GENERAL EDGE ASYMPTOTICS OF SOLUTIONS OF SECOND ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS II.

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Abstract

This is the second of 2 papers in which we study the singularities of solutions of second order linear elliptic boundary value problems at the edges of piecewise analytic domains in $\mathbb{R}^3$. When the opening angle at the edge is variable, there appears the phenomenon of "crossing" of the exponents of singularities. In Part I, we introduced for the Dirichlet problem appropriate combinations of the simple tensor product singularities.

In this second part, we extend the results of Part I to general non-homogeneous boundary conditions. Moreover, we show how these combinations of singularities appear in a natural way as sections of an analytic vector bundle above the edge. In the case when the interior operator is the Laplacian, we give a simpler expression of the combined singular functions, involving divided differences of powers of a complex variable describing the coordinates in the normal plane to the edge.

INTRODUCTION.

We continue in this paper the investigation of asymptotics along analytic edges which we began in Part I [1]. The results are generalizations of those announced in [2].

We use "natural" local coordinates to describe these asymptotics : roughly speaking, y denotes a curvilinear abscissa along an edge, r is the distance to the edge and θ is an angular variable. Coming from corresponding problems in two-dimensional sectors, the Ansatz for the singularity type is

$$\varphi_{\kappa,q}(y,\theta)\, r^{\nu_\kappa(y)} \log^q r \tag{0.1}$$

where the $\nu_\kappa(y)$ are the singularity exponents at the point y (they are analytic functions, combinations of integer numbers and of eigenvalues of the Sturm–Liouville problem $\mathbb{M}_y$ which is associated to the principal conormal part of the operator in y) and where the functions $\varphi_{\kappa,q}$ are analytic (they arise from eigenfunctions of $\mathbb{M}_y$ and from the lower parts of the operator).

We proved in Part I that such an Ansatz is correct away from any "crossing point" of the singularity exponents, i. e. away from any isolated point y_0 where two of the exponents coincide ($\nu_\kappa(y_0) = \nu_{\kappa'}(y_0)$ with $\kappa \neq \kappa'$). We also proved (and this was the main point of our work) that in the neighborhood of such a crossing point, the functions (0.1) have to be replaced by

$$\psi_\alpha(y,\theta)\, S[\mu_1^\alpha(y),\ldots,\mu_{q_\alpha}^\alpha(y);r] \tag{0.2}$$

where the functions ψ_α are analytic and the functions $S[\ldots;r]$ are the divided differences of the function $\lambda\mapsto r^\lambda$ at the points $\mu_1^\alpha(y),\ldots,\mu_{q_\alpha}^\alpha(y)$ which are some of the exponents $\nu_\kappa(y)$ meeting in y_0. Let us recall (compare Part I, § **??**) that, when $\mu_1,\ldots,\mu_K$ are all distinct, the divided difference of w at the K-tuple $\mu_1,\ldots,\mu_K$ is defined by the classical recursion formula :

$$w[\mu_1] = w(\mu_1) \tag{0.3}$$

and for $j=2,\ldots,K$

$$w[\mu_1,\ldots,\mu_j] = \frac{1}{\mu_1-\mu_j}\left(w[\mu_1,\ldots,\mu_{j-1}] - w[\mu_2,\ldots,\mu_j]\right). \tag{0.4}$$

Moreover for analytic functions w one has for any $\mu_1,\ldots,\mu_K$ not necessarily distinct

$$w[\mu_1,\ldots,\mu_K] = \frac{1}{2i\pi}\int_\gamma \frac{w(\lambda)}{\displaystyle\prod_{j=1}^K(\lambda-\mu_j)}\,d\lambda \tag{0.5}$$

where γ is a simple curve surrounding all μ_j. Thus we have

$$S[\mu_1,\ldots,\mu_K;r] = \frac{1}{2i\pi}\int_\gamma \frac{r^\lambda}{\displaystyle\prod_{j=1}^K(\lambda-\mu_j)}\,d\lambda\,. \tag{0.6}$$

We see that $S[\mu_1(y),\ldots,\mu_K(y);r]$ with analytic $\mu_1(y),\ldots,\mu_K(y)$ is a linear combination of terms of the form $r^{\mu_j(y)}\log^q r$ with coefficients that are meromorphic in y. If all $\mu_j(y)$ are equal to the same $\mu(y)$ then

$$S[\underbrace{\mu,\ldots,\mu}_{q+1\text{ times}};r] = \frac{1}{q!}\,r^\mu\log^q r\,. \tag{0.7}$$

Thus the singular functions of type (0.2) are linear combinations of singular functions of type (0.1) with coefficients depending meromorphically on y.

We have proved our results for the special case of the Dirichlet problem for strongly elliptic operators in Part I. In this Part II, we show how all these results can be extended to more general second order elliptic boundary value problems (§ 1).

Afterwards, we investigate more closely the singularity types (0.1) and (0.2). We describe algorithms for their construction (§ 2). We show them from a new point of view, by proving that they can be considered as sections of some analytic vector bundles over the edge (§ 3). We will use certain facts about such bundles which we have gathered in an appendix (§ 5).

In the case when the interior operator is the Laplacian Δ, we show that it is possible to give simpler and more explicit formulas for the singularity types (0.1) and (0.2). Such formulas are inspired by the paper by Maz'ya and Rossmann [11] where they investigate the question of obtaining asymptotics in two-dimensional cones which smoothly depend on the opening angle. These new formulas are based on the divided differences of the function

$$\lambda \mapsto \zeta^\lambda \quad \text{where} \quad \zeta = r\, e^{i\theta} \in \mathbb{C}.$$

Then the functions corresponding to $\varphi_{\kappa,q}(y,\theta)$ and $\psi_\alpha(y,\theta)$ are simply powers of $e^{-2i\theta}$, see Theorem 4.1 in § 4.

We refer to the equation numbers and statement labels of Part I by the adjunction of "I.".

1. GENERAL BOUNDARY CONDITIONS

Just as in Part I, the domains we consider are three-dimensional bounded Lipschitz domains Ω with piecewise analytic boundary and analytic edges. For such a domain, there exists an analytic manifold M of dimension 1 and without boundary such that $\partial\Omega \setminus M$ is the disjoint union of a finite number of connected components $\partial_j\Omega$, which are analytic manifolds of dimension 2 and with boundary. M is the union of the *edges* and the $\partial_j\Omega$ are the *faces*. We assume that near any $y \in M$, Ω is analytically diffeomorphic to a dihedral angle.

In each point y of M, let $\omega(y)$ be the opening of Ω in y : more precisely, $\omega(y)$ is the angle between the two tangent planes to $\partial\Omega$ at y. We also admit some line of discontinuity for the boundary conditions or the boundary data, where $\omega(y) \equiv \pi$ in the whole connected component of M which contains such a line.

Let

$$A(x;\partial_x) = \sum_{|\alpha|\le 2} a_\alpha(x)\partial_x^\alpha$$

be an elliptic second order operator with complex coefficients, analytic on $\overline{\Omega}$. On each face $\partial_j\Omega$ let be given an operator $\mathcal{A}_j$ of order $m_j \in \{0,1\}$ with analytic coefficients. We assume that each of these operators $\mathcal{A}_j$ covers A on $\overline{\partial_j\Omega}$. If $m_j = 0$, we can assume without restriction that $\mathcal{A}_j$ is the identity, thus we have the Dirichlet condition there. If $m_j = 1$, this is in general an oblique derivative operator with possibly a term of order 0. Let us set

$$m = \max_j m_j\,.$$

We want to describe the structure of solutions of the following boundary value problem :

$$\left\{\begin{array}{ll} Au = f & \text{in } \Omega\,, \\ \mathcal{A}_j u = g_j & \text{in } \partial_j\Omega\,. \end{array}\right. \tag{1.1}$$

We assume regularity hypotheses on the right hand sides : for a positive real number s (with $s > 1/2$ if $m = 1$)

$$\left\{\begin{array}{l} f \in H^{s-1}(\Omega)\,, \\ g_j \in H^{s-m_j+1/2}(\partial_j\Omega)\,, \\ \forall j, j' \text{ s.t. } m_j = m_{j'} = 0,\ \forall y \in \overline{\partial_j\Omega} \cap \overline{\partial_{j'}\Omega} \quad g_j(y) = g_{j'}(y)\,. \end{array}\right. \tag{1.2}$$

This compatibility condition between Dirichlet data insures the existence of $u \in H^1(\Omega)$ such that $u = g_j$ in $\partial_j\Omega$ for any j such that $m_j = 0$.

For any nonnegative $s > m - \frac{1}{2}$, the operator $\mathbb{A} := (A, \mathcal{A}_j)$ makes sense on $H^{s+1}(\Omega)$ and it acts continuously from $H^{s+1}(\Omega)$ into the product

$$H^{s-1}(\Omega) \times \prod_j H^{s-m_j+1/2}(\partial_j\Omega) =: \mathbb{H}^{s-1}(\Omega).$$

When $m = 1$ and when a variational formulation is possible on H^1 (for instance for the Laplace operator with Neumann conditions), by the use of duality we change the definitions of $\mathbb{A}$ and $\mathbb{H}^{s-1}(\Omega)$ for $0 \le s < 1/2$ so that $\mathbb{A}$ still acts continuously on $H^{s+1}(\Omega)$. When a "semi-variational" formulation is possible (for instance for the Laplace operator with oblique derivative conditions), there exist also natural definitions for $0 < s < 1/2$. See for instance [5]. We set :

- in the case of a variational formulation : $\beta_0 = -1/2$
- in the case of a semi-variational formulation : $\beta_0 = 0$
- when no such formulation is possible : $\beta_0 = m - 1/2$.

Now we adopt correct definitions for $\mathbb{A}$ and $\mathbb{H}^{s-1}$ so that for any nonnegative $s > \beta_0$, $\mathbb{A}$ is continuous from H^{s+1} into $\mathbb{H}^{s-1}$.

In order to define certain principal conormal operators associated to each point y of an edge $E \subset M$, we need the introduction of special systems of coordinates. We fix y as a curvilinear abscissa along E. We denote by Π_y the orthogonal plane to E at the point y. An *admissible system of coordinates* is a local analytic map $x \mapsto (y, z)$ such that

$$\left\{\begin{array}{rcl} x \in \Pi_y \cap \mathcal{U} & \Longleftrightarrow & y(x) = y \text{ and } z(x) \in \Gamma_y \cap \mathcal{U}_y \\ x \in E \cap \mathcal{U} & \Longleftrightarrow & z(x) = 0 \end{array}\right.$$

where Γ_y is a plane sector, $\mathcal{U}$ is a neighborhood of y and $\mathcal{U}_y$ is a neighborhood of 0. To any such admissible system of coordinates $(y, z) =: (y, z_1, z_2)$ are associated the cylindrical coordinates (y, r, θ), where

$$r = \sqrt{z_1^2 + z_2^2} \quad \text{and} \quad \theta = \operatorname{Arctan} \frac{z_2}{z_1}.$$

Let Γ^{ω} be the plane sector with opening ω. The following two types of choice are possible for the family of plane sectors Γ_y :

1. The sector Γ_y is equal to $\Gamma^{\omega(y)}$. This choice will be used to obtain more information about the case when the interior operator is the Laplacian.

2. The sector Γ_y is equal to Γ^{ω}, where ω is fixed. It is possible to choose the same ω for all points $y \in E$. It suffices that ω belongs to the same set as any of the $\omega(y)$, $y \in E$ among the three following ones $(0, \pi)$, $\{\pi\}$ and $(\pi, 2\pi)$.

Let $(y, z) \mapsto (\tilde{y}, \tilde{z})$ be a change of admissible system of coordinates. We have

$$\tilde{y}(y, z) = y \quad \forall y, z \tag{1.3}$$

and

$$\tilde{z}(y, 0) = 0 \quad \forall y. \tag{1.4}$$

Relations (1.3) yield the following properties for the Jacobian matrix $J(y, z)$ which we write in block form :

$$J(y, z) = \begin{pmatrix} \frac{\partial \tilde{y}}{\partial y} & \frac{\partial \tilde{z}}{\partial y} \\ \frac{\partial \tilde{y}}{\partial z} & \frac{\partial \tilde{z}}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\partial \tilde{z}}{\partial y} \\ 0 & \frac{\partial \tilde{z}}{\partial z} \end{pmatrix}.$$

Moreover, relation (1.4) shows that we have on the edge

$$J(y, 0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\partial \tilde{z}}{\partial z} \end{pmatrix}. \tag{1.5}$$

Now we are able to introduce the three *conormal* operators $\mathbb{B}_0^{\pm}(y; \partial_z)$ and $\mathbb{M}(y; \partial_z)$ which have the following role :

1. The two operators $\mathbb{B}_0^{\pm}(y; \partial_z)$ come from the principal conormal operator valued symbol of $\mathbb{A}$. We will make hypotheses of injectivity on them. Roughly speaking $\mathbb{A}$ is supposed to be *injectively elliptic* along the edge.

