical singularity is tensorized with an affine space, one gets an edge. Regularity results are rather complete in that case [6], [11]. If the operator is translation invariant along the edge, the asymptotics can be derived in a direct way from the asymptotics on the corresponding conical domain [1].

But for physical examples in the ordinary three-dimensional space, when a bounded domain has edges and no corners, then the edges are necessarily curved. The simplest example is a cylinder with circular basis, cut orthogonally to its generating lines. But this example is very particular: The opening of the edge is everywhere $\pi/2$ and the curvature of the edge is constant. If one cuts the cylinder by a plane which is skew with respect to the generating lines, then the edge is elliptic and the opening angle is varying. This gives rise to difficulties for the precise analysis of the structure of the solution, due to the fact that the asymptotics in the corresponding two-dimensional domains depend in a discontinuous way on the opening parameter. In particular, the coefficients of the singular functions along the edge (stress intensity factors etc.) will blow up at certain points. Such a behavior causes difficulties also for numerical approximations.

1.b Geometry of the domain. Let B be an analytic bounded domain in $\mathbb{R}^2$. Let Ψ be an affine function $(x_2, x_3) \mapsto x_1 = \Psi(x_2, x_3)$. We assume that

$$\text{for all } (x_2, x_3) \in \overline{B}, \quad \Psi(x_2, x_3) > 0.$$

We introduce

$$\Omega = \Big\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_2, x_3) \in B,\, 0 < x_1 < \Psi(x_2, x_3) \Big\} \,.$$

This is our skew cylinder. We denote by M the top edge and by $\partial_1\Omega$ the side of the cylinder:

$$\partial_1\Omega = \Big\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_2, x_3) \in \partial B,\, 0 < x_1 < \Psi(x_2, x_3) \Big\} \,.$$

The union of the top and the bottom of the cylinder is denoted by $\partial_2\Omega$.

For what we are going to do, a more general class of domains is admissible: The piecewise analytic domains with edges with the property that in each point of an edge, the domain is locally analytically diffeomorphic to a straight wedge.

For instance, if Ψ_1 and Ψ_2 are analytic on $\mathbb{R}^2$ and such that

$$\forall (x_2, x_3) \in \overline{B}, \quad \Psi_1(x_2, x_3) < \Psi_2(x_2, x_3),$$

we get a cylinder with curved ends:

$$\Omega = \Big\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_2, x_3) \in B,\, \Psi_1(x_2, x_3) < x_1 < \Psi_2(x_2, x_3) \Big\} \,.$$

Such a situation is useful in reaction-diffusion, for instance.

Interesting extensions could be:

1. Include the situation where an isolated point on the edge has an opening angle equal to π. Such a situation appears for instance when two isometric skew cylinders are glued together along their skew tops.

2. Include piecewise C^∞ manifolds with edges.

3. Include higher order singularities (corners of any dimension).

1.c Boundary value problems. We choose two simple examples: a mixed problem and the Neumann problem for the Laplace operator. Let us denote by $(\mathcal{P}_I)$, resp. $(\mathcal{P}_{II})$ these two problems. We write

$$(\mathcal{P}_I) \quad \begin{cases} \Delta u = f \quad \text{on } \Omega \\ u = 0 \quad \text{on } \partial_1\Omega, \quad \frac{\partial u}{\partial n} = h_2 \quad \text{on } \partial_2\Omega. \end{cases}$$

$$(\mathcal{P}_{II}) \quad \begin{cases} \Delta u = f \quad \text{on } \Omega \\ \frac{\partial u}{\partial n} = h_1 \quad \text{on } \partial_1\Omega, \quad \frac{\partial u}{\partial n} = h_2 \quad \text{on } \partial_2\Omega. \end{cases}$$

We take the Laplace operator because it is the simplest possible example of an elliptic operator. Moreover, it is physically relevant. We choose the mixed Dirichlet-Neumann problem because the singularities appear at a low regularity level (in a generic way, $u \notin H^2(\Omega)$). We also consider the Neumann problem, because it does not satisfy the isomorphism condition in weighted Sobolev spaces which is used in [6], [9]. Lastly, the behavior of such problems in a two-dimensional angle is well known [2].

Other boundary value problems behave in a similar way as $(\mathcal{P}_I)$ and $(\mathcal{P}_{II})$. As interior operator, we can take any elliptic second order operator with real analytic coefficients. With such operators, we would in a natural way consider any Dirichlet-Neumann boundary problems (including pure Dirichlet and pure Neumann problems), the Neumann condition being defined using the conormal derivative associated with the interior operator. In this way we obtain, with our above class of piecewise analytic domains, a class of boundary value problems which is invariant with respect to analytic diffeomorphisms.

