On the inf-sup constant of the divergence
alias LBB constant
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Note: This presentation is mainly based on our paper [9]. It is also related with the survey (in preparation) “About the inf-sup constant of the divergence” by C. Bernardi, V. Girault and the authors.

1. The constant of interest and some elementary properties

Here we only consider bounded connected domains $\Omega$ of $\mathbb{R}^d$, $d \geq 1$. Elements of $\mathbb{R}^d$ are denoted by $x = (x_1, \ldots, x_d)$. For such a domain $\Omega$, the inf-sup constant of the divergence associated with Dirichlet boundary conditions, also called LBB constant after Ladyzhenskaya, Babuška [2] and Brezzi [5], is defined as

$\beta(\Omega) = \inf_{q \in L_0^2(\Omega)} \sup_{v \in H_0^1(\Omega)^d} \frac{\langle \text{div} v, q \rangle_{\Omega}}{|v|_{1,\Omega} \|q\|_{0,\Omega}}.$

Here $L_0^2(\Omega)$ stands for the space of square integrable scalar functions $q$ with zero mean value in $\Omega$ endowed with its natural norm $\| \cdot \|_{0,\Omega}$ and natural scalar product $\langle \cdot, \cdot \rangle$, and $H_0^1(\Omega)^d$ is the standard $H^1$ Sobolev space of vector functions $v = (v_1, \ldots, v_d)$ with square integrable gradients and zero traces on the boundary, endowed with its natural semi-norm $|v|_{1,\Omega}$ defined as $(\sum_{k=1}^d \sum_{j=1}^d \|\partial_{x_j} v_k\|_{0,\Omega}^2)^{1/2}$.

Since $\Omega$ is bounded, by virtue of the Poincaré inequality, the above semi-norm is equivalent to the usual norm in $H^1(\Omega)^d$.

We list some elementary properties of $\beta(\Omega)$:

(a) In any dimension $d \geq 1$, $\beta(\Omega) \geq 0$.
(b) In any dimension $d \geq 1$, $\beta(\Omega) \leq 1$, because of the identity

$$\forall v \in H_0^1(\Omega)^d, \quad |v|_{1,\Omega}^2 = \|\text{curl} v\|_{0,\Omega}^2 + \|\text{div} v\|_{0,\Omega}^2.$$

(c) If $d = 1$, $\Omega$ is a finite interval and $\beta(\Omega) = 1$.
(d) In any dimension $d \geq 1$, using a Piola transform it is easy to show that $\beta(\Omega)$ is invariant by translations, dilations, symmetries and rotations. In other words, $\beta(\Omega)$ depends only on the shape of $\Omega$.

2. Positiveness of the LBB constant

The constant $\beta(\Omega)$ is positive for Lipschitz domains [20], weakly Lipschitz domains (see [17, §1.2.1] for the distinction between Lipschitz and weakly Lipschitz), and John domains [1] (which include some domains with a fractal boundary). The proof is based on various constructions of a right inverse for the divergence operator, see [4, 15, 1]. In contrast, domains with an external cusp (or thin peak) satisfy $\beta(\Omega) = 0$, see [24].
3. Relation with the Schur complement of the Stokes operator

The Schur complement $S$ of the Stokes operator is defined as

$$S : L^2_q(\Omega) \to L^2_q(\Omega), \quad q \mapsto \text{div} \Delta^{-1} \nabla q.$$ 

Here $\Delta^{-1}$ is the inverse of the Dirichlet vector Laplacian $\Delta$ acting from $H^1_0(\Omega)^d$ onto $H^{-1}(\Omega)^d$. The operator $S$ is bounded self-adjoint, non-negative. But it is not compact, nor its resolvent. It is of order 0. Let $\sigma(\Omega)$ be the bottom of its spectrum. There holds

$$\sigma(\Omega) = \beta(\Omega)^2.$$ 

The associated eigenvalue problem can be phrased as a spectral Stokes problem—with $v \in H^1_0(\Omega)^d$ and $p \in L^2(\Omega)$,

\[
\begin{align*}
-\Delta v + \nabla p &= 0, \\
\text{div} v &= \sigma p.
\end{align*}
\]

Let $\Sigma(S)$ and $\mathcal{E}_{\text{ess}}(S)$ be the spectrum and the essential spectrum of $S$.