2. The operator $\mathbb{M}(y; \partial_z)$ generates the analytic family of operators (here Sturm–Liouville problems) whose eigenvalues are the leading exponents of the singularities.

In the neighborhood of any point $y_0 \in M$, we choose an admissible system of coordinates : y in an interval I' and z in a neighborhood of 0 in Γ_y. The operator $\mathbb{A}$ is transformed into a triple $\mathbb{B} := (B, \mathcal{B}_1, \mathcal{B}_2)$ where B is the interior operator and $\mathcal{B}_1, \mathcal{B}_2$ are the boundary operators on the two faces which meet the edge in y_0. Let m_j be the order of $\mathcal{B}_j$.

Let $\mathbb{B}_0(y; \partial_y, \partial_z)$ denote the principal part of $\mathbb{B}$ frozen on the edge :

$$\mathbb{B}_0(y; \partial_y, \partial_z) = (pp\, B, pp\, \mathcal{B}_1, pp\, \mathcal{B}_2)(y, 0; \partial_y, \partial_z)$$

where $pp\, B$, resp. $pp\, \mathcal{B}_j$ is the homogeneous part of degree 2, resp. m_j.

We need the following three operators defined on the sector Γ_y :

$$\mathbb{B}_0^{\pm}(y; \partial_z) := \mathbb{B}_0(y; \pm i, \partial_z). \tag{1.6}$$

and

$$\mathbb{M}(y; \partial_z) := \mathbb{B}_0(y; 0, \partial_z). \tag{1.7}$$

From the operator $\mathbb{M}$ we construct in the standard way (see [6]) the holomorphic family of operators

$$\mathbb{M}_y(\lambda) : H^2(0, \omega_y) \longrightarrow L^2(0, \omega_y) \times \mathbb{C}^2.$$

where ω_y is the opening of the sector Γ_y. $\mathbb{M}_y(\lambda)$ consists of an interior operator $\mathcal{M}_y(\lambda)$ and two boundary operators in $\theta = 0$ and $\theta = \omega_y$. The interior operator is constructed as follows : we write the interior operator $M_y(y; \partial_z)$ in cylindrical coordinates

$$r^2\, M(y; \partial_z) = \mathcal{M}(y, \theta; r\partial_r, \partial_\theta) \tag{1.8}$$

and $\mathcal{M}_y(\lambda)$ is the Mellin symbol of $\mathcal{M}(y)$:

$$\mathcal{M}_y(\lambda) = \mathcal{M}(y, \theta; \lambda, \partial_\theta)\,. \tag{1.9}$$

The boundary operators are constructed in the same way, taking into account their order m_j.

For each fixed $y \in I'$, $\mathbb{M}_y(\lambda)$ is invertible except on a countable set, the spectrum, which we denote $\mathrm{Sp}(\mathbb{M}_y)$. The set of the real parts of $\mathrm{Sp}(\mathbb{M}_y)$ is denoted by $\mathrm{ReSp}(\mathbb{M}_y)$. Later in this section, we are going to give information about the structure of this spectrum.

For the injectivity conditions, we need the ordinary Sobolev spaces $H^s(\Gamma)$ and also the weighted spaces $E_0^s(\Gamma)$ (see (I.**??**)) :

$$E_0^s(\Gamma) = \{v \in H^s(\Gamma) \mid r^{|\alpha|-s}\partial_z^\alpha v \in L^2(\Gamma), \quad \forall \alpha \in \mathbb{N}^2,\ |\alpha| \le s\}.$$

Definition 1.1 *Let $y \in I'$ and $\beta \ge 0$. We say that β satisfies the condition $(\mathcal{C}\mathrm{V})$ in y if there holds :*

$$(\mathcal{C}\mathrm{V}) \quad \left\{ \begin{array}{l} \beta \notin \mathrm{ReSp}(\mathbb{M}_y) \\ \beta > \beta_0 \\ \mathbb{B}_0^+(y) \ \textit{and}\ \mathbb{B}_0^-(y) \ \textit{are injective on}\ E_0^{\beta+1}(\Gamma)\,. \end{array} \right.$$

If β is a nonnegative function on $I \subset I'$ we say that β satisfies $(\mathcal{C}\mathrm{V})$ on I if $\forall y \in I$, $\beta(y)$ satisfies the condition $(\mathcal{C}\mathrm{V})$ in y.

Definition 1.2 *Let $y \in I$ and $\beta \ge 0$. We say that β satisfies the condition $(\mathcal{C}\mathrm{H})$ in y if there holds :*

$$(\mathcal{C}\mathrm{H}) \quad \left\{ \begin{array}{l} \beta \notin \mathrm{ReSp}(\mathbb{M}_y) \\ \beta > \beta_0 \\ \mathbb{B}_0^+(y) \ \textit{and}\ \mathbb{B}_0^-(y) \ \textit{are injective on}\ H^{\beta+1}(\Gamma)\,. \end{array} \right.$$

If β is a nonnegative function on $I \subset I'$ we say that β satisfies $(\mathcal{C}\mathrm{H})$ *on I if $\forall y \in I$, $\beta(y)$ satisfies the condition* $(\mathcal{C}\mathrm{H})$ *in y.*

The elements of the kernel of $\mathbb{B}_0^{\pm}(y)$ in any weighted Sobolev space are rapidly decreasing when $r \to \infty$ – *cf* [8]. Therefore we have

Lemma 1.3
(i) If β satisfies $(\mathcal{C}\mathrm{H})$*, then β satisfies* $(\mathcal{C}\mathrm{V})$*.*
(ii) If β satisfies $(\mathcal{C}\mathrm{V})$ *and $\beta' \geq \beta$ is such that*

$$\beta'(y) \notin \mathrm{ReSp}(\mathbb{M}_y) \quad \forall y \in I'$$

then β' satisfies $(\mathcal{C}\mathrm{V})$*.*

Remark 1.4
(i) Concerning Dirichlet, Neumann or mixed Dirichlet-Neumann problems for strongly elliptic operators, it follows from results of Agmon (see [13]) that $\mathbb{B}_0^{\pm}(y)$ are associated to coercive forms on $H^1(\Gamma)$; therefore conditions $(\mathcal{C}\mathrm{V})$ and $(\mathcal{C}\mathrm{H})$ hold for $\beta = 0$. See also [12] for general self-adjoint problems.

(ii) Concerning oblique derivative problems for the Laplace operator, the validity of condition $(\mathcal{C}\mathrm{H})$ depends on the sign of an angle $\vartheta(y)$ which is the difference $\vartheta_2(y) - \vartheta_1(y)$ where $\vartheta_j(y)$ is the angle between the oblique derivative direction and the normal direction (in the normal plane to the edge Π_y). When $\vartheta(y) \leq 0$, condition $(\mathcal{C}\mathrm{H})$ holds for any $\beta > 0$ small enough. When $\vartheta(y) > 0$, condition $(\mathcal{C}\mathrm{H})$ holds for $\beta > \frac{\vartheta(y)}{\omega(y)}$ (compare [9], [5]).

(iii) For any (second order) elliptic boundary value problem, there always exists β large enough so that condition $(\mathcal{C}\mathrm{V})$ holds : see [3].

(iv) Conditions $(\mathcal{C}\mathrm{V})$ and $(\mathcal{C}\mathrm{H})$ are invariant, i. e. they are independent of the choice of admissible coordinates, as can be seen from the form (1.5) of the Jacobian matrix : we have

$$\widetilde{\mathbb{B}}_0^{\pm}(\tilde{y}; \partial_{\tilde{z}}) = \mathbb{B}_0^{\pm}(y; \left(\frac{\partial \tilde{z}}{\partial z}(y,0)\right) \cdot \partial_z).$$

Such *injectivity* conditions yield some a priori estimates and tangential regularity for the operator $\mathbb{B}$. Let us note that [8] and [10] use an *isomorphism* condition for the operators $\mathbb{B}_0^{\pm}(y)$, which is necessary for index results but not for tangential regularity and expansions results.

We assume now that the coordinates are chosen so that $\omega_y = \omega$ does not depend on y. The sector of opening ω is simply denoted by Γ and for $0 < \rho < \rho'$, Γ_ρ, resp. $\Gamma_{\rho'}$ denotes the set of the points in Γ with $r < \rho$, resp. $r < \rho'$. Let us recall from Part I the definition (I.**??**) of the weighted space

$$V_0^s(I \times \Gamma_\rho) = \{v \in H^s(I \times \Gamma_\rho) \mid r^{|\alpha|-s}\partial_z^\alpha v \in L^2(I \times \Gamma_\rho), \quad \forall \alpha \in \mathbb{N}^2, \ |\alpha| \leq s\}.$$

Theorem 1.5 *Let $\beta \geq 0$ be a real number.*
(i) If β satisfies $(\mathcal{C}\mathrm{V})$ on I' then for any $u \in V_0^{\beta+1}(I' \times \Gamma_{\rho'})$ there holds

$$\|u\|_{V_0^{\beta+1}(I\times\Gamma_\rho)} \leq C\left(\|\mathbb{B}u\|_{\mathbb{V}_0^{\beta-1}(I'\times\Gamma_{\rho'})} + \|u\|_{V_0^{\beta}(I'\times\Gamma_{\rho'})}\right).$$

(ii) If β satisfies $(\mathcal{C}\mathrm{H})$ on I' then for any $u \in H^{\beta+1}(I' \times \Gamma_{\rho'})$ there holds

$$\|u\|_{H^{\beta+1}(I\times\Gamma_\rho)} \leq C\left(\|\mathbb{B}u\|_{\mathbb{H}^{\beta-1}(I'\times\Gamma_{\rho'})} + \|u\|_{H^{\beta}(I'\times\Gamma_{\rho'})}\right).$$

A proof of this theorem can be given using standard techniques, following the lines of the proofs for the closed range properties in [4], which are similar to the arguments in [8].

For the tangential regularity, we use the spaces $H^{s,t}$ and $V_0^{s,t}$ with additional regularity t in the direction y which we introduced in Part I, § **??**.

Theorem 1.6 *Let $\beta \geq 0$ and $t > 0$ be real numbers.*
(i) We assume that β satisfies $(\mathcal{C}\mathrm{V})$ on I'. Let $t > 0$. If $u \in V_0^{\beta+1}(I' \times \Gamma_{\rho'})$ is such that $\mathbb{B}u \in \mathbb{V}_0^{\beta-1,t}(I' \times \Gamma_{\rho'})$ then $u \in V_0^{\beta+1,t}(I \times \Gamma_\rho)$.
(ii) We assume that β satisfies $(\mathcal{C}\mathrm{H})$ on I'. Let $t > 0$. If $u \in H^{\beta+1}(I' \times \Gamma_{\rho'})$ is such that $\mathbb{B}u \in \mathbb{H}^{\beta-1,t}(I' \times \Gamma_{\rho'})$ then $u \in H^{\beta+1,t}(I \times \Gamma_\rho)$.

The proof is based upon Theorem 1.5 and follows the same steps as the proof of Proposition I.**??**. It is known [3] that conditions $(\mathcal{C}\mathrm{V})$ and $(\mathcal{C}\mathrm{H})$, respectively, are also necessary for the tangential regularity results *(i)* and *(ii)*.

This property of tangential regularity is a necessary condition for the existence of splittings such as the following ones in Theorem 1.8. Before stating it, we fix some notations for the exponents of singularities. These exponents are constructed from the spectrum of $\mathbb{M}_y$.