It is also possible for us to consider mixed oblique derivative problems (semi-variational problems) in a framework related to that in [7].

1.d Exponents of the singularities. Let us first recall the result in a two-dimensional angle G with opening ω. We use polar coordinates (r,θ) so that G corresponds to $r>0$, $0<\theta<\omega$. For any integer $j\geq 1$, we set

$$\begin{aligned}
\nu_j &= \begin{cases} (2j-1)\dfrac{\pi}{2\omega} & \text{for the mixed problem} \\[2mm] \dfrac{j\pi}{\omega} & \text{for the Neumann problem} \end{cases} \\
\sigma_j(r,\theta) &= r^{\nu_j}\cos\nu_j\theta \\
S_j(r,\theta) &= r^{\nu_j}(\log r\cos\nu_j\theta - \theta\sin\nu_j\theta) \quad \text{if } \nu_j\in\mathbb{N}.
\end{aligned}$$

Proposition 1.1 *Let $s\in\mathbb{R}$, $s>1/2$. Let us assume that the interior datum in G has H^{s-1} regularity and that the Neumann data have $H^{s-\frac12}$ regularity. Then the variational solution w of the mixed Neumann-Dirichlet problem, resp. Neumann problem on G, admits the following decomposition,*

$$w = w_{\text{reg}} + w_{\text{sing}}$$

where

$$\begin{aligned}
w_{\text{reg}} &\in H^{s+1-\varepsilon}(G) \quad \forall\varepsilon>0 \\
w_{\text{sing}} &= \sum_{1\leq\nu_j<s,\,\nu_j\notin\mathbb{N}} c_j\sigma_j + \sum_{1\leq\nu_j<s,\,\nu_j\in\mathbb{N}} c_jS_j.
\end{aligned}$$

Remark 1.2 If $\forall j\in\mathbb{N}$ there holds $\nu_j\neq s$, then $w_{\text{reg}}\in H^{s+1}(G)$. On the other hand, if $\exists j\in\mathbb{N}$ such that $\nu_j=s$, then w_{reg} does not belong to $H^{s+1}(G)$, in general. If we remain in the scale of ordinary Sobolev spaces, it is impossible to remove this ε. If one characterizes the preimage, one finds a space in which H^{s+1} is not closed. Conversely, the image of H^{s+1} is not closed in $H^{s-1}(G)\times H^{s-\frac12}(\partial G)$.

Let us return now to our skew cylinder. We assume that for a fixed real $s>1/2$

$$\begin{aligned}
f &\in H^{s-1}(\Omega) \\
h_j &\in H^{s-\frac12}(\partial_j\Omega), \quad j=1,2
\end{aligned}$$

with the usual compatibility condition for the Neumann problem. Then there exists a variational solution $u \in H^1(\Omega)$.

For any $x \in M$, let $\omega(x)$ be the opening angle of Ω in x. We set

$$\nu(x) = \begin{cases} \dfrac{\pi}{2\omega(x)} & \text{for} \quad (\mathcal{P}_I) \\ \dfrac{\pi}{\omega(x)} & \text{for} \quad (\mathcal{P}_{II}) \end{cases}$$

If $s < \min_{x \in M} \nu(x)$, then $u \in H^{s+1}(\Omega)$. Since, anyway, $\min_{x \in M} \nu(x) > 1/2$, we have $u \in H^{\frac{3}{2}+\varepsilon}(\Omega)$ and the Neumann conditions make sense. From now on we assume that

$$\min_{x \in M} \nu(x) < s.$$

Then u can be split into two parts for any $\varepsilon > 0$,

$$u = u_{\text{reg}} + u_{\text{sing}}$$

where $u_{\text{reg}} \in H^{s+1-\varepsilon}(\Omega)$ and u_{sing} is an asymptotics. Such a decomposition generally depends on $\varepsilon > 0$. Our aim is to describe the structure of such splittings. In particular, we want to separate as much as possible the roles of the different variables, the abscissa y along the edge, the distance r from the edge and the angular variable θ. More precisely, we intend to separate what comes from the geometrical framework (domain and boundary value problem) and what comes from the data (f, h_j). The part that comes from the geometrical framework will be described as an analytic bundle. The part that comes from the data will be described as $H^{s-\varepsilon}$ sections of this bundle. Only the second notion depends on ε. By contrast, the bundle is an object which is associated to the pair (domain, boundary value problem) in a canonical way.

