GENERAL EDGE ASYMPTOTICS OF SOLUTIONS OF SECOND ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS I.

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Abstract

This is the first of two papers in which we study the singularities of solutions of second order linear elliptic boundary value problems at the edges of piecewise analytic domains in $\mathbb{R}^3$. When the opening angle at the edge is variable, there appears the phenomenon of "crossing" of the exponents of singularities. For this case, we introduce the appropriate combinations of the simple tensor product singularities that allow to give estimates in ordinary and weighted Sobolev spaces for the regular part of the solution and for the coefficients of the singularities. These combinations appear in a natural way as sections of an analytic bundle above the edge. Their behavior is described with the help of divided differences of powers of the distance to the edge. The class of operators considered includes second order elliptic operators with analytic complex valued coefficients with mixed Dirichlet, Neumann or oblique derivative conditions. With our description of the singularities we are able to remove some restrictive hypotheses that were previously made in other works. In this first part, we prove the basic facts in a simplified framework. Nevertheless the tools we use are essentially the same in the general situation.

ORIGIN OF THE PROBLEM.

Intersections and unions of simple everyday geometrical objects as cylinders, cones, balls, or half-spaces, usually have edges and corners that give rise to singularities of solutions of elliptic boundary value problems of mathematical physics in such bodies. The precise description of these singularities can be used, for example, for the construction of effective numerical approximation methods.

In general, the opening angles are not constant along the edges. Let us mention two simple examples. The first, the "skew cylinder", is a cylinder with a circular base that is cut by a plane which is not perpendicular to the axis of the cylinder. Here the opening angle is less than $\frac{\pi}{2}$ on the upper half and greater than $\frac{\pi}{2}$ on the lower half of the elliptical edge, and there are precisely two points where this angle is $\frac{\pi}{2}$. The second example is a "knee" obtained by joining two identical skew cylinders at their elliptical top surface. Here one has two points where the angle is π.

It is known that for the mixed Dirichlet-Neumann problem for the Laplacian in a two dimensional sector, the type of the corner singularities changes if the opening angle ω is $\frac{\pi}{2}$. Let us give the formulas when a zero Dirichlet condition is imposed on the side $\theta = \omega$ and the Neumann condition g is prescribed on the side $\theta = 0$. When $\omega \neq \frac{\pi}{2}$ (and $\omega > \frac{\pi}{4}$) the first corner singularity is

$$r^{\frac{\pi}{2\omega}} \cos(\tfrac{\pi}{2\omega}\theta)\,. \tag{0.1}$$

When $\omega = \frac{\pi}{2}$, instead of $r\cos\theta$ we have as singularity :

$$r(\log r\ \cos\theta + (\omega - \theta)\sin\theta)\,. \tag{0.2}$$

Therefore one expects that for the corresponding problem on the skew cylinder, the coefficients of the first singular function will blow up at the two exceptional points mentioned above. The starting point of the present work was this problem and the question, originally posed by I. Babuška, of finding a description of the singularities that allows estimates in Sobolev spaces and is sufficiently explicit to be used for numerical approximations.

From the literature on edge singularities let us mention the works of Kondrat'ev [6] and Nikishkin [12] who considered Dirichlet problems for operators that are equivalent to the Laplace operator (see also the survey [7]). This approach has been generalized by Maz'ya and Roßmann [10] to include the most general elliptic boundary value problems (see also the review of their results in [8]). The hypotheses made in these papers, however, exclude precisely the phenomenon mentioned above for the skew cylinder. Another standard hypothesis is that the regularity index considered for the splitting of the solution into a regular and a singular part does not coincide with any of the exponents of the singular functions. We show that both of these hypotheses can be removed at the expense of an arbitrarily small loss of regularity. The Neumann problem for the Laplacian has also traditionally been excluded from consideration by a certain bijectivity assumption. We show how to get rid of this assumption.

A much more general framework for edge problems has been considered in the approach of Rempel and Schulze [15, 14, 17]. Here the problems arising from varying edge angles and variable coefficients have been attacked on a very general basis. Since it is our aim to give very explicit descriptions and estimates for the singularities, we choose a more direct approach in the spirit of [10].

Some of the results of this paper were announced in [1, 2]. Our work is divided into two parts. In Part I, we present the basic results and proofs. In Part II, we prove some extensions and improvements. In order to deal with the simplest hypotheses and notations possible, we chose to treat in Part I the Dirichlet boundary conditions. We show in Part II how this can be extended to a more general framework, including mixed, Neumann or oblique derivatives conditions.

In Part I, we describe the structure of the singularities with the help of "divided differences" of powers r^λ of the distance r to the edge calculated in λ equal to some singularity exponents of the problem. In Part II, we prove that such a description can be formulated in terms of analytic bundles above the edges, which gives another insight of the structure of the solutions. In the spirit of [11] we also prove that, in the case of the Laplace operator, a correct and simple description can be obtained with divided differences of ζ^λ, where $\zeta \in \mathbb{C}$ is the complex writing of the normal coordinates to the edge. All this is based on an algorithm for the construction of singularities. This algorithm, contained in the proof of Theorem 1.3, will be further explained in Part II. It allows a better understanding of the singularities and may eventually lead to the possibility of proving approximation properties for these singularitites.

The methods of this paper can be used to treat various related problems which will be the subjects of forthcoming papers. In the general situation of higher order operators and systems, the singularity exponents no longer depend on the opening angle in an analytic way : there appear bifurcations. It is nevertheless possible to treat this general case along the lines of this paper (a description of the form of the singular functions in the case of bifurcations has been given by Schmutzler [16]). A second question to be considered concerns estimates in spaces of analytic functions for the solution and the coefficients of the singularities. Thirdly, the assumptions on the geometry of the domain can be generalized in various directions : non-analytic faces, degenerating edges and, more interestingly, polyhedral corners.

1. MAIN RESULTS

1.a Domains. The domains we consider are three-dimensional bounded Lipschitz domains Ω with piecewise analytic boundary and analytic edges. For such a domain, there exists an analytic manifold M of dimension 1 and without boundary such that $\partial\Omega \setminus M$ is the disjoint union of a finite number of connected components $\partial_j\Omega$, which are analytic manifolds of dimension 2 and with boundary. M is the union of the *edges* and the $\partial_j\Omega$ are the *faces*. We assume that near any $y \in M$, Ω is analytically diffeomorphic to a dihedral angle. The example of the skew cylinder obviously satisfies the above properties.

In each point y of M, let $\omega(y)$ be the opening of Ω in y : more precisely, $\omega(y)$ is the angle between the two tangent planes to $\partial\Omega$ at y. The assumption that Ω is Lipschitz excludes that $\omega(y)$ can be equal to 0 or 2π.

We have seen in the example of the knee that the occurrence of the opening π is also natural. But if $\omega(y)$ is equal to π in an isolated point y_0, Ω is *not* diffeomorphic to a dihedral angle in any neighborhood of y_0. Therefore we exclude this case now. We will return to it in a forthcoming paper. On the other hand, if $y_0 \in M$ belongs to a regular part of the boundary of Ω, $\omega(y) \equiv \pi$ in the whole connected component of M which contains y_0 and the assumption holds. Such a part of the edge is a line of discontinuity for the boundary conditions or the boundary data. This will appear in the general boundary conditions we consider in Part II.

Locally near any point y_0 of M, we will use special cylindrical coordinates (y, r, θ) such that in a suitable neighborhood the edge corresponds to $r = 0$, y is the coordinate along the edge and Ω corresponds to $0 < \theta < \omega(y)$. Since Ω is piecewise analytic, we can assume that these coordinates are analytic. Such coordinates are cylindrical coordinates associated with cartesian analytic variables (y, z) : $r = \sqrt{z_1^2 + z_2^2}$ and $\theta = \operatorname{Arctan} \frac{z_2}{z_1}$.

1.b Boundary value problems. As stated above, our first motivation was to treat mixed Dirichlet-Neumann problems for the Laplace operator. It turns out that the type of structure we find is not specific of the Laplace operator and it is natural to extend this work to general elliptic second order boundary value problems. But in this Part I, we will for simplicity only consider the Dirichlet conditions for a class of elliptic second order operators.

Let

$$A(x; \partial_x) = \sum_{|\alpha| \le 2} a_\alpha(x) \partial_x^\alpha$$

be an elliptic second order operator with complex coefficients, analytic on $\overline{\Omega}$. We want to describe the structure of solutions of the following Dirichlet problem :

$$\begin{cases} Au = f & \text{in } \Omega\,, \\ u \in \mathring{H}^1(\Omega)\,. \end{cases} \tag{1.1}$$

We assume some regularity hypotheses on the right hand side : for a positive real number s

$$f \in H^{s-1}(\Omega)\,. \tag{1.2}$$

We suppose that A satisfies some *a priori* estimates along the edges : see Hypothesis 4.2. A more general hypothesis will be presented in Part II, in a more direct and invariant form. It is important to note that any *strongly elliptic* operator A satisfies such an assumption. Let us recall that A is called strongly elliptic if there holds

$$\exists C > 0, \quad \forall \xi \in \mathbb{R}^3, \quad \operatorname{Re} \sum_{|\alpha|=2} a_\alpha(x) \xi^\alpha \ge C|\xi|^2$$

1.c Exponents of singularities. For a solution of (1.1), we expect singularities along the edges which behave as powers $r^{\mu(y)}$ of the distance from the edge. The exponents $\mu(y)$ are determined by the eigenvalues of some Sturm-Liouville problems $\mathcal{M}_y$ on $(0, \omega(y))$ (see (4.5)). It is well known that for the Laplace operator these eigenvalues are $\frac{k\pi}{\omega(y)}$.

According to [4] §14, we obtain that, in the case of second order scalar operators with complex coefficients, the eigenvalues of $\mathcal{M}_y$ are simple and have the following form

$$k\nu(y),\ k \in \mathbb{Z}^* \quad \text{with} \quad \nu(y) \in \mathbb{C} \setminus i\mathbb{R}\,. \tag{1.3}$$

The function ν is analytic on the edge M. We will give a formula for ν in section 4 (see (4.6)).

We also need the translations by integers of the eigenvalues $k\nu$. We write

$$\nu_{kl}(y) := k\nu(y) + l, \quad \text{for} \quad k \geq 0 \quad \text{and} \quad l \geq 0. \tag{1.4}$$

The integer exponents ν_{0l} arise in Taylor expansions. They give rise to singularities when they cross a singularity exponent. We use also the notation

$$\nu_\kappa = \nu_{kl} \quad \text{for} \quad \kappa = (k,l) \in \mathbb{N}^2.$$

1.d Simple asymptotics. What can be expected as asymptotics along the edge in local coordinates (y, r, θ) is

$$\sum_{\kappa,q,n} c_{\kappa,q,n}(y)\, r^{\nu_\kappa(y)} \log^q r\, \varphi_{\kappa,q,n}(y,\theta) \tag{1.5}$$

where only the $c_{\kappa,q,n}$ depend on the data (f, g_j). Actually, such an asymptotics in tensor product form is not convenient in general, since the $c_{\kappa,q,n}$ are not regular enough. Therefore we define the usual (see [6], [4], [10]) regular extension of the coefficients : we introduce a function $\Phi(y,r)$ such that its partial Fourier transform satisfies

$$\mathcal{F}_{y\to\xi}\Phi(\xi,r) = \phi(r|\xi|)$$

where ϕ is a rapidly decreasing function, has a Fourier transform with compact support, and satisfies for a sufficiently large N

$$\phi(0) = 1, \quad \frac{d^n}{dt^n}\phi(0) = 0 \quad (n = 1, \ldots, N). \tag{1.6}$$

We define the convolution with respect to y,

$$(c * \Phi)(y,r) := \int \Phi(y-y', r)\, c(y')\, dy'. \tag{1.7}$$

The following theorem describes our result on the 'simple' edge asymptotics, i. e. when any crossing between exponents ν_κ is excluded. Asymptotics of this type, but under more restrictive hypotheses, are given in [6], [12] and [10]. For the case of a straight edge, see [4], and for a circular edge, see [13].

Theorem 1.1 *We assume that A is strongly elliptic. Let I, I' be intervals such that the local coordinates (y,r,θ) are defined in a neighborhood $\mathcal{U}$ of I' in Ω and $I \subset\subset I'$. We assume that for some $\varepsilon_0 \geq 0$ there holds*

$$\text{for all } \kappa \text{ we have } \ \forall y \in I',\ \operatorname{Re}\nu_\kappa(y) < s \ \text{ or } \ \forall y \in I',\ \operatorname{Re}\nu_\kappa(y) \geq s - \varepsilon_0\,. \tag{1.8}$$

We suppose there is no crossing point in I', i. e.

$$\text{if } \nu_\kappa(y) = \nu_{\kappa'}(y) \text{ and } \operatorname{Re}\nu_\kappa(y) < s \text{ for some } y \in I', \quad \text{then } \nu_\kappa \equiv \nu_{\kappa'}. \tag{1.9}$$

To each κ there exists a finite set of indices (q,n) and analytic functions $\varphi_{\kappa,q,n}(y,\theta)$ such that any solution u of problem (1.1) with $f \in H^{s-1}(\Omega)$ can be decomposed into

$$u = u_{\mathrm{reg}} + u_{\mathrm{sing}}.$$

Here $u_{\mathrm{reg}} \in H^{s+1-\varepsilon}(\mathcal{U})$ and $r^{-s-1+\varepsilon} u_{\mathrm{reg}} \in L^2(\mathcal{U})$ $\forall \varepsilon > \varepsilon_0$ and

$$u_{\mathrm{sing}} = \sum_{\kappa,q,n} (c_{\kappa,q,n} * \Phi)(y,r)\, r^{\nu_\kappa(y)} \log^q r\, \varphi_{\kappa,q,n}(y,\theta)\,. \tag{1.10}$$

The coefficients $c_{\kappa,q,n}(y)$ are defined on I and satisfy $c_{\kappa,q,n} \in H^{s-\operatorname{Re}\nu_\kappa(y)-\varepsilon}(I)$ for all $\varepsilon > 0$. The sum extends over those κ for which $\operatorname{Re}\nu_\kappa < s$ holds on I.

Let us note that the assumption (1.8) is not restrictive. For any interval I such an ε_0 exists ; conversely, for any $y_0 \in M$ and any $\varepsilon_0 > 0$ we can find such an interval I containing y_0. In the works [6], [12] and [10], the corresponding assumption is formulated more restrictively :

$$\forall \kappa,\ \forall y \in M, \quad \operatorname{Re}\nu_\kappa(y) \neq s. \tag{1.11}$$

If (1.11) holds on I, we can take $\varepsilon_0 = 0$ in our statement.

The assumption (1.9) excludes any crossing of the exponents ν_κ, i. e. any isolated point of the edge where two distinct exponents coincide with each other without remaining equal on the whole edge. All the authors quoted above also require this condition. We will see that such a crossing of exponents in general induces the blowing up of coefficients in the expansion (1.10).

For the problem of the skew cylinder with the Laplace operator, it is impossible to avoid such crossings. For y_0 such that $\omega(y_0) = \pi/2$ (there always exist two such points), we have $\nu_\kappa(y_0) = \nu_{\kappa'}(y_0)$ for

$\kappa = (1,0)$ and $\kappa' = (0,2)$ for Dirichlet or Neumann problems
$\kappa = (1,0)$ and $\kappa' = (0,1)$ for the mixed problems.

The points where crossing of exponents will eventually appear (for large s) are dense in M, so this phenomenon occurs in a generic way.

1.e Asymptotics at crossing points. Let y_0 be a crossing point, i. e., a point where there exist κ and κ' such that

$$\nu_\kappa(y_0) = \nu_{\kappa'}(y_0) \text{ with } \operatorname{Re}\nu_\kappa(y_0) < s \quad \text{and} \quad \nu_\kappa \not\equiv \nu_{\kappa'} \text{ in } I. \tag{1.12}$$

Our domain being piecewise analytic, such a point is isolated. We are going to group together the exponents which meet each other at y_0.

Let $\mathcal{K}_{y_0}$ be the set of indices,

$$\mathcal{K}_{y_0} := \{\kappa = (k,l) \mid \operatorname{Re}\nu_\kappa(y_0) < s\}\,.$$

We denote by $\mu_1^0, \ldots, \mu_{j_0}^0$ the distinct elements of the set

$$\{\nu_\kappa(y_0) \mid \kappa \in \mathcal{K}_{y_0}\}.$$

Since y_0 is a crossing point, the cardinality of $\mathcal{K}_{y_0}$ is strictly larger than j_0. For each j, let $\mathcal{K}_{y_0,j}$ be the subset of $\mathcal{K}_{y_0}$,

$$\mathcal{K}_{y_0,j} := \left\{\kappa \in \mathcal{K}_{y_0} \mid \nu_\kappa(y_0) = \mu_j^0\right\}.$$

The μ_j^0 are either crossing exponents (if $\#\mathcal{K}_{y_0,j} > 1$) or simple exponents (if $\#\mathcal{K}_{y_0,j} = 1$).

For each κ, we call multiplicity of ν_κ the maximal power of $\log r$ which appears in the asymptotics (1.10) along with the term $r^{\nu_\kappa(y)}$ for $y \in I \setminus \{y_0\}$. Then we denote by $(\kappa_j^q)_{1\le q\le q_j}$ an enumeration of $\mathcal{K}_{y_0,j}$, repeating each term according to its multiplicity.

Finally, we set for $y \in I$:

$$\mu_j(y) := \max_{\kappa\in\mathcal{K}_{y_0,j}} \operatorname{Re}\nu_\kappa(y). \tag{1.13}$$

What essentially changes from the simple asymptotics (1.10) is the behavior of the functions of r. Instead of having separately the terms $r^{\nu_\kappa(y)}\log^p r$, we have now special combinations of these terms which are globally analytic and cannot be separated without destroying this analyticity. Let us introduce these combinations.

Definition 1.2 Let $q \ge 1$ an integer and $\nu_1, \ldots, \nu_q$ be complex numbers, not necessarily distinct. Let γ be any simple curve surrounding $\nu_1, \ldots, \nu_q$ in the complex plane. Then we define

$$S[\nu_1, \ldots, \nu_q; r] = \frac{1}{2\pi i}\int_\gamma \frac{r^\lambda}{(\lambda-\nu_1)\cdots(\lambda-\nu_q)}\,d\lambda.$$

For each fixed value of r, $S[\nu_1, \ldots, \nu_q; r]$ is nothing else but the divided difference of the function $\lambda \mapsto r^\lambda$ at the points $\nu_1, \ldots, \nu_q$ (see (8.4)).

Here are some examples. We assume that ν_1 is different from ν_2.

$$\begin{aligned}
S[\nu_1; r] &= r^{\nu_1}\\
S[\nu_1, \nu_1; r] &= r^{\nu_1}\log r\\
S[\nu_1, \nu_2; r] &= \frac{r^{\nu_1}-r^{\nu_2}}{\nu_1-\nu_2}\\
S[\nu_1, \nu_1, \nu_2; r] &= \frac{r^{\nu_1}\log r}{\nu_1-\nu_2} - \frac{r^{\nu_1}-r^{\nu_2}}{(\nu_1-\nu_2)^2}\,.
\end{aligned}$$

Theorem 1.3 *We assume the same hypotheses about the boundary value problem and the intervals I and I' as in Theorem* 1.1 *and we take $\varepsilon_0 \geq 0$ satisfying* (1.8). *We suppose that y_0 is a crossing point in I and there is no other crossing point in I'. Then with the above notations, to each $j = 1, \ldots, j_0$ and to each $q = 1, \ldots, q_j$, there exists a finite set of indices n and analytic functions $\psi_{j,q,n}(y, \theta)$ such that any solution u of problem* (1.1) *with $f \in H^{s-1}(\Omega)$ can be decomposed into*

$$u = u_{\text{reg}} + u_{\text{sing}}.$$

Here $u_{\text{reg}} \in H^{s+1-\varepsilon}(\mathcal{U})$ and $r^{-s-1+\varepsilon} u_{\text{reg}} \in L^2(\mathcal{U})$ $\forall \varepsilon > \varepsilon_0 + \delta(I)$ and

$$u_{\text{sing}} = \sum_{j,q,n} (d_{j,q,n} * \Phi)(y, r)\, S[\nu_{\kappa_j^1}(y), \ldots, \nu_{\kappa_j^q}(y); r]\, \psi_{j,q,n}(y, \theta)\,. \tag{1.14}$$

The coefficients $d_{j,q,n}(y)$ are defined on I and satisfy $d_{j,q,n} \in H^{s-\mu_j-\varepsilon}(I)$ for all $\varepsilon > \delta(I)$. Here and in the regular part, $\delta(I)$ is a continuous function of I which tends to 0 *when the length of I tends to* 0*.*

Remark 1.4 The above statement also holds when there is no crossing point in I'. In such a situation, the functions μ_j are the real parts of the ν_κ and the functions $S[\ldots; r]$ are evaluated in $(\nu_\kappa, \ldots, \nu_\kappa)$. The function $\delta(I)$ is 0. Using the formula

$$S[\underbrace{\nu_\kappa, \ldots, \nu_\kappa}_{q+1 \text{ times}}; r] = \frac{1}{q!}\, r^{\nu_\kappa} \log^q r\ ,$$

one obtains in this way Theorem 1.1 as a consequence of Theorem 1.3.