We show that in our situation the spectrum $\mathrm{Sp}(\mathbb{M}_y)$ of $\mathbb{M}_y$ consists of simple eigenvalues which depend analytically on y. This is seen by explicitly solving the Sturm-Liouville eigenvalue problem as follows : one can write the interior differential operator M_y as a product of first order operators

$$(\partial_{z_2} - a_1\partial_{z_1})\,(\partial_{z_2} - a_2\partial_{z_1})$$

with $z = (z_1, z_2)$, and a_1, a_2 complex numbers with $\operatorname{Im} a_1 < 0$ and $\operatorname{Im} a_2 > 0$. Therefore, two independent solutions of the differential equation $\mathcal{M}_y(\lambda)K = 0$ on $(0, \omega)$ are given by

$$K_j(\lambda)(\theta) = e^{\lambda F_j(\theta)},\ j = 1, 2$$

when $\lambda \neq 0$, with

$$F_j(\theta) = \int_0^\theta f_j(\zeta)\,d\zeta \quad \text{where} \quad f_j(\theta) = \frac{a_j\cos\theta - \sin\theta}{a_j\sin\theta + \cos\theta}.$$

Calculating the characteristic determinant with the use of boundary conditions we obtain an equation of the form

$$\lambda^m \sinh\left((\lambda-\gamma)\,\frac{F_2(\omega)-F_1(\omega)}{2}\right)=0$$

where $m\in\mathbb{N}$ and γ depends analytically on a_1, a_2 and the coefficients of the boundary operators. We set

$$\nu=\frac{2i\pi}{F_2(\omega)-F_1(\omega)} \tag{1.10}$$

So $\mathrm{Sp}(\mathbb{M}_y)$ is the set of $k\nu+\gamma$, $k\in\mathbb{Z}$.

As particular cases we find that
- for the Dirichlet problem $\gamma=0$,
- for the Laplace operator $\nu=\pi/\omega$,
- for Δ with mixed Dirichlet–Neumann conditions $\gamma=\pi/(2\omega)$.

Notation 1.7 *Let β be a smooth nonnegative function on I'. We assume that*

$$\forall y\in I',\quad \beta(y)\notin \mathrm{ReSp}(\mathbb{M}_y)$$

For each $y\in I'$ let $\nu_1(y)$ be the only element of $\mathrm{Sp}(\mathbb{M}_y)$ satisfying

$$\mathrm{Re}\,\nu_1(y)>\beta(y)$$

and $\nu_1(y)$ has the least real part satisfying this property in $\mathrm{Sp}(\mathbb{M}_y)$. The map $y\mapsto\nu_1(y)$ is analytic on I'.

The other elements λ of $\mathrm{Sp}(\mathbb{M}_y)$ such that $\mathrm{Re}\,\lambda>\beta$ have the form

$$\nu_1(y)+(k-1)\nu(y):=\nu_k(y),\quad \textit{for } k\ge 1.$$

Their translations by integers are denoted by ν_{kl} :

$$\nu_{kl}(y):=\nu_k(y)+l,\quad \textit{for } l\ge 0.$$

To have a unique notation we set

$$\nu_{0l}:=l,\quad \textit{for } l\ge 0.$$

We can state the main result of this section.

Theorem 1.8 *Let β be a nonnegative function as in the previous Notation 1.7. Let s be a positive number, $s>\beta$. We assume that*

$$\left\{\begin{array}{c} \beta \textit{ satisfies condition } (\mathcal{C}\mathrm{V}),\\ u\in V_0^{\beta+1}(I'\times\Gamma_{\rho'})\quad \textit{and}\quad \mathbb{B}u\in\mathbb{H}^{s-1}(I'\times\Gamma_{\rho'}) \end{array}\right.$$

or

$$\begin{cases} \beta \text{ satisfies condition } (\mathcal{C}\mathrm{H}), \\ u \in H^{\beta+1}(I' \times \Gamma_{\rho'}) \quad \text{and} \quad \mathbb{B}u \in \mathbb{H}^{s-1}(I' \times \Gamma_{\rho'}). \end{cases}$$

Let $\varepsilon_0 > 0$ be given. Then for all $y_0 \in I'$ there exists $I \subset\subset I'$ with $y_0 \in I$ and the following splitting of u :

$$u = w + \sum_{\alpha} (c_\alpha * \Phi)(y, r)\, \psi_\alpha(y, \theta)\, S[\mu_1^\alpha(y), \ldots, \mu_{q_\alpha}^\alpha(y); r] \tag{1.11}$$

with

$$w \in V_0^{s+1-\varepsilon_0}(I \times \Gamma_\rho)\,.$$

Here $\mu_j^\alpha \in \{\nu_{kl} \mid (k, l) \in \mathbb{N}^2;\ \mathrm{Re}\,\nu_{kl} < s\ \forall y \in I\}$ and for all α one has

$$c_\alpha \in H^{s-\mu_\alpha-\varepsilon_0}(I)$$

with $\mu_\alpha(y) = \max\{\mathrm{Re}\,\mu_j^\alpha(y) \mid j = 1, \ldots, q_\alpha\}$.
The ψ_α are analytic functions on $I \times [0, \omega]$ and independent of u.

Proof. Since the proof follows the same lines as that for the Dirichlet problem (Theorem I.**??**), we do not repeat all the details, but indicate only the necessary changes :

• If condition ($\mathcal{C}$H) holds, we want to reduce to the case when u belongs to a weighted space $V_0^{s_0+1}(I' \times \Gamma_{\rho'})$. By localization, we can assume that β is constant. If $\beta \notin \mathbb{N}$, we take $s_0 = \beta$. If not we choose s_0 slightly greater so that $s_0 \notin \mathbb{N}$ and

$$[\beta, s_0] \cap \mathrm{ReSp}(\mathbb{M}_y) = \emptyset, \quad y \in I'.$$

Then $u \in H^{s_0+1}(I \times \Gamma_\rho)$ – *cf* [4]. Then we use the Taylor expansion of u according to Lemma I.**??**. For $|\alpha| < s_0$ we have the traces

$$g_\alpha(y) := \frac{1}{\alpha!}\, \partial_z^\alpha u(y, 0) \in H^{s_0-|\alpha|}(I')\,.$$

Then

$$u_0(y, z) := u(y, z) - \sum_{|\alpha|<s_0} (g_\alpha * \Phi)(y, r)\, z^\alpha \in V_0^{s_0+1}(I \times \Gamma_\rho)\,.$$

Since s_0 satisfies condition ($\mathcal{C}$H) too, Theorem 1.6 yields that

$$u \in H^{s_0+1, s-s_0}(I \times \Gamma_\rho).$$

Therefore the trace g_α belongs to $H^{s-|\alpha|}(I)$. This implies that for the lifting of traces we have :

$$\sum_{|\alpha|<s_0} (g_\alpha * \Phi)(y, r)\, z^\alpha \in H^{s+1}(I \times \Gamma_\rho)\,.$$

Now we see that $u_0 \in V_0^{s_0+1}(I \times \Gamma_\rho)$ is such that $\mathbb{B}u_0 \in \mathbb{H}^{s-1}(I' \times \Gamma_{\rho'})$. Since s_0 satisfies ($\mathcal{C}$V), this reduction is complete.

• We have to perform rather obvious modifications to take into account the different orders of the operators B, $\mathcal{B}_1$ and $\mathcal{B}_2$. As an example let us explain the modifications

in the statements concerning operations about singular functions : Lemmas I.**??** and I.**??** and Proposition I.**??**.
In Lemma I.**??**, $\mathfrak{B}$ has to be replaced by $\mathbb{B}$ and $\mathfrak{M}$ by $\mathbb{M}$. u is the same and f is now the triple (f, h_1, h_2) given by the corresponding formula :

$$u(x) := (c * \Phi)(y, r)\, \psi(y, \theta)\, S[\mu_1(y), \ldots, \mu_J(y); r]$$

and

$$(f, h_1, h_2)(x) := (c * \Phi)(y, r)\, \mathbb{M}\Big[\psi(y, \theta)\, S[\mu_1(y), \ldots, \mu_J(y); r]\Big]\,.$$

Then, with the same μ, $c_{l,p}$, $\varphi_{l,p}$ and $\mu_q^{l,p}$,

$$\mathbb{B}u = f + \sum_{l=1}^{L} \sum_{p} (f_{l,p}, h^1_{l,p}, h^2_{l,p}) + (g, g^1, g^2)\,,$$

where

$$g \in H^\infty_{-\beta+1-\mu+\varepsilon}(I \times \Gamma_\rho) \quad \forall \varepsilon > 0$$

$$g^j \in H^\infty_{-\beta+m_j-1/2-\mu+\varepsilon}(I \times \partial_j\Gamma_\rho) \quad \forall \varepsilon > 0$$

and

$$f_{l,p}(x) := (c_{l,p} * \Phi)(y, r)\, \varphi_{l,p}(y, \theta)\, S[\mu_1^{l,p}(y) - 2 + l, \ldots, \mu_{q_{l,p}}^{l,p}(y) - 2 + l; r]\,;$$

$$h^j_{l,p}(x) := (c_{l,p} * \Phi)(y, r)\, \varphi^j_{l,p}(y)\, S[\mu_1^{l,p}(y) - m_j + l, \ldots, \mu_{q_{l,p}}^{l,p}(y) - m_j + l; r]\,.$$

In Lemma I.**??** the functions $f_{l,p}$ and g are changed exactly as above. Instead of a simple function f, we have to consider in Proposition I.**??** three sorts of right hand sides F : $(f, 0, 0)$, $(0, h_1, 0)$ and $(0, 0, h_2)$ with

$$f(x) := (c * \Phi)(y, r)\, \varphi(y, \theta)\, S[\mu_1(y) - 2, \ldots, \mu_q(y) - 2; r]\,;$$

and

$$h_j(x) := (c * \Phi)(y, r)\, \varphi_j(y)\, S[\mu_1(y) - m_j, \ldots, \mu_q(y) - m_j; r]\,.$$

Then the solution u has the same form and the remainder $\mathbb{B}u - F$ belongs to $\mathbb{H}^\infty_{-\beta+1-\mu+\varepsilon_0}(I \times \Gamma_\rho)$ where

$$\mathbb{H}^\infty_{\delta+1}(I \times \Gamma) := H^\infty_{\delta+1}(I \times \Gamma) \times H^\infty_{\delta+m_1-1/2}(I \times \partial_1\Gamma) \times H^\infty_{\delta+m_2-1/2}(I \times \partial_2\Gamma).$$

■

Remark 1.9 A semi-global version of the previous theorem also holds : compare Theorem I.**??**.

Remark 1.10 We have to use both conditions ($\mathcal{C}$V) and ($\mathcal{C}$H) for the following reasons:
• If we want to obtain an expansion for a variational solution in H^1, we have to use condition ($\mathcal{C}$H) for $\beta = 0$ for the Neumann problem. Condition ($\mathcal{C}$V) would not be sufficient because in general a solution of the Neumann problem, even with flat data, does not belong to the space V^1_0.

• On the other hand for Dirichlet or mixed Dirichlet-Neumann problems, the condition ($\mathcal{C}$V) in $\beta = 0$ is sufficient, because if the Dirichlet data are zero or even with a zero Taylor expansion at the edge, the variational solution belongs to V_0^1.

2. THE STRUCTURE OF THE SINGULARITIES

In this section we take a closer look at the structure of the singular functions that appear in the decomposition (1.11) and in Part I, (I.**??**), (I.**??**) and (I.**??**). In particular, we want to describe the procedure, contained in the proof of Theorem 1.8 – which is based on the proof of Theorem I.**??** –, that generates the *singularity type*, that is the part of the singular functions which depends only on the geometry of the domain and the differential operators but not on the right hand sides.

Thus we consider singular functions

$$\psi_\alpha(y,\theta)\, S_\alpha(y,r) \quad \text{with} \quad S_\alpha(y,r) = S[\mu_1^\alpha(y), \ldots, \mu_{q_\alpha}^\alpha(y); r] \tag{2.1}$$

that appear in the decomposition (1.11) of Theorem 1.8. The index α spans a finite set $\mathfrak{A}(s, I)$. For each α, ψ_α is analytic in $y \in I$ and S_α is the radial part of this singularity type. Thus the set

$$\{\psi_\alpha(y,\theta)\, S_\alpha(y,r) \mid \alpha \in \mathfrak{A}(s,I)\}$$

is a *generating set of singular functions*, by which we mean any set of functions $\psi_\alpha(y,\theta)\, S_\alpha(y,r)$ such that the decomposition (1.11) can be written with the singular part in the form

$$u_{\text{sing}} = \sum_\alpha (c_\alpha * \Phi)(y,r)\, \psi_\alpha(y,\theta)\, S_\alpha(y,r)$$

with coefficients c_α of the appropriate regularity.

Our decomposition Theorem 1.8 merely states the *existence* of the analytic functions $\psi_\alpha(y,\theta)$, but our proofs contain a certain *iterative algorithm* that produces them. The set of the corresponding singularities $\psi_\alpha\, S_\alpha$ is not minimal in general, yet this algorithm still serves its threefold purpose : In the decomposition theorems, it allowed norm estimates for the coefficients and the regular part of the solution, and in the next two sections it will be used to reveal the analytic bundle structure of the singularity type and to describe in a simple way the explicit form of the singular functions in the case of the Laplace equation.