4. Relation with the Cosserat spectrum

Let us introduce the family of operators $\sigma \mapsto L_{\sigma}$

$$L : H^1_0(\Omega)^d \to H^{-1}(\Omega)^d, \quad v \mapsto \sigma \Delta v - \nabla \text{div} v$$

The Cosserat spectrum (after Cosserat brothers [7, 8]) $\Sigma(L)$ [essential spectrum $\mathcal{E}_{\text{ess}}(L)$] is the set of $\sigma \in \mathbb{R}$ such that $L_{\sigma}$ is not invertible [$L_{\sigma}$ is not Fredholm]. There holds

$$\Sigma(L) = \Sigma(S) \cup \{0\} \quad \text{and} \quad \mathcal{E}_{\text{ess}}(L) = \mathcal{E}_{\text{ess}}(S) \cup \{0\}.$$ 

The operator $L$ has non empty essential spectrum: The points 0, $\frac{1}{2}$ and 1 always belong to $\mathcal{E}_{\text{ess}}(L)$ [19]. If the domain $\Omega$ has a smooth boundary, these are the only elements of $\mathcal{E}_{\text{ess}}(L)$. If $\Omega$ is a polygonal domain of $\mathbb{R}^2$, $\mathcal{E}_{\text{ess}}(L)$ is an interval of the form $[\frac{1}{2} - b, \frac{1}{2} + b]$ with a positive $b$ depending on the corner openings of $\Omega$ [10].

A consequence is that for any domain $\Omega$

$$\beta(\Omega)^2 \leq \frac{1}{2}.$$ 

Explicit calculations show that $\beta(\Omega)^2 = \frac{1}{2}$ for the disc $\Omega \subset \mathbb{R}^2$, and more generally $\beta(\Omega)^2 = \frac{1}{3}$ if $\Omega$ is a ball in $\mathbb{R}^d$ [10].

5. Relation with the Friedrichs constant (dimension $d = 2$)

Let $\mathcal{F}(\Omega)$ denote the space of complex valued $L^2(\Omega)$ holomorphic functions and let $\mathcal{F}_0(\Omega)$ be its subspace of functions with mean value 0. After [14] the Friedrichs constant $\Gamma(\Omega) \in \mathbb{R} \cup \{\infty\}$ is the smallest constant $\Gamma$ such that for all $h + ig \in \mathcal{F}_0(\Omega)$

$$\|h\|_{L^2(\Omega)}^2 \leq \Gamma \|g\|_{L^2(\Omega)}^2.$$
Theorem 1 ([18], hypotheses fixed in [9]). Let $\Omega$ be any bounded connected domain in $\mathbb{R}^2$. The LBB constant $\beta(\Omega)$ is positive if and only if $\Gamma(\Omega)$ is finite and
\[
\Gamma(\Omega) + 1 = \frac{1}{\beta(\Omega)^2}.
\]

6. Relation with the Horgan-Payne angle (dimension $d = 2$)

Let $\Omega$ be strictly star-shaped, which means that there is an open ball $B \subset \Omega$ such that any segment with one end in $B$ and the other in $\Omega$, is contained in $\Omega$. Let $O$ be the center of $B$ and $(r, \theta)$ be polar coordinates centered at $O$. Let $\theta \mapsto r = f(\theta)$ be the polar parametrization of the boundary $\partial \Omega$, defined on $\mathbb{R}/2\pi \mathbb{Z} =: \mathbb{T}$. Since $\Omega$ is strictly star-shaped, $f$ belongs to $W^{1,\infty}(\mathbb{T})$. We assume without restriction that $\max_{\theta \in \mathbb{T}} f(\theta) = 1$. After [18], we introduce the function $P$ of $\theta \in \mathbb{T}$ and of a parameter $\alpha \in (0, 1)$ aimed at optimizing an upper bound for $\Gamma(\Omega)$
\[
(0, 1) \times \mathbb{T} \ni (\alpha, \theta) \mapsto P(\alpha, \theta) = \frac{1}{\alpha f(\theta)^2} \left(1 + \frac{f'(\theta)^2}{f(\theta)^2 - \alpha f(\theta)^2}\right).
\]