The loss of regularity $\delta(I)$ comes from the width of the crossings, i. e. the difference between the real parts of the functions ν_κ, $\kappa \in \mathcal{K}_{y_0,j}$ which meet in y_0.

Remark 1.5 It is possible to give a precise formulation for the loss of regularity described by $\delta(I)$. We set (compare (1.13))

$$\underline{\mu}_j(y) := \min_{\kappa \in \mathcal{K}_{y_0,j}} \operatorname{Re} \nu_\kappa(y). \tag{1.15}$$

and

$$\delta(y) := \sup_{1 \leq j \leq j_0} \left(\mu_j(y) - \underline{\mu}_j(y)\right).$$

Then $\delta(I) = \sup_{y \in I} \delta(y)$.

This loss of regularity is a consequence of the induction argument used in the proof of Theorem 1.3 (see § 6).

A careful comparison of both expansions (1.10) and (1.14), using the bundle structure of the singularities developped in Part II, shows that this loss of regularity is an artefact of the induction proof. We can prove that if the interval I is chosen in such a way that $\delta(I) \leq 1$ holds, then there exists a decomposition as in Theorem 1.3 with $\delta(I)$ replaced by 0.

We show in Part II that the expansions in Theorems 1.1 and 1.3 still hold with weaker assumptions on the boundary value problem. Two facts are important :

1. The eigenvalues of the associated "Sturm-Liouville" problems depend on y in an analytic way ; this always holds for second order scalar problems as opposed to higher order problems or systems.
2. Some model operators $\mathbb{B}_0^{\pm}$ defined on the sector Γ_y of opening $\omega(y)$ have to be injective. Such a condition insures the tangential regularity along the edge for the problem (1.1). It is shown in [3] that for operators with constant coefficients, such an injectivity condition is *equivalent* to the existence of expansions of type (1.10).

We formulate in Part II two versions, ($\mathcal{C}$V) and ($\mathcal{C}$H), of such injectivity conditions which insure the validity of an expansion of type (1.10) or (1.14). The range of these conditions includes besides the Dirichlet, Neumann and mixed problems also oblique derivative problems.

1.f An example. Let us illustrate our statements by the simple example we quoted at the beginning. We consider the mixed Dirichlet-Neumann problem for the Laplace operator on a skew cylinder. We have the expansions (1.10) or (1.14) for the solutions of this problem with

$$\nu_{kl} = \begin{cases} k\frac{\pi}{\omega} - \frac{\pi}{2\omega} + l & \text{if } k \geq 1 \\ l & \text{if } k = 0. \end{cases}$$

We could have described a similar example for the Dirichlet problem. But we prefer to consider again the same example that we quoted at the beginning because the first crossing value above the points where the opening is $\frac{\pi}{2}$ is 1, instead of 2 for the Dirichlet problem. So, for the mixed problem the crossing of exponents influences the H^2 regularity of the solution, instead of H^3 concerning the Dirichlet problem.

We take $s \in (1, 2/(1+\alpha))$, where α is the "obliquity" of the skew cylinder. Let us assume that the Dirichlet condition is 0. Then the exponent $\nu_{(0,0)}$ does not appear. With that choice of s, only $\nu_{(1,0)}$ and $\nu_{(0,1)}$ are relevant. For simplicity, let us denote

$$\begin{aligned} \nu_1(y) &:= \nu_{(1,0)}(y) = \frac{\pi}{2\omega(y)} \\ \nu_2(y) &:= \nu_{(0,1)}(y) = 1. \end{aligned}$$

There are exactly two points $y \in M$ where $\nu_1(y) = \nu_2(y)$ holds. These are the two points y_0, y_0' where $\omega(y) = \frac{\pi}{2}$. On $M \setminus \{y_0, y_0'\}$, the simple asymptotics (1.10) holds. Here $q = 0$ and only one value of n is required. We write l instead of $(l, 0, 1)$. Then we can choose

$$\begin{aligned} \varphi_1(y,\theta) &= \cos \nu_1(y)\theta \\ \varphi_2(y,\theta) &= \sin(\omega(y) - \theta). \end{aligned}$$

The function $r^{\nu_1}\varphi_1$ corresponds to the first corner singularity (0.1) and the function $r^{\nu_2}\varphi_2$ is a polynomial. The "simple asymptotics" of u is

$$c_1 * \Phi\, r^{\nu_1}\varphi_1 + c_2 * \Phi\, r^{\nu_2}\varphi_2\ .$$

Here it is possible to compute $c_2(y)$ since it depends only on the pointwise value of the Neumann boundary datum g on the edge :

$$c_2(y) = \frac{g(y,0)}{\cos\omega(y)}.$$

Then $c_2 \in H^{s-1}_{\rm loc}(M \setminus \{y_0, y_0'\})$ and c_2 generally blows up in y_0 and y_0'.

In order to get the direct representation of Theorem 1.3 at the crossing points, we need three basis functions, for instance:

$$\begin{aligned}
S[\nu_1(y); r]\,\psi_{1,1}(y,\theta) &= r^{\nu_1(y)}\ \cos\nu_1(y)\theta \\
S[\nu_1(y); r]\,\psi_{1,2}(y,\theta) &= r^{\nu_1(y)}\ \frac{\sin(\omega(y)-\theta) - \cos\nu_1(y)\theta}{1-\nu_1(y)} \\
S[\nu_1(y), \nu_2(y); r]\,\psi_{2,1}(y,\theta) &= \frac{r - r^{\nu_1(y)}}{1-\nu_1(y)}\ \sin(\omega(y)-\theta).
\end{aligned}$$

Then the asymptotics of u can be written

$$d_{1,1} * \Phi\, S[\nu_1; r]\,\psi_{1,1} + d_{1,2} * \Phi\, S[\nu_1; r]\,\psi_{1,2} + d_{2,1} * \Phi\, S[\nu_1, \nu_2; r]\,\psi_{2,1}.$$

Indeed, only two basis functions are necessary to describe the asymptotics near the crossing points : this is what we call in Part II the "bundle representation". For each $y \in M$, let us introduce the following two spaces $\mathfrak{B}_1(y)$ and $\mathfrak{B}_2(y)$:

$$\begin{aligned}
&\mathfrak{B}_1(y) \text{ is generated by } r^{\nu_1(y)}\ \cos\nu_1(y)\theta, \\
&\mathfrak{B}_2(y) \text{ is generated by } r\ \sin(\omega(y)-\theta).
\end{aligned}$$

We note that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ define analytic bundles above M and that

$$\text{when } y = y_0 \text{ or } y_0', \qquad \text{then} \qquad \mathfrak{B}_1(y) = \mathfrak{B}_2(y).$$

Nevertheless $\mathfrak{B}_1 + \mathfrak{B}_2$ has an analytic extension above M whose a trivialization is given by the two sections

$$\begin{aligned}
X_1(y,r,\theta) &= r^{\nu_1(y)}\cos\nu_1(y)\theta. \\
X_2(y,r,\theta) &= \frac{r\ \sin(\omega(y)-\theta) - r^{\nu_1(y)}\cos\nu_1(y)\theta}{1-\nu_1(y)} \\
&= \frac{r - r^{\nu_1(y)}}{1-\nu_1(y)}\ \sin(\omega(y)-\theta) \\
&\quad + r^{\nu_1(y)}\,\frac{\sin(\omega(y)-\theta) - \cos\nu_1(y)\theta}{1-\nu_1(y)} \\
&= \frac{r - r^{\nu_1(y)}}{1-\nu_1(y)}\ \cos\nu_1(y)\theta \\
&\quad + r\,\frac{\sin(\omega(y)-\theta) - \cos\nu_1(y)\theta}{1-\nu_1(y)}.
\end{aligned}$$

At the limit when $\omega \to \frac{\pi}{2}$, X_2 tends to the logarithmic singularity (0.2). The asymptotics of u can be written as

$$b_1 * \Phi X_1 + b_2 * \Phi X_2 \,.$$

Now we can compare the three representations of a singular part, namely the "simple asymptotics" of Theorem 1.1, the "direct representation" of Theorem 1.3, and the "bundle representation" with the basis X_1, X_2. Assume that we have

$$\begin{aligned} c_1\, r^{\nu_1}\varphi_1 + c_2\, r^{\nu_2}\varphi_2 &= d_{1,1}\, S[\nu_1; r]\, \psi_{1,1} + d_{1,2}\, S[\nu_1; r]\, \psi_{1,2} + d_{2,1}\, S[\nu_1, \nu_2; r]\, \psi_{2,1} \\ &= b_1\, X_1 + b_2\, X_2. \end{aligned}$$

Then there hold the following relations between the coefficients.

$$\begin{array}{lllllllll} b_1 & = & c_1 + c_2 & , \quad c_1 & = & b_1 - \dfrac{b_2}{(1-\nu_1)} \,, & d_{1,1} & = & b_1 \\ b_2 & = & c_2(1-\nu_1)\,, & c_2 & = & \dfrac{b_2}{(1-\nu_1)} \qquad , & d_{1,2} & = & d_{2,1} \;=\; b_2. \end{array}$$

These relations clearly display the blow-up of the coefficients in the simple asymptotics at the points y_0, y_0' and also the necessity for the introduction of the exponents $\mu_j(y)$ in (1.13).

1.g Plan of Part I. The strategy of our paper is as follows :

In section 2 we define the necessary Sobolev spaces with weight. We consider spaces over L^2, but with arbitrary real, even variable, regularity index and weight.

In section 3, we study singular functions of the type appearing in the expansion (1.14). We consider the effects of differential operators acting on such functions and we study the regularizing operator (1.7).

Sections 4, 5 and 6 contain the proof of the main expansion theorem. Our strategy of proof is analogous to that followed by Maz'ya and Rossmann [10].

In section 4, we consider right hand sides whose Taylor expansion at the edge vanishes. Here the phenomenon of crossing of exponents does not yet appear, because we consider an increase of less than 1 in regularity.

In section 5, the right hand sides are themselves of the form of singular functions. Here one observes crossings.

In section 6, we finish the proof for the general right hand sides and a higher increase in regularity.

In section 7, it is shown that the choice of local cylindrical coordinates has no influence on the form of the asymptotic expansion. This justifies the choice made in sections 2–6 of a constant opening angle $\omega = \omega(y)$.

In section 8 we present some basic facts about divided differences which we use in several places for the study of the singularities, especially the Leibniz formula (8.7).

2. NOTATION AND FUNCTION SPACES

2.a Domains. We begin with the notation related to the domains. Since our domains are locally diffeomorphic to dihedral angles, we consider here the case of such a dihedral angle $\mathbb{R} \times \Gamma$, where Γ is a plane sector with opening ω :

$$\Gamma = \{z = (r\cos\theta, r\sin\theta) \mid 0 < r, \quad 0 < \theta < \omega\} .$$

y is the variable in $\mathbb{R}$ and z in Γ and (r,θ) are the polar coordinates in $\mathbb{R}^2$. x denotes the global variable (y,z). Γ_ρ is the finite sector :

$$\Gamma_\rho := \{z \in \Gamma \mid r = |z| < \rho\} \quad \text{for} \quad \rho > 0.$$

I and I' will always denote two open intervals in $\mathbb{R}$ such that the closure of I is contained in I'. ρ and ρ' will always satisfy $0 < \rho < \rho'$. All these objects can be different at each occurence, like constants C.

We will use ordinary Sobolev spaces and weighted ones, with constant or variable exponents or weights.

2.b Ordinary Sobolev spaces. We begin with a convention concerning $\mathring{H}^1(D)$. Generally, $\mathring{H}^1(D)$ will denote as usual the closure of $C_0^\infty(D)$ in $H^1(D)$, except when $D = I \times \Gamma_\rho$ or $D = I' \times \Gamma_{\rho'}$: $\mathring{H}^1(I \times \Gamma_\rho)$ will denote the space of the functions $u \in H^1(I \times \Gamma_\rho)$ such that $u = 0$ on $I \times (\partial\Gamma \cap \overline{\Gamma_\rho})$, and $\mathring{H}^1(I' \times \Gamma_{\rho'})$ is defined similarly.

Let $s > 0$. The characterization of $H^s(\mathbb{R})$ by Fourier transformation is well known. There also exists a characterization by differential quotients. In the usual way, Δ_h denotes the operator :

$$(\Delta_h f)(y) = f(y+h) - f(y)$$

and for $l \in \mathbb{N}$, Δ_h^l denotes its l-th iterate. For a fixed s, let k and l be integers such that

$$0 \le k < s \quad \text{and} \quad l > s - k.$$

Then for any such k and l,

$$\|u\|_{L^2(\mathbb{R})} + \left(\int_{\mathbb{R}} \int_{-1}^{1} \frac{|\Delta_h^l \partial_y^k f(y)|^2}{|h|^{2(s-k)+1}} \, dh \, dy \right)^{\frac{1}{2}} \tag{2.1}$$

is an equivalent norm on $H^s(\mathbb{R})$ [18].

Now, if β is a positive C^∞ function on $\mathbb{R}$, let k and l be integers such that

$$\forall y \in \mathbb{R} \quad 0 \le k < \beta(y) \quad \text{and} \quad l > \beta(y) - k.$$

Then we define equivalent norms on $H^\beta(\mathbb{R})$ in a similar way to (2.1) : $H^\beta(\mathbb{R})$ is the space of the functions $u \in H^k(\mathbb{R})$ such that the following norm is finite

$$\|u\|_{L^2(\mathbb{R})} + \left(\int_{\mathbb{R}} \int_{-1}^{1} \frac{|\Delta_h^l \partial_y^k f(y)|^2}{|h|^{2(\beta(y)-k)+1}}\, dh\, dy\right)^{\frac{1}{2}}. \tag{2.2}$$

Such a space with variable exponents allows to give a global description of the regularity of the function u. Locally, this description corresponds closely to a description in terms of ordinary Sobolev spaces with constant exponents. More precisely, we have :

$$\text{if} \quad \forall y \in I \quad s_1 \le \beta(y) \le s_2 \quad \text{then} \quad H^{s_2}(I) \subset H^\beta(I) \subset H^{s_1}(I)\,. \tag{2.3}$$

Indeed, we will only use this fact in the proofs.

2.c Weighted Sobolev spaces. Let us introduce now weighted spaces on $\mathbb{R}\times\Gamma$. For $s \ge 0$ the space $V_0^s(\mathbb{R}\times\Gamma)$ is the space of the functions u satisfying

$$\forall \alpha \in \mathbb{N}^3 \quad \text{with} \quad |\alpha| \le s, \quad r^{|\alpha|-s}\partial_x^\alpha u \in L^2(\mathbb{R}\times\Gamma) \tag{2.4}$$

and, if s is not an integer, with $\{s\}$ the fractional part of s :

$$\forall \alpha \in \mathbb{N}^3 \quad \text{with} \quad |\alpha| = s - \{s\}, \quad \int_{\mathbb{R}}\int_{\Gamma}\int_{-1}^{1}\int_{\Gamma_1} \frac{|\Delta_h \partial_x^\alpha f(x)|^2}{|h|^{2\{s\}+3}}\, dh\, dx < \infty \tag{2.5}$$

For $\delta \in \mathbb{R}$, $V_\delta^s(\mathbb{R}\times\Gamma)$ is the space of the functions u such that $r^\delta u \in V_0^s(\mathbb{R}\times\Gamma)$. In particular, the condition (2.4) is transformed into :

$$\forall \alpha \in \mathbb{N}^3 \quad \text{with} \quad |\alpha| \le s, \quad r^{|\alpha|-s+\delta}\,\partial_x^\alpha u \in L^2(\mathbb{R}\times\Gamma).$$

As an obvious consequence of the definition, we have :

$$\text{If} \quad \delta, \delta' \in \mathbb{R} \quad \text{and} \quad u \in V_\delta^s(\mathbb{R}\times\Gamma) \quad \text{then} \quad r^{\delta'} u \in V_{\delta-\delta'}^s(\mathbb{R}\times\Gamma). \tag{2.6}$$

For $0 \le \gamma$ and any $\delta \in \mathbb{R}$ one has the inclusion

$$V_{\delta+\gamma}^{s+\gamma}(\mathbb{R}\times\Gamma) \subset V_\delta^s(\mathbb{R}\times\Gamma)\,. \tag{2.7}$$

The weighted spaces $V_\delta^s(\Gamma)$ on Γ are defined in the same way : we use formulas analogous to (2.4) and (2.5), with $|h|^{2\{s\}+3}$ replaced by $|h|^{2\{s\}+2}$ in (2.5).

2.d Partial Fourier transformation. We also need a characterization of the spaces $H^s(\mathbb{R}\times\Gamma)$ and $V_\delta^s(\mathbb{R}\times\Gamma)$ by partial Fourier transformation along the edge.

According to [4] theorem (AA.11), we have :

$$H^s(\mathbb{R}\times\Gamma) = L^2(\mathbb{R}, H^s(\Gamma)) \cap H^s(\mathbb{R}, L^2(\Gamma)). \tag{2.8}$$

It is possible to prove in a similar way that :

$$V_\delta^s(\mathbb{R}\times\Gamma) = L^2(\mathbb{R}, V_\delta^s(\Gamma)) \cap H^s(\mathbb{R}, V_\delta^0(\Gamma)). \tag{2.9}$$

It is now natural to introduce the following norms with parameter $\tau > 0$:

$$\|u\|^2_{H^s(\Gamma,\tau)} := \|u\|^2_{H^s(\Gamma)} + \tau^{2s}\|u\|^2_{L^2(\Gamma)} \tag{2.10}$$

and

$$\|u\|^2_{E_\delta^s(\Gamma,\tau)} := \|u\|^2_{V_\delta^s(\Gamma)} + \tau^{2s}\|u\|^2_{V_\delta^0(\Gamma)} , \tag{2.11}$$

thus $E_\delta^s(\Gamma)$ stands for the space $V_\delta^s(\Gamma) \cap V_\delta^0(\Gamma)$.

We set $<\xi> := \max(|\xi|, 1)$. From (2.8) and (2.10) we find the equivalence of norms :

$$\|u\|^2_{H^s(\mathbb{R}\times\Gamma)} \simeq \int_{\mathbb{R}} \|\hat{u}(\xi,.)\|^2_{H^s(\Gamma,<\xi>)}\, d\xi. \tag{2.12}$$

And from (2.9) and (2.11) :

$$\|u\|^2_{V_\delta^s(\mathbb{R}\times\Gamma)} \simeq \int_{\mathbb{R}} \|\hat{u}(\xi,.)\|^2_{E_\delta^s(\Gamma,<\xi>)}\, d\xi. \tag{2.13}$$

Now, if we set

$$\tilde{u}(\xi, z) := \hat{u}(\xi, \frac{z}{<\xi>}), \tag{2.14}$$

we have

$$\begin{aligned} \|\hat{u}(\xi,.)\|_{H^s(\Gamma,<\xi>)} &= <\xi>^{s-1}\|\tilde{u}(\xi,.)\|_{H^s(\Gamma)} && (2.15)\\ \|\hat{u}(\xi,.)\|_{E_\delta^s(\Gamma,<\xi>)} &= <\xi>^{s-\delta-1}\|\tilde{u}(\xi,.)\|_{E_\delta^s(\Gamma)} && (2.16)\\ \|\hat{u}(\xi,.)\|_{V_\delta^s(\Gamma)} &= <\xi>^{s-\delta-1}\|\tilde{u}(\xi,.)\|_{V_\delta^s(\Gamma)} . && (2.17) \end{aligned}$$

Lemma 2.1 *The following characterizations by partial Fourier transform hold :*

$$\|u\|^2_{H^s(\mathbb{R}\times\Gamma)} \simeq \int_{\mathbb{R}} <\xi>^{2(s-1)}\|\tilde{u}(\xi,.)\|^2_{H^s(\Gamma)}\, d\xi \tag{2.18}$$

and

$$\|u\|^2_{V_\delta^s(\mathbb{R}\times\Gamma)} \simeq \int_{\mathbb{R}} <\xi>^{2(s-\delta-1)}\|\tilde{u}(\xi,.)\|^2_{E_\delta^s(\Gamma)}\, d\xi. \tag{2.19}$$

The first one is a consequence of (2.12) and (2.15) and the second one a consequence of (2.13) and (2.16).