2. SIMPLE ASYMPTOTICS

2.a A simple result. The exponents of the singularities which appear in the asymptotics along the edge depend on the edge parameter y. They are the same as in a two-dimensional problem for the Laplace operator with lower order terms and the opening $\omega(y)$, which is the opening of Ω at the point y. We can enumerate them using a double index $k = (k_1, k_2)$, where $k_1, k_2 \in \mathbb{N}$ and

$$\nu_{(0,k_2)}(y) = k_2$$

and for $k_1 \geq 1$,

$$\nu_k(y) = \begin{cases} (2k_1 - 1)\dfrac{\pi}{2\omega(y)} + k_2 & \text{for } (\mathcal{P}_I) \\ \dfrac{k_1 \pi}{\omega(y)} + k_2 & \text{for } (\mathcal{P}_{II}). \end{cases}$$

We have to include all positive integers in those exponents due to the possible interaction between polynomials and singularities.

What can be expected as asymptotics along the edge is

$$\sum_{k,q,\beta} c_{k,q,\beta}(y)\, r^{\nu_k(y)} \log^q r\, \varphi_{k,q,\beta}(y,\theta) \tag{2.1}$$

where only the $c_{k,q,\beta}$ depend on the data (f, h_j). Actually, such an asymptotics in tensor product form is not convenient in general, since the $c_{k,q,\beta}$ are not regular enough: The best which can be expected is $H^{s-\nu_k(y)}$. Such an asymptotics could be valid only if $s = \infty$, or if the 'regular part' has only half the regularity of the data. Therefore we introduce a function $\Phi(y,r)$ such that its partial Fourier transform satisfies

$$\mathcal{F}_{y\to\xi}\Phi(\xi, r) = \phi(r|\xi|)$$

where ϕ is a rapidly decreasing function, has a Fourier transform with compact support, and satisfies for a sufficiently large N

$$\phi(0) = 1, \quad \frac{d^l}{ds^l}\phi(0) = 0 \quad (l = 1, \ldots, N).$$

We define the convolution with respect to y,

$$(\Phi * c)(y,r) := \int \Phi(y - y', r)\, c(y')\, dy'.$$

The following theorem describes our result on the simple edge asymptotics.

Theorem 2.1 *Let J be an open segment in M and $\tilde{J}$ any open set in M such that $\overline{J} \subset \tilde{J}$ holds and let $\mathcal{U}_J$ be a sufficiently small neighborhood of J in $\overline{\Omega}$. Let χ be an analytic diffeomorphism which defines on $\mathcal{U}_J$ local cylindrical coordinates (y, r, θ) with $0 < \theta < \omega(y)$. In local coordinates, we write I and $\tilde{I}$ for J and $\tilde{J}$. We assume that for some $\varepsilon_0 \geq 0$ there holds*

$$\text{for all } k \text{ we have} \quad \forall y \in \tilde{I},\ \nu_k(y) < s \quad \text{or} \quad \forall y \in \tilde{I},\ \nu_k(y) \geq s - \varepsilon_0 \tag{2.2}$$

$$\text{for all } y \in \tilde{I}, \quad \text{if } \nu_k(y) = \nu_{k'}(y) < s \quad \text{then } k = k'. \tag{2.3}$$

To each k there exists a finite set of indices (q, β) and analytic functions $\varphi_{k,q,\beta}(y,\theta)$ such that any solution u of problem $(\mathcal{P}_I)$ or $(\mathcal{P}_{II})$ can be decomposed into

$$u = u_{\text{reg}} + u_{\text{sing}}.$$

Here $u_{\text{reg}} \in H^{s+1-\varepsilon}(\mathcal{U}_J)$ $\forall \varepsilon > \varepsilon_0$ and

$$u_{\text{sing}} = \chi^{-1}\left(\sum_{k,q,\beta} (\Phi * c_{k,q,\beta})(y,r)\, r^{\nu_k(y)} \log^q r\, \varphi_{k,q,\beta}(y,\theta)\right). \tag{2.4}$$

The coefficients $c_{k,q,\beta}(y)$ are defined on $\tilde{I}$ and satisfy $c_{k,q,\beta} \in H^{s-\nu_k(y)-\varepsilon}(I)$ for all $\varepsilon > 0$. The sum extends over those k for which $\nu_k < s$ holds on $\tilde{I}$.

Remark 2.2 If $k_1 = 0$ then the logarithmic terms are absent and for any fixed $y \in M$, the function $r^{\nu_k(y)} \varphi_{k,0,\beta}(y, \theta)$ is a polynomial in cartesian variables. The corresponding term in the asymptotics can be put into the regular part. We include such terms in our asymptotics as a preparation for the study of crossing points.

Remark 2.3 Assume that $k_2 = 0$ and $\nu_k \notin \mathbb{N}$. Then $q = 0$ and β has only one value. We have

$$\varphi_{k,0,1}(y, \theta) = \cos \nu_k(y)\theta.$$

Remark 2.4 Only the first terms (with $k_2 = 0$) arise directly from the conormal principal symbol of the operator. The other terms come from other parts of the operator and there exists no simple relation between the symbol of the operator and the singularities. For instance, one obtains a logarithmic term by differentiating $r^{\nu_k(y)}$ with respect to y.