We will describe at the end of this section another version of this algorithm that does indeed, for the case of the simple asymptotics, give a minimal set of singular functions and which is necessarily more complicated.

Let us now begin with the description of the elementary algorithm. We use the notations of the previous section, in particular $\mathbb{M}(y; \partial_z)$ for the conormal principal part of the operator $\mathbb{B}$, and $\mathbb{M}_y(\lambda)$ for the corresponding family of Sturm–Liouville operators on $[0, \omega_y]$ defined by Mellin transformation.

Proposition 2.1 *Let I be an interval where a splitting* (1.11) *holds. Let α be an index in $\mathfrak{A}(s,I)$. Then one of the following three possibilities is verified :*

(a) $\mathbb{M}(y;\partial_z)\,(\psi_\alpha\,S_\alpha)=0$,

(b) $\psi_\alpha\,S_\alpha$ is a polynomial in the coordinates (z_1,z_2) with coefficients depending analytically on $y\in I$,

(c) There exist $\alpha_1,\dots,\alpha_m\in\mathfrak{A}(s,I)$ such that with $\alpha_0=\alpha$, the radial parts $S_{\alpha_0},\dots,S_{\alpha_m}$ are independent functions and such that

$$\mathbb{M}(y;\partial_z)\Big(\sum_{j=0}^{m}\psi_{\alpha_j}\,S_{\alpha_j}\Big)=z^\gamma\,\partial_y^p\,\partial_z^\delta\,(\psi_\beta\,S_\beta)\,. \tag{2.2}$$

where $|\gamma|+1\geq|\delta|$, $p+|\delta|\leq 2$, and $\beta\in\mathfrak{A}(s-1,I)$.

Indeed all singular functions $\psi_\alpha\,S_\alpha$ in the decomposition (1.11) *of Theorem* 1.8 *are inductively generated by either (a), (b), or repeated application of (c) with a right hand side $\psi_\beta\,S_\beta$ generated in a previous induction step.*

Proof. One only has to inspect the proofs of the decomposition theorems as given in Part I. In particular, situation *(a)* corresponds to the case of flat right hand sides, see Proposition I.**??**. Situation *(b)* corresponds to the Taylor expansion of the solution u at the edge, and situation *(c)* corresponds to the case of singular right hand sides, see Proposition I.**??**. The latter is repeatedly applied in the final induction proof of Theorem I.**??**. Note that crossings of exponents are only generated by part *(c)* of the algorithm. ∎

In the same way one obtains a certain converse of Proposition 2.1.

Proposition 2.2 *Let $\Psi_s=\{\psi_\alpha(y,\theta)\,S_\alpha(y,r)\mid\alpha\in\mathfrak{A}\}$ be a set of functions of the form* (2.1) *that satisfy :*

(i) $\psi_\alpha(y,\theta)$ is analytic in $y\in I$, $\theta\in[0,\omega_y]$,

(ii) For each $y\in I$, Ψ_s contains a basis for the solutions of steps (a) and (b), i. e. for the eigenfunctions and the polynomials of exponents less than s,

(iii) There is an $\varepsilon>0$ such that for any $\beta\in\mathfrak{A}$, Ψ_s contains $\psi_{\alpha_j}\,S_{\alpha_j}$, $(j=0,\dots,m)$, such that $\sum_{j=0}^m\psi_{\alpha_j}\,S_{\alpha_j}$ is a solution of (2.2) *and*

$$\min\{\underline{\mu}_{\alpha_j}\mid j=0,\dots,m\}\geq\underline{\mu}_\beta+\varepsilon$$

where $\underline{\mu}_\alpha=\inf\{\mu_q^\alpha(y)\mid y\in I,\ q=1,\dots,q_\alpha\}$.

Then Ψ_s is a generating set of singular functions in the sense defined above.

The smaller is the set $\{S_\alpha(y,r)\mid\alpha\in\mathfrak{A}(s,I)\}$ of the radial parts, the better is the accuracy of the algorithm. In the iterative procedure, the radial parts S_{α_j} and S_α in step *(c)* are related in a way which can be described more precisely. We are going to do that when the interval I contains no "crossing points" of the exponents ν_κ.

In Part I, we also described the singularities outside of the "crossing points". These "simple asymptotics", see Theorem I.**??**, are of course also valid in the more general setting of this Part II. Indeed they are a particular situation of the asymptotics in Theorem 1.8 : when there is no crossing point in I, the mixing between different exponents ν_κ in the radial part of a singularity type is no more necessary. So to each $\alpha \in \mathfrak{A}(s, I)$ is associated a triple (κ, q, n) according to the notations of Theorem I.**??** with the following rules

$$\begin{cases} \forall j = 1, \ldots, q_\alpha \quad \mu_j^\alpha = \nu_\kappa \\ q = q_\alpha - 1 \\ n \text{ describes a finite set } \{1, \ldots, n_{\kappa,q}\}. \end{cases}$$

Now we consider the corresponding singular types $\psi_\alpha \, S_\alpha$:

$$\psi_\alpha(y, \theta) \, S_\alpha(y, r) = \varphi_{\kappa,q,n}(y, \theta) \, r^{\nu_\kappa(y)} \log^q r$$

– compare with (0.7) –, that appear in the decomposition (I.**??**) of Theorem I.**??**. The function $\varphi_{\kappa,q,n}$ is analytic in y and θ for $y \in I$. In this situation, we can write the elementary algorithm of Proposition 2.1 in a more explicit way.

Proposition 2.3 *Let I be an interval without crossing point where a splitting* (I.**??**) *holds. Let $\alpha \equiv (\kappa, q, n)$ be an index in $\mathfrak{A}(s, I)$. Then one of the following three possibilities is verified :*

(a) $\kappa = (k, 0)$, $q = 0$, $n = 1$ and $\mathbb{M}(y; \partial_z) \, (\varphi_{\kappa,q,n} \, r^{\nu_k}) = 0$,

(b) $\kappa = (0, l)$, $q = 0$ and $\varphi_{\kappa,q,n} \, r^l$ is a polynomial in the coordinates (z_1, z_2) with coefficients depending analytically on $y \in I$,

(c) $\kappa = (k, l)$ with $k \neq 0$ and $l \neq 0$, and there exist $m \geq q$ and $n_0, \ldots, n_m$ such that

$$\mathbb{M}(y; \partial_z) \Big(\varphi_{\kappa,q,n}(y, \theta) \, r^{\nu_\kappa} \log^q r + \sum_{\substack{j=0 \\ j \neq q}}^{m} \varphi_{\kappa,j,n_j} \, r^{\nu_\kappa} \log^j r \Big) = z^\gamma \, \partial_y^p \, \partial_z^\delta \, (\varphi_{\tilde\kappa,\tilde q,\tilde n} \, r^{\nu_{\tilde\kappa}(y)} \log^{\tilde q} r) \tag{2.3}$$

where $|\gamma| + 1 \geq |\delta|$, $p + |\delta| \leq 2$, and

$$\nu_\kappa = \nu_{\tilde\kappa} + |\gamma| + 2 - |\delta| \quad \textit{i. e.} \quad k = \tilde k \text{ and } l = \tilde l + |\gamma| + 2 - |\delta|.$$

The situation *(a)* simply means that $\varphi(y, \cdot)$ is an eigenfunction of $\mathbb{M}_y$ i. e., a solution of

$$\mathbb{M}_y(\nu_k(y)) \, \varphi(y, \cdot) = 0.$$

In this case φ can be chosen such that it has an analytic extension to any interval $I' \supset I$ on which admissible coordinates are defined.

Also in case *(b)* these coefficients can be chosen in such a way that φ has an analytic extension to such an I'.

The situation *(c)* can also be described explicitly.

Corollary 2.4 *Let I be an interval without crossing point where a splitting* (I.??) *holds. Let $I' \supset I$ be an interval on which admissible coordinates are defined. Let us choose $\varphi_{\kappa,q,n}(y,\theta)$ analytic in y and θ for $y \in I'$ in the initiating steps (a) and (b) of the above algorithm (i. e. for $\kappa = (k,0)$ or $(0,l)$). Then for any index $\alpha \equiv (\kappa, q, n) \in \mathfrak{A}(s,I)$, $\varphi_{\kappa,q,n}(y,\theta)$ has an extension to I' which is analytic in θ and meromorphic in y.*

Proof. One only has to describe the step *(c)* of the algorithm explicitly. The right hand side of (2.3) is of the form

$$\sum_{j=0}^{\tilde{q}+k} \psi_j(y,\theta)\, r^{\mu(y)-2} \log^j r \quad \text{if} \quad \mu(y) = \tilde{\mu}(y) + |\gamma| - |\delta| + 2.$$

where μ stands for ν_κ. Now we can construct the solutions $\varphi_l(y,\theta)$, $l = 0, \ldots, q_0$ of

$$\mathbb{M}(y;\partial_z)\Big(\sum_{l=0}^{q_0} \varphi_l(y,\theta)\, r^{\mu(y)} \log^l r\Big) = \sum_{j=0}^{\tilde{q}+k} \psi_j(y,\theta)\, r^{\mu(y)-2} \log^j r \tag{2.4}$$

explicitly. Our function $\varphi_{\kappa,q,n}(y,\theta)$ is then given by $\varphi_q(y,\theta)$. We solve (2.4) by Mellin transformation :

$$\sum_{l=0}^{q_0} \frac{\varphi_l(y,\theta)\, l!}{(\lambda - \mu(y))^{l+1}} = \sum_{j=0}^{\tilde{q}+k} \frac{\mathbb{M}_y(\lambda)^{-1}\psi_j(y,\cdot)\, j!}{(\lambda-\mu(y))^{j+1}}\,. \tag{2.5}$$

From this it follows that $\varphi_q(y,\theta)$ is given by the coefficients of the Laurent expansion of $\mathbb{M}_y(\lambda)^{-1}\psi_j(y,\cdot)$ at $\lambda = \mu(y)$:

$$\varphi_q(y,\theta) = \operatorname*{Res}_{\lambda=\mu(y)} \sum_{j=0}^{\tilde{q}+k} \frac{j!}{q!}\, \mathbb{M}_y(\lambda)^{-1}\psi_j(y,\cdot)\cdot(\lambda-\mu(y))^{q-j-1}\,. \tag{2.6}$$

Now it suffices to check the following lemma. ∎

Lemma 2.5 *If $y \mapsto \psi(y)$ and $y \mapsto \mu(y)$ are analytic on I' and if $m \in \mathbb{N}$, then the function*

$$y \mapsto \varphi(y) := \operatorname*{Res}_{\lambda=\mu(y)} \mathbb{M}_y(\lambda)^{-1}\psi(y,\cdot)\cdot(\lambda-\mu(y))^{-m}$$

is meromorphic on I'. If ψ is only meromorphic on I', then φ is still meromorphic on I'.

Proof. • If $\mu \equiv \nu_{k,0}$, then $(\lambda-\mu(y))\mathbb{M}_y(\lambda)^{-1}$ is analytic in y and λ for $y \in I'$ and λ in a neighborhood of $\mu(y)$. Then

$$m!\,\varphi(y) = \partial_\lambda^m\Big((\lambda-\mu(y))\mathbb{M}_y(\lambda)^{-1}\Big)\Big|_{\lambda=\mu(y)}\cdot\psi(y)$$

Then φ is analytic if ψ is analytic.

• If there is a crossing in y_0, i. e. if $\mu(y_0) = \nu_k(y_0)$ and $\mu(y) \neq \nu_k(y)$ for $y \in I' \setminus \{y_0\}$, we see that for $y \neq y_0$ (if $m \geq 1$)

$$(m-1)!\,\varphi(y) = \partial_\lambda^{m-1}\mathbb{M}_y(\lambda)^{-1}\Big|_{\lambda=\mu(y)}\cdot\psi(y)\,.$$

As $\partial_\lambda^d \mathbb{M}_y(\lambda)^{-1}$ is obtained by composing $\mathbb{M}_y(\lambda)^{-1}$ and derivatives $\partial_\lambda^{d'} \mathbb{M}_y(\lambda)$ of $\mathbb{M}_y(\lambda)$, we obtain the lemma. ∎

Whereas this simple form of the algorithm was sufficient for our purposes, there is a more precise version for the case of the simple asymptotics which we will describe now. For simplicity, we consider the Dirichlet problem only.