2.e Negative exponents. For negative s, the space $V_\delta^s(\mathbb{R}\times\Gamma)$ is defined by duality, as the dual space of $\overset{o}{V}{}_{-\delta}^{-s}(\mathbb{R}\times\Gamma)$ which is the closure of $C_0^\infty(\mathbb{R}\times\Gamma)$ in $V_{-\delta}^{-s}(\mathbb{R}\times\Gamma)$. Then formulas (2.8) and (2.9) are replaced by

$$H^s(\mathbb{R}\times\Gamma) = L^2(\mathbb{R}, H^s(\Gamma)) + H^s(\mathbb{R}, L^2(\Gamma)) \tag{2.20}$$

and

$$V_\delta^s(\mathbb{R}\times\Gamma) = L^2(\mathbb{R}, V_\delta^s(\Gamma)) + H^s(\mathbb{R}, V_\delta^0(\Gamma)) \tag{2.21}$$

Moreover formulas (2.18) and (2.19) still hold. It is important to notice that for any $\rho > 0$ there holds, compare [4] Theorem (AE.7),

$$\forall s < 0 \quad V_0^s(\Gamma_\rho) = H^s(\Gamma_\rho). \tag{2.22}$$

As a consequence of (2.20) and (2.21) we get

$$\forall s < 0 \quad V_0^s(\mathbb{R}\times\Gamma_\rho) = H^s(\mathbb{R}\times\Gamma_\rho). \tag{2.23}$$

For any s, the spaces $V_\delta^s(\Gamma)$ can be characterized by the Mellin transform

$$\hat f(\lambda) = \int_0^\infty r^{-\lambda-1} f(r)\, dr\,.$$

The following result holds

Lemma 2.2 *Let s and δ be real numbers.*
(i) If $f \in V_\delta^s(\Gamma)$, then $\hat f(\lambda)$ is defined for $\operatorname{Re}\lambda = s-1-\delta$ with values in $H^s((0,\omega))$ and we have the equivalence of norms :

$$\|f\|^2_{V_\delta^s(\Gamma)} \simeq \int_{\operatorname{Re}\lambda = s-1-\delta} \|\hat f(\xi,.)\|^2_{H^s((0,\omega),|\lambda|)}\, d\lambda.$$

(ii) If $f \in V_\delta^s(\Gamma)$ with compact support, then $\hat f(\lambda)$ is defined for $\operatorname{Re}\lambda \le s-1-\delta$ and is analytic with values in $H^s((0,\omega))$.

2.f Anisotropic spaces. In order to describe the tangential regularity along the edge of our operators, we will need anisotropic spaces on $\mathbb{R}\times\Gamma$, with additional regularity in the variable $y\in\mathbb{R}$. Let $s\in\mathbb{R}$ and $\delta\in\mathbb{R}$. For $t\in\mathbb{N}$, $V_\delta^{s,t}(\mathbb{R}\times\Gamma)$ is the space of the u satisfying

$$\forall k \in \{0,\dots,t\} \quad \partial_y^k u \in V_\delta^s(\mathbb{R}\times\Gamma)$$

and similarly $H^{s,t}(\mathbb{R}\times\Gamma)$. For any $t \ge 0$, starting from (2.8) and (2.9) we set for $s \ge 0$

$$H^{s,t}(\mathbb{R}\times\Gamma) = H^t(\mathbb{R}, H^s(\Gamma)) \cap H^{s+t}(\mathbb{R}, L^2(\Gamma)). \tag{2.24}$$

and

$$V_\delta^{s,t}(\mathbb{R}\times\Gamma) = H^t(\mathbb{R}, V_\delta^s(\Gamma)) \cap H^{s+t}(\mathbb{R}, V_\delta^0(\Gamma)). \tag{2.25}$$

As a consequence of (2.18) and (2.19) we get

$$\|u\|^2_{H^{s,t}(\mathbb{R}\times\Gamma)} \simeq \int_{\mathbb{R}} <\xi>^{2(s+t-1)} \|\tilde{u}(\xi,.)\|^2_{H^s(\Gamma)} \, d\xi \tag{2.26}$$

and

$$\|u\|^2_{V_\delta^{s,t}(\mathbb{R}\times\Gamma)} \simeq \int_{\mathbb{R}} <\xi>^{2(s+t-\delta-1)} \|\tilde{u}(\xi,.)\|^2_{E_\delta^s(\Gamma)} \, d\xi. \tag{2.27}$$

The above formulas work as a definition for $s < 0$. The following technical result will be useful later. Its proof is a straightforward consequence of (2.17) and interpolation between spaces V_δ^s [18].

Lemma 2.3 *Let s, t and δ be real numbers. Let $\gamma > 0$. Then for all $\beta \in [0,\gamma]$ we have the inclusion*

$$H^t(\mathbb{R}, V_\delta^s(\Gamma)) \cap H^{t+\gamma}(\mathbb{R}, V_\delta^{s-\gamma}(\Gamma)) \subset H^{t+\beta}(\mathbb{R}, V_\delta^{s-\beta}(\Gamma)).$$

We also have the following embeddings for any s, δ, $t \geq 0$ and $\gamma \geq 0$

$$H^{s,t}(\mathbb{R}\times\Gamma) \subset H^{s-\gamma,t+\gamma}(\mathbb{R}\times\Gamma) \tag{2.28}$$

and

$$V_\delta^{s,t}(\mathbb{R}\times\Gamma) \subset V_\delta^{s-\gamma,t+\gamma}(\mathbb{R}\times\Gamma). \tag{2.29}$$

2.g Variable weights and exponents. For a variable weight $\delta \in C^\infty(\mathbb{R})$, we define $V_\delta^s(\mathbb{R}\times\Gamma)$ as the space of the functions u such that $r^{\delta(y)}u(y,z) \in V_0^s(\mathbb{R}\times\Gamma)$.

As a consequence of the definitions, any function $u \in V_{s+\delta}^s(\mathbb{R}\times\Gamma)$ satisfies for any $s \geq 0$

$$r^{\delta(y)}\, u \in L^2(\mathbb{R}\times\Gamma).$$

We introduce the following limit of these spaces :

$$H_\delta^\infty(\mathbb{R}\times\Gamma) := \bigcap_{s\geq 0} V_{s+\delta}^s(\mathbb{R}\times\Gamma)\,.$$

It is also possible to define weighted spaces with variable exponents $\beta(y)$ and weights $\delta(y)$. We first introduce $V_0^\beta(\mathbb{R}\times\Gamma)$ using a formula like (2.2), then we proceed as above. All these spaces can be defined on any $I\times\Gamma_\rho$ and if

$$\forall y \in I \quad s_1 \leq \beta(y) \leq s_2 \quad \text{and} \quad \delta_1 \geq \delta(y) \geq \delta_2\,,$$

we have

$$V_{\delta_2}^{s_2}(I\times\Gamma_\rho) \subset V_\delta^\beta(I\times\Gamma_\rho) \subset V_{\delta_1}^{s_1}(I\times\Gamma_\rho)\,.$$

3. SINGULAR FUNCTIONS IN WEIGHTED SOBOLEV SPACES

Let us recall that the singular functions have locally the following form (compare with expansion (1.14)) :

$$\psi(y,\theta)\, S[\mu_1(y),\ldots,\mu_J(y);r]\,(c*\Phi)(y,r). \tag{3.1}$$

where ψ is analytic, $\mu_1,\ldots,\mu_J$ are analytic with complex values and c is a function defined on an interval of $\mathbb{R}$. We are going to prove differentiability and regularity properties for functions of this form.

We begin with two preparatory lemmas, the first one about the regular extension $c*\Phi$ on the dihedral angle $\mathbb{R}\times\Gamma$ of given functions c along the edge and the second one about the regularity of the "radial" part S of the singular functions (3.1).

Lemma 3.1 *We assume that $\beta\in\mathbb{R}$ and $\beta<N+1$, where N is the integer we introduced in the definition of Φ. Let $c\in H^\beta(I')$. Then for any $I\subset\subset I'$ there exists $\rho>0$ such that for any $\varepsilon>0$ we have the four following propositions*

(i) $\forall\alpha\in\mathbb{N}^2, |\alpha|\geq 1,\ \forall k\in\mathbb{N}$:

$$\partial_y^k\partial_z^\alpha(c*\Phi-c)=\partial_y^k\partial_z^\alpha(c*\Phi)\in H^\infty_{-\beta-1+k+|\alpha|}(I\times\Gamma_\rho);$$

(ii) if $\beta\geq 0$, then $c*\Phi-c\in V^\beta_{-1+\varepsilon}(I\times\Gamma_\rho)$, more precisely,

$$c*\Phi-c\in H^\beta(I;H^\infty_{-1+\varepsilon}(\Gamma_\rho))\cap L^2(I;H^\infty_{-\beta-1+\varepsilon}(\Gamma_\rho))\ ;$$

(iii) if $\beta\geq 0$, then $c*\Phi\in H^\infty_{-1+\varepsilon}(I\times\Gamma_\rho)$;

(iv) if $\beta<0$, then $c*\Phi\in H^\infty_{-\beta-1}(I\times\Gamma_\rho)$.

Remark 3.2 The operator $c\mapsto c*\Phi$ works like a lifting of traces.

Proof. Using a cut-off function, we can assume that $c\in H^\beta(\mathbb{R})$ and c has compact support in I'. Let $g=c*\Phi$. We apply partial Fourier transformation to g with respect to the variable y :

$$\hat g(\xi,z)=\hat c(\xi)\phi(r|\xi|),\quad \xi\in\mathbb{R},\quad 0<r=|z|<\rho.$$

Then for the derivatives of g one finds the Fourier transforms :

$$\left|\widehat{(\partial_y^{\alpha_1}\partial_r^{\alpha_2})}g(\xi,z)\right|=\left||\xi|^{|\alpha|}\hat c(\xi)\partial^{\alpha_2}\phi(r|\xi|)\right|.$$

Now one has :

$$\int_0^\rho\left|r^{\delta+|\alpha|}|\xi|^{|\alpha|}\hat c(\xi)\partial^{\alpha_2}\phi(r|\xi|)\right|^2 rdr=|\xi|^{-2\delta-2}|\hat c(\xi)|^2\int_0^{\rho|\xi|}t^{2\delta+2|\alpha|+1}|\partial^{\alpha_2}\phi(t)|^2dt. \tag{3.2}$$

In order to show *(iii)*, we choose $\delta=-1+\varepsilon$ and estimate the right hand side of (3.2) by $C|\hat c(\xi)|^2$ for any $\alpha\in\mathbb{N}^2$. Thus with $|\alpha|=\alpha_1+|\alpha'|$:

$$r^{-1+\varepsilon+|\alpha|}\partial_y^{\alpha_1}\partial_z^{\alpha'}g\in L^2(\mathbb{R}\times\Gamma_\rho)$$

for any $\alpha \in \mathbb{N}^3$, hence

$$g \in H^{\infty}_{-1+\varepsilon}(\mathbb{R} \times \Gamma_\rho).$$

When we have $\beta < 0$, the integral on the right hand side of (3.2) converges if we choose $\delta = -\beta - 1$. So we can estimate the right hand side of (3.2) by $C|\xi|^{2\beta}|\hat{c}(\xi)|^2$. Hence *(iv)*.

In order to show *(i)*, we first reduce to the case $k = 0$ since

$$\partial_y(c * \Phi) = (\partial_y c) * \Phi.$$

Next we note that for $\alpha_2 \geq 1$ (compare (1.6)) :

$$|\partial^{\alpha_2}\phi(t)| \leq Ct^{N+1-\alpha_2}.$$

Thus the integral on the right hand side of (3.2) converges for any $\alpha_1 \in \mathbb{N}$, $\alpha_2 \geq 1$ if $\delta > -N - 2$. The choice $\delta = -\beta - 1$ is possible if $\beta < N + 1$ holds. Then one finds

$$r^{-\beta-1+|\alpha|}\partial_y^{\alpha_1}\partial_z^{\alpha'} g \in L^2(\mathbb{R} \times \Gamma_\rho) \quad \text{for any} \quad \alpha \in \mathbb{N}^3 \quad \text{with} \quad |\alpha'| \geq 1,$$

hence for any $\beta < N + 1$ and $|\alpha'| \geq 1$ there holds

$$\partial_z^{\alpha'} g \in H^{\infty}_{-\beta-1+|\alpha'|}(\mathbb{R} \times \Gamma_\rho),$$

hence *(i)*.

We show *(ii)* for $\beta \in \mathbb{N}$. The case of general $\beta \in \mathbb{R}_+$ then follows by interpolation. We have to show for $\alpha = (\alpha_1, \alpha')$, $\alpha_1 \leq \beta$:

$$r^{-\beta-1+\varepsilon+|\alpha|}\partial_y^{\alpha_1}\partial_z^{\alpha'}(c * \Phi - c) \in L^2(\mathbb{R} \times \Gamma_\rho).$$

For $|\alpha'| \geq 1$ we have shown this above.

For $\alpha' = 0$ we write similarly as in (3.2) :

$$\int_0^\rho |r^{\delta+\alpha_1}|\xi|^{\alpha_1}\hat{c}(\xi)(\phi(r|\xi|) - 1)|^2 r\,dr = |\xi|^{-2\delta-2}|\hat{c}(\xi)|^2 \int_0^{\rho|\xi|} t^{2\delta+2\alpha_1+1}|\phi(t) - 1|^2 dt. \tag{3.3}$$

For $\delta = -\beta - 1 + \varepsilon$ and $\beta < N + 1$ the integral on the right hand side exists, since $|\phi(t) - 1| \leq Ct^{N+1}$ due to (1.6). From the boundedness of $\phi(t) - 1$ follows

$$|\xi|^{-2\delta-2}|\hat{c}(\xi)|^2 \int_0^{\rho|\xi|} t^{2\delta+2\alpha_1+1}|\phi(t) - 1|^2 dt \leq C(|\xi|^{2\alpha_1} + |\xi|^{2\beta-2\varepsilon})|\hat{c}(\xi)|^2.$$

This is bounded by $|\xi|^{2\beta}|\hat{c}(\xi)|^2$ and therefore integrable if $\alpha_1 \leq \beta$ holds. Hence *(ii)* follows.

∎

Lemma 3.3 *We assume that $u \in V^\beta_\delta(I' \times \Gamma_{\rho'})$ with $\beta \in \mathbb{R}$ and $\delta \in C^\infty(I')$. Let $\mu_1, \ldots, \mu_J \in C^\infty(I')$, let $I \subset\subset I'$ and :*

$$\mu = \inf\{\operatorname{Re}\mu_1(y), \ldots, \operatorname{Re}\mu_J(y) \mid y \in I\}.$$

Then, for any $\varepsilon > 0$:

$$u(y, z)\, S[\mu_1(y), \ldots, \mu_J(y); r] \in V^\beta_{\delta-\mu+\varepsilon}(I \times \Gamma_\rho).$$

Proof. By definition of the singular function $S[\ldots]$, we can write the function

$$g(x) = u(y,z)\, S[\mu_1(y), \ldots, \mu_J(y); r]$$

as a contour integral :

$$g(x) = \int_\gamma g_\lambda(x) d\lambda.$$

Here γ is a simply closed contour around the set

$$\{\mu_1(y), \ldots, \mu_J(y) \mid y \in I\} \subset \mathbb{C}.$$

It can be chosen such that

$$\min\{\mathrm{Re}\,\lambda \mid \lambda \in \gamma\} \geq \mu - \varepsilon.$$

The function g_λ is given by

$$g_\lambda(x) = \frac{1}{2\pi i}\, u(y,z)\, \frac{r^\lambda}{(\lambda - \mu_1(y)) \cdots (\lambda - \mu_J(y))}\,.$$

It follows (2.6) that $g_\lambda \in V^\beta_{\delta - \mathrm{Re}\,\lambda}(I \times \Gamma_\rho) \subset V^\beta_{\delta - \mu + \varepsilon}(I \times \Gamma_\rho)$ and

$$\|g_\lambda\|_{V^\beta_{\delta-\mu+\varepsilon}(I\times\Gamma_\rho)} \leq C \|u\|_{V^\beta_\delta(I'\times\Gamma_{\rho'})}$$

for all $\lambda \in \gamma$. Hence $g \in V^\beta_{\delta-\mu+\varepsilon}(I \times \Gamma_\rho)$ follows. ∎

We will also need analytic expressions for the derivatives of the singular functions S.

Lemma 3.4 *Let $\mu_1, \ldots, \mu_J$ be analytic functions. Then :*

(i)

$$\partial_y S[\mu_1(y), \ldots, \mu_J(y); r] = \sum_{j=1}^J (\partial_y \mu_j)(y)\, S[\mu_1(y), \ldots, \mu_j(y), \mu_j(y), \ldots, \mu_J(y); r]$$

(ii) For any k there exist analytic coefficients $c_{j,k}(y)$ such that :

$$\partial_r^k S[\mu_1(y), \ldots, \mu_J(y); r] = \sum_{j=1}^J c_{j,k}(y)\, S[\mu_1(y) - k, \ldots, \mu_j(y) - k; r]$$

(iii) If $w(y,\theta)$ is analytic, one has :

$$\partial_z^\alpha (w(y,\theta) S[\mu_1(y), \ldots, \mu_J(y); r]) = \sum_{j=1}^J c_{j,\alpha}(y,\theta) S[\mu_1(y) - |\alpha|, \ldots, \mu_j(y) - |\alpha|; r]$$

with analytic functions $c_{j,\alpha}(y,\theta)$.

Proof. The first formula is obvious and the third one is a straightforward consequence of the second when one writes the derivatives with respect to z in polar coordinates. The proof of *(ii)* is based on an argument about divided differences. We have

$$\partial_r^k S[\mu_1(y),\dots,\mu_J(y);r] = \int_\gamma \frac{p_k(\lambda)\, r^{\lambda-k}}{(\lambda-\mu_1(y))\dots(\lambda-\mu_J(y))}\, d\lambda$$

where γ is a simply closed contour around the set $\{\mu_1(y),\dots,\mu_J(y)\}$ and p_k is a polynomial of degree k. Setting $u_r(\lambda) = r^{\lambda-k}$ and $v(\lambda) = p_k(\lambda)$ we have according to (8.4)

$$\partial_r^k S[\mu_1(y),\dots,\mu_J(y);r] = (u_r v)\,[\mu_1(y),\dots,\mu_J(y)]\,.$$

Now, due to the Leibniz formula (8.7), we get *(ii)* if we set

$$c_{j,k}(y) = p_k[\mu_j(y),\dots,\mu_J(y)]\,.$$

∎

We are now able to state an important result of this section which gives precise information about the regularity of a complete singular function (3.1).

Lemma 3.5 *We assume that $\beta \in C^\infty(I')$ such that $\beta(y) < N+1$ for all $y \in I'$. Let $c \in H^\beta(I')$. Let $\psi \in C^\infty(I' \times [0,\omega])$. Let $\mu_1,\dots,\mu_J \in C^\infty(I')$, and :*

$$\mu(y) = \inf\{\operatorname{Re}\mu_1(y),\dots,\operatorname{Re}\mu_J(y)\}.$$

Then for any $I \subset\subset I'$ there exists $\rho > 0$ such that for any $\varepsilon > 0$ we have the four following propositions

(i) $\forall \alpha \in \mathbb{N}^2, |\alpha| \geq 1$:

$$\psi(y,\theta)\, S[\mu_1(y),\dots,\mu_J(y);r]\, \partial_z^\alpha (c * \Phi - c) \in H^\infty_{-\beta-1+|\alpha|-\mu+\varepsilon}(I \times \Gamma_\rho)$$

(ii) if $\beta \geq 0$, then

$$\psi(y,\theta)\, S[\mu_1(y),\dots,\mu_J(y);r]\, (c * \Phi - c) \in V^{\beta-\varepsilon}_{-1-\mu}(I \times \Gamma_\rho)$$

and for any $l \in [0,\beta]$

$$\psi(y,\theta)\, S[\mu_1(y),\dots,\mu_J(y);r]\, (c * \Phi - c) \in H^l(I; H^\infty_{-\beta-1+l-\mu+\varepsilon}(\Gamma_\rho))$$

(iii) if $\beta \geq 0$, then

$$\psi(y,\theta)\, S[\mu_1(y),\dots,\mu_J(y);r]\; c * \Phi \in H^\infty_{-1-\mu+\varepsilon}(I \times \Gamma_\rho)$$

(iv) if $\beta \leq 0$, then

$$\psi(y,\theta)\, S[\mu_1(y),\dots,\mu_J(y);r]\; c * \Phi \in H^\infty_{-\beta-1-\mu+\varepsilon}(I \times \Gamma_\rho)$$

Proof. By localization around any point $y_0 \in I'$, we can reduce the statements to the case when β is a constant and the function $\mu(y)$ satisfies

$$\mu(y) < \inf\{\mu(y') \mid y' \in I\} + \varepsilon \quad \forall y \in I.$$

In this case, the corollary follows by a direct combination of Lemma 3.1 and Lemma 3.3. For the proof of *(ii)* we also use Lemma 2.3. ∎

We need to know the effect of a partial differential operator on any function of the form (3.1). Later in Proposition 5.1 we will solve the converse problem of solving a boundary value problem with singular right hand side. The following lemma has to be compared with Lemma 5.3.

