SOLVABILITY OF A SYSTEM OF INTEGRAL EQUATIONS FOR CLAMPED PLATES

MARTIN COSTABEL & MONIQUE DAUGE
U.R.A. 305 du C.N.R.S. – IRMAR – Université de Rennes 1
Campus de Beaulieu – 35042 Rennes Cedex 03, France

1. Introduction

We study the unique solvability of a 2×2 system of boundary integral equations arising from a single layer potential representation for the biharmonic Dirichlet problem. We answer the question *"for which curves does this system have a unique solution ?"* as follows: For each curve Γ there are between 1 and 4 *exceptional scale factors* ρ such that on the scaled curve $\rho\Gamma$ there is no unique solvability. We show results of numerical computations of the exceptional scale factors for several classes of curves.

This paper is a short version of [1], where detailed proofs are given. We present here some additional numerical results.

2. The biharmonic single layer operator

Let $\Omega^- \subset \mathbb{R}^2$ be a bounded domain with boundary Γ and exterior $\Omega^+ = \mathbb{R}^2 \setminus \overline{\Omega^-}$. The biharmonic Dirichlet problem

$$\left\{ \begin{array}{rcll} \Delta^2 u &=& 0 & \text{in } \Omega^- \quad (\text{or in } \Omega^+) \\ u &=& g_0 & \text{on } \Gamma \\ \partial_n u &=& g_1 & \text{on } \Gamma, \end{array} \right. \tag{1}$$

can be solved using the system of integral equations on Γ

$$\left\{ \begin{array}{rcl} \displaystyle\int_\Gamma \Big\{ G(x,y)\,\varphi_0(y) + \partial_{n(y)} G(x,y)\,\varphi_1(y) \Big\}\, ds(y) &=& g_0(x) \\ \displaystyle\int_\Gamma \Big\{ \partial_{n(x)} G(x,y)\,\varphi_0(y) + \partial_{n(x)}\partial_{n(y)} G(x,y)\,\varphi_1(y) \Big\}\, ds(y) &=& g_1(x), \end{array} \right. \tag{2}$$

whose solution (φ_0, φ_1) provides a couple of densities allowing the representation of u as the following "single layer potential":

$$u(x) = \int_\Gamma \Big\{ G(x,y)\,\varphi_0(y) + \partial_{n(y)} G(x,y)\,\varphi_1(y) \Big\}\, ds(y), \quad x \in \Omega^- \cup \Omega^+. \tag{3}$$

In (2) and (3), G is a fundamental solution for Δ^2 in $\mathbb{R}^2$. Here, we take the standard fundamental solution

$$G(x,y) = \frac{1}{8\pi}\,|x-y|^2\log|x-y|.$$

We write system (2) in the condensed form $V_\Gamma\boldsymbol{\varphi} = \boldsymbol{g}$ and we call the integral operator V_Γ *"the biharmonic single layer potential operator"* on the curve Γ. This operator appears not only in the solution of the clamped plate problem by a single layer potential ansatz, as in (2), but also in the solution of that problem by a direct method and also in discretizations of the Poincaré-Steklov operator for the bilaplacian, used for example in coupling methods of finite and boundary elements or in other domain decomposition methods (see [3]).

In this paper, we exhibit conditions on the curve Γ that guarantee the invertibility of V_Γ. Our results are a generalization to the biharmonic equation of results well-known for Symm's integral equation of potential theory:

$$S_\Gamma\varphi \equiv -\frac{1}{2\pi}\int_\Gamma \log|x-y|\,\varphi(y)\;ds(y) = g(x).$$

There, the operator S_Γ is invertible if and only if Γ has a logarithmic capacity $\operatorname{cap}\Gamma \neq 1$ (i.e. if Γ is not a "Γ contour" in Symm's language). Thus for each Γ there is exactly one ρ, namely $\rho = 1/\operatorname{cap}\Gamma$, such that $S_{\rho\Gamma}$ is not invertible.

