Numerical Approximation of a Singularly Perturbed Contact Problem

Martin COSTABEL, Monique DAUGE and Manil SURI *

Abstract. *As a simplified model for contact problems, we study a mixed Neumann-Robin boundary value problem for the Laplace operator in a smooth domain in* $\mathbb{R}^2$ *. The Robin condition contains a small parameter* ε *inducing boundary layers of corner type at the transition points as proved in* [4]*. We present an integral equation for the numerical solution of this problem together with estimates of the error. We investigate the improvement obtained if we add to the discrete space some special functions.*

1 INTRODUCTION

The problem of a rigid punch making contact with an elastic body Ω is a classical one, which was considered for example in [12]. It is known that the displacement induced in Ω exhibits singular behavior at the limiting point of contact ($\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ in Figure 1) of the form $r^{\frac{1}{2}}$, where r is the distance from this point. The rigid punch problem can be considered to be a limiting case of the problem of *elastic* contact, where the impinging body is not quite rigid, but is modeled by a system of Winkler springs with elastic constant $k = 1/\varepsilon$. Here, $\varepsilon \in [0, a], a > 0$, with $\varepsilon = 0$ corresponding to the rigid punch. Such contact problems have a wide range of applications, see e.g. [16], [17].

For the case that ε is large (and fixed), the solution behaves like $r \log r$ instead of $r^{\frac{1}{2}}$ near the points of contact — this follows e.g. from [11]. Such information about the strength of the singularity is essential in the design of numerical methods, since the accuracy and convergence of the approximations is governed by the

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strength of the singularity. For instance, a common strategy is to enrich the finite element subspace with an $r^{\frac{1}{2}}$ function for the case $\varepsilon = 0$ and with an $r \log r$ function for the case of ε large — this can dramatically improve the convergence of low-order (h version) elements (see e.g. [15]).

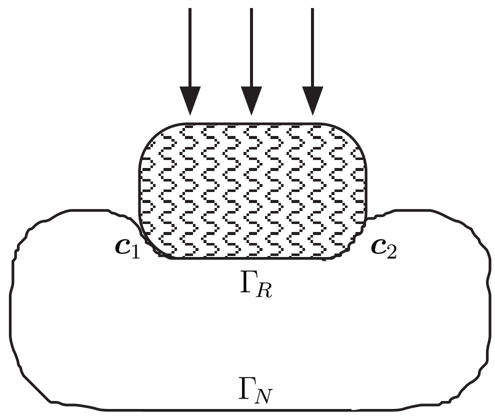

Figure 1. Contact

Suppose now that a *robust* numerical method is desired, i.e. one which approximates well the limiting case of the rigid punch and of ε large, as well as all cases in between. Then it is desirable to understand how the singularity behaves as $\varepsilon \to 0$, so that one can design the numerical method appropriately (perhaps by adding the correct singularity functions). One might be tempted to assume that the limiting cases and their numerical treatment are the only important factors to consider, but as we show in this paper, both the singular behavior and the design of a robust method are rather delicate questions when the whole range $\varepsilon \in [0, a]$ is of interest.

If the contact occurs along portion Γ_R of the boundary $\Gamma = \Gamma_R \cup \Gamma_N$ of Ω (Figure 1), the equations of the displacement $\boldsymbol{u}_\varepsilon$ inside Ω can be written as follows

$$\left\{ \begin{array}{rcll} L\boldsymbol{u}_\varepsilon & = & f & \text{in } \Omega \\ T_\varepsilon & = & 0 & \text{on } \Gamma_N \\ \varepsilon\, T_\varepsilon \cdot \boldsymbol{n} + \boldsymbol{u}_\varepsilon \cdot \boldsymbol{n} & = & g & \text{on } \Gamma_R \\ T_\varepsilon \times \boldsymbol{n} & = & 0 & \text{on } \Gamma_R \end{array} \right.$$

with L the elasticity operator, $\boldsymbol{n}$ the exterior unit normal and T_ε the associated traction.

We see that although the operator L inside Ω is uniformly elliptic, the boundary conditions (on Γ_R) cover the operator L for each $\varepsilon \geq 0$, but not uniformly with respect to ε . In fact, at the limit $\varepsilon = 0$, the boundary operator has a *lower* order, which is characteristic of a *singular perturbation.* As we show in Sections 3 and 4, this induces a *corner layer* behavior in the structure of the singularity, which behaves like $r \log r$ in an $\mathscr{O}(\varepsilon)$ neighborhood of $\boldsymbol{c}_i$ and then changes to $r^{\frac{1}{2}}$ outside this region. The numerical method must now be designed to resolve this special singular behavior.

Our goal in this paper is to (1) characterize the singular components of the solution for $\varepsilon \in [0, a]$ and (2) obtain numerical methods that approximate these singular components robustly for $\varepsilon \in [0, a]$. In order to simplify the exposition, we treat the case of the Laplace operator Δ , since the singular behavior as well as numerical approximation are similar to those for the elasticity system L . The domain Ω is supposed to be smooth and its boundary Γ is split into the two parts Γ_N and Γ_R . On Γ_N the Neumann conditions are prescribed and on Γ_R , a Robin-type condition $\varepsilon\partial_n + I$ is prescribed, with a small ε :

$$
(\mathcal{P}_\varepsilon) \qquad \left\{ \begin{array}{rcll} \Delta u_\varepsilon & = & f & \text{in } \Omega \\ \partial_n u_\varepsilon & = & 0 & \text{on } \Gamma_N \\ \varepsilon\, \partial_n u_\varepsilon + u_\varepsilon & = & g & \text{on } \Gamma_R, \end{array} \right.
$$

the variational form of which reads:

$$
\forall v \in H^1(\Omega) : \quad \int_\Omega \nabla u_\varepsilon \cdot \nabla v \; dx \; + \; \frac{1}{\varepsilon} \int_{\Gamma_R} u_\varepsilon \; v \; ds = \int_\Omega f \; v \; dx \; + \; \frac{1}{\varepsilon} \int_{\Gamma_R} g \; v \; ds \, . \tag{1.1}
$$

At the limit as $\varepsilon \to 0$, problem $(\mathcal{P}_\varepsilon)$ tends to the following mixed Dirichlet-Neumann problem

$$
(\mathcal{P}_0) \qquad \left\{ \begin{array}{rcll} \Delta u_0 & = & f & \text{in } \Omega \\ \partial_n u_0 & = & 0 & \text{on } \Gamma_N \\ u_0 & = & g & \text{on } \Gamma_R \, . \end{array} \right.
$$

Again, the singular behavior at $\boldsymbol{c}_i$ for problem $(\mathcal{P}_0)$ is of the form $r^{\frac{1}{2}}$, and for problem $(\mathcal{P}_\varepsilon)$, of the form $r \log r$. In Section 2, we present two decompositions of the solution into smooth and singular components, the first for $\varepsilon = 0$ and the second for $\varepsilon > 0$, from which the singular behavior for $(\mathcal{P}_0)$ and $(\mathcal{P}_\varepsilon)$ can be read off.