Lemma 3.6 *Let $\mathfrak{B}$ be a second order operator with smooth coefficients on $I' \times \Gamma_{\rho'}$*

$$\mathfrak{B}(y,z;\partial_y,\partial_z) = \sum_{k=0}^{2} \sum_{|\alpha|+k\le 2} \mathfrak{b}_{\alpha,k}(y,z)\,\partial_y^k \partial_z^\alpha \,.$$

Let $\mathfrak{M}$ be the principal conormal part of $\mathfrak{B}$ on the edge

$$\mathfrak{M}(y;\partial_z) = \sum_{|\alpha|=2} \mathfrak{b}_{\alpha,0}(y,0)\,\partial_z^\alpha \,.$$

As in the previous lemma we take $c \in H^\beta(I')$, $\psi \in C^\infty(I' \times [0,\omega])$, $\mu_1,\dots,\mu_J \in C^\infty(I')$ and $I \subset\subset I'$. We define $\mu(y)$ in the same way and assume that $\beta > 0$. L denotes an integer such that for all $y \in I'$, $L \ge [\beta]$. We set

$$u(x) := (c * \Phi)(y,r)\,\psi(y,\theta)\,S[\mu_1(y),\dots,\mu_J(y);r]$$

and

$$f(x) := (c * \Phi)(y,r)\,\mathfrak{M}\Big[\psi(y,\theta)\,S[\mu_1(y),\dots,\mu_J(y);r]\Big]\,.$$

Then

$$\mathfrak{B}u = f + \sum_{l=1}^{L} \sum_{p} f_{l,p} + g\,,$$

with

$$g \in H^\infty_{-\beta+1-\mu+\varepsilon}(I \times \Gamma_\rho) \quad \forall \varepsilon > 0$$

and

$$f_{l,p}(x) := (c_{l,p} * \Phi)(y,r)\,\varphi_{l,p}(y,\theta)\,S[\mu_1^{l,p}(y) - 2 + l,\dots,\mu_{q_{l,p}}^{l,p}(y) - 2 + l; r]\,;$$

the $c_{l,p}$ are derivatives of c of order at most l, the $\varphi_{l,p}$ are analytic and the $\mu_q^{l,p}$ are in $\{\mu_j \mid j = 1,\dots,J\}$.

Proof. We set

$$u^0 := \psi(y,\theta)\,S[\mu_1(y),\dots,\mu_J(y);r]\,.$$

So $u = (c * \Phi)\,u^0$ and $f = (c * \Phi)\,\mathfrak{M}u^0$ and we have

$$\mathfrak{B}u - f = \sum_{\substack{k,k',\alpha,\alpha' \\ k+k'+|\alpha|+|\alpha'|\le 2}} a_{k,k',\alpha,\alpha'}(x)\,\partial_y^k \partial_z^\alpha (c * \Phi)\,\partial_y^{k'} \partial_z^{\alpha'} u_y^0\,. \tag{3.4}$$

Here the coefficients $a_{k,k',\alpha,\alpha'}$ are analytic functions of $x \in I' \times \Gamma_{\rho'}$. The function f contributes to the terms such that $k, k', \alpha = 0$ and $|\alpha'| = 2$ and we have

$$\forall \alpha', \ |\alpha'| = 2, \quad a_{0,0,0,\alpha'}(y,z) = \mathfrak{b}_{\alpha'}(y,z) - \mathfrak{b}_{\alpha'}(y,0)\,. \tag{3.5}$$

The possible terms in (3.4) will be considered differently according to the length of α.

(1). $|\alpha| \geq 1$. Then

$$\partial_y^k \partial_z^\alpha (c * \Phi) = \partial_z^\alpha \Big((\partial_y^k c) * \Phi \Big) \tag{3.6}$$

with $\partial_y^k c \in H^{\beta-k}(I')$; due to to Lemma 3.4 *(i)* and *(iii)*

$$\partial_y^{k'} \partial_z^{\alpha'} u_y^0 = \sum_{j=1}^{J'} c_{k'\alpha' j}(y,\theta)\, S[\mu_1^{k'\alpha' j} - |\alpha'|, \ldots, \mu_q^{k'\alpha' j} - |\alpha'|; r]\,. \tag{3.7}$$

Lemma 3.5 *(i)* yields that the product of (3.6) and (3.7) belongs to the space $H^\infty_{-(\beta-k)-1+|\alpha|-(\mu-|\alpha'|)+\varepsilon}(I \times \Gamma_\rho)$. Since $k + |\alpha| + |\alpha'| \leq 2$, this space is included in $H^\infty_{-\beta+1-\mu+\varepsilon}(I \times \Gamma_\rho)$. Hence all the terms with $|\alpha| \geq 1$ contribute to the "regular part" g.

(2). $|\alpha| = 0$.

We have a generic term of the form

$$a(x)\, \partial_y^k (c * \Phi)\, \partial_y^{k'} \partial_z^{\alpha'} u_y^0\,. \tag{3.8}$$

Here we expand the function $a(x)$ into a Taylor sum with respect to r in $r = 0$. We find terms of the form

$$r^m a_m(y,\theta)\, \partial_y^k (c * \Phi)\, \partial_y^{k'} \partial_z^{\alpha'} u_y^0 \tag{3.9}$$

and a remainder term of the form

$$r^m a'_m(r,y,\theta)\, \partial_y^k (c * \Phi)\, \partial_y^{k'} \partial_z^{\alpha'} u_y^0\,. \tag{3.10}$$

Here a_m and a'_m are analytic functions.

For the terms (3.9), we find they are sums of terms of the form

$$(\partial_y^k c * \Phi)\, \varphi(y,\theta)\, S[\mu_{n_1} - |\alpha'| + m, \ldots, \mu_{n_q} - |\alpha'| + m; r] \tag{3.11}$$

Here we use in particular the form of the derivatives of the functions $S[\ldots]$ (see Lemma 3.4). Hence the terms (3.9) are of the form

$$\sum_p f_{l,p} \quad \text{with} \quad l = m - |\alpha'| + 2.$$

We see that for the terms (3.9), we have only the following two cases to consider : $|\alpha'| = 2$ and $|\alpha'| \leq 1$.

a) If $|\alpha'| = 2$ then $k, k', \alpha = 0$ and (3.5) gives $m \geq 1$. Thus $l \geq 1$.

b) If $|\alpha'| \leq 1$ we obtain again $l \geq 1$.

Anyway, $l \geq m + k + k'$. Hence $l \geq k$. Thus we have always $l \geq 1$ and $k \leq l$ as claimed for the $f_{l,p}$. For the regularity of the functions $f_{l,p}$, we find from Lemma 3.5

(iii), resp. *(iv)*

$$\text{if } k \le \beta \quad : \quad f_{l,p} \in H^\infty_{1-\mu-l+\varepsilon}(I \times \Gamma_\rho) \tag{3.12}$$

$$\text{if } k \ge \beta \quad : \quad f_{l,p} \in H^\infty_{-\beta+k+1-\mu-l+\varepsilon}(I \times \Gamma_\rho)\,. \tag{3.13}$$

When $l > L$ i.e. $l > \beta$, (3.12) gives $f_{l,p} \in H^\infty_{1-\mu-\beta+\varepsilon}(I \times \Gamma_\rho)$; using the fact that $k \le l$ we find that (3.13) gives also $f_{l,p} \in H^\infty_{1-\mu-\beta+\varepsilon}(I \times \Gamma_\rho)$.

Finally, for the terms of the form (3.10), we can choose $m \ge \beta + |\alpha'| - 2$ and we obtain that these terms are included in the regular part g. ∎

We also need an expression of the Taylor expansion at the edge in a form which is similar to the form (3.1) of singular functions.

Lemma 3.7 *Let $\beta \in \mathbb{R}_+ \setminus \mathbb{N}$ and $f \in H^\beta(I' \times \Gamma_{\rho'})$. Then for the traces on the edge I we have*

$$g_\alpha(y) := \frac{1}{\alpha!}\,\partial_z^\alpha f(y,0) \in H^{\beta-1-|\alpha|}(I)$$

for any $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| < \beta - 1$. Let f_0 be the Taylor remainder :

$$f_0(y,z) := f(y,z) - \sum_\alpha (g_\alpha * \Phi)(y,r)\, z^\alpha. \tag{3.14}$$

Then $f_0 \in V_0^\beta(I \times \Gamma_\rho)$.

Proof. We can assume that $f \in H^\beta(\mathbb{R} \times \Gamma)$ and f has compact support. Applying partial Fourier transformation in y, we obtain the expansion (3.14) in the form

$$\hat{f}_0(\xi,z) := \hat{f}(\xi,z) - \sum_\alpha \hat{g}_\alpha(\xi)\,\phi(r|\xi|)\,z^\alpha. \tag{3.15}$$

Now with $<\xi> := \max\{1, |\xi|\}$ as above in (2.14), we introduce the transformation :

$$\tilde{z} = z{<}\xi{>}, \quad \tilde{r} = r{<}\xi{>}, \quad \tilde{f}(\xi,\tilde{z}) = \hat{f}(\xi, \frac{\tilde{z}}{<\xi>}), \quad \tilde{f}_0(\xi,\tilde{z}) = \hat{f}_0(\xi, \frac{\tilde{z}}{<\xi>}),$$

and obtain for for $|\xi| \ge 1$:

$$\tilde{f}_0(\xi,\tilde{z}) := \tilde{f}(\xi,\tilde{z}) - \sum_\alpha |\xi|^{-|\alpha|}\,\hat{g}_\alpha(\xi)\,\phi(\tilde{r})\,\tilde{z}^\alpha.$$

We show here only the estimates for $|\xi| \ge 1$, since the estimates for small ξ are easily obtained and do not affect the regularity of f_0 and g_α. For fixed ξ, we have

$$\hat{f}(\xi,.) \in H^\beta(\Gamma),$$

hence we have

$$\begin{aligned} \hat{g}_\alpha(\xi) &= \frac{1}{\alpha!}\,\partial_z^\alpha \hat{f}(\xi,0) \\ &= \frac{1}{\alpha!}\,|\xi|^{|\alpha|}\partial_{\tilde{z}}^\alpha \tilde{f}(\xi,0). \end{aligned}$$

Thus there is an estimate

$$|\hat{g}_\alpha(\xi)| \le C|\xi|^{|\alpha|}\|\tilde{f}(\xi,.)\|_{H^\beta(\Gamma)}\,.$$

Thus

$$<\xi>^{\beta-1-|\alpha|}\,|\hat{g}_\alpha(\xi)| \le C<\xi>^{\beta-1}\|\tilde{f}(\xi,.)\|_{H^\beta(\Gamma)}\,.$$

According to the characterization (2.18) of the norm in $H^\beta(\mathbb{R}\times\Gamma)$, this is in $L^2(\mathbb{R})$, hence $g_\alpha(y)\in H^{\beta-1-|\alpha|}(\mathbb{R})$. From [4] Theorem (AA.7), follows the estimate, uniformly in $|\xi|\ge 1$

$$\|\tilde{f}_0(\xi,.)\|_{E^0_\beta(\Gamma)} + \|\tilde{f}_0(\xi,.)\|_{L^2(\Gamma)} \le C\|\tilde{f}(\xi,.)\|_{H^\beta(\Gamma)}\,.$$

Multiplying both sides by $<\xi>^{\beta-1}$, integrating over $\xi\in\mathbb{R}$, and using both characterizations (2.18) and (2.19) we find the estimate

$$\|f_0\|_{V^\beta_0(\mathbb{R}\times\Gamma)} \le C\|f\|_{H^\beta(\mathbb{R}\times\Gamma)}\,.$$

∎

4. FLAT RIGHT HAND SIDES

Since our domains are locally diffeomorphic to dihedral angles, we use local coordinates (y,z) as in the beginning of section 2. The interior operator $A(x;\partial_x)$ in the boundary value problem (1.1) can be written :

$$B(y,z;\partial_y,\partial_z) = \sum_{k=0}^{2}\sum_{|\alpha|+k\le 2} b_{\alpha,k}(y,z)\,\partial_y^k\partial_z^\alpha\,. \tag{4.1}$$

We will use the following splitting of B

$$\begin{aligned} B &= M(y;\partial_z) + N(y,z;\partial_y,\partial_z) \\ \text{with} \qquad & \\ M(y;\partial_z) &= \sum_{|\alpha|=2} b_{\alpha,0}(y,0)\,\partial_z^\alpha\,. \end{aligned} \tag{4.2}$$

The operator N can be split into N_1+N_2 with

$$\begin{aligned} N_1(y,z;\partial_y,\partial_z) &= \sum_{k=0}^{2}\sum_{|\alpha|\le 1} b_{\alpha,k}(y,z)\,\partial_y^k\partial_z^\alpha \\ N_2(y,z;\partial_y,\partial_z) &= \sum_{|\alpha|=2}\left(b_{\alpha,0}(y,z)-b_{\alpha,0}(y,0)\right)\partial_z^\alpha\,. \end{aligned} \tag{4.3}$$

According to the standard method [5], we define the family of Sturm-Liouville problems associated to B as follows. For each $y\in I$, we introduce $\mathcal{M}_y(\theta;r\partial_r,\partial_\theta)$ as

$$r^2 M(y;\partial_z) = \mathcal{M}_y(\theta;r\partial_r,\partial_\theta)\,. \tag{4.4}$$

This defines the family $\mathcal{M}_y(\theta;\lambda,\partial_\theta)$ as operators

$$\mathcal{M}_y(\lambda)\ :\ \overset{\circ}{H}{}^1(0,\omega)\to H^{-1}(0,\omega)\,. \tag{4.5}$$

The spectrum of $\mathcal{M}_y$, i. e. the set of the $\lambda \in \mathbb{C}$ such that $\mathcal{M}_y(\lambda)$ is not invertible is of the form

$$\{k\nu(y) \mid k \in \mathbb{Z}^*\},$$

where ν is an analytic function of y (see [4], §14). Let us recall from [4] the formulas which give $\nu(y)$.

For each fixed y, one can write the differential operator $M(y;\partial_z)$ as product of first order operators

$$(\partial_{z_2} - a_1\partial_{z_1})(\partial_{z_2} - a_2\partial_{z_1})$$

with $z = (z_1, z_2)$, and a_1, a_2 complex numbers with $\operatorname{Im} a_1 < 0$ and $\operatorname{Im} a_2 > 0$. Then we set

$$F_j(\theta) = \int_0^\omega f_j(\zeta)\,d\zeta \quad \text{where} \quad f_j(\theta) = \frac{a_j\cos\theta - \sin\theta}{a_j\sin\theta + \cos\theta},$$

and ν is given by

$$\nu = \frac{2i\pi}{F_2(\omega) - F_1(\omega)}. \tag{4.6}$$

The following statement is a hypoellipticity result in weighted spaces. This is a direct consequence of the ellipticity of B and can be proved in a classical way by dyadic partitions along the edge. See [9] for instance.

Lemma 4.1 *Let $\beta \in \mathbb{R}_+$ and $\delta \in \mathbb{R}$. We assume that :*

$$u \in V^0_\delta(I' \times \Gamma_{\rho'}), \quad u \in H^1(I' \times (\Gamma_{\rho'} \setminus \Gamma_\varepsilon)) \ \forall\varepsilon > 0 \quad \textit{and} \quad u\Big|_{I\times\partial\Gamma} = 0,$$

and that

$$Bu \in V^{\beta-1}_{\delta+\beta+1}(I' \times \Gamma_{\rho'}).$$

The following regularity of the derivatives of u then holds :

$$u \in V^{\beta+1}_{\delta+\beta+1}(I \times \Gamma_\rho).$$

We assume we have some *a priori* estimates in weighted spaces V^s_0 for the Dirichlet problem associated with the operator B.

Hypothesis 4.2 *Let $\beta \geq 0$ be such that $\beta \neq k\operatorname{Re}\nu(y)\ \forall k \geq 1,\ \forall y \in I'$.*
We assume that for any such β, for all u in $\mathring{H}^1(I' \times \Gamma_{\rho'}) \cap V^{\beta+1}_0(I' \times \Gamma_{\rho'})$, we have the a priori estimate

$$\|u\|_{V_0^{\beta+1}(I\times\Gamma_\rho)} \leq C\left(\|Bu\|_{V_0^{\beta-1}(I'\times\Gamma_{\rho'})} + \|u\|_{V_0^{\beta}(I'\times\Gamma_{\rho'})}\right) \tag{4.7}$$

Remark 4.3 If the operator B is strongly elliptic, then this hypothesis holds.

Hypothesis 4.2 allows to get tangential regularity in the anisotropic spaces $V^{s,t}_0$ introduced in subsection 2.f.

Proposition 4.4 *Let $\beta \geq 0$ be such that $\beta \neq k \operatorname{Re} \nu(y)$ $\forall k \geq 1$, $\forall y \in I'$. We assume that $u \in \mathring{H}^1(I' \times \Gamma_{\rho'})$, $u \in V_0^{\beta+1}(I' \times \Gamma_{\rho'})$ and $Bu \in V_0^{\beta-1,t}(I' \times \Gamma_{\rho'})$ for a $t > 0$. Then $u \in V_0^{\beta+1,t}(I \times \Gamma_\rho)$ with the a priori estimate*

$$\|u\|_{V_0^{\beta+1,t}(I\times\Gamma_\rho)} \leq C\left(\|Bu\|_{V_0^{\beta-1,t}(I'\times\Gamma_{\rho'})} + \|u\|_{V_0^{\beta+1}(I'\times\Gamma_{\rho'})}\right). \tag{4.8}$$

Proof. **a)** *Induction step.* Let t, t' such that $0 \leq t' < t < t'+1$. Let us assume we have the following *a priori* estimate : for $u \in \mathring{H}^1(I' \times \Gamma_{\rho'})$, $u \in V_0^{\beta+1,t'}(I' \times \Gamma_{\rho'})$ we have

$$\|u\|_{V_0^{\beta+1,t'}(I\times\Gamma_\rho)} \leq C\left(\|Bu\|_{V_0^{\beta-1,t'}(I'\times\Gamma_{\rho'})} + \|u\|_{V_0^{\beta,t'}(I'\times\Gamma_{\rho'})}\right). \tag{4.9}$$

Hypothesis 4.2 corresponds to the above estimate for $t' = 0$. Let us show the following : if $u \in \mathring{H}^1(I' \times \Gamma_{\rho'})$, $u \in V_0^{\beta+1,t'}(I' \times \Gamma_{\rho'})$ and $Bu \in V_0^{\beta-1,t}(I' \times \Gamma_{\rho'})$ then $u \in V_0^{\beta+1,t}(I \times \Gamma_\rho)$ with the a priori estimate

$$\|u\|_{V_0^{\beta+1,t}(I\times\Gamma_\rho)} \leq C\left(\|Bu\|_{V_0^{\beta-1,t}(I'\times\Gamma_{\rho'})} + \|u\|_{V_0^{\beta+1,t'}(I'\times\Gamma_{\rho'})}\right). \tag{4.10}$$

We define $\Delta_h u(y,z) := (u(y+h,z) - u(y,z))$ for $|h|$ small enough. Then we have

$$B(\Delta_h u) = \Delta_h(Bu) + [B, \Delta_h]u\,.$$

Now $\Delta_h Bu \in V_0^{\beta-1,t'}(I' \times \Gamma_{\rho'})$. The second order differential operator $[B, \frac{1}{h}\Delta_h]$ has coefficients which converge for $h \to 0$ to the coefficients of the second order operator $[B, \partial_y]$. Hence

$$\|[B,\Delta_h]u\|_{V_0^{\beta-1,t'}(I'\times\Gamma_{\rho'})} \leq C\,|h|\,\|u\|_{V_0^{\beta+1,t'}(I'\times\Gamma_{\rho'})}$$

With (4.9) we get

$$\|\Delta_h u\|_{V_0^{\beta+1,t'}(I\times\Gamma_\rho)} \leq \tag{4.11}$$
$$C\left(\|\Delta_h Bu\|_{V_0^{\beta-1,t'}(I'\times\Gamma_{\rho'})} + \|[B,\Delta_h]u\|_{V_0^{\beta-1,t'}(I'\times\Gamma_{\rho'})} + \|\Delta_h u\|_{V_0^{\beta,t'}(I'\times\Gamma_{\rho'})}\right)$$

Now we use the following equivalence of norms : for $\varepsilon > 0$ we have

$$\|u\|^2_{V_0^{s,t}(I\times\Gamma_\rho)} \simeq \int_{-\varepsilon}^{\varepsilon} \frac{\|\Delta_h u\|^2_{V_0^{s,t'}(I\times\Gamma_\rho)}}{|h|^{2(t-t')+1}}\,dh \tag{4.12}$$

(compare with (2.2) and (AF.3) in [4]).