3. The finite dimensional reduction

For $\boldsymbol{f} = (f_0, f_1)$ in the Sobolev space $H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$ and $\boldsymbol{\varphi} = (\varphi_0,\varphi_1)$ in its dual $H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma)$, the natural product of duality is given by

$$\langle\boldsymbol{\varphi},\boldsymbol{f}\rangle = \int_\Gamma \varphi_0\, f_0\;ds + \int_\Gamma \varphi_1\, f_1\;ds.$$

Let $P_0 = 1$, $P_1 = x_1$ and $P_2 = x_2$ be a basis of the space of first degree polynomials and let $\boldsymbol{p}_j$ denote the pair $(P_j, \partial_n P_j)$. It is known, see [2], that for $\boldsymbol{\varphi} = (\varphi_0,\varphi_1)$ in the subspace of codimension 3, defined by 3 equilibrium conditions,

$$\Big\{(\varphi_0,\varphi_1)\in H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma) \mid \quad \langle\boldsymbol{\varphi},\boldsymbol{p}_j\rangle = 0,\quad j=0,1,2\Big\} \tag{4}$$

the operator V_Γ defines a positive bilinear form:

$$\langle\boldsymbol{\varphi}, V_\Gamma\boldsymbol{\varphi}\rangle = \int_{\Omega^-\cup\Omega^+} |\Delta u|^2\,dx,$$

with u defined by (3). From this, one can show that the following augmented system (with $P_3 = x_1^2 + x_2^2$ and $\boldsymbol{p}_3 = (P_3, \partial_n P_3)$)

$$\begin{cases} V_\Gamma \boldsymbol{\varphi} = \sum_{j=0}^{3} \omega_j \boldsymbol{p}_j + \boldsymbol{g} \\ \langle \varphi, \boldsymbol{p}_k \rangle = \xi_k \quad (k = 0, \dots, 3) \end{cases} \tag{5}$$

has, for given $(\boldsymbol{g}, \vec{\xi})$ in $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma) \times \mathbb{R}^4$, always a unique solution $(\boldsymbol{\varphi}, \vec{\omega})$ in $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \times \mathbb{R}^4$. For $\boldsymbol{g} = 0$, this gives a linear relation between $\vec{\xi}$ and $\vec{\omega}$, written as:

$$\vec{\omega} = B_\Gamma \vec{\xi}.$$

We added the fourth polynomial P_3 and the fourth equilibrium condition in (5) in order to get the simple scaling behavior of B_Γ that follows. One can show

Theorem: (i) *For any curve* Γ *(open, closed, regular or not), the operator* V_Γ *is invertible if and only if the* 4×4 *matrix* B_Γ *is invertible.*
(ii) *For* $\rho > 0$ *one has*

$$B_{\rho\Gamma} = \mathrm{diag}(\rho, 1, 1, \rho^{-1}) \left(B_\Gamma + \frac{\log \rho}{8\pi} C \right) \mathrm{diag}(\rho, 1, 1, \rho^{-1}).$$

with the matrix

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(iii) ρ *is an exceptional scale factor for* Γ *if and only if*

$$\rho = e^{-8\pi\lambda},$$

where λ *is a real eigenvalue of the matrix* $C^{-1} B_\Gamma$.

4. Numerical results

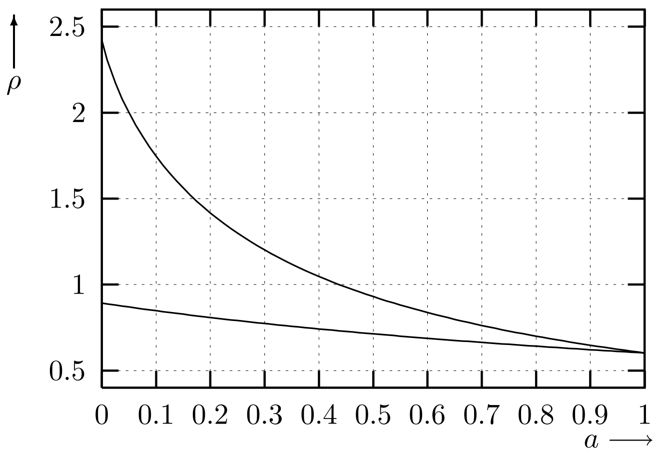
Analytical results. We have the following:
(i) Unit circle: 1 exceptional scale factor $\rho = 1/e$
(ii) Unit interval $\Gamma = [0, 1]$: 2 exceptional scale factors $\rho_1 = 4e^{-1/2} = 2.426...$ and $\rho_2 = 4e^{-3/2} = 0.8925...$
Numerical computations. For a given curve Γ, we solve the integral equation

$$\begin{cases} V_\Gamma \boldsymbol{\varphi} - \sum_{j=0}^{3} \omega_j \boldsymbol{p}_j = 0 \\ \langle \boldsymbol{\varphi}, \boldsymbol{p}_k \rangle = \delta_{kl} \quad (k = 0, \dots, 3)\,. \end{cases} \tag{6}$$

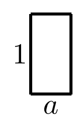
For $l = 0, \ldots, 3$, the vectors $\vec{\omega} \in \mathbb{R}^4$ constitute the columns of the matrix B_Γ whose eigenvalues determine the exceptional scale factors.