The decomposition in Section 2 for the case $\varepsilon > 0$ is unrelated to that for $\varepsilon = 0$. Therefore in the next two sections we obtain decompositions into smooth and singular components that are *uniform* in ε . In Section 3, we present an exposition of results proved in [4] about the asymptotics of u_ε as $\varepsilon \to 0$. In particular we state optimal estimates for $u_\varepsilon - u_0$. In Section 4, we return to the question of singular solutions: the asymptotics described in Section 3 allow a better understanding of the transformation of singular functions as $\varepsilon \to 0$. The reader is pointed in particular to Proposition 4.10 which summarizes our main decomposition results.

Next, in Section 5, we present a boundary integral equation scheme for the approximation of u_ε and give in Section 6 theoretical and numerical results for the estimation of the error when the discrete space is based on piecewise linears. Finally, in Section 7, we investigate the approximation properties of a special singular function method where we add a finite number of singular functions to the discrete $\mathbb{P}_1$ space. We show how doing this allows singular components to be approximated at an *exponential* rate.

Let us mention that the numerical approximation of the family of boundary value problems $(\mathcal{P}_\varepsilon)$ has a long history (see [2] and also [4]). The motivation in [2] for studying $(\mathcal{P}_\varepsilon)$ was a means of regularizing problem $(\mathcal{P}_0)$. Our results show that the regularization is only local, in an $\mathscr{O}(\varepsilon)$ neighborhood of $\boldsymbol{c}_i$, and so may not be very helpful numerically. Another practical situation where $(\mathcal{P}_\varepsilon)$ occurs is when Dirichlet boundary conditions are imposed naturally via penalization (as is done in some commercial programs), with ε being the penalization parameter. The analysis in this paper can help in determining the correct size of this parameter to ensure optimal convergence of the underlying numerical method.

2 SINGULARITIES

As is well known from standard elliptic theory, the regularity of the solution u_ε of $(\mathcal{P}_\varepsilon)$ increases with the regularity of the data, except at the transition points $\boldsymbol{c}_i$ where the boundary conditions change. In the neighborhood of such points, it is also well known that the solution admits an expansion into a regular part $(u_\varepsilon)_{\mathrm{reg}}$ and a singular part [11, 7, 6]: the singular part is a linear combination of *model singular solutions*, which are ε-dependent.

We are going to describe them in the neighborhood of a fixed transition point $\boldsymbol{c}$, where we introduce polar coordinates (r,θ) centered in $\boldsymbol{c}$ and such that the intervals $\{r \in (0,r_0),\ \theta = 0\}$ and $\{r \in (0,r_0),\ \theta = \pi\}$ are contained in Γ_R and Γ_N respectively for r_0 small enough.

The situation is easier for problem $(\mathcal{P}_0)$, where all involved operators are homogeneous. Its singular solutions are ($j \geq 1$ integer):

$$S^j(r,\theta) = r^{j-1/2}\sin(j-\frac{1}{2})\theta\,. \tag{2.1}$$

For $f \in H^{s-2}(\Omega)$ [1] and $g \in H^{s-1/2}(\Gamma_R)$, the solution u_0 of $(\mathcal{P}_0)$ splits into:

$$u_0 = (u_0)_{\text{reg}} + \sum_{1 \le j < s-1} c^j[u_0]\,\chi(r)\,S^j(r,\theta). \tag{2.2}$$

Here, $c^j[u_0]$ are coefficients depending on u_0, χ is a smooth cut-off function equal to 1 in a neighborhood of $\boldsymbol{c}$, and

$$(u_0)_{\text{reg}} \in H^s(\Omega)\,.$$

From (2.1), it is seen that the worst singularity ($j=1$) behaves like $r^{\frac{1}{2}}$.

The singular solutions of $(\mathcal{P}_\varepsilon)$ are not homogeneous like the S^j, but they admit expansions in increasing powers of r. Their principal parts are the following Σ^j ($j \ge 1$ integer) :

$$\Sigma^j(r,\theta) = \frac{1}{\pi}\,r^j\Big((\theta-\pi)\sin(j\theta) - \log r\cos(j\theta)\Big)\,. \tag{2.3}$$

For $f \in H^{s-2}(\Omega)$ and $g \in H^{s-3/2}(\Gamma_R)$, the solution u_ε of $(\mathcal{P}_\varepsilon)$ splits into:

$$u_\varepsilon = (u_\varepsilon)_{\text{reg}} + \sum_{1 \le j < s-1} \gamma^j[u_\varepsilon]\,\chi(r)\Big(\Sigma^j + \sum_{1 \le \ell < s-1-j} \varepsilon^{-\ell}\,\Sigma^{j,\ell}\Big) \tag{2.4}$$

with

$$(u_\varepsilon)_{\text{reg}} \in H^s(\Omega) \quad \text{and} \quad \Sigma^{j,\ell} = \mathscr{O}\left(r^{j+\ell}\log^{\ell+1} r\right)\,.$$

We see from (2.3) that the worst singularity now behaves like $r\log r$. However, as $\varepsilon \to 0$, the coefficients in the expansion (2.4) blow up, so that the singularity does not behave uniformly as $r\log r$. Also, there is no obvious link between the expansions (2.2) and (2.4). In the next two sections, we obtain decompositions uniform in ε.