We divide each side of (4.11) by $|h|^{2(t-t')+1}$ and integrate on $(-\varepsilon, \varepsilon)$. Then (4.12) yields :

$$\|u\|_{V_0^{\beta+1,t}(I\times\Gamma_\rho)} \leq C\left(\|Bu\|_{V_0^{\beta-1,t}(I'\times\Gamma_{\rho'})} + \|u\|_{V_0^{\beta+1,t'}(I'\times\Gamma_{\rho'})} + \|u\|_{V_0^{\beta,t}(I'\times\Gamma_{\rho'})}\right).$$

Since with (2.29), $V_0^{\beta+1,t'}(I' \times \Gamma_{\rho'}) \subset V_0^{\beta,t'+1}(I' \times \Gamma_{\rho'}) \subset V_0^{\beta,t}(I' \times \Gamma_{\rho'})$ we have obtained (4.10).

b) *General situation.* We choose as many values of t' as necessary so that the first one is 0, the difference of two neighboring t' is less than 1 and the last step allows to reach t. The proof of Proposition 4.4 is complete. ∎

Here is now the first decomposition result : we assume the right hand side is flat and we get the first terms in the asymptotics. Thanks to the condition $s - s_0 \le 1$, we do not yet meet crossings of singularity exponents.

Proposition 4.5 *Let s and s_0 be nonnegative numbers such that*

$$s,\ s_0 \neq k \operatorname{Re} \nu(y) \quad \forall k \ge 1,\ \forall y \in I' \quad \text{and} \quad 0 < s - s_0 \le 1.$$

We assume that

$$u \in \overset{\circ}{H}{}^1(I' \times \Gamma_{\rho'}) \cap V_0^{s_0+1}(I' \times \Gamma_{\rho'}) \quad \text{and} \quad Bu \in V_0^{s-1}(I' \times \Gamma_{\rho'}).$$

Then for any $I \subset\subset I'$ we have two sorts of splittings for u.
(i) Tensor product form :

$$u = v + \sum_{k,\, s_0 < k \operatorname{Re} \nu(y) < s} c_k(y)\, \varphi_k(y, \theta)\, r^{k\nu(y)} \tag{4.13}$$

with

$$c_k \in H^{s - k \operatorname{Re} \nu - \varepsilon}(I), \quad \forall \varepsilon > 0,$$

and

$$v \in L^2(I; V_0^{s+1}(\Gamma_\rho)).$$

(ii) With the regular extension of the coefficients :

$$u = w + \sum_{k,\, s_0 < k \operatorname{Re} \nu(y) < s} (c_k * \Phi)(y, r)\, \varphi_k(y, \theta)\, r^{k\nu(y)} \tag{4.14}$$

with the same coefficients c_k as in (4.13) *and*

$$w \in V_0^{s+1-\varepsilon}(I \times \Gamma_\rho) \quad \forall \varepsilon > 0.$$

In both expansions (4.13) *and* (4.14) *the functions φ_k only depend on the operator M and are analytic.*

Since $s_0 = 0$ is admissible in the above proposition, we get the following regularity result when no singularity is present :

Corollary 4.6 *If $0 < s < \inf\{\operatorname{Re} \nu(y) \mid y \in I\}$, then from $u \in \overset{\circ}{H}{}^1(I' \times \Gamma_{\rho'})$ and $Bu \in V_0^{s-1}(I' \times \Gamma_{\rho'})$ follows $u \in V_0^{s+1}(I' \times \Gamma_{\rho'})$.*

Here the ε present in Proposition 4.5 can be taken equal to 0. This can be checked from the proof of the point *(ii)* of this proposition.

Proof of Proposition 4.5. In order to prove this proposition, we begin with a preliminary result of tangential regularity concerning $\partial_y u$.

Lemma 4.7 *We assume the same hypotheses about s, s_0 and u as in the previous proposition* 4.5. *Then $\partial_y u \in V_0^s(I \times \Gamma_\rho)$ with the estimate*

$$\|\partial_y u\|_{V_0^s(I \times \Gamma_\rho)} \le C \left(\|Bu\|_{V_0^{s-1}(I' \times \Gamma_{\rho'})} + \|u\|_{V_0^{s_0+1}(I' \times \Gamma_{\rho'})} \right).$$

Proof. Since by (2.29) $V_0^{s-1}(I'\times\Gamma_{\rho'})\subset V_0^{s_0-1,s-s_0}(I'\times\Gamma_{\rho'})$, Proposition 4.4 yields that $u\in V_0^{s_0+1,s-s_0}(I\times\Gamma_\rho)$. This space being embedded in $V_0^{s,1}(I\times\Gamma_\rho)$ we get the wanted result. ∎

The next step in the proof of Proposition 4.5 is to show regularity properties for the function

$$g := Mu. \tag{4.15}$$

Lemma 4.8 *With the above notations and assumptions, we have*

$$
\begin{aligned}
& g \in V_0^{s-1}(I\times\Gamma_\rho) && (4.16)\\
& g \in H^1(I, V_0^{s-2}(\Gamma_\rho)) && (4.17)\\
& g \in L^2(I, V_0^{s-1}(\Gamma_\rho)) && (4.18)\\
\forall\beta\in[s_0,s]\quad & g \in H^{s-\beta}(I, V_0^{\beta-1}(\Gamma_\rho)) && (4.19)
\end{aligned}
$$

Proof. **a)** Since $Mu = Bu - Nu$, to prove (4.16) it suffices to show that $Nu\in V_0^{s-1}(I\times\Gamma_\rho)$ holds. We use the splitting $N = N_1 + N_2$ introduced above in (4.3). From the previous lemma follows that for $|\alpha|\le 1$, $\partial_z^\alpha u$, $\partial_y\partial_z^\alpha u$ and $\partial_y^2 u$ all belong to $V_0^{s-1}(I\times\Gamma_\rho)$, hence

$$N_1 u\in V_0^{s-1}(I\times\Gamma_\rho).$$

For N_2, we conclude from $u\in V_0^s(I'\times\Gamma_{\rho'})$, $Bu\in V_0^{s-1}(I'\times\Gamma_{\rho'})\subset V_1^{s-1}(I'\times\Gamma_{\rho'})$ and Lemma 4.1 for $\beta = s$, $\delta = -s$:

$$u\in V_1^{s+1}(I\times\Gamma_\rho).$$

The coefficients of N_2 are smooth and vanish for $z = 0$, hence

$$N_2 u\in V_0^{s-1}(I\times\Gamma_\rho).$$

Thus we have shown (4.16).

b) To prove (4.17), we first note that, since $u\in V_0^{s_0+1}(I'\times\Gamma_{\rho'})$, we have $u\in L^2(I', V_0^{s_0+1}(\Gamma_{\rho'}))$, thus

$$g\in L^2(I', V_0^{s_0-1}(\Gamma_{\rho'}))\subset L^2(I', V_0^{s-2}(\Gamma_\rho)).$$

It remains to prove that $\partial_y g$ also belongs to $L^2(I', V_0^{s-2}(\Gamma_\rho))$. We have $\partial_y g = M(\partial_y u) + M_1 u$ where M_1 is a second order operator acting only in the z variable. Due to Lemma 4.7, $\partial_y u\in L^2(I', V_0^s(\Gamma_{\rho'}))$. u also belongs to the same space. Thus $M(\partial_y u)$ and $M_1 u$ are in $L^2(I', V_0^{s-2}(\Gamma_{\rho'}))$. This is what we want.

c) If $s\ge 1$ (4.18) is a straightforward consequence of (4.16) and (2.9).
If $0<s<1$, we use (2.22) and (2.23). So $g\in H^{s-1}(I\times\Gamma_\rho)\cap H^1(I, H^{s-2}(\Gamma_\rho))$. By the use of a cut-off function, we can assume that

$$g\in H^{s-1}(\mathbb{R}\times\Gamma)\cap H^1(\mathbb{R}, H^{s-2}(\Gamma)).$$

Due to a Calderón extension operator from $H^\sigma(\Gamma)$ to $H^\sigma(\mathbb{R}^2)$ (for $\sigma = s-2,\ s-1,\ 0$) we can suppose

$$g\in H^{s-1}(\mathbb{R}^3)\cap H^1(\mathbb{R}, H^{s-2}(\mathbb{R}^2)).$$

Using the Fourier transform in the 3 variables, it is easy to show that

$$H^{s-1}(\mathbb{R}^3)\cap H^1(\mathbb{R}, H^{s-2}(\mathbb{R}^2))\subset L^2(\mathbb{R}, H^{s-1}(\mathbb{R}^2)).$$

Thus, we have got (4.18).

d) Now, since $s-\beta \in [0,1]$, we obtain (4.19) from (4.17) and (4.18) using Lemma 2.3.

∎

End of the proof of Proposition 4.5. We can write (4.15) as a family of boundary value problems on Γ :

$$M_y u_y = g_y \quad \text{on} \quad \Gamma, \quad u_y \in \overset{\circ}{H}{}^1(\Gamma) \tag{4.20}$$

where

$$M_y(\partial_z) = M(y,\partial_z), \quad u_y(z) = u(y,z), \quad g_y(z) = g(y,z).$$

We can assume that u_y and g_y have compact support in Γ_ρ and are defined throughout Γ. From (4.18) we find

$$g_y \in V_0^{s-1}(\Gamma) \quad \text{a.e. on} \ \ I.$$

For fixed $y \in I$, we derive from (4.20) a decomposition of u_y into regular and singular parts as follows (see (4.25)).

Let us recall that $\mathcal{M}_y(\theta; r\partial_r, \partial_\theta)$ is defined by

$$r^2 M_y(\partial_z) = \mathcal{M}_y(\theta; r\partial_r, \partial_\theta)\,.$$

This defines the family $\mathcal{M}_y(\theta;\lambda,\partial_\theta)$ of differential operators in θ belonging to the family of Sturm-Liouville problems which are obtained by Mellin transform from (4.20), where we recall that the Mellin transform is defined by

$$\hat{f}(\lambda) = \int_0^\infty r^{-\lambda-1} f(r)\, dr\,.$$

Then we have

$$\mathcal{M}_y(\lambda)\, \hat{u}_y(\lambda) = \hat{g}_y(\lambda-2) \quad \text{for} \quad \operatorname{Re}\lambda \le s_0\,.$$

Now we define $\mathcal{R}_y(\lambda)$ as the inverse of

$$\mathcal{M}_y(\lambda)\ :\ \overset{\circ}{H}{}^1(0,\omega) \to H^{-1}(0,\omega)\,.$$

Then $\mathcal{R}_y(\lambda)$ is an operator-valued function which is meromorphic in $\lambda \in \mathbb{C}$ for each $y \in I$. The poles of $\mathcal{R}_y(\lambda)$ are situated at $\lambda = k\nu(y)$, $k \in \mathbb{Z}^*$. They are all simple and, evidently, they depend analytically on $y \in I$. (This is the general situation for second order problems.) Therefore near a pole $k\nu(y)$, one has a Laurent expansion of $\mathcal{R}_y(\lambda)$ of the form

$$\mathcal{R}_y(\lambda) = \frac{P_y}{\lambda - k\nu(y)} + Q_y(\lambda)\,. \tag{4.21}$$

Here $Q_y(\lambda)$ is analytic for λ near $k\nu(y)$ and P_y is an analytic function of $y \in I$ whose values are one-dimensional projectors :

$$(P_y h)(\theta) = \varphi_k(y,\theta) \int_0^\omega \psi_k(y,\tau)\, h(\tau)\, d\tau\,. \tag{4.22}$$

Here $\varphi_k(y,\theta)$ and $\psi_k(y,\theta)$ are analytic functions. The function $\varphi_k(y,\theta)$ is an eigenfunction of $\mathcal{M}_y(\lambda)$ for the eigenvalue $\lambda = k\nu(y)$ and $\psi_k(y,\theta)$ is an eigenfunction of the adjoint eigenvalue problem.

Now we define

$$c_k(y) = -\int_0^\omega \psi_k(y,\theta)\, \hat{g}_y(k\nu(y)-2,\theta)\, d\theta \tag{4.23}$$

and $v(y,z)$ such that

$$\hat{v}_y(\lambda) = \mathcal{R}_y(\lambda)\hat{g}_y(\lambda-2) \quad \text{for} \quad \operatorname{Re}\lambda = s\,. \tag{4.24}$$

Then the Mellin inversion formula and the Cauchy residue theorem yield

$$u_y = v_y + \sum_{\operatorname{Re}\lambda\in(s_0,s)} \operatorname*{Res}_{\lambda=k\nu(y)} \Big(r^\lambda\, \mathcal{R}_y(\lambda)\, \hat{g}_y(\lambda-2) \Big)\,.$$

From the formula

$$c_k(y)\, \varphi_k(y,\theta)\, r^{k\nu(y)} = -\operatorname*{Res}_{\lambda=k\nu(y)} r^\lambda \left(\frac{P_y \hat{g}_y(\lambda-2)(\theta)}{\lambda - k\nu(y)} \right)\,,$$

one obtains the decomposition

$$u_y = v_y + \sum_{k\operatorname{Re}\nu(y)\in(s_0,s)} c_k(y)\, \varphi_k(y,\theta)\, r^{k\nu(y)}\,. \tag{4.25}$$

This is the well-known decomposition into regular and singular parts for the operator M_y in the plane sector Γ. From the definitions (4.23) and (4.24), we can now derive estimates for c_k and v, uniformly in $y\in I$. The first estimate will give the regularity of the coefficient functions c_k :

$$\|c_k\|_{H^{s-k\operatorname{Re}\nu-\varepsilon}(I)} \le C\|g\|_{I\times\Gamma_\rho} \quad \forall\varepsilon>0\,, \tag{4.26}$$

where $\|g\|_{I\times\Gamma_\rho}$ is a suitable norm of g (compare Lemma 4.8). In order to show this, we first deduce from (4.23) the estimate

$$|c_k(y)| \le \|\psi_k(y,.)\|_{H^1(0,\omega)}\ \|\hat{g}_y(k\nu(y)-2,.)\|_{H^{-1}(0,\omega)}\,. \tag{4.27}$$

We use (4.19) for $\beta = k\operatorname{Re}\nu+\varepsilon'$ and obtain for any small enough $\varepsilon'>0$

$$g \in H^{s-k\operatorname{Re}\nu-2\varepsilon'}(I, V_0^{k\operatorname{Re}\nu-1+\varepsilon'}(\Gamma_\rho))\,. \tag{4.28}$$

Thus, Lemma 2.2 allows to deduce from (4.27)

$$|c_k(y)| \le C\|g_y\|_{V_0^{k\operatorname{Re}\nu(y)-1+\varepsilon'}(\Gamma)}\,. \tag{4.29}$$

Integrating over I one obtains

$$\|c_k\|_{L^2(I)} \le C\|g\|_{L^2(I,V_0^{k\operatorname{Re}\nu-1+\varepsilon'}(\Gamma_\rho))}\,. \tag{4.30}$$

In order to estimate the $H^{s-k\operatorname{Re}\nu-\varepsilon}$-seminorm $|c_k|_{H^{s-k\operatorname{Re}\nu-\varepsilon}}$, we deduce from (4.23)

$$\begin{aligned} |c_k(y)-c_k(y')| \ &\le\ C\Big(\|\hat{g}_y(k\nu(y)-2,.) - \hat{g}_{y'}(k\nu(y')-2,.)\|_{H^{-1}(0,\omega)} + \\ &\qquad + |y-y'|\, \|\hat{g}_y(k\nu(y)-2,.)\|_{H^{-1}(0,\omega)} \Big)\,. \end{aligned} \tag{4.31}$$

We have to estimate the two terms of the right hand side of (4.31). For the second one, we have as above

$$|y-y'|\, \|\hat{g}_y(k\nu(y)-2,.)\|_{H^{-1}(0,\omega)} \le C\,|y-y'|\, \|g_y\|_{V_0^{k\operatorname{Re}\nu(y)-1+\varepsilon'}(\Gamma)}\,. \tag{4.32}$$

Concerning the first term, we split it into two terms (for $|y-y'|$ small enough)

$$
\begin{aligned}
\|\hat{g}_y(k\nu(y)-2,.)-\hat{g}_{y'}(k\nu(y')-2,.)\|_{H^{-1}(0,\omega)} &\leq \\
C\Big(\|[\hat{g}_y-\hat{g}_{y'}](k\nu(y)-2,.)\|_{H^{-1}(0,\omega)} &+ \qquad (4.33)\\
+\|\hat{g}_{y'}(k\nu(y)-2,.)-\hat{g}_{y'}(k\nu(y')-2,.)\|_{H^{-1}(0,\omega)}\Big)&. \qquad (4.34)
\end{aligned}
$$

As previously, Lemma 2.2 allows to majorize (4.33) by

$$
C\,\|g_y-g_{y'}\|_{V_0^{k\,\mathrm{Re}\,\nu(y)-1+\varepsilon'}(\Gamma)}\,. \tag{4.35}
$$

We estimate (4.34) by

$$
C\,|y-y'|\sup_{\lambda\in(k\nu(y),k\nu(y'))}\|\frac{\partial\hat{g}_{y'}}{\partial\lambda}(\lambda-2,.)\|_{H^{-1}(0,\omega)}\,.
$$

Lemma 2.2 and the Cauchy formula together with the fact that $g_{y'}$ has a compact support allow to majorize the previous expression by

$$
C\,|y-y'|\,\|g_{y'}\|_{V_0^{k\,\mathrm{Re}\,\nu(\tilde{y})-1+\varepsilon'}(\Gamma)} \tag{4.36}
$$

where $\tilde{y}$ is such that $\forall y''\in(y,y')$, $\mathrm{Re}\,\nu(\tilde{y})\geq\mathrm{Re}\,\nu(y'')$.

Now we use a localization argument allowing to use constant Sobolev exponents, compare (2.3). From (4.31) to (4.36), for any $y_0\in I$, for a sufficiently small interval I_0 around y_0 and suitable $0<\varepsilon'<\varepsilon''<\varepsilon/2$, this gives

$$
\begin{aligned}
|c_k(y)-c_k(y')| &\leq \qquad (4.37)\\
C\Big(|y-y'|\,\|g_y\|_{V_0^{k\,\mathrm{Re}\,\nu(y_0)-1+\varepsilon''}(\Gamma)} &+ |y-y'|\,\|g_{y'}\|_{V_0^{k\,\mathrm{Re}\,\nu(y_0)-1+\varepsilon''}(\Gamma)} +\\
+\|g_y-g_{y'}\|_{V_0^{k\,\mathrm{Re}\,\nu(y_0)-1+\varepsilon''}(\Gamma)}\Big)&.
\end{aligned}
$$

We integrate (4.37) over $I_0\times I_0$

$$
\begin{aligned}
|c_k|^2_{H^{s-k\,\mathrm{Re}\,\nu(y_0)-\varepsilon}(I_0)} &\equiv \int_{I_0}\int_{I_0}\frac{|c_k(y)-c_k(y')|^2}{|y-y'|^{2(s-k\,\mathrm{Re}\,\nu(y_0)-\varepsilon)+1}}\,dy\,dy'\\
&\leq C\int_{I_0}\int_{I_0}\Bigg\{2|y-y'|^2\,\frac{\|g_y\|^2_{V_0^{k\,\mathrm{Re}\,\nu(y_0)-1+\varepsilon''}(\Gamma)}}{|y-y'|^{2(s-k\,\mathrm{Re}\,\nu(y_0)-\varepsilon)+1}} +\\
&\qquad +\frac{\|g_y-g_{y'}\|^2_{V_0^{k\,\mathrm{Re}\,\nu(y_0)-1+\varepsilon''}(\Gamma)}}{|y-y'|^{2(s-k\,\mathrm{Re}\,\nu(y_0)-\varepsilon)+1}}\Bigg\}\,dy\,dy'\\
&\leq C\|g\|^2_{L^2(I_0,V_0^{k\,\mathrm{Re}\,\nu(y_0)-1+\varepsilon''}(\Gamma))}+\\
&\qquad C|g|^2_{H^{s-k\,\mathrm{Re}\,\nu(y_0)-\varepsilon}(I_0,V_0^{k\,\mathrm{Re}\,\nu(y_0)-1+\varepsilon''}(\Gamma))}\,.
\end{aligned}
$$

Together with (4.28) this yields (4.26).