Proposition 2.6 *Let the hypotheses of Theorem* I.?? *be satisfied. Then for the singular part of the solution u there exists the following more precise description*

$$u_{\text{sing}} = \sum_{|\beta|<s} (c_\beta * \Phi)\, \frac{z^\beta}{\beta!} \;\; + \sum_{k,l\,:\,\nu_{k,l}<s} \; \sum_{j=0}^{l} (d_y^j c_k * \Phi)\, \mathcal{S}_{kl,j} \tag{2.7}$$

where $c_\beta \in H^{s-|\beta|-\varepsilon}(I)$ *and* $c_k \in H^{s-\nu_k-\varepsilon}(I)$ *for all* $\varepsilon > 0$ *and*

$$\mathcal{S}_{kl,j} = \sum_q \varphi_{kl,jq}(y,\theta)\, r^{\nu_{kl}} \log^q r$$

with functions $\varphi_{kl,jq}(y,\theta)$ *analytic in* $y \in I$, $\theta \in [0,\omega_y]$.

Our algorithm gives a formula of recursion for the singular functions $\mathcal{S}_{kl,j}$. In order to formulate this, we introduce the following decomposition of the operator B into homogeneous components : Let $b_{\alpha m}$ be the coefficients of B and $b_{\alpha m\beta}$ be their Taylor expansions according to :

$$B = \sum_{|\alpha|+m\le 2} b_{\alpha m}(y,z)\, \partial_z^\alpha \partial_y^m \quad \text{and} \quad b_{\alpha m\beta}(y) = \partial_z^\beta b_{\alpha m}(y,z)\Big|_{z=0}.$$

Then we set

$$B_{m\tau}(y,z;\partial_z) := \sum_{|\beta|+2-|\alpha|=\tau} b_{\alpha m\beta}(y)\, \frac{z^\beta}{\beta!}\, \partial_z^\alpha. \tag{2.8}$$

Note that if $\tau = 0$ then $m = 0$ and that $B_{00} = M$.

Proposition 2.7 *Let the hypotheses of the previous Proposition* 2.6 *be satisfied. Then we have the following recursion formula for any* $k \in \mathbb{N}^*$ *and* $l \ge 1$ *such that* $\nu_{kl} < s$ *and any* $j \le l$ *:*

$$\mathbb{M}(y,\partial_z)\mathcal{S}_{kl,j} = -\sum_{\tau=1}^{l} \sum_{m=0}^{\min(2,j)} \sum_{n=0}^{m} \binom{m}{n} B_{m\tau}\Big(\partial_y^{m-n} \mathcal{S}_{k,l-\tau,j-n}\Big). \tag{2.9}$$

Here $\mathbb{M}(y,\partial_z)\mathcal{S} = \mathcal{T}$ *means the interior equation* $M(y,\partial_z)\mathcal{S} = \mathcal{T}$ *and the zero Dirichlet conditions.*

This algorithm corresponds to case *(c)* of Proposition 2.1. The case *(b)* obviously corresponds to the first part of the asymptotics (2.7). The initiation of the algorithm comes from $\mathcal{S}_{k0,0}$ which is in the situation *(a)*. We note that the function $\varphi_{\kappa,jq}$ is a linear combination with analytic coefficients $c_n(y)$ in I of the functions $\varphi_{\kappa,q,n}$ in Proposition(2.3).

3. SINGULAR FUNCTIONS AS SECTIONS OF ANALYTIC VECTOR BUNDLES

The singular functions are separated into coefficients $c(y)$ depending on the right hand sides of the differential equations and functions

$$\psi(y,\theta)\ S[\mu_1(y),\ldots,\mu_q(y);r]$$

depending only on the geometry of the domain and the differential operators. This separation which is by no means unique, can be described by associating invariant objects, namely a family of analytic vector bundles over the edge, to the geometry and the differential operators, and regarding the actual singular functions as sections of these bundles.

This point of view allows a better understanding of the asymptotics in the neighborhood of crossing points. There we have the phenomenon that the number of basis functions $\psi_\alpha\, S_\alpha$ of the form (2.1) needed to describe the singularities of a function u is, in general, higher than the number of basis functions $\varphi_{\kappa q} r^{\nu_\kappa} \log^q r$ needed to describe the "simple asymptotics" in an interval not containing any crossing point. This was seen in the simplest example in § I.**??**. There the simple asymptotics was described by two basis functions for each y different from the points where the two corresponding exponents coincide. In the crossing points themselves, the singularities are still described by two coefficients and two basis functions. Using the form (2.1) for the singularities, however, one introduces three basis functions. There are, however, still only two linearly independent coefficients. This situation is a simple case of that one described in Proposition 3.4, namely that the bundle defined by the simple asymptotics away from the crossing points (Theorem I.**??**) is a subbundle of the bundle defined by the the asymptotics at the crossing points as described in Theorems 1.8, I.**??** and I.**??**.

The bundles defined by the simple asymptotics outside the crossing points, have a unique extension as analytic bundles over the whole edge including the crossing points. The sections of these extensions can be written in terms of the functions $\psi(y,\theta)\ S[\mu_1(y),\ldots,\mu_q(y);r]$ (Theorem 3.3). Thus these functions, and the larger bundles generated by them, appear here in a very natural way as function theoretic objects.

Let us now define the bundles in question. We fix an interval J on the edge for which an admissible system of coordinates exists. We shall consider two kinds of vector bundles over J, bundles which are generated by functions of (y,θ) only and bundles which are generated by functions of (y,r,θ). Thus we consider two "large" bundles with Hilbert space fibers in which all our analytic vector bundles with finite fiber dimension will be embedded as subbundles. For the first one, we require that continuous functions $\varphi(y,\theta)$ $(y\in J,\ \theta\in[0,\omega_y])$ are sections of this bundle $\mathfrak{X}_1$ with Hilbert space fiber $\mathfrak{X}_1(y)$ which can be taken for instance as $\mathfrak{X}_1(y) = L^2(0,\omega_y)$. For the second case, we choose the Hilbert space fiber $\mathfrak{X}_2(y)$, for example, as $L^2(\mathbb{R}_+\times[0,\omega_y];d\mu)$ with $d\mu(r,\theta) = e^{-r-\frac{1}{r}}\,dr\,d\theta$, or also as $L^2([1,2]\times[0,\omega_y])$. In any case, for continuous functions $\varphi(y,\theta)$ ($y \in J$, $\theta \in [0,\omega_y]$), $\mu(y)$ ($y \in J$) and q in $\mathbb{N}$, the function $\varphi(y,\theta)\, r^{\mu(y)} \log^q r$ will appear as a section of this bundle.

Now we consider a subinterval $I \subset J$. We fix integers k, l and q and set $\kappa = (k,l)$. We assume that there is no crossing point for the exponent $\nu_\kappa(y)$ in I, i. e.

$$\nu_\kappa(y) \neq \nu_{\kappa'}(y) \quad \forall y \in I \text{ if } \kappa \neq \kappa'. \tag{3.1}$$

We say that an analytic function $\varphi\, r^{\nu_\kappa} \log^q r$ "appears as a singular function" if the following condition is satisfied :
There exists a finite number of analytic functions $\psi_{\kappa' q'}$ with $(\kappa', q') \neq (\kappa, q)$ such that the function

$$u(y,r,\theta) = \varphi(y,\theta) r^{\nu_\kappa(y)} \log^q r + \sum_{\kappa' q'} \psi_{\kappa' q'}(y,\theta) r^{\nu_{\kappa'}(y)} \log^{q'} r \tag{3.2}$$

is locally a solution of the boundary value problem with regular right hand side :
For some $s > \nu_\kappa(y)\ \ \forall y \in I$:

$$\mathbb{B}u \in \mathbb{V}_0^{s-1} := V_0^{s-1} \times \prod_j V_0^{s-m_j+1/2}\,. \tag{3.3}$$

Let now $\Psi_{\kappa q}(I)$ be the set of all analytic functions $\varphi(y,\theta)$ such that the function $\varphi\, r^{\nu_\kappa} \log^q r$ appears as a singular function. According to our decomposition Theorem I.**??**, there is a finite number of analytic functions

$$\varphi_{\kappa q n} \quad \text{for} \quad n = 1, \ldots, n_{\kappa q}$$

such that every $\varphi \in \Psi_{\kappa q}(I)$ can be written as

$$\varphi(y,\theta) = \sum_{n=1}^{n_{\kappa q}} c_n(y)\, \varphi_{\kappa q n}(y,\theta) \tag{3.4}$$

with analytic coefficients $c_n(y)$.

This shows that for any $y \in I$, the space $\operatorname{span}\{\varphi(y,\cdot) \mid \varphi \in \Psi_{\kappa q}(I)\}$ is a subspace of the finite–dimensional fiber at y of the bundle

$$\overline{\operatorname{span}}\{\varphi_{\kappa q n} \mid n = 1, \ldots, n_{\kappa q}\}$$

(see Definition 5.4).

One can therefore select $\varphi_1, \ldots, \varphi_n \in \Psi_{\kappa q}(I)$ such that $\varphi_1(y), \ldots, \varphi_n(y)$ span this subspace for all y except for a finite number of points. It is easy to see that the subbundle $\overline{\operatorname{span}}\{\varphi_1, \ldots, \varphi_n\}$ is the unique analytic bundle that extends $\operatorname{span}\{\varphi \mid \varphi \in \Psi_{\kappa q}(I)\}$. In this sense we define

Definition 3.1 $\mathfrak{B}_{\kappa q}(I) = \overline{\operatorname{span}}\{\Psi_{\kappa q}(I)\}$.

Remark 3.2 By definition, the analytic sections of $\mathfrak{B}_{\kappa q}(I)$ contain the functions necessary to describe the edge asymptotics in the case where the coefficients c_k are *analytic*. The precise version of the algorithm for the construction of the singularity types given in Proposition 2.7 shows that these same functions suffice to describe the ("simple") edge asymptotics also in the case where the coefficients have a *finite* regularity. The bundles $\mathfrak{B}_{\kappa q}$ are *minimal* in this sense. Note, however, that the fiber dimension of the $\mathfrak{B}_{\kappa q}$ does, in general, not give the right number of independent coefficients. Namely, it is impossible to capture, in the framework of vector bundles, the dependence among the coefficients that is given by differential operators, see Proposition 2.6.

Now we use Corollary 2.4 which implies that the generating sections $\varphi_1, \dots, \varphi_n$ of $\mathfrak{B}_{\kappa q}(I)$ can be chosen in such a way that they have a meromorphic extension on the whole interval J. By Remark 5.3, this defines a unique extension of $\mathfrak{B}_{\kappa q}(I)$ to an analytic vector bundle over J which we denote by $\mathfrak{B}_{\kappa q}(J)$.

These bundles $\mathfrak{B}_{\kappa q}(J)$ and their sums contain now a description of the singular functions also at the crossing points in J, and this in a minimal way. We will now show how this description is related to that one given in our decomposition theorems.

To this purpose we define first some more bundles. Firstly, we reattach the functions $r^{\nu_\kappa(y)} \log^q r$ to the bundles $\mathfrak{B}_{\kappa q}(J)$ and look at this in the neighborhood of a crossing point. Thus let now the interval I contain one crossing point y_0 with the corresponding exponents $\mu_1^0, \dots, \mu_{j_0}^0$ – see Section I.**??** in Part I. We recall the notations of this § I.**??**, in particular $\mathcal{K}_{y_0,j}$ for the indices of the exponents ν_κ that satisfy $\nu_\kappa(y_0) = \mu_j^0$, and $\mathcal{K}_{y_0} = \bigcup_{j=1}^{j_0} \mathcal{K}_{y_0,j}$. We define

$$
\begin{aligned}
\mathfrak{B}_j \;&=\; \overline{\operatorname{span}}\Big\{ r^{\nu_\kappa(y)} \log^q r \cdot \varphi(y,\theta) \mid \kappa \in \mathcal{K}_{y_0,j}, \;\; q \in \mathbb{N}, \\
&\qquad\qquad \varphi \text{ is an analytic section of } \mathfrak{B}_{\kappa q}(I) \Big\} \\
&=\; \overline{\operatorname{span}}\Big\{ r^{\nu_\kappa(y)} \log^q r \cdot \mathfrak{B}_{\kappa q}(y) \mid \kappa \in \mathcal{K}_{y_0,j}, \;\; q \in \mathbb{N} \Big\}.
\end{aligned}
\tag{3.5}
$$

Next, we repeat the above constructions where we replace now the simple asymptotics by the asymptotics as we described them for the crossing points in Theorems 1.8, I.**??**, I.**??**. Since we do not strive for minimality here, we can simply define the bundles $\mathfrak{D}_j$ as follows :
Let Σ_j be the set of all functions $\psi_\alpha \, S_\alpha$ for $\alpha \in \mathfrak{A}(s, I)$ – see § 2 where they are constructed according to the algorithm in Proposition 2.1 – and such that

$$
S_\alpha = S[\mu_1^\alpha, \dots, \mu_{q_\alpha}^\alpha; r] \quad \text{with} \quad \mu_1^\alpha(y_0), \dots, \mu_{q_\alpha}^\alpha(y_0) = \mu_j^0 .
$$

We consider these functions as sections of the bundle $\mathfrak{X}_2$, and we define

$$
\mathfrak{D}_j \;:=\; \overline{\operatorname{span}}\{\Sigma_j\} . \tag{3.6}
$$

The relation between the bundles $\mathfrak{B}_j$ and $\mathfrak{D}_j$ is given by the following description of the sections of $\mathfrak{B}_j$.