2.b The question of optimality. Results related to Theorem 2.1 have been proven for the Dirichlet problem by Kondratev [5] concerning the first singularity, Nikishkin [10] for all singularities, Maz'ya and Roßmann [9] for more general operators. We have to mention some differences between our result and those of the above authors: The first one (but not essential) is our use of spaces with non-integer exponents. The second one is that our results do not need the isomorphism assumption in weighted Sobolev spaces; such an assumption excludes the Neumann problem, for instance.

The third difference lies in our localization. All the above authors make the following assumption

$$\forall k,\ \forall y \in M, \quad \nu_k(y) \neq s. \tag{2.5}$$

In such a case ε_0 may be taken as 0 in our theorem. It is interesting to realize that such an assumption means a severe restriction on the possible range for the couple (α, s) where α describes the slope of the top of the skew cylinder: $0 < \alpha < 1$ and $\beta = \alpha\pi/2$ is the angle between the top of the cylinder and the horizontal (x_2, x_3)-plane.

Let us illustrate this restriction. The maximal opening is $(1 + \alpha)\pi/2$ and the minimal one is $(1 - \alpha)\pi/2$. It is easy to check that if

$$\nu_{(1,0)}\left((1-\alpha)\frac{\pi}{2}\right) \geq \nu_{(1,1)}\left((1+\alpha)\frac{\pi}{2}\right) \tag{2.6}$$

then the union of the numerical ranges of all the ν_k's on M is a full half axis. The condition (2.6) is fulfilled in the example of the Neumann problem $(\mathcal{P}_{II})$ if

$$\frac{2}{1-\alpha} \geq \frac{2}{1+\alpha} + 1,$$

i. e., if $\alpha \geq \sqrt{5} - 2 \simeq 0.236$.

As we already explained for two-dimensional problems, if the condition (2.5) does not hold, there is no optimal regularity for the regular part (see Remark 1.2). Even if condition (2.5) holds, it is still difficult to obtain the optimality. Only [9] claim the optimality when there is no logarithmic term is the asymptotics and moreover $s - \nu_{(1,0)} < 1$.

Concerning the regularity of the coefficients, even in the straight edge case with a translation invariant operator, when logarithmic terms appear, there also appear coefficients with less regularity than $H^{s-\nu_k}$. Such coefficients have the property that for a certain number $Q > 0$

$$\mathcal{F}^{-1}_{\xi\to y}\hat{c}(\xi)\,\log^{-Q}(|\xi|+2)\,\in H^{s-\nu_k}.$$

We think that this problem of optimality has a similar level of complexity as the considerations about the localization of coefficients in §16 of [1].

2.c The crossing of exponents. In Theorem 2.1 we made the assumption (see (2.3)) that there is no crossing of exponents, i.e., there are no points y such that for some k, k' with $k \neq k'$ there holds $\nu_k(y) = \nu_{k'}(y)$. All the authors quoted above also require this condition. We will see that such a crossing of exponents in general induces the blowing up of coefficients in the expansion (2.4).

For our problem of the skew cylinder, it is impossible to avoid such crossings. For y_0 such that $\omega(y_0) = \pi/2$ (there always exist two such points), we have $\nu_k(y_0) = \nu_{k'}(y_0)$ for

$$\begin{array}{lllll} k=(1,0) & \text{and} & k'=(0,1) & \text{for} & (\mathcal{P}_I) \\ k=(1,0) & \text{and} & k'=(0,2) & \text{for} & (\mathcal{P}_{II}). \end{array}$$

The points where crossing of exponents will eventually appear (for large s) are dense in M, so this phenomenon occurs in a generic way.

Our situation has one special feature, however: It is possible to choose the exponents as analytic functions; here this choice $\nu_k(y)$ is obvious, as it is also obvious for the oblique derivative problems. There are other cases where such an analytic choice is not obvious but still possible. Here is an example.

