On the inf-sup constant of the divergence
alias LBB constant

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Note: This presentation is mainly based on our paper [9]. It is also related with the survey (in preparation) *"About the inf-sup constant of the divergence"* by C. BERNARDI, V. GIRAULT and the authors.

1. THE CONSTANT OF INTEREST AND SOME ELEMENTARY PROPERTIES

Here we only consider *bounded connected* domains Ω of $\mathbb{R}^d$, $d \geq 1$. Elements of $\mathbb{R}^d$ are denoted by $\boldsymbol{x} = (x_1, \ldots, x_d)$. For such a domain Ω, the inf-sup constant of the divergence associated with Dirichlet boundary conditions, also called LBB constant after LADYZHENSKAYA, BABUŠKA [2] and BREZZI [5], is defined as

$$\beta(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{\big\langle \operatorname{div} \boldsymbol{v}, q \big\rangle_\Omega}{|\boldsymbol{v}|_{1,\Omega} \, \|q\|_{0,\Omega}} \,. \tag{1}$$

Here $L^2_\circ(\Omega)$ stands for the space of square integrable scalar functions q with zero mean value in Ω endowed with its natural norm $\|\cdot\|_{0,\Omega}$ and natural scalar product $\langle\cdot,\cdot\rangle$, and $H^1_0(\Omega)^d$ is the standard H^1 Sobolev space of vector functions $\boldsymbol{v} = (v_1, \ldots, v_d)$ with square integrable gradients and zero traces on the boundary, endowed with its natural semi-norm $|\boldsymbol{v}|_{1,\Omega}$ defined as $(\sum_{k=1}^d \sum_{j=1}^d \|\partial_{x_j} v_k\|^2_{0,\Omega})^{1/2}$. Since Ω is bounded, by virtue of the Poincaré inequality, the above semi-norm is equivalent to the usual norm in $H^1(\Omega)^d$.

We list some elementary properties of $\beta(\Omega)$:

(a) In any dimension $d \geq 1$, $\beta(\Omega) \geq 0$,
(b) In any dimension $d \geq 1$, $\beta(\Omega) \leq 1$, because of the identity

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^d, \quad |\boldsymbol{v}|^2_{1,\Omega} = \|\operatorname{curl} \boldsymbol{v}\|^2_{0,\Omega} + \|\operatorname{div} \boldsymbol{v}\|^2_{0,\Omega}$$

(c) If $d = 1$, Ω is a finite interval and $\beta(\Omega) = 1$,
(d) In any dimension $d \geq 1$, using a Piola transform it is easy to show that $\beta(\Omega)$ is invariant by translations, dilations, symmetries and rotations. In other words, $\beta(\Omega)$ depends only on the *shape* of Ω.

2. POSITIVENESS OF THE LBB CONSTANT

The constant $\beta(\Omega)$ is positive for Lipschitz domains [20], weakly Lipschitz domains (see [17, §1.2.1] for the distinction between Lipschitz and weakly Lipschitz), and John domains [1] (which include some domains with a fractal boundary). The proof is based on various constructions of a right inverse for the divergence operator, see [4, 15, 1]. In contrast, domains with an external cusp (or thin peak) satisfy $\beta(\Omega) = 0$, see [24].

3. Relation with the Schur complement of the Stokes operator

The Schur complement $\mathcal{S}$ of the Stokes operator is defined as

$$\begin{array}{rccc} \mathcal{S}: & L^2_\circ(\Omega) & \longrightarrow & L^2_\circ(\Omega) \\ & q & \longmapsto & \mathsf{div}\,\Delta^{-1}\,\nabla q\,. \end{array}$$

Here Δ^{-1} is the inverse of the Dirichlet vector Laplacian Δ acting from $H^1_0(\Omega)^d$ onto $H^{-1}(\Omega)^d$. The operator $\mathcal{S}$ is bounded self-adjoint, non-negative. But it is not compact, nor its resolvent. It is of order 0. Let $\sigma(\Omega)$ be the bottom of its spectrum. There holds

$$\sigma(\Omega) = \beta(\Omega)^2. \tag{2}$$

The associated eigenvalue problem can be phrased as a spectral Stokes problem—with $\boldsymbol{v} \in H^1_0(\Omega)^d$ and $p \in L^2_\circ(\Omega)$,

$$\left\{ \begin{array}{rcl} -\Delta\boldsymbol{v} + \nabla p &=& 0\,, \\ \mathsf{div}\,\boldsymbol{v} &=& \sigma p\,. \end{array} \right. \tag{3}$$

Let $\mathfrak{S}(\mathcal{S})$ and $\mathfrak{S}_{\mathsf{ess}}(\mathcal{S})$ be the spectrum and the essential spectrum of $\mathcal{S}$.