In order to estimate v, we need a bound on the operator norm for

$$\mathcal{R}_y(\lambda) : H^{s-1}(0,\omega) \to H^{s+1}(0,\omega).$$

We use the norms with parameters $\|\,.\,\|_{H^{s-1}((0,\omega),|\lambda|)}$ and $\|\,.\,\|_{H^{s+1}((0,\omega),|\lambda|)}$ (see (2.10) for the corresponding definition) and denote the corresponding operator norms by $|||\,.\,|||_{s,\lambda}$. With the method of a priori estimates with parameters of [4], we obtain for any $\xi_1 < \xi_2 \in \mathbb{R}$ that there exists $C > 0$ such that :

$$|||\,\mathcal{R}_y(\lambda)\,|||_{s,\lambda} \le C \quad \text{for all } \lambda \text{ with } \operatorname{Re}\lambda \in (\xi_1, \xi_2),\ |\operatorname{Im}\lambda| \ge C.$$

Due to the continuity with respect to y, the constant C does not depend on $y \in I$. For all λ with $\operatorname{Re}\lambda \in (\xi_1, \xi_2)$ one obtains then

$$|||\,\mathcal{R}_y(\lambda)\,|||_{s,\lambda} \le C \max\{d_y(\lambda)^{-1}, 1)\},$$

where $d_y(\lambda)$ is the distance of λ to the set $\{k\nu(y) \mid k \in \mathbb{Z}^*\}$, the spectrum of $\mathcal{M}_y$. Using this estimate for $\operatorname{Re}\lambda = s$ and the characterization of the norms in $V_0^s(\Gamma)$ by Mellin transforms (Lemma 2.2), we obtain from the definition (4.24) of v

$$\|v_y\|_{V_0^{s+1}(\Gamma)} \le C\|g_y\|_{V_0^{s-1}(\Gamma)}\ .$$

Hence we have immediately

$$\|v\|_{L^2(I;V_0^{s+1}(\Gamma))} \le C\|g\|_{L^2(I;V_0^{s-1}(\Gamma))} \tag{4.38}$$

Thus with (4.18) the proof of *(i)* is complete.

Now we consider the decomposition *(ii)* with the regular extension of the coefficients of the singular functions. We have

$$w = v + \sum_k (c_k - c_k * \Phi)\,\varphi_k\, r^{k\nu}\ .$$

According to (4.38), we have

$$r^{-s-1}v \in L^2(I \times \Gamma_\rho).$$

For the term $(c_k - c_k * \Phi)\,\varphi_k\, r^{k\nu}$, we find from Lemma 3.5*(ii)* that it belongs to $V^{s-k\operatorname{Re}\nu-\varepsilon}_{-1-k\operatorname{Re}\nu}(I \times \Gamma_\rho)$, hence to $V^0_{-1-s+\varepsilon}(I \times \Gamma_\rho)$. Thus we have also

$$w \in V^0_{-1-s+\varepsilon}(I \times \Gamma_\rho) \quad \text{for any} \quad \varepsilon > 0. \tag{4.39}$$

Next we want to show that

$$Bw \in V_0^{s-1-\varepsilon}(I \times \Gamma_\rho) \tag{4.40}$$

holds for any $\varepsilon > 0$.

We know that $Bu \in V_0^{s-1}(I \times \Gamma_\rho)$ holds. Therefore we have to show that

$$B((c_k * \Phi)\,\varphi_k\, r^{k\nu}) \in V_0^{s-1-\varepsilon}(I \times \Gamma_\rho) \tag{4.41}$$

holds.

We write $B = M + N$ as above in (4.2). We use Lemma 3.6 for $\beta = s - k\operatorname{Re}\nu - \varepsilon$, $\psi = \varphi_k$ and $S[\ldots] = r^{k\nu}$. Since $0 < \beta < 1$, there are no terms of the form $f_{l,\alpha}$.

Hence, we get

$$B((c_k * \Phi)\, \varphi_k\, r^{k\nu}) = (c_k * \Phi)\, M(\varphi_k\, r^{k\nu}) + g,$$

where $g \in H^{\infty}_{-(s-k\,\mathrm{Re}\,\nu)+1-k\,\mathrm{Re}\,\nu+\varepsilon}(I \times \Gamma_\rho) = H^{\infty}_{-s+1+\varepsilon}(I \times \Gamma_\rho)$ which is contained in $V_0^{s-1-\varepsilon}(I \times \Gamma_\rho)$. But we know that

$$M(\varphi_k\, r^{k\nu}) = 0$$

holds, hence

$$B((c_k * \Phi)\, \varphi_k\, r^{k\nu}) = g \in V_0^{s-1-\varepsilon}(I \times \Gamma_\rho)\,,$$

hence (4.41) and (4.40).

Now, (4.39) and (4.40) together allow the application of Lemma 4.1 which gives

$$w \in V_0^{s+1-\varepsilon}(I \times \Gamma_\rho)$$

as desired. Thus *(ii)* is proven. ∎

We are going to prove now tangential regularity for the functions and coefficients which appear in Proposition 4.5.

Proposition 4.9 *We suppose the same hypotheses as in Proposition* 4.5. *In addition we assume that $f = Bu$ has a higher tangential regularity of order $t \in \mathbb{R}$:*

$$f \in V_0^{s-1,t}(I' \times \Gamma_{\rho'})\,.$$

Then u has the additional tangential regularity

$$u \in V_0^{s_0+1,t+s-s_0}(I \times \Gamma_\rho)\,.$$

Moreover, in the splittings (4.13) *and* (4.14) *we have the following regularity*

$$c_k \in H^{s-k\,\mathrm{Re}\,\nu+t-\varepsilon}(I), \quad \forall \varepsilon > 0\,,$$

$$v \in H^t(I; V_0^{s+1-\varepsilon}(\Gamma_\rho))$$

and

$$w \in V_0^{s+1-\varepsilon,t}(I \times \Gamma_\rho)\,.$$

Proof. Concerning the regularity of u, since

$$V_0^{s-1,t}(I' \times \Gamma_{\rho'}) \subset V_0^{s_0-1,t+s-s_0}(I' \times \Gamma_{\rho'})$$

it is just an application of Proposition 4.4.

As for c_k, v and w, it is possible to follow all the steps of the proof of Proposition 4.5 in the anisotropic spaces. We find more interesting to give a proof (for $t = L \in \mathbb{N}$) using other arguments.

This proof proceeds by induction on L. We define for $l \in \mathbb{N}$:

$$\tilde{u}^l = \partial_y^l w + \sum_{k,\, s_0 < k\,\mathrm{Re}\,\nu(y) < s} (\partial_y^l c_k * \Phi)(y,r)\, \varphi_k(y,\theta)\, r^{k\nu(y)}\,. \tag{4.42}$$

Then we will show that for $0 \le l \le L$ there hold the following 3 assertions

$$\begin{array}{rl} (i) & \partial_y^{l-m} B\tilde{u}^m \in V_0^{s-1-\varepsilon}(I\times\Gamma_\rho) \quad \text{for} \quad 0 \le m \le l \\ (ii) & \tilde{u}^l \in \mathring{H}^1(I\times\Gamma_\rho) \cap V_0^{s_0+1}(I\times\Gamma_\rho) \\ (iii) & c_k \in H^{s-k\operatorname{Re}\nu+l-\varepsilon}(I) \quad \text{and} \quad \partial_y^l w \in V_0^{s+1-\varepsilon}(I\times\Gamma_\rho)\,. \end{array}$$

For $l = 0$, since $\tilde{u}^0 = u$, *(i)* and *(ii)* are given by the assumptions of the proposition and *(iii)* is given by the results of Proposition 4.5.

We assume that the 3 assertions hold for $l < L$ and we are going to prove them for $l+1$. We have the relation

$$\tilde{u}^{l+1} = \partial_y \tilde{u}^l - \sum_k (\partial_y^l c_k * \Phi)\, \partial_y(\varphi_k\, r^{k\nu})\,. \tag{4.43}$$

(i) We prove that

$$\partial_y^{l+1-m} B\tilde{u}^m \in V_0^{s-1-\varepsilon}(I\times\Gamma_\rho) \tag{4.44}$$

by induction on $m = 0, \ldots, l+1$.

For $m = 0$, $\partial_y^{l+1} B\tilde{u}^0 = \partial_y^{l+1} f \in V_0^{s-1}(I\times\Gamma_\rho)$ by the assumption.

Next, we suppose that (4.44) holds for m, with $m \le l$. We want to prove that $\partial_y^{l-m} B\tilde{u}^{m+1} \in V_0^{s-1-\varepsilon}(I\times\Gamma_\rho)$. We let B operate on (4.43). We have

$$B\tilde{u}^{m+1} = B(\partial_y \tilde{u}^m) - \sum_k B\Big((\partial_y^m c_k * \Phi)\, \partial_y(\varphi_k\, r^{k\nu})\Big)\,.$$

Hence

$$B\tilde{u}^{m+1} = \partial_y B\tilde{u}^m + [B,\partial_y]\tilde{u}^m - \sum_k B\Big((\partial_y^m c_k * \Phi)\, \partial_y(\varphi_k\, r^{k\nu})\Big)\,. \tag{4.45}$$

We are going to expand the second and the third term of the right hand side of (4.45).

a) We have by the definition (4.42)

$$[B,\partial_y]\tilde{u}^m = [B,\partial_y]\partial_y^m w + \sum_k [B,\partial_y]\Big((\partial_y^m c_k * \Phi)\, \varphi_k\, r^{k\nu}\Big)\,. \tag{4.46}$$

We apply Lemma 3.6 to the last term and we use that by the induction hypothesis *(iii)* for l, $\partial_y^m c_k \in H^{s-k\operatorname{Re}\nu+l-m-\varepsilon}(I)$:

$$\begin{aligned} [B,\partial_y]\Big((\partial_y^m c_k * \Phi)\, \varphi_k\, r^{k\nu}\Big) \quad = \quad & (\partial_y^m c_k * \Phi)\, [M,\partial_y]\, \varphi_k\, r^{k\nu} \\ & + \sum_{n,p} f_{npk} \\ & + g_k\,. \end{aligned} \tag{4.47}$$

- Since $M(\varphi_k\, r^{k\nu}) = 0$,

$$(\partial_y^m c_k * \Phi)\, [M,\partial_y]\, \varphi_k\, r^{k\nu} = (\partial_y^m c_k * \Phi)\, M\, \partial_y(\varphi_k\, r^{k\nu})\,. \tag{4.48}$$

- In the sum $\sum_{n,p} f_{npk}$ we have

$$1 \le n \le s - k\operatorname{Re}\nu + l - m - \varepsilon, \qquad \text{i.e.} \qquad 1 \le n \le l - m\,.$$

• The "regular part" g_k belongs to $H^{\infty}_{-(s-k\operatorname{Re}\nu+l-m)+1-k\operatorname{Re}\nu+\varepsilon}(I)$ which is equal to $H^{\infty}_{-s-l+m+1+\varepsilon}(I)$. Hence $g_k \in V_0^{l-m+s-1-\varepsilon}(I\times\Gamma_\rho)$.

b) In the same way we find that

$$
\begin{aligned}
B\Big((\partial_y^m c_k * \Phi)\,\partial_y\,\varphi_k\,r^{k\nu}\Big) \;=\;& (\partial_y^m c_k * \Phi)\,M\,\partial_y\,(\varphi_k\,r^{k\nu}) \\
&+ \sum_{n,p} f'_{npk} \\
&+ g'_k\,.
\end{aligned} \tag{4.49}
$$

The equalities (4.45) to (4.49) yield

$$
\begin{aligned}
\partial_y^{l-m} B\tilde u^{m+1} \;=\;& \partial_y^{l-m+1} B\tilde u^{m} \\
&+ \partial_y^{l-m}[B,\partial_y]\partial_y^m w \\
&+ \partial_y^{l-m}\sum_{n=1}^{l-m}\sum_{p}(f_{npk}-f'_{npk}) \\
&+ \partial_y^{l-m}(g_k-g'_k)\,.
\end{aligned} \tag{4.50}
$$

We analyse each of the 4 groups of terms.

• The induction hypothesis for $l+1$ and m yields that

$$\partial_y^{l-m+1} B\tilde u^{m} \in V_0^{s-1-\varepsilon}(I\times\Gamma_\rho)\,.$$

• For some second order operators $\mathfrak{B}_j$ we have

$$\partial_y^{l-m}[B,\partial_y]\partial_y^m w = \sum_{j=0}^{l}\mathfrak{B}_j\partial_y^j w\,.$$

The induction hypothesis *(iii)* for l gives that for $0\le j\le l$

$$\mathfrak{B}_j\partial_y^j w \in V_0^{s-1-\varepsilon}(I\times\Gamma_\rho)\,.$$

• Lemma 3.4 and Lemma 3.6 give

$$\partial_y^{l-m}(f_{npk}-f'_{npk}) = \sum_{\delta}(\partial_y^i c_k * \Phi)\,\psi_\delta\, S[k\nu-2+n,\ldots,k\nu-2+n;r]$$

with $i\le n+l$. So $\partial_y^i c_k \in H^\beta(I)$ with $\beta = s-k\operatorname{Re}\nu-n<0$. Thus Lemma 3.5 *(iv)* yields

$$\partial_y^{l-m}(f_{npk}-f'_{npk}) \in H^{\infty}_{-s+1+\varepsilon}(I\times\Gamma_\rho)\subset V_0^{s-1-\varepsilon}(I\times\Gamma_\rho)\,.$$

Hence (4.50) gives what we want, and *(i)* is shown for $l+1$.

(ii) From the induction hypothesis and the previous step we have

$$
\begin{aligned}
&\tilde u^l \in \overset{\circ}{H}{}^1(I\times\Gamma_\rho)\cap V_0^{s_0+1}(I\times\Gamma_\rho)\,,\\
&B\tilde u^l \in V_0^{s-1-\varepsilon}(I\times\Gamma_\rho)\quad\text{and}\quad \partial_y B\tilde u^l \in V_0^{s-1-\varepsilon}(I\times\Gamma_\rho)\,.
\end{aligned}
$$

If we look at the proof of Lemma 4.7 we see that $\partial_y\tilde u^l$ belongs to $V_0^{s_0+1}(I\times\Gamma_\rho)$. Then (4.43) gives that $\tilde u^{l+1}$ belongs to $V_0^{s_0+1}(I\times\Gamma_\rho)$, because

$$(\partial_y^l c_k * \Phi)\,\partial_y(\varphi_k\,r^{k\nu}) \in H^{\infty}_{-1-k\operatorname{Re}\nu+\varepsilon}(I\times\Gamma_\rho)\subset V_0^{s_0+1}(I\times\Gamma_\rho)$$

(remember that by the induction hypothesis $\partial_y^l c_k$ is in $H^{s-k\,\mathrm{Re}\,\nu-\varepsilon}(I)$).

(iii) From *(i)* and *(ii)* we see that $\tilde{u}^{l+1}$ satisfies the assumptions of Proposition 4.5. Hence there is a decomposition

$$\tilde{u}^{l+1} = \tilde{w}^{l+1} + \sum(\tilde{c}_k^{l+1} * \Phi)\,\varphi_k\, r^{k\nu}$$

with $\tilde{w}^{l+1} \in V_0^{s-1-\varepsilon}(I \times \Gamma_\rho)$ and $\tilde{c}_k^{l+1} \in H^{s-k\,\mathrm{Re}\,\nu-\varepsilon}(I)$.

On the other hand, by definition,

$$\tilde{u}^{l+1} = \partial_y^{l+1} w + \sum(\partial_y^{l+1} c_k * \Phi)\,\varphi_k\, r^{k\nu}$$

with $\partial_y^{l+1} w \in H^{-1}(I; V_0^{s+1-\varepsilon}(\Gamma_\rho))$.

The identification of both asymptotics above yields that

$$\tilde{c}_k^{l+1} = \partial_y^{l+1} c_k \quad \text{and} \quad \tilde{w}^{l+1} = \partial_y^{l+1} w\,.$$

Hence *(iii)* for $l+1$. Thus we have finally obtained the claimed regularity for the regular part w and the coefficients c_k.

To obtain the regularity of the semi-regular part v, since

$$v = w - \sum_k (c_k * \Phi - c_k)\,\varphi_k\, r^{k\nu}$$

it suffices to apply Lemma 3.5 *(ii)* to

$$(c_k * \Phi - c_k)\,\varphi_k\, r^{k\nu}\,.$$

We get that it belongs to $H^L(I; H^\infty_{-s-1+\varepsilon}(\Gamma_\rho)) \subset H^L(I; V_0^{s+1-\varepsilon}(\Gamma_\rho))$. ∎

5. SINGULAR RIGHT HAND SIDES

In the previous section we have considered the case of flat right hand sides. However the difference between the conormal regularity of the solution u and of the regular part w was less than 1. Such a restriction always occurs when the operator has variable coefficients : to go further in the decomposition of u we have to give an approximate solution of the Dirichlet problem with singular right hand sides.

After the first decomposition (4.14) of u we will have to solve the Dirichlet problem with right hand sides $S[k\nu + l - 2, \ldots, k\nu + l - 2; r]$ where k is like in Proposition 4.5, $l \in \mathbb{N}^*$ and the argument $k\nu + l - 2$ appears one, two or three times in $S[\ldots; r]$ (see the proof of Lemma 6.2 and Remark 6.4). Here a crossing of singularity exponents can occur. This happens when the powers corresponding to the right hand side ($k\nu(y) + l$ in the above situation) cross a pole $k'\nu(y)$ of the inverse $\mathcal{R}_y$. The solution of such a problem will be described by a function S with arguments $k\nu(y)+l$ and $k'\nu(y)$; moreover, it will have to be considered as a possible right hand side in the next step of the splitting of u. This is why we consider right hand sides of the form $S[\ldots; r]$ with general arguments.

Proposition 5.1 *Let $\beta \in \mathbb{R}_+$. We denote by $[\beta]$ the integer part of β. Let $c \in H^\beta(I')$. Let $\varphi(y,\theta)$ be analytic in y, θ on $I' \times [0,\omega]$. Let $\mu_j(y)$ $(j = 1,\dots,J)$ be analytic functions on I', not necessarily different from each other; let γ be a contour in $\mathbb{C}$ such that $\{\mu_j(y) \mid j = 1,\dots,J,\ y \in I'\} \subset \operatorname{int}\gamma$; let $\mu(y)$ be $\min_j \operatorname{Re}\mu_j(y)$. We assume that $\mu(y) \geq 0$ holds for any $y \in I'$. We define the singular right hand side f on $I' \times \Gamma_{\rho'}$ by*

$$f(x) := (c * \Phi)(y,r)\,\varphi(y,\theta)\,S[\mu_1(y) - 2, \dots, \mu_J(y) - 2; r]$$

Then, $\forall \varepsilon_0 > 0$, $\forall y_0 \in I'$, $\exists I \subset\subset I'$ containing y_0 and there exists u of the form

$$u(x) := \sum_{l=0}^{[\beta]} \sum_{p=0}^{p_l} (d_{l,p} * \Phi)(y,r)\,\psi_{l,p}(y,\theta)\,S[\mu_1^{l,p}(y), \dots, \mu_{q_{l,p}}^{l,p}(y); r]$$

where $d_{l,p}(y) \in H^{\beta-l}(I)$ are derivatives of c of order at most l, $\psi_{l,p}$ are analytic on $I \times [0,\omega]$ and the $\mu_q^{l,p}$ are of the form either $\mu_j + l$ or $k\nu + l'$ with $k \in \mathbb{N}^*$, $l' \in \mathbb{N}, l' \leq l$ s.t. $\forall y \in I$, $k\nu(y) + l' \in \operatorname{int}\gamma + l$
and u is such that

$$u\Big|_{I \times \partial\Gamma_\rho} = 0 \quad \textit{and} \quad Bu - f \in H^\infty_{-\beta+1-\mu+\varepsilon_0}(I \times \Gamma_\rho)\,.$$

Remark 5.2 If $\beta \leq 0$ holds, we have $f \in H^\infty_{-\beta+1-\mu+\varepsilon_0}(I \times \Gamma_\rho)$, compare Lemma 3.5 *(iv)*. Therefore we can take $u = 0$ in this case. If $\beta > 0$, Lemma 3.5 *(iii)* gives only $f \in H^\infty_{1-\mu+\varepsilon_0}(I \times \Gamma_\rho)$.

The proof of the above proposition is based upon the following induction lemma.

Lemma 5.3 *We assume the same hypotheses about f as in the previous proposition. Then, $\forall \varepsilon_0 > 0$, $\forall y_0 \in I'$, $\exists I \subset\subset I'$ containing y_0 and there exists u of the form*

$$u(x) := (c * \Phi)(y,r) \sum_p \varphi_p(y,\theta)\,S[\mu_1^p(y), \dots, \mu_{q_p}^p(y); r]$$

where φ_p are analytic on $I \times [0,\omega]$ and the μ_j^p are of the form either μ_j or $k\nu$ with $k \in \mathbb{N}^*$ s.t. $\forall y \in I$, $k\nu(y) \in \operatorname{int}\gamma$
and u is such that

$$u\Big|_{I \times \partial\Gamma_\rho} = 0 \quad \textit{and} \quad Bu = f + g + \sum_{l=1}^{[\beta]} \sum_p f_{l,p}\,,$$

with

$$g \in H^\infty_{-\beta+1-\mu+\varepsilon_0}(I \times \Gamma_\rho)\,;$$

$$f_{l,p}(x) := (c_{l,p} * \Phi)(y,r)\,\varphi_{l,p}(y,\theta)\,S[\mu_1^{l,p}(y) - 2, \dots, \mu_{q_{l,p}}^{l,p}(y) - 2; r]\,;$$

$c_{l,p}$ are derivatives of c of order at most l ; $\varphi_{l,p}$ are analytic on $I \times [0,\omega]$;

$$\mu_q^{l,p} \in \{\mu_j + l \mid j = 1,\dots,J\} \cup \{k\nu + l \mid k \in \mathbb{N}^*\ ;\ k\nu(y) \in \operatorname{int}\gamma\ \ \forall y \in I\}\,.$$

Remark 5.4 In Proposition 5.1 and in Lemma 5.3 it is also possible to have a variable Sobolev exponent β for the coefficient c as in Lemma 3.6.