Let us consider a domain $\Gamma \subset \mathbb{R} \times \mathbb{R}^n$ with an edge and such that

$$\Gamma = \{(y,z) \in \mathbb{R} \times \mathbb{R}^n \mid z \in K(y)\}$$

where for each y, $K(y)$ is a cone. Let $G(y)$ denote $K(y) \cap S^{n-1}$. We suppose that the family $(G(y))_y$ is analytic. And we consider, for instance, the Neumann problem for Δ on Γ. Then the exponents will be

$$\nu_{(k_1,k_2)}(y) = 1 - \frac{n}{2} + \sqrt{(1-\frac{n}{2})^2 + \lambda_{k_1}(y)} + k_2$$

where $k_1, k_2 \in \mathbb{N}$ and $(\lambda_{k_1}(y))_{k_1 \in \mathbb{N}}$ is the increasing sequence of eigenvalues of the Neumann problem for the (positive) Laplace-Beltrami operator on $G(y)$. As a consequence of a result of [3], there exists an analytic choice for the eigenvalues , i. e., analytic functions $\tilde{\lambda}_{k_1}$ such that for any y, the sequence $(\tilde{\lambda}_{k_1}(y))_{k_1 \in \mathbb{N}}$ is an enumeration of the eigenvalues with repetition according to the multiplicities. In general, this does not coincide with the enumeration in increasing order. Let us quote as an example the case when the $G(y)$ are spherical caps with opening $\alpha(y)$ where $0 < \alpha(y) < \pi$ (see §18 of [1]).

Our results can be applied to such a geometrical situation, with the corresponding appropriate choice of the exponents.

2.d Motivations. In Section 3, we will present the main results of this paper. Our motivations for their presentation are the following.

1. To give an asymptotics in the neighborhood of crossing points which is as explicit and as simple as possible
2. To eliminate as many technical hypotheses as possible.

To achieve these aims, we have chosen to treat in a first stage a class of problems which is restricted by the following two requirements:

1. Analyticity for the coefficients and the faces of the domain
2. No bifurcation points (see below).

This class of problems is sufficiently large to contain the examples described above, in particular the skew cylinder problems.

As already said, for second order operators with real coefficients, there exist analytic choices for the exponents. Indeed, this is also true for general elliptic second order operators with complex coefficients, because the poles of the resolvent of the associated operator pencil are always simple (see [1], §14 for instance). But such an analytic choice is generically impossible for fourth order operators such as the bilaplacian. The basic problem is the expression of the roots of a polynomial whose coefficients depend analytically on a parameter. The roots are algebraic but, in general, non-analytic functions of the parameter.

Such situations of bifurcations are studied in [12]. It would be interesting to give the actual structure of asymptotics for general elliptic boundary value problems. In the general case there appear combinations of both crossings and bifurcations. We think that even then it will be possible to reach the aims we described at the end of the first section, i. e., to separate all that can be separated.

3. ASYMPTOTICS AT CROSSING POINTS

3.a Ordering the exponents. Let y_0 be a crossing point, i. e., a point where there exist distinct k and k' such that

$$\nu_k(y_0) = \nu_{k'}(y_0) < s. \tag{3.1}$$

Since we assume that our cylinder is actually skew, crossing points are isolated, so there exist open intervals I and $\tilde{I}$ with $y_0 \in I$, $\overline{I} \subset \tilde{I}$, and there is no other crossing point in $\overline{\tilde{I}}$.

If the opening angle along the edge is constant (as it happens for the base of our cylinder or as it would be in the case of a plane circular crack), then if condition (3.1) is satisfied, it holds along the entire edge. In such a case we have a superposition and not a crossing, and the simple asymptotics of Theorem 2.1 is valid.

Let $\mathcal{K}_{y_0}$ be the set of indices,

$$\mathcal{K}_{y_0} := \{k = (k_1, k_2) \mid \nu_k(y_0) < s\}.$$

We denote by $\mu_1, \ldots, \mu_{j_0}$ the distinct elements of the set

$$\{\nu_k(y_0) \mid k \in \mathcal{K}_{y_0}\}.$$

Since y_0 is a crossing point, the cardinality of $\mathcal{K}_{y_0}$ is strictly larger than j_0. For each j, let $\mathcal{K}_{y_0,j}$ be the subset of $\mathcal{K}_{y_0}$,

$$\mathcal{K}_{y_0,j} := \{k \in \mathcal{K}_{y_0} \mid \nu_k(y_0) = \mu_j\}.$$

The μ_j are either crossing exponents (if $\#\mathcal{K}_{y_0,j} > 1$) or simple exponents (if $\#\mathcal{K}_{y_0,j} = 1$).

For each k, we call multiplicity of ν_k the maximal power of $\log r$ which appears in the asymptotics (2.4) along with the term $r^{\nu_k(y)}$ for $y \in I \setminus \{y_0\}$. Then we denote by $(k_j^q)_{1 \le q \le q_j}$ an enumeration of $\mathcal{K}_{y_0,j}$, repeating each term according to its multiplicity.

Finally, we set for $y \in \tilde{I}$:

$$\mu_j(y) := \max_{k \in \mathcal{K}_{y_0,j}} \nu_k(y). \tag{3.2}$$

3.b Direct formulation of asymptotics. What essentially changes from the simple asymptotics (2.4) is the behavior of the functions of r. Instead of having separately the terms $r^{\nu_l(y)} \log^p r$, we have now special combinations of these terms which cannot be separated. Let us introduce these combinations.