Theorem 3.3 *Let $\mathfrak{C}_{\kappa q}$ be analytic subbundles of $\mathfrak{X}_1$ over I with finite fiber dimensions and define the subbundle of $\mathfrak{X}_2$*

$$\mathfrak{C}_j := \overline{\operatorname{span}}\Big\{ r^{\nu_\kappa(y)} \log^q r \cdot \mathfrak{C}_{\kappa q}(y) \mid \kappa \in \mathcal{K}_{y_0,j}, \ \ q = 1, \ldots, q_0 \Big\} . \tag{3.7}$$

Then any analytic section of $\mathfrak{C}_j$ is of the form

$$v(y,r,\theta) = \sum_{l=1}^{K} S[\nu_{\kappa_1}, \ldots, \nu_{\kappa_l}; r] \cdot \psi_l(y,\theta) \tag{3.8}$$

with $\kappa_l \in \mathcal{K}_{y_0,j}$ for $l = 1, \ldots, K$ and the functions ψ_l are analytic sections of

$$\overline{\operatorname{span}}\Big\{ \mathfrak{C}_{\kappa q}(y) \mid \kappa \in \mathcal{K}_{y_0,j}, \ \ q = 1, \ldots, q_0 \Big\} .$$

Proof. Let v be such a section. Then v is an analytic function in the variables r, y, θ. For $y \neq y_0$, v is of the form :

$$v(y,r,\theta) = \sum_{\kappa,q} r^{\nu_\kappa(y)} \log^q r \cdot \varphi_{\kappa q}(y,\theta) \tag{3.9}$$

with analytic sections $\varphi_{\kappa q}$ of $\mathfrak{C}_{\kappa q}(I \setminus \{y_0\})$. This follows from the linear independence of the functions

$$(r,\theta) \mapsto r^{\nu_\kappa(y)} \log^q r \cdot \varphi_{\kappa q}(y,\theta)$$

for different (κ, q) if $y \in I \setminus \{y_0\}$.

Now let

$$w(y,\lambda,\theta) = \int_0^1 r^{-\lambda-1}\, v(y,r,\theta)\, dr \tag{3.10}$$

be the Mellin transform of $\chi_{[0,1]}(r) \cdot v(y,r,\theta)$. Then $w(y,\lambda,\theta)$ is analytic for $y \in I$, $\theta \in [0,\omega_y]$, $\operatorname{Re}\lambda < 0$. From (3.9) follows that for a differential operator D of the form

$$D = \prod_\kappa \Big(r\frac{d}{dr} - \nu_\kappa(y) \Big)^{q_\kappa + 1}$$

there holds $Dv = 0$, first for $y \neq y_0$, and then by analyticity for all $y \in I$. It follows that $D(\chi_{[0,1]}(r) \cdot v)$ is a distribution supported in $r = 1$, depending analytically on $y \in I$, $\theta \in [0,\omega_y]$. Mellin transformation gives a function $\eta(y,\lambda,\theta)$, analytic in $y \in I$, $\theta \in [0,\omega_y]$ and holomorphic for all $\lambda \in \mathbb{C}$, such that

$$w(y,\lambda,\theta) \ = \ \frac{\eta(y,\lambda,\theta)}{\prod_\kappa (\lambda - \nu_\kappa(y))^{q_\kappa+1}} \ . \tag{3.11}$$

Inverse Mellin transformation gives for $r \in (0,1)$,

$$\begin{aligned} v(y,r,\theta) \ &= \ \frac{1}{2i\pi} \int_\gamma r^\lambda\, w(y,\lambda,\theta)\, d\lambda \\ &= \ \frac{1}{2i\pi} \int_\gamma r^\lambda\, \frac{\eta(y,\lambda,\theta)}{(\lambda - \nu_{\kappa_1}(y)) \cdots (\lambda - \nu_{\kappa_K}(y))}\, d\lambda \ . \end{aligned} \tag{3.12}$$

Now we use the Leibniz rule for the divided differences which gives

$$v(y,r,\theta) \;=\; \sum_{l=1}^{K} S[\nu_{\kappa_1}(y),\ldots,\nu_{\kappa_l}(y);r]\,\psi_l(y,\theta)\,, \tag{3.13}$$

where $\psi_l(y,\theta)$ are the divided differences of the function $\lambda \mapsto \eta(y,\lambda,\theta)$ at the points $\lambda = \nu_{\kappa_1}(y),\ldots,\nu_{\kappa_l}(y)$.

Comparison of (3.9) and (3.11) shows that for $y \neq y_0$,

$$\eta(y,\lambda,\theta) \;=\; \sum_{\kappa,q} q!\,(\lambda-\nu_\kappa(y))^{q_\kappa-q}\,\varphi_{\kappa q}(y,\theta)\,.$$

Hence the functions $\psi_l(y,\theta)$ are, over $I\setminus\{y_0\}$, analytic sections of

$$\overline{\mathrm{span}}\Big\{\mathfrak{C}_{\kappa q}(y) \mid \kappa\in\mathcal{K}_{y_0,j},\ \ q=1,\ldots,q_0\Big\}\,.$$

Since they are analytic in y_0, they are sections of this bundle over all of I. ∎

Corollary 3.4 *The analytic vector bundles $\mathfrak{B}_j$ are subbundles of the bundles $\mathfrak{D}_j$.*

Remark 3.5 The individual terms in the representation (3.8) of the section v are, in general, not themselves sections of the bundle $\mathfrak{B}_j$. Taking the bundle generated by all these functions $S[\nu_{\kappa_1}(y),\ldots,\nu_{\kappa_l}(y);r]\psi_l(y,\theta)$, one obtains therefore a larger bundle $\hat{\mathfrak{B}}_j$. These bundles $\hat{\mathfrak{B}}_j$ are minimal possible choices for the bundles $\mathfrak{D}_j$ whose sections produce the asymptotics near crossing points in the form we chose for our decomposition theorems. There is, however, no canonical construction for $\hat{\mathfrak{B}}_j$. In our proof of Proposition 3.3, the construction depends on the arbitrary choice of an order of the elements of $\mathcal{K}_{y_0,j}$ when applying the Leibniz rule for the divided differences. All this can be clearly seen in the example in § I.**??**.

4. A SIMPLE COMPLEX VARIABLE FORM OF THE SINGULAR FUNCTIONS

If the conormal principal part $\mathbb{M}(y;\partial_z)$ is the two-dimensional Laplace operator, we can use a complex coordinate in the normal plane to give a simple and explicit description for the singular functions (2.1) at crossing points. This formulation was inspired by a recent paper by Maz'ya and Rossmann [11] where a different but related problem was treated, namely the problem of writing the corner singularities of a *two-dimensional* Dirichlet problem for the Laplacian in a form that is stable with respect to variations of the corner angle. It is in fact not hard to see that our formulation is equivalent to that one given by Maz'ya and Rossmann, if we consider the angle $\omega(y)$ as independent unknown instead of the edge variable y. Thus we have an equivalent and, in some respect, simpler solution also for this problem of stable asymptotics in two dimensions.

The restriction to the case of the Laplace operator as conormal principal part is actually not as serious a limitation as it may look: If the coefficients of our operator A are *real*, we can always achieve this form locally by the choice of suitable admissible coordinates. In general, we will then have a variable opening angle ω_y .

Thus we assume that the coefficients of A are real and we fix an admissible system of coordinates such that $\mathbb{M}(y;\partial_z) = \Delta_z$ for $y \in I$. We will consider Dirichlet, Neumann and mixed Dirichlet-Neumann boundary conditions. In these cases, the exponents $\nu_\kappa(y)$ are well known (see (1.10)): For $\kappa = (k,l) \in \mathbb{N}^2$ we have

$$\nu_\kappa(y) = k\tfrac{\pi}{\omega_y} + l \quad \text{for Dirichlet and for Neumann conditions,} \tag{4.1}$$
$$\nu_\kappa(y) = k\tfrac{\pi}{\omega_y} + \tfrac{\pi}{2\omega_y} + l \quad \text{for mixed boundary conditions.} \tag{4.2}$$

The corresponding eigenfunctions of the Sturm-Liouville problem are also well known. They are

$$\text{either} \quad r^{\nu_k(y)}\sin(\nu_k(y)\theta) \quad \text{or} \quad r^{\nu_k(y)}\cos(\nu_k(y)\theta)\,,$$

depending on the boundary conditions.

We introduce the complex variable

$$\zeta := r\,e^{i\theta} \tag{4.3}$$

and we can then write these simplest singular functions as

$$\text{real or imaginary parts of } \zeta^{\nu_k(y)}\,. \tag{4.4}$$

In order to describe the singular functions near crossing points, we fix a regularity index $s > 0$ and consider, as in Theorems 1.8 and I.**??**, the situation of an interval I that contains exactly one crossing point y_0 (or no crossing point at all, in which case $y_0 \in I$ is arbitrary). We use the notations introduced above, such as $\mu_1^0, \dots, \mu_{j_0}^0$ for the different exponents less than s at y_0 and $\mathcal{K}_{y_0,j}$ for the set of indices κ such that $\nu_\kappa(y_0) = \mu_j^0$.

We define furthermore

$$\ell_j := \max\{l \in \mathbb{N} \mid \exists k \in \mathbb{N} :\ \nu_{kl}(y_0) = \mu_j^0\}\,. \tag{4.5}$$

As we have seen, the singular functions come in "clusters" corresponding to the clusters $\mathcal{K}_{y_0,j}$ of exponents that coincide at y_0 .We will describe a *generating set* of singular functions (see § 2) for each cluster.

We introduce the divided differences of the function $\lambda \mapsto \zeta^\lambda$:

$$S[\mu_1,\dots,\mu_K;\zeta] = \frac{1}{2i\pi}\int_\gamma \frac{\zeta^\lambda}{\prod_{j=1}^{K}(\lambda-\mu_j)}\,d\lambda \tag{4.6}$$

where γ is a simple closed curve around the complex numbers $\mu_1,\dots,\mu_K$.

Theorem 4.1 *A generating set of singular functions is given by the real and imaginary parts of*

$$S_{n;\,\kappa_1,\ldots,\kappa_K}(y,z) \;:=\; e^{-2in\theta}\, S[\nu_{\kappa_1}(y),\ldots,\nu_{\kappa_K}(y);\zeta]\,, \tag{4.7}$$

where $n \in \{0,1,\ldots,2\ell_j\}$ *and* $\kappa_1,\ldots,\kappa_K \in \mathcal{K}_{y_0,j}$, $j = 1,\ldots,j_0$.

Proof. We use the construction of the singular functions as given in the proofs of our decompositions theorems and as summarized in § 2, Propositions 2.1 and 2.2.

Thus we have to consider three cases (a), (b), (c):

(a) Solutions of $\mathbb{M}(y;\partial_z)\,S = 0$. As we have seen, these are of the form:

$$\text{Re or Im of} \quad \zeta^{\nu_k} = S[\nu_k;\zeta] = S_{0;(k0)}\ .$$

(b) Polynomials of degree $l < s$. These are generated by monomials of the form

$$\zeta^m\overline{\zeta}^{m'} = e^{-2im'\theta}\,\zeta^{m+m'} = S_{m';\,(0l)}(y,z)$$

with $m + m' \le l < s$.

Thus in cases (a) and (b) we find the form (4.7).

(c) Here we have to consider solutions S' of the two-dimensional boundary value problem

$$\mathbb{M}(y;\partial_z)\,S' = z^\gamma\,\partial_y^p\,\partial_z^\delta\,S \tag{4.8}$$

with

$$|\gamma| + 1 \ge |\delta|\,, \quad p + |\delta| \le 2\,, \tag{4.9}$$

and where S has been constructed previously.

We are looking for solutions of (4.8) that are, for $y = y_0$, sums of terms of the form $r^{\mu_j^0}\,\log^q r\,\psi(y_0,\theta)$. Homogeneity and condition (4.9) show that the exponent μ_j^0 in S' is at least one plus the corresponding exponent in S. Therefore we can consider (4.8) as an induction on this exponent.