3 EXPANSION WITH RESPECT TO ε

The family of problems $(\mathcal{P}_\varepsilon)$ belongs to a class of singular perturbation problems similar to those studied by Nazarov [13, 14] and Il'in [9]. Problems $(\mathcal{P}_\varepsilon)$ are characterized by the appearance of "corner layers" of size ε at the transition points $\overline{\Gamma_R} \cap \overline{\Gamma_N}$. In contrast to most cases studied in [9, 13, 14, 10], here these corner layers are not exponentially decaying but of long range type. We give a

[1] We use standard Sobolev space notation. For s a non-negative integer, $H^s(S)$ consists of all functions with s generalized derivatives on the set S, with $L^2(S) = H^0(S)$ being the space of square-integrable functions. For other values of s, these spaces are defined by standard interpolation and duality procedures.

description of the construction of the first terms in the expansion with respect to ε. For more details, see [4].

The simplest Ansatz is a power series in ε:

$$u_\varepsilon \simeq u^0 + \varepsilon\, u^1 + \cdots \varepsilon^k\, u^k + \cdots \tag{3.1}$$

Putting this Ansatz into $(\mathcal{P}_\varepsilon)$ and identifying the powers of ε, we first obtain problem $(\mathcal{P}_0)$ for $u^0 = u_0$. To find the next terms u^k, we have to solve for any $k \geq 1$

$$(\mathcal{R}^k) \qquad \left\{ \begin{array}{rcll} \Delta u^k &=& 0 & \text{in } \Omega \\ \partial_n u^k &=& 0 & \text{on } \Gamma_N \\ u^k &=& -\partial_n u^{k-1} & \text{on } \Gamma_R\,. \end{array} \right.$$

But, since according to (2.2) $u^0 = (u^0)_{\rm reg} + c^1[u_0]\, S^1$, there holds

$$\begin{array}{rcl} \partial_n u^0 &=& c^1[u_0]\,\partial_n S^1 \\ &=& -\frac{1}{2}\, c^1[u_0]\, r^{-1/2} \quad \text{mod. } H^{1/2}(\Gamma_R), \end{array}$$

and problem $(\mathcal{R}^1)$ has no solution in H^1. Due to the existence of a non-trivial kernel in $H^{1/2-\delta}(\Omega)$, $\delta > 0$, problem $(\mathcal{R}^1)$ has even infinitely many solutions in $H^{1/2-\delta}$.

This failure of the power series Ansatz leads to the idea of introducing the following *"model problem"* (3.2) on the homogeneous domain (the half plane) $\Pi = \mathbb{R} \times \mathbb{R}^+$ whose limiting boundary is tangent to Ω at $\boldsymbol{c}$ (see Figure 2), so that its solution can appear as a "corner layer":

$$\left\{ \begin{array}{rcll} \Delta w^1_\varepsilon &=& 0 & \text{in } \Pi \\ \partial_n w^1_\varepsilon &=& 0 & \text{on } \Gamma^+_N \\ \varepsilon\,\partial_n w^1_\varepsilon + w^1_\varepsilon &=& -\partial_n S^1 & \text{on } \Gamma^+_R\,. \end{array} \right. \tag{3.2}$$

Indeed, we can see that if we have a profile W^1, solution of

$$\left\{ \begin{array}{rcll} \Delta W^1 &=& 0 & \text{in } \Pi \\ \partial_n W^1 &=& 0 & \text{on } \Gamma^+_N \\ \partial_n W^1 + W^1 &=& -\partial_n S^1 & \text{on } \Gamma^+_R\,, \end{array} \right. \tag{3.3}$$

then the homogenized function:

$$w^1_\varepsilon(r,\theta) = \varepsilon^{-1/2} W^1(\frac{r}{\varepsilon},\theta) \tag{3.4}$$

solves problem (3.2) (see §6, Figure 4, where an approximate representation of the trace of w^1_ε on $\mathbb{R}^+$ in a neighborhood of 0 for several values of ε is shown).

The following result is proved in [4, §5]:

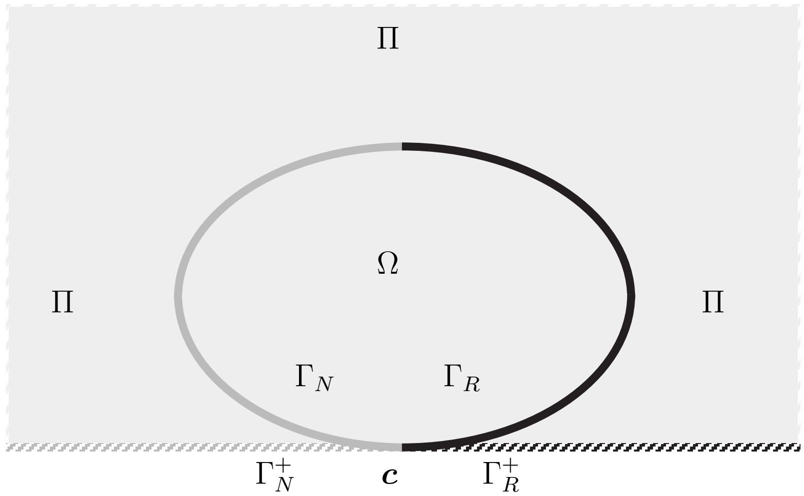

Figure 2. Homogeneous tangent domain

Lemma 3.1 *There exists a unique solution* K^1 *in* $H^1_{\mathrm{loc}}(\Pi)$ *of the homogeneous problem*

$$\begin{cases} \Delta K^1 &= 0 \qquad \textit{in } \Pi \\ \partial_n K^1 &= 0 \qquad \textit{on } \Gamma_N^+ \\ \partial_n K^1 + K^1 &= 0 \qquad \textit{on } \Gamma_R^+ \,, \end{cases}$$

such that

$$K^1 = S^1 + \mathscr{O}\left(r^{-1/2}\log r\right) \quad \textit{as} \quad r \to +\infty\,.$$

More precisely, as $r \to +\infty$ *we have*

$$K^1 = S^1 + \frac{1}{2\pi} r^{-1/2}\left((\pi-\theta)\cos\frac{\theta}{2} + \log r \sin\frac{\theta}{2}\right) + c\, r^{-1/2}\sin\frac{\theta}{2} + \mathscr{o}\left(r^{-1/2}\right)\,.$$

We deduce from this lemma that $W^1 = K^1 - S^1$ solves (3.3). Therefore, w^1_ε given by (3.4) solves (3.2) and a better beginning for our Ansatz is $u^0 + \varepsilon\, c^1[u_0]\,\chi(r)\,w^1_\varepsilon$, i.e.