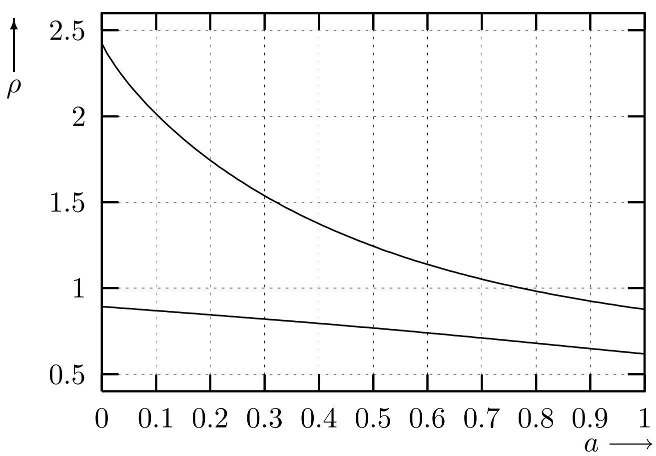
For the numerical approximation of (6), we use a very simple method: for the approximation of $\boldsymbol{\varphi}$ we choose for φ_0 Dirac delta functions in the mesh points of some partition of Γ, and for φ_1 piecewise constant functions with break points in the same mesh points. If Γ has corners, we use a mesh refinement at the corners. The first integral equation in (6) (this corresponds to the first line of the system (2)), is discretized by a Galerkin method using as test functions the same delta functions as for the approximation of φ_0. This corresponds therefore to nodal collocation. For the second integral equation we do not test by the piecewise constant functions, but we use the simpler mid-point collocation.

For the following computations, we chose typically about $N = 160$ nodes and a mesh refinement with exponent 2 at the corners of the polygon.

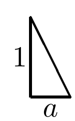


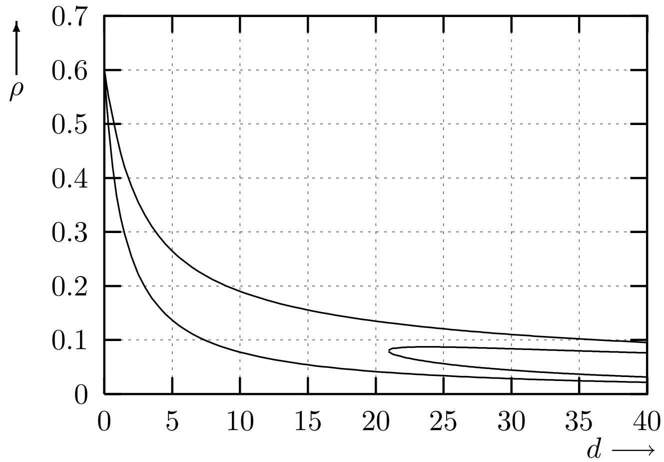
Rectangle, side lengths 1 and $a \in [0,1]$.



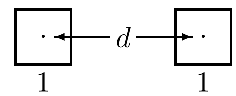


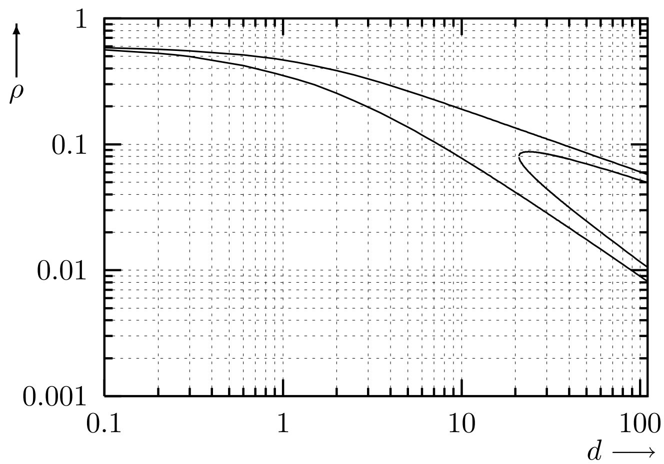
Rectangular triangle, side lengths 1 and $a \in [0,1]$.