4. Relation with the Cosserat spectrum

Let us introduce the family of operators $\sigma \mapsto \mathcal{L}_\sigma$

$$\begin{array}{rccc} \mathcal{L}: & H^1_0(\Omega)^d & \longrightarrow & H^{-1}(\Omega)^d \\ & \boldsymbol{v} & \longmapsto & \sigma\Delta\boldsymbol{v} - \nabla\,\mathsf{div}\,\boldsymbol{v} \end{array}$$

The Cosserat spectrum (after Cosserat brothers [7, 8]) $\mathfrak{S}(\mathcal{L})$ [essential spectrum $\mathfrak{S}_{\mathsf{ess}}(\mathcal{L})$] is the set of $\sigma \in \mathbb{R}$ such that $\mathcal{L}_\sigma$ is not invertible [$\mathcal{L}_\sigma$ is not Fredholm]. There holds

$$\mathfrak{S}(\mathcal{L}) = \mathfrak{S}(\mathcal{S}) \cup \{0\} \quad \text{and} \quad \mathfrak{S}_{\mathsf{ess}}(\mathcal{L}) = \mathfrak{S}_{\mathsf{ess}}(\mathcal{S}) \cup \{0\}. \tag{4}$$

The operator $\mathcal{L}$ has non empty essential spectrum: The points 0, $\frac{1}{2}$ and 1 always belong to $\mathfrak{S}_{\mathsf{ess}}(\mathcal{L})$ [19]. If the domain Ω has a smooth boundary, these are the only elements of $\mathfrak{S}_{\mathsf{ess}}(\mathcal{L})$. If Ω is a polygonal domain of $\mathbb{R}^2$, $\mathfrak{S}_{\mathsf{ess}}(\mathcal{L})$ is an interval of the form $[\frac{1}{2} - b, \frac{1}{2} + b]$ with a positive b depending on the corner openings of Ω [10].

A consequence is that for any domain Ω

$$\beta(\Omega)^2 \leq \frac{1}{2}\ .$$

Explicit calculations show that $\beta(\Omega)^2 = \frac{1}{2}$ for the disc $\Omega \subset \mathbb{R}^2$, and more generally $\beta(\Omega)^2 = \frac{1}{d}$ if Ω is a ball in $\mathbb{R}^d$ [10].

5. Relation with the Friedrichs constant (dimension $d = 2$)

Let $\mathfrak{F}(\Omega)$ denote the space of complex valued $L^2(\Omega)$ holomorphic functions and let $\mathfrak{F}_\circ(\Omega)$ be its subspace of functions with mean value 0. After [14] the *Friedrichs constant* $\Gamma(\Omega) \in \mathbb{R}\cup\{\infty\}$ is the smallest constant Γ such that for all $h+ig \in \mathfrak{F}_\circ(\Omega)$

$$\|h\|^2_{L^2(\Omega)} \leq \Gamma \|g\|^2_{L^2(\Omega)}\ .$$

Theorem 1 ([18], hypotheses fixed in [9])**.** *Let Ω be any bounded connected domain in $\mathbb{R}^2$. The LBB constant $\beta(\Omega)$ is positive if and only if $\Gamma(\Omega)$ is finite and*