$$u^0 + \varepsilon^{1/2}\, c^1[u_0]\,\chi(r)\, W^1(\frac{r}{\varepsilon})\,.$$

And the next term can now be searched as a solution of the following problem $(\mathcal{R}^{1'})$ instead $(\mathcal{R}^1)$:

$$(\mathcal{R}^{1'}) \qquad \begin{cases} \Delta u^1 &= f^{1,1}\log\varepsilon + f^{1,0} \qquad \text{in } \Omega \\ \partial_n u^1 &= 0 \qquad \text{on } \Gamma_N \\ u^1 &= -\partial_n (u^0)_{\mathrm{reg}} \qquad \text{on } \Gamma_R\,, \end{cases}$$

where the terms $f^{1,1}$ and $f^{1,0}$ come from the cut-off error. The above boundary value problem has a unique solution in $H^1(\Omega)$, of the form $u^{1,1}\log\varepsilon + u^{1,0}$. Whence the improved expansion

$$u_\varepsilon \;=\; u^0 \;+\; \varepsilon^{1/2}\, c^1[u_0]\,\chi\, W^1(\frac{r}{\varepsilon}) \;+\; \varepsilon(u^{1,1}\log\varepsilon + u^{1,0}) \;+\; r^1_\varepsilon\,. \tag{3.5}$$

In [4, §6], the following estimate for the remainder r^1_ε is proven:

$$\|r^1_\varepsilon\|_{H^1(\Omega)} = \mathscr{O}\left(\varepsilon^{-\delta+3/2}\right), \quad \forall\delta > 0.$$

Whence we can deduce the (optimal) error estimates

Proposition 3.2 *There hold the following estimates between the solutions u_ε and u_0 of problems $(\mathcal{P}_\varepsilon)$ and $(\mathcal{P}_0)$:*

$$\begin{aligned}
\|u_\varepsilon - u_0\|_{H^1(\Omega)} \;&=\; \mathscr{O}\left(\varepsilon^{1/2}\right)\\
\|u_\varepsilon - u_0\|_{L^2(\Omega)} \;&=\; \mathscr{O}\left(\varepsilon\log\varepsilon\right)\\
\|u_\varepsilon - u_0\|_{L^2(\Gamma_R)} \;&=\; \mathscr{O}\left(\varepsilon\,|\log\varepsilon|^{1/2}\right)\,.
\end{aligned}$$

By induction arguments [4, §7], we can obtain a complete expansion of u_ε, i.e. expansions with remainders of arbitrarily high order:

$$u_\varepsilon \;=\; u^0 \;+\; \sum_{j=1}^{N} \varepsilon^j\, u^j[\log\varepsilon] \;+\; \sum_{j=1}^{N} \varepsilon^{j-1/2}\, w^j(\varepsilon) + r^N_\varepsilon\,. \tag{3.6}$$

Here, $u^j[\log\varepsilon]$ denotes a polynomial of degree j in $\log\varepsilon$, and w^j has the form:

$$w^j(\varepsilon) = \sum_{\ell=1}^{j} c^j[u^0,\cdots,u^{j-\ell}]\,\chi(r)\,W^\ell(\frac{r}{\varepsilon}) \tag{3.7}$$

where

$$W^\ell = \mathscr{O}\left(r^{-1/2}\log^\ell r\right) \quad \text{as} \quad r\to+\infty, \tag{3.8}$$

and the remainder satisfies the estimate:

$$\|r^N_\varepsilon\|_{H^1(\Omega)} = \mathscr{O}\left(\varepsilon^{-\delta+N+1/2}\right), \quad \forall\delta > 0. \tag{3.9}$$

4 INTERACTION BETWEEN SINGULARITIES AND ε-EXPANSION

4.a Asymptotics

Before stating general results, let us explain the situation of the first singularity in the expansions (2.2) and (2.4) of the solution. Recall that for u_ε, $\varepsilon > 0$, this first term is $\gamma^1[u_\varepsilon]\,\Sigma^1$, whereas for u_0, this is $c^1[u_0]\,S^1$.

Using again that $u^0 = (u^0)_{\rm reg} + c^1[u_0]\,S^1$ and recalling that $W^1 = K^1 - S^1$, where K^1 is the "canonical" object introduced in Lemma 3.1, we can write the beginning of the expansion of u_ε in two different ways:

$$u^0 + \varepsilon^{1/2}\,c^1[u_0]\,\chi(r)\,W^1(\frac{r}{\varepsilon}) \;=\; (u^0)_{\rm reg} + \varepsilon^{1/2}\,c^1[u_0]\,\chi(r)\,K^1(\frac{r}{\varepsilon})\,. \tag{4.1}$$

Thus the principal part as $\varepsilon \to 0$ of the first singularity of u_ε belongs to $K^1(\frac{r}{\varepsilon})$.

An expansion (2.4) for K^1 yields $K^1 = (K^1)_{\rm reg} + \gamma^1[K^1]\,\Sigma^1$, therefore

$$\begin{aligned} K^1(\frac{r}{\varepsilon}) &= (K^1)_{\rm reg}(\frac{r}{\varepsilon}) + \gamma^1[K^1]\,\Sigma^1(\frac{r}{\varepsilon}) \\ &= (K^1(\frac{r}{\varepsilon}))_{\rm reg} + \varepsilon^{-1}\,\gamma^1[K^1]\,\Sigma^1(r)\,. \end{aligned}$$

Whence the principal part as $\varepsilon \to 0$ of $\gamma^1[u_\varepsilon]$ is $\varepsilon^{-1/2}\,c^1[u_0]\,\gamma^1[K^1]$. Examining the next terms in the expansion (3.5) of u_ε yields

$$\gamma^1[u_\varepsilon] = \varepsilon^{-1/2}\,c^1[u_0]\,\gamma^1[K^1] + \mathscr{O}\left(\varepsilon^{1/2}\log\varepsilon\right)\,. \tag{4.2}$$

Remark 4.1 1. The jump of $\partial_n K^1$ in $\boldsymbol{c}$ being equal to $K^1(0)$, we can see that $\gamma^1[K^1]$ is equal to $K^1(0)$.