Definition 3.1 Let $q \ge 1$ an integer and $\nu_1, \ldots, \nu_q$ be complex numbers, not necessarily distinct. Let γ be any simple curve surrounding $\nu_1, \ldots, \nu_q$ in the complex plane. Then we define

$$S[\nu_1, \ldots, \nu_q; r] = \frac{1}{2\pi i} \int_\gamma \frac{r^\lambda}{(\lambda - \nu_1) \cdots (\lambda - \nu_q)} \, d\lambda.$$

Here are some examples. We assume that ν_1 is different from ν_2.

$$S[\nu_1; r] = r^{\nu_1} \tag{3.3}$$

$$S[\nu_1, \nu_1; r] = r^{\nu_1} \log r \tag{3.4}$$

$$S[\nu_1, \nu_2; r] = \frac{r^{\nu_1} - r^{\nu_2}}{\nu_1 - \nu_2} \tag{3.5}$$

$$S[\nu_1, \nu_1, \nu_2; r] = \frac{r^{\nu_1} \log r}{\nu_1 - \nu_2} - \frac{r^{\nu_1} - r^{\nu_2}}{(\nu_1 - \nu_2)^2} \tag{3.6}$$

When all the ν_l are distinct, we obtain

$$S[\nu_1,\ldots,\nu_q;r] = \sum_{l=1}^{q} \frac{r^{\nu_l}}{\prod\limits_{\substack{k=1\\k\neq l}}^{q}(\nu_l-\nu_k)}. \tag{3.7}$$

Remark 3.2 Example (3.5) gives (3.4) as a limit case for $\nu_2 \to \nu_1$. More generally, the function S is analytic in all its arguments on $\mathbb{C}^q \times (0,\infty)$. On the other hand, example (3.7) shows that the coefficients of the powers r^{ν_l} blow up near the points where two ν_l's coincide.

Theorem 3.3 *Let J, $\tilde{J}$, I and $\tilde{I}$ be defined as in Section* 3.a *and in Theorem* 2.1. *Let also $\mathcal{U}_J$ and χ be as in Theorem* 2.1. *We still assume that for some $\varepsilon_0 \geq 0$ condition* (2.2) *holds. To each $j=1,\ldots,j_0$ and to each $q=1,\ldots,q_j$, there exists a finite set of indices γ and analytic functions $\psi_{j,q,\gamma}(y,\theta)$ such that any solution u of problem $(\mathcal{P}_I)$ or $(\mathcal{P}_{II})$ can be decomposed into*

$$u = u_{\mathrm{reg}} + u_{\mathrm{sing}}.$$

Here $u_{\mathrm{reg}} \in H^{s+1-\varepsilon}(\mathcal{U}_J)$ $\forall\varepsilon > \varepsilon_0$ and

$$u_{\mathrm{sing}} = \sum_{j=1}^{j_0} v_j$$

with

$$v_j = \chi^{-1}\left(\sum_{q,\gamma}(\Phi * d_{j,q,\gamma})(y,r)\, S[\nu_{k_j^1}(y),\ldots,\nu_{k_j^q}(y);r]\,\psi_{j,q,\gamma}(y,\theta)\right). \tag{3.8}$$

The coefficients $d_{j,q,\gamma}(y)$ are defined on $\tilde{I}$ and satisfy $d_{j,q,\gamma} \in H^{s-\mu_j(y)-\varepsilon}(I)$ for all $\varepsilon > 0$.

Remark 3.4 If there is no crossing in $\tilde{I}$, then this statement yields the same result as Theorem 2.1. Indeed, the sets $\mathcal{K}_{y_0,j}$ are all reduced to one element and the functions in r are all of the form $S[\nu,\ldots,\nu;r]$, i. e. $r^\nu \log^q r$.

Remark 3.5 In a generic way, the coefficients $d_{j,k,\gamma}$ do not vanish at y_0 (see the example in Subsection 3.d below). As a consequence, if $\mathcal{K}_{y_0,j}$ has more than one element, the coefficients $c_{k,q,\beta}$ for $k \in \mathcal{K}_{y_0,j}$ tend to infinity at y_0, in general.

3.c Bundle formulation of asymptotics. As in Subsection 3.a, we denote by I a neighborhood of the crossing point y_0. For any $y \in I \setminus \{y_0\}$ and $k \in \mathcal{K}_{y_0}$, let $B_k(y)$ be the vector space spanned by the functions of (r,θ) which occur in the simple asymptotics (2.4) in the terms corresponding to the exponents $\nu_k(y)$:

$$B_k(y) := \operatorname{span}\left\{r^{\nu_k(y)} \log^q r\, \varphi_{k,q,\beta}(y,\theta)\right\}.$$

The $y \mapsto B_k(y)$ define analytic bundles over $I \setminus \{y_0\}$. The following theorems give another description of what happens at the crossing points.