Proof of Lemma 5.3. By choosing the interval I around y_0 small enough, we can achieve that there is at most one point in I where two of the functions $\mu_1, \dots, \mu_J$ take the same value without being identically equal or one of the μ_j and a function $k\nu$ ($k \in \mathbb{N}$) take the same value without being identically equal. We can assume that this point ("crossing point") is the point y_0. We can then decompose the function $S[\mu_1(y) - 2, \dots, \mu_J(y) - 2; r]$ into a sum of contour integrals over contours γ such that inside each γ, we have the following situation :

$$\begin{cases} \text{All the functions } \mu_j \text{ or } k\nu \text{ on } I \\ \text{which meet the interior of } \gamma \\ \text{coincide at the point } y_0. \end{cases}$$

By linearity, it suffices to consider only one such contour γ. Hence we can assume that
(i) $\lambda_0 := \mu_1(y_0) = \cdots = \mu_J(y_0)$,
(ii) there is at most one $k \in \mathbb{N}$ such that $k\nu(y_0) = \lambda_0$,
(iii) any two of the μ_j ($j = 1, \dots, J$) or $k\nu$ are either different on $I \setminus \{y_0\}$ or identical on I,
(iv) for any y in I, $k\nu(y) \notin \gamma$ and $\mu_j(y) \notin \gamma$ ($j = 1, \dots, J$).

We assume further that

$$\forall y \in I, \quad k \operatorname{Re} \nu(y) \geq \mu(y) - \varepsilon_0/2\,.$$

We denote the multiplicity of y_0 by K : $K = J$ or $K = J + 1$ according to

$$\begin{cases} \text{If } \nexists k \in \mathbb{N} \text{ such that } k\nu(y_0) = \lambda_0 & \text{then } K = J \\ \text{If } \exists k \in \mathbb{N} \text{ such that } k\nu(y_0) = \lambda_0 & \text{then } K = J + 1. \end{cases}$$

If $K = J + 1$, we define

$$\mu_K := k\nu\,.$$

The function f is then given by

$$f(x) = (c * \Phi)(y, r)\, \frac{\varphi(y, \theta)}{2i\pi} \int_\gamma \frac{r^{\lambda - 2}\, d\lambda}{\prod_{j=1}^{J} (\lambda - \mu_j(y))} \tag{5.1}$$

Now we recall the resolvent $\mathcal{R}_y(\lambda)$ of the operator $\mathcal{M}_y(\lambda)$ from the proof of Lemma 4.5. We define a function $u_y^0(r, \theta)$ by

$$u_y^0(r, \theta) = \frac{1}{2i\pi} \int_\gamma \Big(\mathcal{R}_y(\lambda)\varphi(y, .)\Big)(\theta)\, \frac{r^\lambda\, d\lambda}{\prod_{j=1}^{J} (\lambda - \mu_j(y))}\,. \tag{5.2}$$

Note that the meromorphic function of λ, $\mathcal{R}_y(\lambda)\varphi(y, .)$ has no poles for $\lambda \in \gamma$. When $K = J + 1$ it has one simple pole inside γ for $\lambda = k\nu$ (see (4.21) for an expression of $\mathcal{R}_y(\lambda)$ in that case).

We will then define the function u by

$$u(x) = (c * \Phi)(y,r)\; u_y^0(r,\theta)\,. \tag{5.3}$$

In order to describe it more precisely, we introduce the function χ

$$\begin{cases} \text{If } K = J, & \chi(\lambda,y,\theta) := \Big(\mathcal{R}_y(\lambda)\varphi(y,.)\Big)(\theta) \\ \text{If } K = J+1, & \chi(\lambda,y,\theta) := (\lambda - k\nu(y))\Big(\mathcal{R}_y(\lambda)\varphi(y,.)\Big)(\theta) \end{cases} \tag{5.4}$$

This function χ is analytic in all its variables, for λ in a neighborhood of γ and the interior of γ. Thus we have

$$\Big(\mathcal{R}_y(\lambda)\varphi(y,.)\Big)(\theta)\; \frac{1}{\prod\limits_{j=1}^{J}(\lambda - \mu_j(y))} = \frac{\chi(\lambda,y,\theta)}{\prod\limits_{j=1}^{K}(\lambda - \mu_j(y))}\,. \tag{5.5}$$

Together with (5.2), this yields

$$u_y^0(r,\theta) = \frac{1}{2i\pi}\int_\gamma \frac{\chi(\lambda,y,\theta)\; r^\lambda}{\prod\limits_{j=1}^{K}(\lambda - \mu_j(y))}\, d\lambda\,. \tag{5.6}$$

For each fixed y, r and θ, this expression appears as the divided difference (8.4) of the product of the two functions

$$R_r : \lambda \mapsto r^\lambda \tag{5.7}$$

and

$$\chi_{y,\theta} : \lambda \mapsto \chi(\lambda,y,\theta)\,. \tag{5.8}$$

at the points $\mu_1(y),\ldots,\mu_K(y)$.

Now we use the Leibniz formula (8.7) : (5.6) can be written as

$$u_y^0(r,\theta) = \sum_{j=1}^{K} R_r[\mu_1(y),\ldots,\mu_j(y)]\;\chi_{y,\theta}[\mu_j(y),\ldots,\mu_K(y)]\,. \tag{5.9}$$

But

$$R_r[\mu_1(y),\ldots,\mu_j(y)] = S[\mu_1(y),\ldots,\mu_j(y);r]\,. \tag{5.10}$$

and

$$\chi_{y,\theta}[\mu_j(y),\ldots,\mu_K(y)] \equiv w_{K-j}(y,\theta)\,. \tag{5.11}$$

is analytic in y and θ. The identities (5.3), (5.6) and (5.7) to (5.11) give

$$u(y,r,\theta) = (c*\Phi)(y,r)\sum_{j=1}^{K} w_{K-j}(y,\theta)\,S[\mu_1(y),\ldots,\mu_j(y);r] \tag{5.12}$$

as desired.

Remark 5.5 If one does not use this Leibniz formula for divided differences, one can invoke the Weierstraß Preparation Theorem to prove identity (5.12) : χ is decomposed into

$$\chi(\lambda, y, \theta) = a(\lambda, y, \theta) \prod_{j=1}^{K} (\lambda - \mu_j(y)) + \sum_{j=1}^{K-1} b_j(y, \theta)(\lambda - \lambda_0)^j \tag{5.13}$$

where a and b_j are analytic in all their variables. If we write the polynomial part in Newton form (8.3) with the interpolation nodes $\mu_K(y), \ldots, \mu_1(y)$, we find again the coefficients w_k which appear in formula (5.12). The advantage of the Leibniz formula is to allow non-analytic functions μ_j.

End of the proof of Lemma 5.3. From the definitions (5.1) and (5.2) of f and u_y^0, we find

$$(c * \Phi)\, M_y u_y^0 = f_y\,.$$

Applying Lemma 3.6 to $u = (c * \Phi)\, u_y^0$, we find that

$$Bu = (c * \Phi)\, M_y u_y^0 + \sum f_{l,p} + g$$

with $f_{l,p}$ as in Lemma 5.3 and $g \in H^{\infty}_{-\beta+1-\mu'+\varepsilon}(I \times \Gamma_\rho)$ for

$$\mu'(y) = \min\{\operatorname{Re} \mu_j(y) \mid 1 \le j \le K\}$$

and any $\varepsilon > 0$. By construction, $\mu'(y) \ge \mu(y) - \varepsilon_0/2$. Hence

$$g \in H^{\infty}_{-\beta+1-\mu+\varepsilon_0}(I \times \Gamma_\rho)$$

as claimed in the lemma. ∎

Proof of Proposition 5.1. Apply Lemma 5.3 repeatedly with f replaced by $f_{l,p}$. Each application increases the lower bound for l by 1. Repeat until this lower bound is $\ge \beta$. ∎

Proposition 5.1 is a local statement, because the size of the interval I depends on the choice of ε_0 and on the point y_0. There exists a semi-global version of Lemma 5.3 and Proposition 5.1 which also avoids the reference to the contour γ : for any given y_0, we want to construct a solution u defined by a unique formula on the largest possible interval I. For doing so it suffices to require that I is such that there is no crossing point inside $I \setminus \{y_0\}$. We obtain a correct description of the regularity of the remainder g with the help of a variable weight. Here are the two statements corresponding respectively to Lemma 5.3 and Proposition 5.1. We everywhere understand that we only consider as crossing points the *isolated* ones.

Lemma 5.6 *We assume the same hypotheses about f as in Proposition* 5.1. *Let $y_0 \in I'$ and let $I \subset\subset I'$ be an interval containing y_0 and such that there is no crossing in $I \setminus \{y_0\}$ between the μ_j, or between the μ_j and the functions $k\nu$. We set*

$$\mathfrak{K}_0 := \{k \in \mathbb{N}^* \mid \exists j \in 1, \ldots, J :\, k\nu(y_0) = \mu_j(y_0)\,\}$$

and

$$\underline{\mu}_0(y) := \min\Big(\{\operatorname{Re} \mu_j(y) \mid 1 \le j \le J\} \cup \{\operatorname{Re} k\nu(y) \mid k \in \mathfrak{K}_0\}\Big)\,.$$

Then, there exists u of the form

$$u(x) := (c * \Phi)(y,r) \sum_p \varphi_p(y,\theta)\, S[\mu_1^p(y), \ldots, \mu_{q_p}^p(y); r]$$

where φ_p are analytic on $I \times [0,\omega]$ and the μ_j^p are of the form either μ_j or $k\nu$ with $k \in \mathfrak{K}_0$ and u is such that

$$u\Big|_{I\times\partial\Gamma_\rho} = 0 \quad and \quad Bu = f + g + \sum_{l=1}^{[\beta]} \sum_p f_{l,p}\,,$$

with

$$g \in H^\infty_{-\beta+1-\underline{\mu}_0+\varepsilon}(I \times \Gamma_\rho) \quad \forall \varepsilon > 0\,;$$

and $f_{l,p}$ as in Lemma 5.3 with

$$\mu_q^{l,p} \in \{\mu_j + l \mid j = 1, \ldots, J\} \cup \{k\nu + l \mid k \in \mathfrak{K}_0\}\,.$$

The proof of this lemma follows the same steps as the proof of Lemma 5.3. The only difference is that we have to consider a contour $\gamma = \gamma(y)$ which depends analytically (or piecewise analytically) on $y \in I$ and such that $\mathrm{int}\gamma(y)$ contains the same μ_j and $k\nu$ for all $y \in I$. By induction, we obtain the following :

Proposition 5.7 *We assume the same hypotheses about f as in Proposition 5.1. Let $y_0 \in I'$ and let $I \subset\subset I'$ be an interval containing y_0. We set for any $l = 0, \ldots, [\beta]$:*

$$\mathfrak{K}_l := \{k \in \mathbb{N}^* \mid \exists j \in 1, \ldots, J :\, k\nu(y_0) = \mu_j(y_0) + l\,\}$$

and

$$\underline{\mu}_l(y) := \min\Big(\{\underline{\mu}_{l-1}(y)\} \cup \{\mathrm{Re}\, k\nu(y) - l \mid k \in \mathfrak{K}_l\}\Big)\,.$$

We assume that there is no crossing in $I \setminus \{y_0\}$ between the $\mu_j + l$, the $k'\nu + l'$ ($k' \in \mathfrak{K}_{l-l'}$) and the functions $k\nu$.

Then there exists u of the form

$$u(x) := \sum_{l=0}^{[\beta]} \sum_{p=0}^{p_l} (d_{l,p} * \Phi)(y,r)\, \psi_{l,p}(y,\theta)\, S[\mu_1^{l,p}(y), \ldots, \mu_{q_{l,p}}^{l,p}(y); r]$$

where $d_{l,p}(y) \in H^{\beta-l}(I)$ are derivatives of c of order at most l, $\psi_{l,p}$ are analytic on $I \times [0,\omega]$ and the $\mu_q^{l,p}$ are of the form either $\mu_j + l$ or $k\nu + l'$ with $k \in \mathfrak{K}_{l-l'}$ and u is such that

$$u\Big|_{I\times\partial\Gamma_\rho} = 0 \quad and \quad Bu - f =: g \in H^\infty_{-\beta+1-\underline{\mu}_l+\varepsilon}(I \times \Gamma_\rho) \;\; \forall \varepsilon > 0\,.$$

Remark 5.8 As a consequence we obtain that if there is no crossing between the $\mu_j + l$ and the $k\nu$ on the interval I,

$$g \in H^\infty_{-\beta+1-\mu+\varepsilon}(I \times \Gamma_\rho) \;\; \forall \varepsilon > 0\,.$$

(Note that the sets $\mathfrak{K}_l$ are empty.)

6. GENERAL RIGHT HAND SIDES

In Theorem 6.1 we give expansion formulas which hold locally in the neighborhood of small intervals of the edge. In Theorem 6.5 we extend the validity of such formulas to larger intervals (a sufficient condition on the interval is to contain at most one crossing point).

As in Section 4, we assume in this whole section that B is a second order elliptic operator with analytic coefficients such that Hypothesis 4.2 holds.

Theorem 6.1 *Let s be a positive number. We assume that*

$$u \in \overset{o}{H}{}^1(I' \times \Gamma_{\rho'}) \quad \textit{and} \quad Bu \in H^{s-1}(I' \times \Gamma_{\rho'}).$$

Let $\varepsilon_0 > 0$ be given. Then for all $y_0 \in I'$ there exists $I \subset\subset I'$ with $y_0 \in I$ and the following splitting of u :

$$u = w + \sum_p (c_p * \Phi)(y,r)\, \varphi_p(y,\theta)\, S[\mu_1^p(y), \ldots, \mu_{q_p}^p(y); r] \tag{6.1}$$

with

$$w \in V_0^{s+1-\varepsilon_0}(I \times \Gamma_\rho)$$

Here $\mu_j^p \in \{k\nu + l \mid (k,l) \in \mathbb{N}^2;\ k \operatorname{Re}\nu(y) + l < s\ \forall y \in I\}$ and for all p one has

$$c_p \in H^{s-\mu_p-\varepsilon_0}(I)$$

with $\mu_p(y) = \max\{\operatorname{Re}\mu_j^p(y) \mid j = 1, \ldots, q_p\}$.
The φ_p are analytic functions on $I \times [0,\omega]$.

The proof of the above theorem is based upon the following induction lemma. In the Remarks 6.3 and 6.4 below, we discuss the choice of the exponents μ_j^p and the regularity of the coefficients c_p.

Lemma 6.2 *Let s_0, s_1 and s be nonnegative numbers such that $s_0 < s_1 \le s$ and*

$$s_0, s_1 \neq k \operatorname{Re}\nu(y)\ \ \forall k \in \mathbb{N}^*,\ \forall y \in I' \quad \textit{and} \quad s_1 - s_0 \le 1\,.$$

We assume that

$$u \in \overset{o}{H}{}^1(I' \times \Gamma_{\rho'}) \cap V_0^{s_0+1}(I' \times \Gamma_{\rho'}) \quad \textit{and} \quad Bu \in V_0^{s-1}(I' \times \Gamma_{\rho'}).$$

Let $\varepsilon_0 > 0$ be given. Then for all $y_0 \in I'$ there exists $I \subset\subset I'$ with $y_0 \in I$ and a splitting of u :

$$u = w + \sum_p (c_p * \Phi)(y,r)\, \varphi_p(y,\theta)\, S[\mu_1^p(y), \ldots, \mu_{q_p}^p(y); r] \tag{6.2}$$

with

$$w \in V_0^{s_1+1}(I \times \Gamma_\rho) \quad \textit{and} \quad Bw \in V_0^{s-1-\varepsilon}(I \times \Gamma_\rho)\ \forall \varepsilon > \varepsilon_0\,.$$

Here $\mu_j^p \in \{k\nu + l \mid k \in \mathbb{N}^*,\ l \in \mathbb{N};\ s_0 < k \operatorname{Re}\nu(y) + l < s\ \forall y \in I\}$ *;*
for each p*, all* $\mu_j^p(y)$ $(j = 1, \dots, q_p)$*,* $y \in I$ *are contained in a ball of radius* ε_0 *; the* φ_p *are analytic on* $I \times [0, \omega]$ *;*
and with $\mu_p(y) = \max\{\operatorname{Re}\mu_j^p(y) \mid j = 1, \dots, q_p\}$ *one has*

$$c_p \in H^{s-\mu_p-\varepsilon}(I) \quad \forall \varepsilon > 0\,.$$

Remark 6.3 This lemma gives an improved result, compared with what we obtained in § 4. The regular part w obtained in Proposition 4.5 only satisfies $Bw \in V_0^{s_1-1}(I \times \Gamma_\rho)$ and not $Bw \in V_0^{s-1}(I \times \Gamma_\rho)$ in general.

In Theorem 6.1, we could have chosen any of the $\operatorname{Re}\mu_j^p$ instead of the maximum μ_p to describe the regularity of c_p. The choice of the μ_j^p is related to the allowed loss of regularity ε_0. In Lemma 6.2, the stated regularity for c_p is sharper. Indeed we will show in Remark 6.4 that there exists $j \in \{1, \dots, q_p\}$ such that

$$c_p \in H^{s-\operatorname{Re}\mu_j^p-\varepsilon}(I) \quad \forall \varepsilon > 0\,.$$

This μ_j^p has the form $k\nu(y) + l$ with the smallest possible k.

Proof of Lemma 6.2. We begin by applying Proposition 4.5 with s replaced by s_1. We obtain a splitting

$$u = w_1 + u_0$$

with

$$w_1 \in V_0^{s_1+1}(I \times \Gamma_\rho)$$

and

$$u_0 = \sum_{k, s_0 < k \operatorname{Re}\nu < s_1} (c_k * \Phi)\, \varphi_k\, r^{k\nu}. \tag{6.3}$$

According to Proposition 4.9 we have the regularity

$$c_k \in H^{s-k\operatorname{Re}\nu-\varepsilon}(I) \quad \forall \varepsilon > 0\,.$$

Now we apply Lemma 3.6 to Bu_0 and obtain

$$Bu_0 = f + \sum_{l,p} f_{l,p} + g_0$$

with

$$f = \sum_k (c_k * \Phi)\, M(\varphi_k\, r^{k\nu}) = 0$$

$$g_0 \in H^\infty_{-s+1+\varepsilon}(I \times \Gamma_\rho) \subset V_0^{s-1-\varepsilon}(I \times \Gamma_\rho) \quad \forall \varepsilon > 0\,.$$

The $f_{l,p}$ have the form (with $l \geq 1$ and some index k_p appearing in the sum (6.3))

$$f_{l,p} = (c_{l,p} * \Phi)\, \varphi_{l,p}\, S[k_p\nu - 2 + l, \dots, k_p\nu - 2 + l; r] \tag{6.4}$$

where the $c_{l,p}$ are derivatives of c_{k_p} of order at most l, the $\varphi_{l,p}$ are analytic, and in the argument of $S[\dots; r]$ the same index is repeated a certain number of times. Hence

$$c_{l,p} \in H^{s-k_p\operatorname{Re}\nu-l-\varepsilon}(I) \quad \forall \varepsilon > 0\,.$$

Up to now, I was just any interval $\subset\subset I'$, and ε_0 did not yet appear.