2. Similarly $\gamma^1[u_\varepsilon]$ is equal to $\varepsilon^{-1}\,u_\varepsilon(0)$.
3. The expansion (3.5) at the point $\boldsymbol{c}$ yields that $u_\varepsilon(0) \sim \varepsilon^{1/2}\,c^1[u_0]\,K^1(0)$. This is consistent with the above items and (4.2).
4. To understand the transformation of singularities as $\varepsilon \to 0$, we have to consider the whole function K^1: the main singular part of u_ε is $\varepsilon^{1/2}c^1[u_0]\,K^1(\frac{r}{\varepsilon})$ and since $K^1 = S^1 + \mathcal{O}(r^{-1/2}\log r)$ as $r \to +\infty$, this main singular part tends to $\varepsilon^{1/2}c^1[u_0]\,S^1(\frac{r}{\varepsilon}) = c^1[u_0]\,S^1$.

∎

By induction arguments again [4, §8], we generalize (4.1): we split the expansion (3.6) of u_ε so that each piece of the new expansion has (at least) the regularity $H^{2-\delta}$ for any $\delta > 0$ — instead of $H^{-\delta+3/2}$ in (3.6):

$$u_\varepsilon \;=\; (u^0)_{\rm reg} \;+\; \sum_{j=1}^{N} \varepsilon^j\,(u^j)_{\rm reg}[\log\varepsilon] \;+\; \sum_{j=1}^{N+1} \varepsilon^{j-1/2}\,\tilde{w}^j(\varepsilon) + \tilde{r}_\varepsilon^N\,, \tag{4.3}$$

with

$$\tilde{w}^j(\varepsilon) = \sum_{\ell=1}^{j} c^j[u^0,\cdots,u^{j-\ell}]\,\chi(r)\,\tilde{W}^\ell(\frac{r}{\varepsilon}) \tag{4.4}$$

where

$$\tilde{W}^1 = K^1 \quad \text{and} \quad \forall\ell,\;\; \tilde{W}^\ell = \mathscr{O}\left(r^{1/2}\log^{\ell-1} r\right) \;\;\text{as}\;\; r\to+\infty, \tag{4.5}$$

and the remainder satisfies the estimate:

$$\|\tilde{r}_\varepsilon^N\|_{H^{2-\delta}(\Omega)} = \mathscr{O}\left(\varepsilon^{-\delta'+N+1/2}\right), \quad \forall\delta,\delta'>0. \tag{4.6}$$

Here the regular parts $(u^0)_{\rm reg}$ and $(u^j)_{\rm reg}$ belong to $H^{5/2-\delta}(\Omega)$ for any $\delta > 0$.

4.b Estimates in Sobolev norms

As a consequence of the expansions (3.5) and (4.3), we can prove the following estimates for the norms $H^{2-\delta}(\Omega)$ of u_ε. These estimates will be used to bound the approximation error in a boundary element method, see §6.

Proposition 4.2 *The solution u_ε of $(\mathcal{P}_\varepsilon)$ belongs to $H^{2-\delta}(\Omega)$ for any $\delta > 0$ and satisfies the estimates*

$$\|u_\varepsilon\|_{H^{2-\delta}(\Omega)} \;=\; \begin{cases} \mathscr{O}(1)\,, & \delta > 1/2 \\ \mathscr{O}\left(\log^{1/2}\varepsilon\right), & \delta = 1/2 \\ \mathscr{O}\left(\varepsilon^{\delta-1/2}\right), & \delta < 1/2\,. \end{cases} \tag{4.7}$$

We can deduce from this proposition suitable estimates for the norm $H^s(\Gamma)$ of the trace v_ε of u_ε. Moreover, if we restrict the trace to Γ_R, along with the last inequality in Proposition 3.2, we have improved estimates for the difference $v_\varepsilon - v_0$ which coincides with $v_\varepsilon - g$:

Proposition 4.3 *The trace v_ε of the solution u_ε of $(\mathcal{P}_\varepsilon)$ satisfies the estimates on Γ for any $s < 3/2$*

$$\|v_\varepsilon\|_{H^s(\Gamma)} \;=\; \begin{cases} \mathscr{O}(1)\,, & s < 1 \\ \mathscr{O}\left(\log^{1/2}\varepsilon\right), & s = 1 \\ \mathscr{O}(\varepsilon^{1-s})\,, & s > 1\,, \end{cases} \tag{4.8}$$

and the improved estimates on Γ_R

$$\|v_\varepsilon - g\|_{H^s(\Gamma_R)} = \begin{cases} \mathscr{O}(\varepsilon)\,, & s<0 \\ \mathscr{O}\left(\varepsilon \log^{1/2}\varepsilon\right), & s=0 \\ \mathscr{O}(\varepsilon^{1-s})\,, & s>0\,. \end{cases} \tag{4.9}$$

As a conclusion of this section we gather in one statement the information relating to the "first" singular part of u_ε .

Proposition 4.4 *The solution* u_ε *of* $(\mathcal{P}_\varepsilon)$ *admits the following expansion, that we write in two different ways*

$$u_\varepsilon = \begin{cases} u^0 & + \ \varepsilon^{1/2}\, c^1[u_0]\, \chi(r)\, W^1(\frac{r}{\varepsilon}) \\ (u^0)_{\text{reg}} & + \ \varepsilon^{1/2}\, c^1[u_0]\, \chi(r)\, K^1(\frac{r}{\varepsilon}) \end{cases} + \ \tilde r_\varepsilon\,, \tag{4.10}$$

where K^1 *is the profile introduced in Lemma* 3.1, *and* $W^1 = K^1 - S^1$ *is a boundary layer profile which decays like* $\mathscr{O}\left(r^{-1/2}\log r\right)$ *at infinity. The remainder satisfies, compare with* (4.7)*:*

$$\|\tilde r_\varepsilon\|_{H^{2-\delta}(\Omega)} = \mathscr{O}\left(\varepsilon^{-\delta'+1/2}\right), \quad \forall \delta, \delta' > 0\,. \tag{4.11}$$

Thus, the principal part of the difference between u_ε and u_0 behaves like $\varepsilon^{1/2}\, W^1(\frac{r}{\varepsilon})$: in the transition point 0 , it is equal to $\varepsilon^{1/2}\, K^1(0)$, on Γ_R it decays like $\mathscr{O}(\varepsilon)$, and on Γ_N like $\mathscr{O}(\varepsilon \log \varepsilon)$. In Figure 4, §6, we plot a numerical result which is a good approximation to this layer function $\varepsilon^{1/2}\, W^1(\frac{r}{\varepsilon})$ divided by ε , so that we can see the behavior on Γ_R and Γ_N .