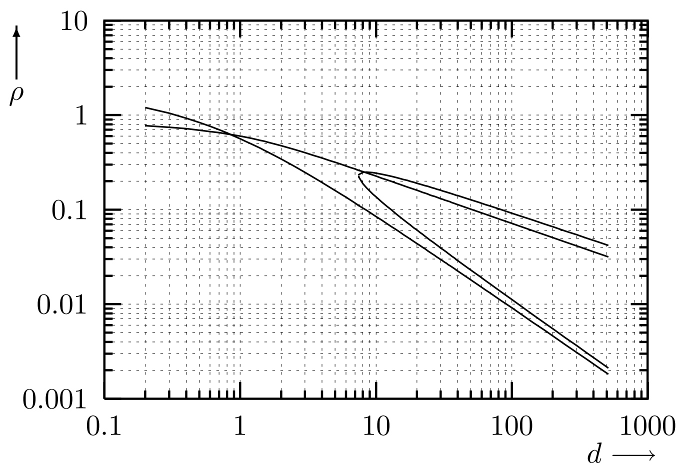


2 unit squares,
distance
$d \in [0, 40]$.

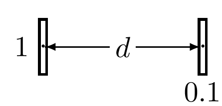




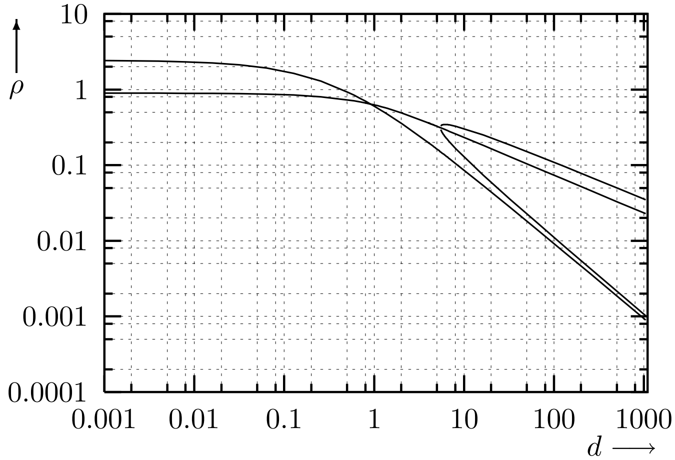
2 unit squares
(log scale)
distance
$d \in [0.1, 100]$.



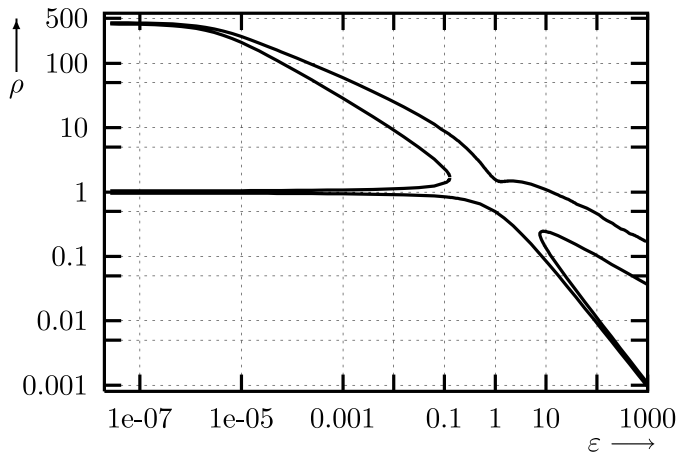
2 rectangles,
height 1,
width 0.1



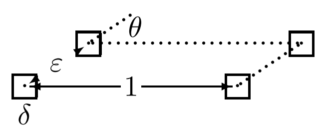
horizontal
distance
$d \in [0.2, 500]$.



2 “intervals”,
height 1,
width 10^{-6}
horizontal
distance
$d \in [10^{-3}, 10^{3}]$.



4 squares,
side lengths
$\delta = 2 \cdot 10^{-6}$,
$\varepsilon \in [2 \cdot 10^{-8}, 10^{3}]$
$\theta = 30°$



The results show that between 1 and 4 values of ρ can appear and that double or triple points exist. They show also the extraordinary robustness of this simple boundary element method: One and the same simple program with the same number of nodes works fine while one of the dimensions of the domain varies over 10 orders of magnitude!

References

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3. G. Schmidt, B. N. Khoromskij. Boundary integral equations for the biharmonic Dirichlet problem on non smooth domains. Preprint 129, WIAAS, Berlin 1994.