We denote by $m(\Omega)$ the original bound of [18]
\[
(5) \quad m(\Omega) = \sup_{\theta \in \mathbb{T}} \left\{ \inf_{\alpha \in (0, \frac{1}{\sqrt{\Gamma(\Omega) + 1}})} P(\alpha, \theta) \right\}
\]
and by $M(\Omega)$ our modified Horgan-Payne like bound
\[
(6) \quad M(\Omega) = \inf_{\alpha \in (0, 1)} \left\{ \sup_{\theta \in \mathbb{T}} P(\alpha, \theta) \right\}
\]
The quantity $M(\Omega)$ is always larger than $m(\Omega)$.

Let $\omega(\Omega)$ be the “Horgan-Payne angle” introduced by [23] $\omega(\Omega) = \arccos \left(\frac{m(\Omega) - 1}{m(\Omega) + 1}\right)$.

This angle has a simple geometrical interpretation as the minimal angle between radius $[OA]$ and tangent along $\partial \Omega$ at $A$, for $A$ running in $\partial \Omega$. It is easy to see that $\sin \frac{\omega(\Omega)}{2} = (m(\Omega) + 1)^{-1/2}$. Then, by virtue of Theorem 1, $\Gamma(\Omega) \leq m(\Omega)$ if and only if $\beta(\Omega) \geq \sin \frac{\omega(\Omega)}{2}$.

Theorem 2 ([9]). Any strictly star-shaped domain $\Omega$ satisfies the bounds
\[
(7) \quad \Gamma(\Omega) \leq M(\Omega) \quad \text{and} \quad \beta(\Omega) \geq \frac{1}{\sqrt{M(\Omega) + 1}}.
\]
If $\Omega$ is an ellipse, a triangle, a rectangle or a regular polygon, then $m(\Omega)$ coincides with $M(\Omega)$. Therefore
\[
(8) \quad \Gamma(\Omega) \leq m(\Omega) \quad \text{and} \quad \beta(\Omega) \geq \frac{1}{\sqrt{m(\Omega) + 1}} = \sin \frac{\omega(\Omega)}{2}.
\]

As a matter of fact, there exist strictly star-shaped domains such that $m < M$. And even more:
Theorem 3 ([9]). There exists a strictly star-shaped domain $\Omega \subset \mathbb{R}^2$ such that

\begin{equation}
\Gamma(\Omega) > m(\Omega) \quad \text{i.e.} \quad \beta(\Omega) < \sin \frac{\omega(\Omega)}{2}.
\end{equation}

Counterexamples are provided by symmetric domains with a narrow pass for which we have proved an upper bound for $\beta(\Omega)$ (this can be related to the fact that elongated domains have a small $\beta$ [6, 21, 11, 12]). This proves that the original result of [18] stating that (8) is valid for any strictly star-shaped domains is erroneous. Nevertheless our positive result of Theorem 2 is still in the spirit of [18] and allows to prove a general simple bound from below for $\beta(\Omega)$ that realizes an improvement of [13] for strictly star-shaped two-dimensional domains.

Though related, discrete inf-sup conditions are a rather different story. Now the choice of distinct discrete spaces for scalar and vector unknowns comes into play, see [16, 3, 22] among many others...

References

[13] R.G. Durán, An elementary proof of the continuity from $L^2_0(\Omega)$ to $H^1_0(\Omega)$ of Bogovski’s right inverse of the divergence, Revista de la Unión Matemática Argentina 53(2) (2012), 59–78.