As already said, *cf* Remark 4.1, the basic information about the singular behavior of u_ε is provided by $\varepsilon^{1/2}\, K^1(\frac{r}{\varepsilon})$. As $\varepsilon \to 0$, this term tends to S^1 (2.1), and for "large" ε its singular part is Σ^1 (2.3). But the "uniform" singular part is neither S^1 nor Σ^1 , but the whole $\varepsilon^{1/2}\, K^1(\frac{r}{\varepsilon})$. In Figure 3, §6, we plot a numerical result which is a good approximation for this function $\varepsilon^{1/2}\, K^1(\frac{r}{\varepsilon})$.

4.c Estimates in countably normed spaces

In view of an enrichment of the discrete space by singular functions, we state the following analytic weighted L^2 estimates about the model function K^1 , relying on a nested open sets technique, *cf* [1, §4]:

$$\|\, r^{\gamma+|\alpha|} D^\alpha (K^1 - K^1(0))\, \|_{L^2(\Pi)} \le \ C^{|\alpha|+1} \alpha!\, \|\, r^\gamma (K^1 - K^1(0))\, \|_{L^2(\Pi)}\ , \tag{4.12}$$

from which we can deduce the uniform estimates

$$\| \, r^{\gamma+1+|\alpha|} D^\alpha (K^1 - K^1(0)) \, \|_{L^\infty(\Pi)} \leq \; C^{|\alpha|+1} \alpha! \, \| \, r^\gamma (K^1 - K^1(0)) \, \|_{L^2(\Pi)} \, . \qquad (4.13)$$

Since the right hand side is bounded for any $-2 < \gamma < -\frac{3}{2}$, we deduce

Proposition 4.5 *For any* $\frac{1}{2} < \delta < 1$, *there exists* $M > 0$ *such that for any* $\alpha \in \mathbb{N}^2$ *and any* $x \in \overline{\Pi}$ *there holds*

$$\Big| D^\alpha (K^1(x) - K^1(0)) \Big| \; \leq \; M^{|\alpha|+1} \alpha! \, |x|^{\delta - |\alpha|}. \qquad (4.14)$$

Similar estimates hold for the profiles $\tilde{W}^\ell$ introduced in (4.5), which, combined with a splitting of higher order of the outer expansion, allows for proving the following

Proposition 4.6 *If* g *is smooth, for any* $N \in \mathbb{N}$, u_ε *can be split into three parts*

$$u_\varepsilon = u^{[N]}_{\text{reg}}(\varepsilon) + u^{[N]}_{\text{sing}}(\varepsilon) + \tilde{r}^N_\varepsilon,$$

where the first part $u^{[N]}_{\text{reg}}(\varepsilon)$ *belongs to* $H^N(\Omega)$ *uniformly in* ε, *the second part* $u^{[N]}_{\text{sing}}(\varepsilon)$ *satisfies estimates* (4.14) *with the following behavior in* ε*: For any* $\frac{1}{2} < \delta < 1$, *there exists* $M > 0$ *such that for any* $\varepsilon \in (0,1)$, *any* $\alpha \in \mathbb{N}^2$ *and any* $x \in \overline{\Omega}$ *there holds*

$$\Big| D^\alpha \Big(u^{[N]}_{\text{sing}}(\varepsilon)(x) - u^{[N]}_{\text{sing}}(\varepsilon)(0) \Big) \Big| \; \leq \; \varepsilon^{-\delta+1/2} \, M^{|\alpha|+1} \alpha! \, |x|^{\delta - |\alpha|}. \qquad (4.15)$$

The third part is the remainder in (4.3)*. It satisfies estimates* (4.6)*.*

5 APPROXIMATION BY A BOUNDARY ELEMENT METHOD

We choose to present a BEM for the numerical approximation of problems $(\mathcal{P}_\varepsilon)$ and $(\mathcal{P}_0)$ for several reasons:

- The reduction to the boundary substantially decreases the number of unknowns, thus making easier the observation of the dependence with respect to ε.
- The discrete spaces live on the boundary and the investigation of the influence of the addition of certain singular functions to a standard $\mathbb{P}_1$ approximation space is much easier too.

There always exist plenty of ways to (try to) reduce a boundary value problem to an integral equation on the boundary. Here we only describe the method that we use for numerical experiments. This method has the twofold advantage to be simple and to have good coerciveness properties.

5.a Basic notations

We make use of the standard boundary operators V , K and W associated with the usual elementary solution of the Laplace operator in two-dimensional spaces [3, 18, 8]

$$G(x,y) = -\frac{1}{2\pi}\ \log|x-y|, \quad x,y \in \mathbb{R}^2.$$

Let us recall that

- V is the single layer potential:

$$V\varphi(x) = \int_\Gamma G(x,y)\,\varphi(y)\,d\sigma(y), \quad x,y \in \Gamma\,;$$

  the operator V is a self-adjoint isomorphism from $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$ if $\operatorname{cap}(\Gamma) \neq 1$.

- K is the double layer potential:

$$Kv(x) = \int_\Gamma \partial_{n(y)} G(x,y)\,v(y)\,d\sigma(y), \quad x,y \in \Gamma\,;$$

  the operator K is continuous from $H^{1/2}(\Gamma)$ into itself and compact if Γ is smooth enough.

- W is the hypersingular operator:

$$Wv(x) = -\partial_{n(x)} \int_\Gamma \partial_{n(y)} G(x,y)\,v(y)\,d\sigma(y), \quad x,y \in \Gamma\,;$$

  the operator W is a self-adjoint isomorphism from $H^{1/2}(\Gamma)/\mathbb{R}$ onto its dual space.

Everywhere $d\sigma(y)$ is the Lebesgue measure on Γ , and $n(y)$ is the exterior unit normal to Γ at y .