The next step is to apply Proposition 5.1 to the singular right hand side

$$f_0 := \sum_{l,p} f_{l,p}\,.$$

We find a function u_2 of the form

$$u_2 = \sum_\delta (d_\delta * \Phi)\,\psi_\delta\, S[\mu_1^\delta, \ldots, \mu_{q_\delta}^\delta; r]\,. \tag{6.5}$$

where $\mu_j^\delta \in \{\nu_{kl} \mid (k,l) \in \mathbb{N}^* \times \mathbb{N};\ s_0 + 1 < \operatorname{Re}\nu_{kl}(y) < s\ \forall y \in I\}$ and

$$d_\delta \in H^{s-\operatorname{Re}\nu_{k_\delta l_\delta}-\varepsilon}(I) \quad \forall \varepsilon > 0\,,$$

where $\nu_{k_\delta l_\delta}(y)$ is an element of the set $\{\mu_1^\delta(y), \ldots, \mu_{q_\delta}^\delta(y)\}$ which is contained in a circle around $\nu_{k_\delta l_\delta}(y)$ of radius ε_0.
The ψ_δ are analytic, and for

$$g_1 := Bu_2 - f_0$$

we have the regularity

$$g_1 \in H^\infty_{-s+1+\varepsilon_0+\varepsilon}(I \times \Gamma_\rho) \subset V_0^{s-1-\varepsilon_0-\varepsilon}(I \times \Gamma_\rho) \quad \forall \varepsilon > 0\,.$$

Here the interval I depends on y_0 and ε_0.
Finally, we set

$$w := w_1 + u_2\,.$$

Then the singular part of u has the form

$$u - w = u_0 - u_2$$

which has the properties claimed in the lemma.

From Lemma 3.5 *(iii)* we obtain the regularity of u_2 :

$$u_2 \in H^\infty_{-1-(s_0+1)}(I \times \Gamma_\rho) \subset V_0^{s_0+2}(I \times \Gamma_\rho) \subset V_0^{s_1+1}(I \times \Gamma_\rho)$$

Hence $w \in V_0^{s_1+1}(I \times \Gamma_\rho)$ as claimed.
For Bw, we obtain

$$\begin{aligned} Bw &= Bw_1 + Bu_2 \\ &= Bu - f_0 - g_0 + g_1 + f_0 \\ &= Bu - g_0 - g_1 \\ &\in V_0^{s-1-\varepsilon}(I \times \Gamma_\rho) \quad \varepsilon > \varepsilon_0\,. \end{aligned}$$

∎

Remark 6.4 In the statement of Lemma 6.2 we avoided all information not necessary for the proof of Theorem 6.1. Indeed, from the previous proof (see (6.4) and (6.5)) we can obtain more precise results about the splitting (6.2). To each p there corresponds an integer k_p such that $s_0 < k_p \operatorname{Re}\nu < s_1$, c_p is a derivative of c_{k_p} and the corresponding μ_j^p are grouped in the neighborhood of $k_p\nu + l_p$ where l_p is an integer such that $k_p \operatorname{Re}\nu + l_p \le s$. It is possible to write the singular part in (6.2) in the following form :

$$\sum_{s_0 < k \operatorname{Re}\nu < s_1} \ \sum_{0 \le l \le s - k \operatorname{Re}\nu} u_{k,l} \tag{6.6}$$

with

$$u_{k,0} = (c_k * \Phi)\, \varphi_k\, r^{k\nu} \quad \text{with} \quad c_k \in H^{s-k\operatorname{Re}\nu-\varepsilon}(I) \quad \forall \varepsilon > 0 \tag{6.7}$$

and for $l \ge 1$

$$u_{k,l} = \sum_p (d_p * \Phi)\, \varphi_p\, S[\mu_1^p(y), \ldots, \mu_{q_p}^p; r] \quad \text{with} \quad d_p \in H^{s-k\operatorname{Re}\nu-l-\varepsilon}(I) \quad \forall \varepsilon > 0 \tag{6.8}$$

- d_p is a derivative of order $\le l$ of c_k,
- φ_p is analytic,
- $\mu_j^p(y) \in \{k\nu + l\} \cup \{k'\nu + l' \mid l' \le l - 1\}$ and

$$|\mu_j^p(y) - (k\nu(y) + l)| \le \varepsilon_0 \,.$$

Proof of Theorem 6.1. Let ε_0 and y_0 be chosen. Then we can find an interval I containing y_0 and numbers $s_0,\ s_1, \ldots, s_n$ with the following properties :

$$\begin{aligned}
&0 < s_0 < s_1 < \ldots < s_n = s\,;\\
&s_0 < \min\{\operatorname{Re}\nu(y) \mid y \in I'\}\,;\\
&s_{j+1} - s_j \le 1 \quad (j = 0, \ldots, n-1)\,;\\
&\operatorname{Re}\nu_{kl}(y) \ne s_j \quad \forall j = 1, \ldots, n,\ \forall (k,l) \in \mathbb{N}^2,\ \forall y \in I\,.
\end{aligned}$$

The latter condition includes the condition $\operatorname{Re}\nu_{kl}(y) \ne s\ \forall y \in I$ which can be achieved for I small enough by slightly decreasing s and adjusting ε_0 correspondingly. We replace ε_0 by a suitable chosen smaller number, $\frac{\varepsilon_0}{n+1}$ will do.

As a first step, we write the Taylor expansion of f according to Lemma 3.7 :

$$f = f_0 + \sum_{|\alpha| < s-2} (g_\alpha * \Phi)(y, r)\, z^\alpha \,.$$

Here $f_0 \in V_0^{s-1}(I \times \Gamma_\rho)$ and $g_\alpha \in H^{s-2-|\alpha|}(I)$.

Then we use Proposition 5.1 for the “singular” right hand side

$$f_1 = \sum_\alpha (g_\alpha * \Phi)(y, r)\, z^\alpha$$

and obtain a function u_1 of the form

$$u_1 := \sum_{\alpha, lp} (g_{\alpha,lp} * \Phi)\, \psi_{\alpha,lp}\, S[\mu_1^{\alpha,lp}, \ldots, \mu_{q_{\alpha,lp}}^{\alpha,lp}; r]$$

with $Bu_1 = f_1 + f_2,\ f_2 \in H^\infty_{-s+1+\varepsilon_0}(I) \subset V_0^{s-1-\varepsilon_0}(I)$.
The coefficients $g_{\alpha,lp}$ are derivatives of order at most l of the traces g_α of f and have the regularity

$$g_{\alpha,lp} \in H^{s-2-|\alpha|-l}(I)\,.$$

Now we define

$$\tilde{u} = u - u_1 \in \overset{o}{H}{}^1(I \times \Gamma_\rho)$$

and we see that the hypotheses of the induction Lemma 6.2 are satisfied :

$$\begin{aligned} \tilde{u} = u - u_1 &\in \overset{o}{H}{}^1(I \times \Gamma_\rho) \cap V_0^{s_0+1}(I \times \Gamma_\rho) \\ B\tilde{u} = f_0 - f_2 &\in V_0^{s-1-\varepsilon_0}(I \times \Gamma_\rho)\,. \end{aligned}$$

Here we used the regularity result of Corollary 4.6. Now we can use Lemma 6.2 repeatedly to split off singularities as in (6.2) and to obtain regular parts corresponding to the regularities $s_1, \ldots, s_n = s$. Note that in each step the interval I and the radius ρ will be decreased and there will be a loss of regularity of the order of ε_0, so in order to achieve a total loss of not more than ε_0 we had to start with a much smaller number for ε_0. ∎

If we rely on the semi-global statements of Proposition 5.7 and Remark 5.8, we obtain the following semi-global version of Theorem 6.1.

Theorem 6.5 *Let s be a positive number. We assume that*

$$u \in \overset{o}{H}{}^1(I' \times \Gamma_{\rho'}) \quad \textit{and} \quad Bu \in H^{s-1}(I' \times \Gamma_{\rho'}).$$

Let $y_0 \in I'$. Let $I \subset\subset I'$ contain y_0 such that on I, y_0 is the only possible crossing point between the ν_{kl} and $s \neq \operatorname{Re}\nu_k$ on I. Then we have the expansion (6.1) *on $I \times \Gamma_\rho$ with*

$$w \in V_0^{s+1-\varepsilon}(I \times \Gamma_\rho) \quad \forall \varepsilon > \delta(I)$$

and

$$c_p \in H^{s-\mu_p-\varepsilon}(I) \quad \forall \varepsilon > \delta(I)\,.$$

Here $\delta(I)$ is a continuous function of I which tends to 0 *when the length of I tends to* 0*. If moreover there is no crossing in y_0, $\delta(I) \equiv 0$.*

7. CHANGE OF NORMAL COORDINATES

We consider a class of diffeomorphisms which leave the edge invariant and we will show that these coordinate transformations do not change the form of the decomposition of a function into regular and singular parts. Thus we can derive the statements of Theorems 1.1 and 1.3, where the opening angle $\omega(y)$ was variable, from Theorem 6.5, where we used a constant angle ω.

Such a diffeomorphism $\mathcal{T}$ is defined locally on $I \times \Gamma_\rho$ with values in a domain $\mathcal{D}_\rho$. We write :

$$\mathcal{T}(y,z) = (Y,Z) \quad \text{or in polar coordinates} \quad \mathcal{T}(y,r,\theta) = (Y,R,\Theta)\,.$$

We assume that

$$Y = y, \quad Z = \mathcal{T}_y z, \quad \mathcal{T}_y 0 = 0\,. \tag{7.1}$$

The family of transformations $\mathcal{T}_y$ satisfies :

$$\begin{cases} \forall y \in I,\ \mathcal{T}_y \text{ is a linear isomorphism of } \mathbb{R}^2 \\ \text{which depends analytically on } y \in I. \end{cases} \tag{7.2}$$

The invariance properties of the singular functions with respect to such diffeomorphisms will be used in Part II to get more explicit formulas for the Laplace operator (compare [11]).

Here are some simple properties of these transformations.

Lemma 7.1 *Let $\mathcal{T}$ be a diffeomorphism of type* (7.2). *Then for any s and δ, $\mathcal{T}$ induces an isomorphism from $V^s_\delta(I \times \Gamma_\rho)$ onto $V^s_\delta(\mathcal{D}_\rho)$, from $H^\infty_\delta(I \times \Gamma_\rho)$ onto $H^\infty_\delta(\mathcal{D}_\rho)$ and from $H^s(I \times \Gamma_\rho)$ onto $H^s(\mathcal{D}_\rho)$.*

Lemma 7.2 *Let $\mathcal{T}$ be a diffeomorphism of type* (7.2). *For analytic functions $\varphi(y,\theta)$ and $\mu_1(y), \ldots, \mu_K(y)$, and a function c let*

$$u(y,r,\theta) := c(y)\,\varphi(y,\theta)\,S[\mu_1(y),\ldots,\mu_K(y);r]\,.$$

Then

$$u \circ \mathcal{T}^{-1}(Y,R,\Theta) = \sum_{j=1}^{K} c(Y)\,\psi_j(Y,\Theta)\,S[\mu_1(Y),\ldots,\mu_j(Y);R]\,.$$

Proof. It suffices to note that

$$\Theta = \Theta(y,\theta) \quad \text{and} \quad R = \delta(y,\theta)\,r,$$

where Θ and δ are analytic functions of their arguments. Then we use the Leibniz formula (8.7). ∎

The following result means that the expression $c * \Phi$ is "almost" invariant under $\mathcal{T}$, i. e. the difference $(c*\Phi)(y,r) - (c*\Phi)(y, |\mathcal{T}_y(r,\theta)|)$ is regular.

Lemma 7.3 *Let $\mathcal{T}$ be a diffeomorphism of type* (7.2). *Let $c \in H^\beta(I')$. Then*

$$(c*\Phi)(Y,R) - (c*\Phi)\circ\mathcal{T}^{-1}(Y,Z) \in H^\infty_{-\beta-1+\varepsilon}(\mathcal{D}_\rho) \quad \forall\varepsilon > 0\,.$$

Proof. We set

$$g(Y,Z) = (c*\Phi)(Y,R) - (c*\Phi)\circ\mathcal{T}^{-1}(Y,Z)\,.$$

Let $\tilde{I}$ be an interval and $\tilde{\Gamma}$ a sector such that $\mathcal{D}_\rho \subset \tilde{I} \times \tilde{\Gamma}_{\rho'}$. It suffices to show that $\forall k \in \mathbb{N}$

$$\partial^k_Y g \in L^2(\tilde{I}, H^\infty_{-\beta-1+k+\varepsilon}(\tilde{\Gamma}_{\rho'})).$$

• If $k \le \beta$, we write

$$g = g_1 - g_2$$

with

$$g_1 = c*\Phi - c \quad \text{and} \quad g_2 = (c*\Phi)\circ\mathcal{T}^{-1} - c\,.$$

Lemma 3.1 *(ii)* yields that $\partial_Y^k g_1$ belongs to the correct space.
We have $g_2 = (c * \Phi - c) \circ \mathcal{T}^{-1}$ and $\partial_Y^k g_2$ is a sum of terms of the form

$$\Big(d_\alpha \, \partial^\alpha (c * \Phi - c)\Big) \circ \mathcal{T}^{-1} \,,$$

with $|\alpha| \leq k$ and d_α smooth functions. Lemma 3.1 *(i)* and *(ii)*, and Lemma 7.1 yield the wanted result.

• If $k > \beta$, we write

$$g = g_1 - g_2$$

with

$$g_1 = c * \Phi \quad \text{and} \quad g_2 = (c * \Phi) \circ \mathcal{T}^{-1} \,.$$

Now we use similar arguments, using Lemma 3.1 *(iv)* instead of *(ii)*. ∎

8. APPENDIX : DIVIDED DIFFERENCES

In this whole section $\mu_1, \ldots, \mu_K$ denote complex numbers, which do not need to be all distinct from each other ; w denotes a continuous complex valued function, defined on $\mathbb{C}$.

When $\mu_1, \ldots, \mu_K$ are all distinct, the divided difference of w at the K-tuple $\mu_1, \ldots, \mu_K$ is defined by the classical recursion formula :

$$w[\mu_1] = w(\mu_1) \tag{8.1}$$

and for $j = 2, \ldots, K$

$$w[\mu_1, \ldots, \mu_j] = \frac{1}{\mu_1 - \mu_j} \left(w[\mu_1, \ldots, \mu_{j-1}] - w[\mu_2, \ldots, \mu_j] \right) . \tag{8.2}$$

To the divided differences $w[\mu_1], \ldots, w[\mu_1, \ldots, \mu_K]$ is associated a polynomial $p \in \mathbb{P}_{K-1}$ by

$$p(\lambda) = w[\mu_1] + w[\mu_1, \mu_2]\,(\lambda - \mu_1) + \cdots + w[\mu_1, \ldots, \mu_K]\,(\lambda - \mu_1) \cdots (\lambda - \mu_{K-1}) \,. \tag{8.3}$$

This is the unique polynomial in $\mathbb{P}_{K-1}$ which coincides with w at the K points $\mu_1, \ldots, \mu_K$ (Newton form of the interpolation polynomial).

It is easily seen that for analytic functions w one has

$$w[\mu_1, \ldots, \mu_K] = \frac{1}{2i\pi} \int_\gamma \frac{w(\lambda)}{\prod_{j=1}^{K} (\lambda - \mu_j)} \, d\lambda \tag{8.4}$$

where γ is a simple curve surrounding all μ_j : it suffices to check that the integral formula satisfies the recursion formulas (8.1) and (8.2). Formula (8.4) allows to define $w[\mu_1, \ldots, \mu_K]$ for any points $\mu_1, \ldots, \mu_K$. It also proves that $w[\mu_1, \ldots, \mu_K]$ does not depend on the order of the points $\mu_1, \ldots, \mu_K$.

When all the μ_j are distinct, due to (8.4) we obtain

$$w[\mu_1, \ldots, \mu_K] = \sum_{j=1}^{K} \frac{w(\mu_j)}{\prod\limits_{\substack{k=1\\k\neq j}}^{K} (\mu_j - \mu_k)} \,. \tag{8.5}$$

When all the μ_j are equal to $\mu \in \mathbb{C}$, due to (8.4) we obtain

$$w[\mu_1, \ldots, \mu_K] = \frac{w^{(K-1)}(\mu)}{(K-1)!} \,. \tag{8.6}$$

For the divided differences of the product of two functions, we have a formula which is a natural extension of the Leibniz formula.

Lemma 8.1 Leibniz formula. *Let u and v be two functions, $K \in \mathbb{N}$, $\mu_1, \ldots, \mu_K$ distinct complex numbers. Then there holds*

$$(uv)[\mu_1, \ldots, \mu_K] = \sum_{j=1}^{K} u[\mu_1, \ldots, \mu_j]\, v[\mu_j, \ldots, \mu_K] \,. \tag{8.7}$$

If u and v are K-times differentiable, formula (8.7) *still holds for any $\mu_1, \ldots, \mu_K$ (not necessarily distinct).*

Remark 8.2 To each ordering of the set $\{\mu_1, \ldots, \mu_K\}$ is associated a different Leibniz formula.

Proof. We use induction over K. For $K = 1$, relation (8.7) is obvious. Let us assume it holds for K terms. We compute

$$\begin{aligned}(uv)[\mu_1, \ldots, \mu_{K+1}] &= \\ &= \frac{1}{\mu_1 - \mu_{K+1}} \Big((uv)[\mu_1, \ldots, \mu_K] - (uv)[\mu_2, \ldots, \mu_{K+1}] \Big) \,. \end{aligned}\tag{8.8}$$

Due to the induction assumption, (8.8) is equal to

$$\begin{aligned} &= \frac{1}{\mu_1 - \mu_{K+1}} \left\{ \sum_{j=1}^{K} u[\mu_1, \ldots, \mu_j] v[\mu_j, \ldots, \mu_K] - \sum_{j=2}^{K+1} u[\mu_2, \ldots, \mu_j] v[\mu_j, \ldots, \mu_{K+1}] \right\} \\ &= \frac{1}{\mu_1 - \mu_{K+1}} \sum_{j=1}^{K} \Big(u[\mu_1, \ldots, \mu_j] v[\mu_j, \ldots, \mu_K] - u[\mu_2, \ldots, \mu_{j+1}] v[\mu_{j+1}, \ldots, \mu_{K+1}] \Big) \,. \end{aligned}$$

But $u[\mu_2, \ldots, \mu_{j+1}] = u[\mu_1, \ldots, \mu_j] - (\mu_1 - \mu_{j+1})\, u[\mu_1, \ldots, \mu_{j+1}]$. Thus (8.8) is equal to

$$= \frac{1}{\mu_1 - \mu_{K+1}} \sum_{j=1}^{K} \Big\{ u[\mu_1, \ldots, \mu_j]\, (v[\mu_j, \ldots, \mu_K] - v[\mu_{j+1}, \ldots, \mu_{K+1}]) +$$

$$
\begin{aligned}
&\qquad + (\mu_1 - \mu_{j+1})\, u[\mu_1, \ldots, \mu_{j+1}] v[\mu_{j+1}, \ldots, \mu_{K+1}]\Big\} \\
= \quad & \frac{1}{\mu_1 - \mu_{K+1}} \left\{ \sum_{j=1}^{K} u[\mu_1, \ldots, \mu_j]\, (\mu_j - \mu_{K+1}) v[\mu_j, \ldots, \mu_{K+1}] + \right. \\
&\qquad \left. + \sum_{j=2}^{K+1} (\mu_1 - \mu_j)\, u[\mu_1, \ldots, \mu_j] v[\mu_j, \ldots, \mu_{K+1}] \right\} \\
= \quad & \frac{1}{\mu_1 - \mu_{K+1}} \sum_{j=1}^{K+1} u[\mu_1, \ldots, \mu_j] v[\mu_j, \ldots, \mu_{K+1}]\, (\mu_j - \mu_{K+1} + \mu_1 - \mu_j)\,.
\end{aligned}
$$

■

Remark 8.3 We give another proof of (8.7) using the integral representation (8.4). We set

$$\mathcal{J} := (uv)[\mu_1, \ldots, \mu_K]\,.$$

We have

$$\mathcal{J} = \frac{1}{2i\pi} \int_\gamma \frac{u(\lambda)\, v(\lambda)}{q(\lambda)}\, d\lambda$$

where $q(\lambda) = \prod_{j=1}^{K} (\lambda - \mu_j)$ is in $\mathbb{P}_K$.

Let $b \in \mathbb{P}_{K-1}$ be the interpolation polynomial (8.3) of u at the points $\mu_1, \ldots, \mu_K$. We have

$$b(\lambda) = \sum_{j=1}^{K} u[\mu_1, \ldots, \mu_j]\, (\lambda - \mu_1) \cdots (\lambda - \mu_{j-1})\,. \tag{8.9}$$

Since all the roots of q are roots of $u - b$, the function a such that

$$u(\lambda) = a(\lambda) q(\lambda) + b(\lambda)\,.$$

is analytic. Thus

$$
\begin{aligned}
\mathcal{J} \quad &= \quad \frac{1}{2i\pi} \int_\gamma \frac{b(\lambda)}{q(\lambda)}\, v(\lambda)\, d\lambda \\
&= \quad \frac{1}{2i\pi} \sum_{j=1}^{K} u[\mu_1, \ldots, \mu_j]\, \frac{v(\lambda)}{(\lambda - \mu_j) \cdots (\lambda - \mu_K)}\, d\lambda \\
&= \quad \sum_{j=1}^{K} u[\mu_1, \ldots, \mu_j] v[\mu_j, \ldots, \mu_K]\,.
\end{aligned}
$$

Formula (8.7) is proven.

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