Theorem 3.6 *The sum*

$$y \mapsto \bigoplus_{k \in \mathcal{K}_{y_0}} B_k(y)$$

which is defined on $I \setminus \{y_0\}$, extends as an analytic bundle on I.

It is possible to consider smaller sums, each of which corresponds to a single crossing exponent μ_j: Define C_j by

$$C_j(y) := \bigoplus_{k \in \mathcal{K}_{y_0,j}} B_k(y).$$

Theorem 3.7 *For any $j = 1, \ldots, j_0$, the bundle $y \mapsto C_j(y)$ extends as an analytic bundle on I.*

Remark 3.8 It is an open problem whether the bundles B_k themselves extend to analytic bundles on I. We have solved this problem in one special case: Let B_{k_1} and B_{k_2} be one-dimensional bundles over $I \setminus \{y_0\}$, given by

$$B_{k_l}(y) = \operatorname{span}\left\{r^{\nu_{k_l}(y)} \varphi_{k_l}(y, \theta)\right\} \quad (l = 1, 2).$$

If their sum extends to an analytic bundle on I, then both B_{k_1} and B_{k_2} are analytic on I.

Even if we knew that the bundles B_k extend to analytic bundles on I, the statements of the Theorems 3.6 and 3.7 would not be trivial: The sum of two analytic bundles is not always an analytic bundle. There may occur a collapse in dimension. For instance, if $B_1(y)$ is generated by $r^{\nu_1(y)}\varphi(\theta)$ and $B_2(y)$ by $r^{\nu_2(y)}\varphi(\theta)$, and if $\nu_1(y) = \nu_2(y)$ only in $y = y_0$, then $B_1 + B_2$ collapses in y_0. But a sum of analytic bundles can always be extended as an analytic bundle. In our example, $B_1(y_0) + B_2(y_0)$ has to be replaced by $\operatorname{span}\left\{r^{\nu_1(y)}\varphi(\theta),\, r^{\nu_1(y)} \log r\, \varphi(\theta)\right\}$. A trivialization of this extension is given by

$$\left\{r^{\nu_1(y)}\varphi(\theta),\, \frac{r^{\nu_1(y)} - r^{\nu_2(y)}}{\nu_1(y) - \nu_2(y)}\, \varphi(\theta)\right\}.$$

This extension property does not hold, in general, for C^∞-bundles. Let us give a simple example of bundles with values in $\mathbb{R}^3$:

$$\begin{aligned} B(y) &= \operatorname{span}\{(1,0,0)\} \\ B'(y) &= \operatorname{span}\left\{(1, e^{-1/y^2}\cos\frac{1}{y}, e^{-1/y^2}\sin\frac{1}{y})\right\}. \end{aligned}$$

The sum $B(y) + B'(y)$ cannot be extended to a C^∞-bundle in $y = 0$.

We start now from the analytic extension $\tilde{C}_j(y)$ of C_j. Let $y \mapsto X_{j,\alpha}(y, \cdot, \cdot)$ for $\alpha = 1, \ldots, A_j$ define a trivialization of $\tilde{C}_j$.

As a consequence of the form of the B_k's, we get the following lemma.

Lemma 3.9 *There exist analytic functions $\psi^{j,\alpha}_{q,\gamma}(y,\theta)$ such that*

$$X_{j,\alpha}(y,r,\theta) = \sum_{q,\gamma} S[\nu_{k^1_j}(y),\ldots,\nu_{k^q_j}(y);r]\,\psi^{j,\alpha}_{q,\gamma}(y,\theta).$$

All the objects discussed in this subsection up to now are only linked with the geometrical framework (domain and boundary value problem). The singular parts v_j in Theorem 3.3 can now be written as a kind of regularized $H^{s-\varepsilon}$-sections of the bundles $\tilde{C}_j$:

$$v_j = \chi^{-1}\left(\sum_{\alpha=1}^{A_j}(\Phi * b_{j,\alpha})(y,r)\,X_{j,\alpha}(y,r,\theta)\right)$$

with $b_{j,\alpha}(y) \in H^{s-\mu_j(y)-\varepsilon}(I)$.