Our integral equation is a *direct* formulation, i.e. the unknown is one of the Cauchy data of the solution u of the bvp. Here the natural choice is the Dirichlet trace v , and the variational space is

$$\mathcal{V} = H^{1/2}(\Gamma)\,.$$

For any elements v , $w \in \mathcal{V}$, we denote by v_R , w_R their restrictions to Γ_R and V_{RR} denotes the restriction of V to Γ_R :

$$V_{RR}\, v_R(x) = \int_{\Gamma_R} G(x,y)\,v(y)\,d\sigma(y), \quad x,y \in \Gamma_R\,.$$

5.b First formulation

We can suppose without restriction that the interior right hand side f in $(\mathcal{P}_\varepsilon)$ is zero. We write the integral equation in variational form

$$(\mathcal{B}_\varepsilon) \qquad v_\varepsilon \in \mathcal{V}, \quad \forall w \in \mathcal{V}, \quad b_\varepsilon(v_\varepsilon, w) = G_\varepsilon(g)[w],$$

where b_ε is the bilinear form (5.1) and $G_\varepsilon(g)$ is the function (5.2) of the Robin datum $g = g_R \in H^{1/2}(\Gamma_R)$:

$$b_\varepsilon(v,w) = \langle Wv, w\rangle - \frac{1}{\varepsilon}\langle v_R, Kw\rangle + \frac{1}{\varepsilon}\langle Kv, w_R\rangle + \frac{1}{\varepsilon}\langle v_R, w_R\rangle + \frac{1}{\varepsilon^2}\langle V_{RR}\, v_R, w_R\rangle \tag{5.1}$$

and

$$G_\varepsilon(g)[w] = -\frac{1}{\varepsilon}\langle g_R, Kw\rangle + \frac{1}{2\varepsilon}\langle g_R, w_R\rangle + \frac{1}{\varepsilon^2}\langle V_{RR}\, g_R, w_R\rangle\,. \tag{5.2}$$

Everywhere $\langle\cdot,\cdot\rangle$ denotes suitable extensions of the $L^2(\Gamma)$ duality. We can prove that problems $(\mathcal{P}_\varepsilon)$ and $(\mathcal{B}_\varepsilon)$ are equivalent: the unique solution v_ε of $(\mathcal{B}_\varepsilon)$ is the Dirichlet trace of u_ε .

Let $\|\cdot\|_{\mathcal{V},\varepsilon}$ be the family of (non uniformly) equivalent norms on $\mathcal{V}$

$$\|v\|_{\mathcal{V},\varepsilon} = \Big(\|v\|^2_{H^{1/2}(\Gamma)} + \frac{1}{\varepsilon^2}\|v\|^2_{\tilde H^{-1/2}(\Gamma_R)}\Big)^{1/2},$$

with $\tilde H^{-1/2}(\Gamma_R)$ the dual space of $H^{1/2}(\Gamma_R)$. We have the continuity and coercivity estimates:

Lemma 5.1 *With constants c , c' independent of ε , we have for any v , $w \in \mathcal{V}$*

$$|b_\varepsilon(v,w)| \;\le\; c\,\|v\|_{\mathcal{V},\varepsilon}\,\|w\|_{\mathcal{V},\varepsilon}\,, \tag{5.3}$$

$$b_\varepsilon(v,v) \;\ge\; c'\,\|v\|^2_{\mathcal{V},\varepsilon}\,. \tag{5.4}$$

Concerning the right hand side of equation $(\mathcal{B}_\varepsilon)$, we have

Lemma 5.2 *With a constants c independent of ε , we have for any $w \in \mathcal{V}$*

$$|G_\varepsilon(g)[w]| \;\le\; \frac{c}{\varepsilon}\,\|g_R\|_{\tilde H^{-1/2}(\Gamma_R)}\,\|w\|_{\mathcal{V},\varepsilon}\,, \tag{5.5}$$

and for any $\breve g \in \mathcal{V}$ which extends g_R to the whole boundary Γ

$$|G_\varepsilon(g)[w]| \;\le\; c\,\|\breve g\|_{\mathcal{V},\varepsilon}\,\|w\|_{\mathcal{V},\varepsilon}\,. \tag{5.6}$$

Let $\mathcal{V}_N \subset \mathcal{V}$ a conforming discretization of $\mathcal{V}$. The associated discrete problem is

$$v_{\varepsilon,N} \in \mathcal{V}_N, \quad \forall w_N \in \mathcal{V}_N, \quad b_\varepsilon(v_{\varepsilon,N}, w_N) = G_\varepsilon(g)[w_N]\,, \tag{5.7}$$

and Cea's lemma joined with Lemma 5.1 yields immediately the error estimate

$$\|v_\varepsilon - v_{\varepsilon,N}\|_{\mathcal{V},\varepsilon} \;\le\; c\,\|v_\varepsilon - w_N\|_{\mathcal{V},\varepsilon}\,, \quad \forall w_N \in \mathcal{V}_N\,. \tag{5.8}$$

5.c A variant

We obtain a variant of problem $(\mathcal{B}_\varepsilon)$ by setting $\breve{v}_\varepsilon = v_\varepsilon - \breve{g}$ where $\breve{g}$ is a smooth extension of g to Γ. Thus problem $(\mathcal{B}_\varepsilon)$ is equivalent to

$$(\breve{\mathcal{B}}_\varepsilon) \qquad \breve{v}_\varepsilon \in \mathcal{V}, \quad \forall w \in \mathcal{V}, \quad b_\varepsilon(\breve{v}_\varepsilon, w) = \breve{G}_\varepsilon(g)[w],$$

where $\breve{G}_\varepsilon(g)[w] = G_\varepsilon(g)[w] - b_\varepsilon(\breve{g}, w)$. We can check that $\breve{G}_\varepsilon(g)[w]$ satisfies estimate (5.6). The associated discrete problem reads

$$\breve{v}_{\varepsilon,N} \in \mathcal{V}_N, \quad \forall w_N \in \mathcal{V}_N, \quad b_\varepsilon(\breve{v}_{\varepsilon,N}, w_N) = \breve{G}_\varepsilon(g)[w_N]\,, \tag{5.9}$$

and satisfies the error estimates

$$\|\breve{v}_\varepsilon - \breve{v}_{\varepsilon,N}\|_{\mathcal{V},\varepsilon} \;\le\; c\,\|\breve{v}_\varepsilon - w_N\|_{\mathcal{V},\varepsilon}, \quad \forall w_N \in \mathcal{V}_N\,. \tag{5.10}$$

6 DISCRETIZATION BY PIECEWISE LINEARS

6.a Estimates

We investigate the case when we have a family of uniform meshes $\mathcal{T}_h$ on Γ, so the discretization parameter is h, and $\mathcal{V}_h$ is the the space of continuous functions whose restriction to each element is $\mathbb{P}_1$.