$$\Gamma(\Omega) + 1 = \frac{1}{\beta(\Omega)^2}\ .$$

6. Relation with the Horgan-Payne angle (dimension $d = 2$)

Let Ω be *strictly star-shaped*, which means that there is an open ball $B \subset \Omega$ such that any segment with one end in B and the other in Ω, is contained in Ω. Let O be the center of B and (r, θ) be polar coordinates centered at O. Let $\theta \mapsto r = f(\theta)$ be the polar parametrization of the boundary $\partial\Omega$, defined on $\mathbb{R}/2\pi\mathbb{Z} =: \mathbb{T}$. Since Ω is strictly star-shaped, f belongs to $W^{1,\infty}(\mathbb{T})$. We assume without restriction that $\max_{\theta\in\mathbb{T}} f(\theta) = 1$. After [18], we introduce the function P of $\theta \in \mathbb{T}$ and of a parameter $\alpha \in (0, 1)$ aimed at optimizing an upper bound for $\Gamma(\Omega)$

$$(0,1)\times\mathbb{T} \ni (\alpha,\theta) \longmapsto P(\alpha,\theta) = \frac{1}{\alpha f(\theta)^2}\left(1 + \frac{f'(\theta)^2}{f(\theta)^2 - \alpha f(\theta)^4}\right).$$

We denote by $m(\Omega)$ the original bound of [18]

$$m(\Omega) = \sup_{\theta\in\mathbb{T}} \Big\{ \inf_{\alpha\in\left(0,\frac{1}{f(\theta)^2}\right)} P(\alpha,\theta) \Big\} \tag{5}$$

and by $M(\Omega)$ our modified Horgan-Payne like bound

$$M(\Omega) = \inf_{\alpha\in(0,1)} \Big\{ \sup_{\theta\in\mathbb{T}} P(\alpha,\theta) \Big\} \tag{6}$$

The quantity $M(\Omega)$ is always larger than $m(\Omega)$.

Let $\omega(\Omega)$ be the “Horgan-Payne angle” introduced by [23]

$$\omega(\Omega) = \arccos\left(\frac{m(\Omega) - 1}{m(\Omega) + 1}\right).$$

This angle has a simple geometrical interpretation as the minimal angle between radius $[OA]$ and tangent along $\partial\Omega$ at A, for A running in $\partial\Omega$. It is easy to see that $\sin\frac{\omega(\Omega)}{2} = (m(\Omega) + 1)^{-1/2}$. Then, by virtue of Theorem 1, $\Gamma(\Omega) \le m(\Omega)$ if and only if $\beta(\Omega) \ge \sin\frac{\omega(\Omega)}{2}$.

Theorem 2 ([9])**.** *Any strictly star-shaped domain Ω satisfies the bounds*

$$\Gamma(\Omega) \le M(\Omega) \quad \textit{and} \quad \beta(\Omega) \ge \frac{1}{\sqrt{M(\Omega) + 1}}\ . \tag{7}$$

If Ω is an ellipse, a triangle, a rectangle or a regular polygon, then $m(\Omega)$ coincides with $M(\Omega)$. Therefore

$$\Gamma(\Omega) \le m(\Omega) \quad \textit{and} \quad \beta(\Omega) \ge \frac{1}{\sqrt{m(\Omega) + 1}} = \sin\frac{\omega(\Omega)}{2}\ . \tag{8}$$

As a matter of fact, there exist strictly star-shaped domains such that $m < M$. And even more:

Theorem 3 ([9]). *There exists a strictly star-shaped domain $\Omega \subset \mathbb{R}^2$ such that*

$$\Gamma(\Omega) > m(\Omega) \quad \textit{i.e.} \quad \beta(\Omega) < \sin\frac{\omega(\Omega)}{2} \ . \tag{9}$$

Counterexamples are provided by symmetric domains with a narrow pass for which we have proved an upper bound for $\beta(\Omega)$ (this can be related to the fact that elongated domains have a small β [6, 21, 11, 12]). This proves that the original result of [18] stating that (8) is valid for any strictly star-shaped domains is erroneous. Nevertheless our positive result of Theorem 2 is still in the spirit of [18] and allows to prove a general simple bound from below for $\beta(\Omega)$ that realizes an improvement of [13] for strictly star-shaped two-dimensional domains.

Though related, discrete inf-sup conditions are a rather different story. Now the choice of distinct discrete spaces for scalar and vector unknowns comes into play, see [16, 3, 22] among many others...

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