Since all coefficients are real, we can write (4.8) using the complex variable ζ. By induction, we can assume that S has the form (4.7). We have then to show that S' is composed of terms of this form, too. Now derivatives with respect to y do not change this form, they increase only the multiplicity of some of the exponents ν_{κ_r} and introduce analytic coefficients coming from the derivatives of ν_{κ_r}. Therefore we can omit ∂_y^p and we can simplify (4.8) to the form

$$\Delta_z\,S'(y,z) = \zeta^m\overline{\zeta}^{m'}\,\partial_\zeta^d\partial_{\overline{\zeta}}^{d'}\left\{\left(\frac{\overline{\zeta}}{\zeta}\right)^n S[\nu_{\kappa_1}(y),\ldots,\nu_{\kappa_K}(y);\zeta]\right\} \tag{4.10}$$

with

$$m + m' + 1 \ge d + d' \tag{4.11}$$

with the appropriate boundary conditions for $\theta = 0$ and $\theta = \omega(y)$.

The right hand side of (4.10) is given by

$$\int_\gamma \frac{F(\zeta,\overline{\zeta},\lambda)}{\prod_{j=1}^{K}(\lambda-\nu_{\kappa_j}(y))}\,d\lambda\,,$$

where (up to a constant)

$$\begin{aligned} F(\zeta,\overline{\zeta},\lambda) &= \overline{\zeta}^{m'+n-d'}\,\zeta^{m+\lambda-n-d} \\ &= r^{\tau-2}\,e^{i\sigma\theta} \end{aligned} \tag{4.12}$$

with

$$\begin{aligned} \tau &= \lambda+m+m'+2-d-d' =: \lambda+l\,, \\ \sigma &= \lambda+m-m'-d+d'-2n\,. \end{aligned} \tag{4.13}$$

We keep now y fixed and set $\omega := \omega(y)$. We have to solve

$$\Delta\, v \;=\; r^{\tau-2}\,e^{i\sigma\theta} \tag{4.14}$$

for $0<\theta<\omega$ with the appropriate boundary conditions for $\theta=0$ and $\theta=\omega$. The solution we look for is of the form

$$v \;=\; r^\tau\, w(\theta)\;.$$

For v we obtain the Sturm-Liouville problem

$$(\partial_\theta^2+\tau^2)\,w \;=\; e^{i\sigma\theta} \quad + \;\text{boundary conditions.} \tag{4.15}$$

It is, of course, very easy to write the solution w explicitly. The solution is unique and meromorphic in τ with simple poles at $\tau=\nu_k(y)$, $k\in\mathbb{N}$. The form of the solution is particularly simple outside the two resonance points $\tau=\pm\sigma$. Now $\tau=\sigma$ does not appear since this is equivalent to $m'+n+2=d'$ which is impossible since $d'\le n$. The other resonance $\tau=-\sigma$ appears for $\lambda=n+d-m-1$, which can be avoided for $\lambda\in\gamma$ by a slight deformation of γ. Thus for $\lambda\in\gamma$ we can write the solution w of (4.15) as

$$w(\theta) \;=\; \frac{1}{\tau^2-\sigma^2}\left(e^{i\sigma\theta}+a(y,\tau)\,e^{i\tau\theta}+b(y,\tau)\,e^{-i\tau\theta}\right)\;. \tag{4.16}$$

Here the coefficients a and b have simple poles at most at $\tau=\nu_k(y)$, $k\in\mathbb{N}$, and are otherwise holomorphic. (For the Dirichlet problem we have $a=\frac{e^{-i\tau\omega}-e^{-i\sigma\omega}}{2i\sin\tau\omega}$, $b=\frac{-e^{i\tau\omega}+e^{-i\sigma\omega}}{2i\sin\tau\omega}$).

Now we can choose the contour γ in such a way that there is at most one $k\in\mathbb{N}$ such that the curve $\nu_k(y)-l$, $y\in I$, meets the interior of γ, with the integer l as defined in (4.13). In the interior of γ, there might be a pole related to $\tau=-\sigma$, i. e., $\lambda=n+d-m-1$. This is relevant only if this integer coincides with μ_j^0, the exponent at y_0, common value of $\nu_{\kappa_1}(y_0),\dots,\nu_{\kappa_K}(y_0)$.

Therefore if we define

$$G \;:=\; (\lambda-\mu_j^0)(\lambda+l-\nu_k(y))\,v\,,$$

we have

$$G \;=\; g_1(y,\tau)\, r^\tau\, e^{i\sigma\theta} + g_2(y,\tau)\, r^\tau\, e^{i\tau\theta} + g_3(y,\tau)\, r^\tau\, e^{-i\tau\theta} \tag{4.17}$$

with functions $g_i(y,\tau)$ that are analytic in $y \in I$ and in τ in the interior of the shifted contour $\gamma + l$.

The solution S' of (4.10)is given by

$$S' \;=\; \int_\gamma \frac{v\,d\lambda}{\prod_{j=1}^{K}(\lambda - \nu_{\kappa_j}(y))} \;=\; \int_\gamma \frac{G\,d\lambda}{\prod_{j=1}^{K'}(\lambda - \nu_{\kappa_j}(y))} \tag{4.18}$$

where $K \le K' \le K+2$, since at most two poles are added as we have seen, and by a slight abuse of notation, we have set $\nu_{\kappa_{K+1}} = \nu_k - l$.

Now we consider the three terms of G in (4.17) separately.

The simplest one is

$$g_2(y,\tau)\, r^\tau\, e^{i\tau\theta} = g_2(y,\tau)\,\zeta^\tau \;\;:$$

The integral

$$\int_\gamma \frac{g_2(y,\tau)\,\zeta^\tau}{\prod_{j=1}^{K'}(\lambda - \nu_{\kappa_j}(y))}\, d\lambda \;=\; \int_{\gamma\,+\,l} \frac{g_2(y,\tau)\,\zeta^\tau}{\prod_{j=1}^{K'}(\tau - (\nu_{\kappa_j}(y) + l))}\, d\tau$$

can be decomposed, by the use of the Leibniz formula (I.**??**) for divided differences, as a sum

$$\sum_{q=1}^{K'} g_{2q}(y)\, S[\nu_{\kappa_1}(y) + l, \ldots, \nu_{\kappa_q}(y) + l; \zeta]$$

with analytic coefficients $g_{2q}(y)$. Here we have therefore the desired form (4.7), even with $n = 0$.

Secondly, the term

$$g_3(y,\tau)\, r^\tau\, e^{-i\tau\theta} = g_3(y,\tau)\,\overline{\zeta}^\tau$$

leads to the complex conjugates of the basis functions just considered. Again $n = 0$ here.

Finally, we consider the term

$$g_1(y,\tau)\, r^\tau\, e^{i\sigma\theta} \;=\; g_1(y,\tau)\, e^{-2in'\theta}\,\zeta^\tau$$

with

$$n' = (\tau - \sigma)/2 = n + m' + 1 - d'\,. \tag{4.19}$$

Using again the Leibniz formula, we obtain terms of the form (4.7) with the increased value of $n' =: n + l'$.

It remains to prove the bound on n: $0 \le n \le 2\ell_j$. To this purpose we show that in the last part of the above induction step we have always

$$l' \;\le\; 2l\,. \tag{4.20}$$

This will suffice, since l is the increment in the exponent μ_j^0, and l' is the increment in n, and for the beginning of the induction, namely (a) and (b) above, one has obviously $0 \le n \le 2\ell_j$ (namely $n = 0$ in (a) and $n = m' \le m + m' = \ell_j$ in (b)).

We have, according to (4.13) and (4.19),

$$l' \,=\, m' + 1 - d' \quad \text{ and } \quad l \,=\, m + m' + 2 - d - d' \,.$$

Hence $l' = l - m - 1 + d$, and since $d + d' \le 2$, the only case where $l' > l$ can appear is for $m = 0$ and $d = 2$.

In this case necessarily $d' = 0$ and $m' \ge 1$, hence

$$l' \,=\, m' + 1 \,\le\, 2m' \,=\, 2l \,.$$

Thus (4.20) is shown and the proof is complete. ∎

Remark 4.2 For every fixed $y \in I$, the singular function (4.7) is a linear combination of terms of the form

$$\overline{\zeta}^n \, \zeta^{\nu_\kappa(y) - n} \, \log^q \zeta \,. \tag{4.21}$$

This is the classical complex-variable form of the singularities for Laplace's equation on a two-dimensional sector [7]. Note, however, that logarithmic terms show up in the edge singularities not only, as in the two-dimensional case, together with integer exponents ν_κ (that is, at crossing points), but generally for any exponent ν_{kl} with $l \ge 1$. This is due to the presence of tangential derivatives in the differential operator A.

Remark 4.3 If y and the multiplicities in (4.7) are such that all $\nu_{\kappa_1}, \ldots, \nu_{\kappa_K}$ are different, then one can write the singular function (4.7) as

$$S_{n;\,\kappa_1,\ldots,\kappa_K} \,:=\, \overline{\zeta}^n \, \zeta^{-n} \, \sum_{q=1}^{K} a_q(y) \, \zeta^{\nu_{\kappa_q}(y)} \tag{4.22}$$

with the coefficients

$$a_q(y) \,=\, \prod_{\substack{r=1 \\ r \ne q}}^{K} \frac{1}{\nu_{\kappa_q}(y) - \nu_{\kappa_r}(y)} \,. \tag{4.23}$$

5. APPENDIX : SOME FACTS ON ANALYTIC VECTOR BUNDLES

We will consider real analytic vector bundles on a compact interval $J \subset \mathbb{R}$. In fact, we will need subbundles of finite fiber dimension of some fixed analytic vector bundle $\mathfrak{X}$ on J, whose fibers we assume to be Hilbert spaces.

Since J is contractible, there exists a global trivialization of $\mathfrak{X}$. The structure of $\mathfrak{X}$ can therefore be described as follows : there is a Hilbert space X (whose inner product we denote by $<\cdot,\cdot>$) and for each $y \in J$ one has a Hilbert space $\mathfrak{X}(y)$ and an isomorphism

$$T(y) : \mathfrak{X}(y) \longrightarrow X.$$

An *analytic section* $\varphi : J \to \mathfrak{X}$ is then defined by a mapping $y \mapsto \varphi(y) \in \mathfrak{X}(y)$ such that the associated section φ_T of the trivial bundle $J \times X$ is analytic, i. e. the function

$$\varphi_T = (y \mapsto \varphi_T(y) = T(y)\varphi(y)) : J \longrightarrow X$$

is analytic. (Analytic functions on J with values in the Hilbert space X can be defined in a variety of equivalent ways. One possible definition is : $\psi : J \to X$ is analytic if for any $w \in X$ the function $y \mapsto <w, \psi(y)> : J \to \mathbb{C}$ is real analytic).

An *analytic subbundle* of $\mathfrak{X}$ of finite fiber dimension n is given by a set $\{\varphi_1, \ldots, \varphi_n\}$ of analytic sections of $\mathfrak{X}$ such that for each $y \in J$, the vectors $\{\varphi_1(y), \ldots, \varphi_n(y)\}$ are linearly independent in $\mathfrak{X}(y)$. We denote such a subbundle $\mathfrak{B}$ by

$$\mathfrak{B} = \operatorname{span}\{\varphi_1, \ldots, \varphi_n\}.$$

Since every analytic vector bundle over J is trivial, we obtain every subbundle of $\mathfrak{X}$ of finite dimension in this way. More precisely, let $\{\mathfrak{B}(y)\}_{y \in J}$ be a family of finite dimensional subspaces of $\mathfrak{X}(y)$ with the property that to every point in J there exists a neighborhood $\mathcal{U}$ and a set $\{\varphi_1^{\mathcal{U}}, \ldots, \varphi_n^{\mathcal{U}}\}$ of analytic sections of $\mathfrak{X}\Big|_{\mathcal{U}}$ such that $\{\varphi_1^{\mathcal{U}}(y), \ldots, \varphi_n^{\mathcal{U}}(y)\}$ is a basis of $\mathfrak{B}(y)$ for any $y \in \mathcal{U}$. In this case there exists a global basis $\{\varphi_1, \ldots, \varphi_n\}$ of analytic sections of $\mathfrak{X}$ on J such that $\mathfrak{B} = \operatorname{span}\{\varphi_1, \ldots, \varphi_n\}$.

Our first result concerns the span of a finite number of analytic sections of $\mathfrak{X}$. This is not a subbundle of $\mathfrak{X}$, in general, since its dimension can collapse in a finite number of points, but it determines in fact a unique subbundle.