3.d An example. Let us illustrate our statements by a simple example. We consider the mixed problem $(\mathcal{P}_I)$. We take $s \in (1, 2/(1+\alpha))$, where α is the "obliquity" of the skew cylinder (see Section 2.b). Since there is one zero Dirichlet condition, $\nu_{(0,0)}$ does not appear. With that choice of s, only $\nu_{(1,0)}$ and $\nu_{(0,1)}$ are relevant. For simplicity, let us denote

$$\begin{aligned} \nu_1(y) &:= \nu_{(1,0)}(y) = \frac{\pi}{2\omega(y)} \\ \nu_2(y) &:= \nu_{(0,1)}(y) = 1. \end{aligned}$$

There are exactly two points $y \in M$ where $\nu_1(y) = \nu_2(y)$ holds. These are the two points y_0, y_0' where $\omega(y) = \pi/2$. On $M \setminus \{y_0, y_0'\}$, the simple asymptotics (2.4) holds. Here $q = 0$ and only one value of β is required. We write l instead of $(l,0,1)$. Then we can choose

$$\begin{aligned} \varphi_1(y,\theta) &= \cos\nu_1(y)\theta \\ \varphi_2(y,\theta) &= \sin(\omega(y)-\theta). \end{aligned}$$

Here it is possible to compute $c_2(y)$ since it depends only on the pointwise value of the boundary datum h_2 on the edge:

$$c_2(y) = \frac{h_2(y,0)}{\cos\omega(y)}.$$

We have the precise regularity result $c_2 \in H^{s-1}_{\mathrm{loc}}(M \setminus \{y_0, y_0'\})$.

Concernig the bundle representation of the asymptotics, we observe the following facts.

$B_1(y)$ is generated by $r^{\nu_1(y)}\cos\nu_1(y)\theta$.
$B_2(y)$ is generated by $r\,\sin(\omega(y)-\theta)$.
When $y = y_0$ or y_0', then $B_1(y) = B_2(y)$.

A basis of the analytic extension of $B_1 + B_2$ is given by

$$
\begin{aligned}
X_1(y,r,\theta) &= r^{\nu_1(y)} \cos \nu_1(y)\theta. \\
X_2(y,r,\theta) &= \frac{r \sin(\omega(y)-\theta) - r^{\nu_1(y)} \cos \nu_1(y)\theta}{1-\nu_1(y)} \\
&= \frac{r - r^{\nu_1(y)}}{1-\nu_1(y)} \sin(\omega(y)-\theta) \\
&\quad + r^{\nu_1(y)} \frac{\sin(\omega(y)-\theta) - \cos \nu_1(y)\theta}{1-\nu_1(y)} \\
&= \frac{r - r^{\nu_1(y)}}{1-\nu_1(y)} \cos \nu_1(y)\theta \\
&\quad + r \frac{\sin(\omega(y)-\theta) - \cos \nu_1(y)\theta}{1-\nu_1(y)}.
\end{aligned}
$$

The different forms of X_2 correspond to different possible orders of enumeration in the representation of Lemma 3.9.

If we want to get the direct representation of Theorem 3.3, we need 3 basis functions, for instance:

$$
\begin{aligned}
S[\nu_1(y); r]\, \psi_{1,1}(y,\theta) &= X_1(y,r,\theta) \\
S[\nu_1(y); r]\, \psi_{1,2}(y,\theta) &= r^{\nu_1(y)} \frac{\sin(\omega(y)-\theta) - \cos \nu_1(y)\theta}{1-\nu_1(y)} \\
S[\nu_1(y), \nu_2(y); r]\, \psi_{2,1}(y,\theta) &= \frac{r - r^{\nu_1(y)}}{1-\nu_1(y)} \sin(\omega(y)-\theta).
\end{aligned}
$$

Now we can compare the three representations of a singular part, namely the "simple asymptotics" of Theorem 2.1, the "direct representation" of Theorem 3.3, and the "bundle representation" with the basis X_1, X_2. Assume that we have

$$
\begin{aligned}
c_1\, r^{\nu_1} \varphi_1 + c_2\, r^{\nu_2} \varphi_2 &= d_{1,1}\, S[\nu_1; r]\, \psi_{1,1} + d_{1,2}\, S[\nu_1; r]\, \psi_{1,2} + d_{2,1}\, S[\nu_1, \nu_2; r]\, \psi_{2,1} \\
&= b_1\, X_1 + b_2\, X_2.
\end{aligned}
$$

Then there hold the following relations between the coefficients.

$$
\begin{aligned}
b_1 &= c_1 + c_2 \\
b_2 &= c_2(1-\nu_1) \\
c_1 &= b_1 - b_2/(1-\nu_1) \\
c_2 &= b_2/(1-\nu_1) \\
d_{1,1} &= b_1 \\
d_{1,2} &= d_{2,1} = b_2.
\end{aligned}
$$

These relations clearly display the blow-up of the coefficients in the simple asymptotics at the points y_0, y_0' and also the necessity for the introduction of the exponents $\mu_j(y)$ in (3.2).

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