Lemma 5.1 *Let $\{\psi_1, \ldots, \psi_n\}$ be not identically vanishing analytic sections of $\mathfrak{X}$ over J. Then there exist analytic sections $\{\varphi_1, \ldots, \varphi_m\}$ such that $\varphi_1(y), \ldots, \varphi_m(y)$ are linearly independent for all $y \in J$ and a finite set $\{y_1, \ldots, y_k\} \subset J$ such that*

$$\operatorname{span}\{\psi_1(y), \ldots, \psi_n(y)\} = \operatorname{span}\{\varphi_1(y), \ldots, \varphi_m(y)\}$$

for all $y \in J \setminus \{y_1, \ldots, y_k\}$. That is, the subset $\operatorname{span}\{\psi_1, \ldots, \psi_n\}$ of $\mathfrak{X}$ coincides on $J \setminus \{y_1, \ldots, y_k\}$ with the subbundle $\mathfrak{B} := \operatorname{span}\{\varphi_1, \ldots, \varphi_m\}$.

Proof. By application of the global trivialization T of $\mathfrak{X}$, we can assume that $\mathfrak{X}$ is trivial, i. e., $\mathfrak{X}(y) = X$ for all $y \in J$. The proof uses induction on n.

For $n = 1$, the number of zeros of ψ_1 is finite and there exists a polynomial p_1 vanishing at these zeros with the appropriate multiplicity such that

$$\varphi_1(y) := \frac{\psi_1(y)}{p_1(y)}$$

is an analytic section on all of J that vanishes nowhere.

Now suppose $\{\psi_1, \ldots, \psi_n\}$ were given and $\{\varphi_1, \ldots, \varphi_m\}$ have been constructed appropriately. Let ψ_{n+1} be given and consider the Gram determinant of the vectors $\varphi_1, \ldots, \varphi_m, \psi_{n+1}$:

$$D(y) = \det\Big(<\tilde{\varphi}_i(y), \tilde{\varphi}_j(y)>_{i,j=1,\ldots,m+1}\Big)$$

where $\tilde{\varphi}_j = \varphi_j$ for $j = 1, \ldots, m$ and $\tilde{\varphi}_{m+1} = \psi_{n+1}$.
If $D(y)$ vanishes identically, the bundle $\mathfrak{B}_m = \mathrm{span}\{\varphi_1, \ldots, \varphi_m\}$ is already suitable for $\{\psi_1, \ldots, \psi_{n+1}\}$.
Otherwise, the analytic function $D(y)$ vanishes at most on a finite set of points in J. Let y_0 be such a point and let k be the order of the zero y_0. Thus

$$(y - y_0)^{-k} D(y) \text{ is analytic and different from 0 in } y_0.$$

We construct φ_{m+1} by induction on k.
Since $D(y_0) = 0$, there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ such that

$$\psi_{n+1}(y_0) = \sum_{j=1}^{m} \lambda_j \, \varphi_j(y_0) \, .$$

If we set

$$\varphi^{(1)}_{m+1}(y) := \frac{1}{y - y_0} \Big(\psi_{n+1}(y) - \sum_{j=1}^{m} \lambda_j \, \varphi_j(y)\Big),$$

then $D^{(1)}(y)$, the Gram determinant of $\varphi_1(y), \ldots, \varphi_m(y), \varphi^{(1)}_{m+1}(y)$ satisfies

$$D^{(1)}(y) = \frac{1}{y - y_0} \, D(y) \, .$$

Thus the order of the zero has decreased by one. We construct $\varphi^{(j)}_{m+1}$ (for $j = 1 \ldots, k$) analogously. For

$$\varphi_{m+1} := \varphi^{(k)}_{m+1}$$

we have then

$$D^{(k)}(y) = \det\Big(<\varphi_i(y), \varphi_j(y)>_{i,j=1,\ldots,m+1}\Big) = (y - y_0)^{-k} D(y) \, .$$

This means that $\{\varphi_1(y), \ldots, \varphi_{m+1}(y)\}$ are linearly independent for y in a neighborhood of y_0. By construction we have also

$$\mathrm{span}\{\psi_1(y), \ldots, \psi_{n+1}(y)\} = \mathrm{span}\{\varphi_1(y), \ldots, \varphi_{m+1}(y)\}$$

for y in a neighborhood of y_0, $y \neq y_0$.
Thus we have constructed the bundle $\mathfrak{B}_{m+1}$ locally. By the above remark, there exists also a global basis. ∎

Remark 5.2 If we replace "analytic" by "C^∞", the corresponding result does not hold, as the following example shows. The space $\mathbb{R}^3$ is taken as space X and $J = [-1, 1]$. We set

$$\begin{aligned} \psi_1(y) &= (1, 0, 0) \\ \psi_2(y) &= (1, e^{-1/y^2} \cos \frac{1}{y}, e^{-1/y^2} \sin \frac{1}{y}) \, . \end{aligned}$$

When $y \neq 0$, the dimension of $\Psi(y) := \text{span}\{\psi_1(y), \psi_2(y)\}$ is 2. In $y = 0$ this dimension is 1. But in a certain sense the bidimensional space $\Psi(y)$ tends to $\mathbb{R}^3$ when $y \to 0$, because

$$\forall x \in \mathbb{R}^3, \ \forall \varepsilon > 0 \ \exists y, \ |y| < \varepsilon : \quad x \in \Psi(y) \, .$$

No smooth functions φ_1 and φ_2 exist such that $\Psi(y) = \text{span}\{\varphi_1(y), \varphi_2(y)\}$ in a neighborhood of 0.

Remark 5.3 Lemma 5.1 remains true if the given sections $\psi_1, \ldots, \psi_n$ are only *meromorphic* on J. This means that the ψ_j are analytic sections outside of a finite number of points ("poles") and that there exist non identically vanishing polynomials $p_j : J \to \mathbb{C}$ such that $p_j(y) \cdot \psi_j(y)$ are analytic sections on J.

Definition 5.4 *In the situation of Lemma* 5.1 *and Remark* 5.3*, we will also say that the bundle*

$$\text{span}\{\psi_1, \ldots, \psi_n\}\Big|_{J \setminus \{y_1, \ldots, y_k\}}$$

can be extended as an analytic bundle $\mathfrak{B} = \text{span}\{\varphi_1, \ldots, \varphi_m\}$ *on* J *and we write*

$$\mathfrak{B} = \overline{\text{span}}\{\psi_1, \ldots, \psi_n\}.$$

Corollary 5.5 *Let* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *be two analytic subbundles of* $\mathfrak{X}$ *with finite fiber dimensions. Then their sum* $\mathfrak{B}_1 + \mathfrak{B}_2$ *can be extended to a unique analytic subbundle of* $\mathfrak{X}$*, denoted by*

$$\overline{\text{span}}\{\mathfrak{B}_1, \mathfrak{B}_2\}.$$

Proof. Consider trivializations $\{\varphi_1^j, \ldots, \varphi_{m_j}^j\}$ of $\mathfrak{B}_j$, i. e. analytic sections such that for each $y \in J$, $\varphi_1^j(y), \ldots, \varphi_{m_j}^j(y)$ are a basis of $\mathfrak{B}_j(y)$. Then with Lemma 5.1 and Definition 5.4 we can simply define

$$\overline{\text{span}}\{\mathfrak{B}_1, \mathfrak{B}_2\} := \overline{\text{span}}\{\varphi_1^1, \ldots, \varphi_{m_1}^1, \varphi_1^2, \ldots, \varphi_{m_2}^2\}.$$

∎

We end this section by a result which shows a relation between the sum of analytic vector bundles and divided differences. We consider special bundles which are "generated" by an analytic function w in the sense we explain below.

So let $\mathcal{D}$ be a domain in $\mathbb{C}$ and $\mathfrak{X}$ a Hilbert space. Let $w : \mathcal{D} \to \mathfrak{X}$ be a holomorphic function. Let us denote by $w^{(k)}$ the k-th derivative of w. Let $\mu_1, \ldots, \mu_n$ be analytic functions from a compact interval J into $\mathcal{D}$. We assume that the functions μ_j for $j = 1, \ldots, n$ are all distinct i. e. :

$$\mu_i \equiv \mu_j \text{ on } J \Longrightarrow i = j \, .$$

Let $q_1, \ldots, q_n$ be positive integers. According to Lemma 5.1

$$\overline{\text{span}}\{w(\mu_j(y)), \ldots, w^{(q_j - 1)}(\mu_j(y))\}$$

defines for each $j = 1, \ldots, n$ an analytic vector bundle $\mathfrak{B}_j$ over J and the sum

$$\overline{\operatorname{span}}\{\mathfrak{B}_1, \ldots, \mathfrak{B}_n\} := \mathfrak{B}$$

is an analytic vector bundle.

We want to find analytic sections of $\mathfrak{B}$. We assume the following hypothesis.

Hypothesis 5.6 *Let N be the sum $\sum_{j=1}^{n} q_j$. For any finite set of distinct points $\{\lambda_1, \ldots, \lambda_n\}$ in $\mathcal{D}$, the nN vectors $w^{(k)}(\lambda_j)$ for $k = 0, \ldots, N-1$ and $j = 1, \ldots, n$ are linearly independent.*

Proposition 5.7 *We assume Hypothesis 5.6. Then a trivialization of $\mathfrak{B}$ is given by*

$$w[\tilde{\mu}_1(y)],\, w[\tilde{\mu}_1(y), \tilde{\mu}_2(y)], \ldots,\, w[\tilde{\mu}_1(y), \ldots, \tilde{\mu}_N(y)]$$

where

$$\tilde{\mu}_k = \mu_j \quad \textit{for} \quad \underbrace{q_1 + \cdots + q_{j-1}}_{=0 \ \textit{if}\ j=1} < k \le q_1 + \cdots + q_j$$

(this is the repetition according to the multiplicity).

Remark 5.8 Compare with Theorems 3.3 and 4.1 where the generating function is, respectively, $w(\lambda) = r^{\lambda}$ and $w(\lambda) = \zeta^{\lambda}$.

The proof of Proposition 5.7 relies upon the following lemma.

Lemma 5.9 *Let $w : \mathcal{D} \to \mathfrak{X}$ be holomorphic and $\nu_1, \ldots, \nu_N$ be elements of $\mathcal{D}$ (not necessarily distinct). Let $k_1, \ldots, k_n$ be distinct integers in $\{1, \ldots, N\}$. Then*

$$w[\nu_{k_1}, \ldots, \nu_{k_n}]$$

is a linear combination of

$$w[\nu_1], \ldots, w[\nu_1, \ldots, \nu_N]\,.$$

Proof. Let γ be a contour around $\nu_1, \ldots, \nu_N$. We have according to (0.5)

$$\begin{aligned} w[\nu_{k_1}, \ldots, \nu_{k_n}] &= \frac{1}{2i\pi} \int_{\gamma} \frac{w(\lambda)}{\prod_{j=1}^{n} (\lambda - \nu_{k_j})}\, d\lambda \\ &= \frac{1}{2i\pi} \int_{\gamma} \frac{Q(\lambda) w(\lambda)}{P(\lambda)}\, d\lambda \end{aligned}$$

where $P(\lambda) = \prod_{j=1}^{N} (\lambda - \nu_j)$ and $Q(\lambda)$ is the polynomial

$$Q(\lambda) = \frac{P(\lambda)}{\prod_{j=1}^{n} (\lambda - \nu_{k_j})}\,.$$

The Lemma is then a consequence of the Leibniz formula (I.??). ∎

Proof of Proposition 5.7. A first application of the Lemma 5.9 proves that for any $y \in J$ the space $\mathfrak{C}(y)$ generated by

$$w[\tilde{\mu}_1(y)], \ldots, w[\tilde{\mu}_1(y), \ldots, \tilde{\mu}_N(y)]$$

contains all the $w^{(k)}(\mu_j)$ for $j = 1, \ldots, n$ and $k = 0, \ldots, q_j$. Now it suffices to prove that the dimension of $\mathfrak{C}(y)$ is equal to N in every point $y \in J$. Let us fix $y_0 \in J$ and let $\{\nu_1, \ldots, \nu_m\}$ be the distinct values of $\{\tilde{\mu}_1(y_0), \ldots, \tilde{\mu}_N(y_0)\}$ and p_j be the multiplicity of ν_j for $j = 1, \ldots, m$. As a consequence of Lemma 5.9,

$$w^{(p-1)}(\nu_j) \in \mathfrak{C}(y_0) \quad \forall j = 1, \ldots, m \quad 1, \ldots, p_j.$$

Hypothesis 5.6 implies that these N vectors are independent. ∎

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