Standard approximation results by piecewise linears yield that there exists $w_h \in \mathcal{V}_h$ satisfying the error estimates (with constants c independent of h)

$$\|v - w_h\|_{H^{1/2}(\Gamma)} \;\le\; c\,h^{s-1/2}\,\|v_\varepsilon\|_{H^s(\Gamma)}, \quad s \le 2, \tag{6.1}$$

$$\|v - w_h\|_{\tilde{H}^{-1/2}(\Gamma_R)} \;\le\; c\,h^{s-1/2}\,\|v_\varepsilon\|_{H^{s-1}(\Gamma_R)}, \quad s \le 2. \tag{6.2}$$

These approximation estimates give better results for the error $\breve{v}_\varepsilon - \breve{v}_{\varepsilon,h}$, due to improved estimates (4.9) for $v_\varepsilon - g = \breve{v}_\varepsilon$ on Γ_R. That is why we begin with the study of the discretization error for problem $(\breve{\mathcal{B}}_\varepsilon)$. Combining the above estimates (6.1)-(6.2) with (5.10), we obtain

$$\|\breve{v}_\varepsilon - \breve{v}_{\varepsilon,h}\|_{\mathcal{V},\varepsilon} \;\le\; c\,h^{s-1/2}\Big(\|\breve{v}_\varepsilon\|_{H^s(\Gamma)} + \frac{1}{\varepsilon}\,\|\breve{v}_\varepsilon\|_{H^{s-1}(\Gamma_R)}\Big)\,. \tag{6.3}$$

From the definition of the norm $\|\cdot\|_{\mathcal{V},\varepsilon}$, we see that also $\|\breve{v}_\varepsilon - \breve{v}_{\varepsilon,h}\|_{H^{1/2}(\Gamma)}$ and $\frac{1}{\varepsilon}\,\|\breve{v}_\varepsilon - \breve{v}_{\varepsilon,h}\|_{\tilde{H}^{-1/2}(\Gamma_R)}$ are estimated by the right hand side of (6.3).

Concerning L^2 estimates, by interpolation between estimates in $H^{1/2}(\Gamma_R)$ and in $\tilde{H}^{-1/2}(\Gamma_R)$, we obtain (6.4) below and by a Aubin-Nitsche duality argument we arrive at (6.5):

$$\|\breve{v}_\varepsilon - \breve{v}_{\varepsilon,h}\|_{L^2(\Gamma_R)} \leq c\,\varepsilon^{1/2}\,\|\breve{v}_\varepsilon - \breve{v}_{\varepsilon,h}\|_{\mathcal{V},\varepsilon}\,, \tag{6.4}$$

$$\|\breve{v}_\varepsilon - \breve{v}_{\varepsilon,h}\|_{L^2(\Gamma)} \leq c\,h^{1/2}\,\|\breve{v}_\varepsilon - \breve{v}_{\varepsilon,h}\|_{\mathcal{V},\varepsilon}\,. \tag{6.5}$$

Relying on Proposition 4.3 we obtain sharp estimates of the norms of $\breve{v}_\varepsilon$ on Γ and Γ_R: We give in Table 1 the convergence rates of the L^2 norm of the error $\breve{v}_\varepsilon - \breve{v}_{\varepsilon,h}$ in Γ_R and Γ, obtained by optimizing the estimates provided by (6.1)-(6.5) and (4.8)-(4.9) when $h \geq \varepsilon$ and $h \leq \varepsilon$ respectively:
- For $h \geq \varepsilon$, we use $s = 1$ in (6.3). Using $s = 1 - \delta$ allows to get rid of the logarithmic term $\log^{1/2}\varepsilon$, but leads to a loss of $h^{-\delta}$.
- For $h \leq \varepsilon$, we consider the limit $s \to 3/2$ in (4.8)-(4.9), taking into account the singular behavior in $r\log r$ of $\breve{v}_\varepsilon$.

	$h \geq \varepsilon$	$h \leq \varepsilon$
Error in $L^2(\Gamma_R)$	$\varepsilon^{1/2}\,h^{1/2}\log^{1/2}\varepsilon$	$\varepsilon^{-1/2}\,h^{3/2}\log^{3/2}h$
Error in $L^2(\Gamma)$	$h\,\log^{1/2}\varepsilon$	$\varepsilon^{-1/2}\,h^{3/2}\log^{3/2}h$

Table 1. Theoretical convergence rates for problem $(\breve{\mathscr{B}}_\varepsilon)$

Remark 6.1 A Finite Element Method in Ω with regular $\mathbb{P}_1$ elements based on the variational formulation (1.1) would give the following error estimates: In the $H^1(\Omega)$ norm, one obtains

$$\|u_\varepsilon - u_{\varepsilon,h}\|_{H^1(\Omega)} = \begin{cases} \mathscr{O}\left(h^{1/2}\right), & \text{if } h \geq \varepsilon \\ \mathscr{O}\left(\varepsilon^{-1/2}\,h\,\log^{3/2}h\right), & \text{if } h \leq \varepsilon\,. \end{cases}$$

In the $L^2(\Gamma_R)$ norm for the traces on Γ_R, one obtains the same convergence rates as those given in Table 1 for the Boundary Element Method. ■

6.b Numerical experiments

We performed numerical experiments using the following convenient modification of the Galerkin method (5.7) associated with the spaces $\mathcal{V}_h$ of piecewise linears:

If $\mathcal{I}_h g$ denotes the interpolate of g at the nodes of $\mathcal{V}_h$, our formulation reads

$$\hat{v}_{\varepsilon,h} \in \mathcal{V}_h, \quad \forall w_h \in \mathcal{V}_h, \quad b_\varepsilon(\hat{v}_{\varepsilon,h}, w_h) = G_\varepsilon(\mathcal{I}_h g)[w_h]\,. \tag{6.6}$$

The additional interpolation error introduced by this modification is of higher order than the errors shown in Table 1.

In Figure 3, 4 and 5, we present some results of computations. As domain, we consider the triangle Γ with interior Ω, whose corners are $(-3,0)$, $(2,0)$ and $(0,4)$. The part Γ_N is the segment $[-2,0]$ in the x axis. The transition point of interest is $\boldsymbol{c} = (0,0)$. The right hand side g is chosen such that the exact solution u_0 of $(\mathcal{P}_0)$ is just the singular function $S^1 = r^{1/2}\sin(\theta/2)$.

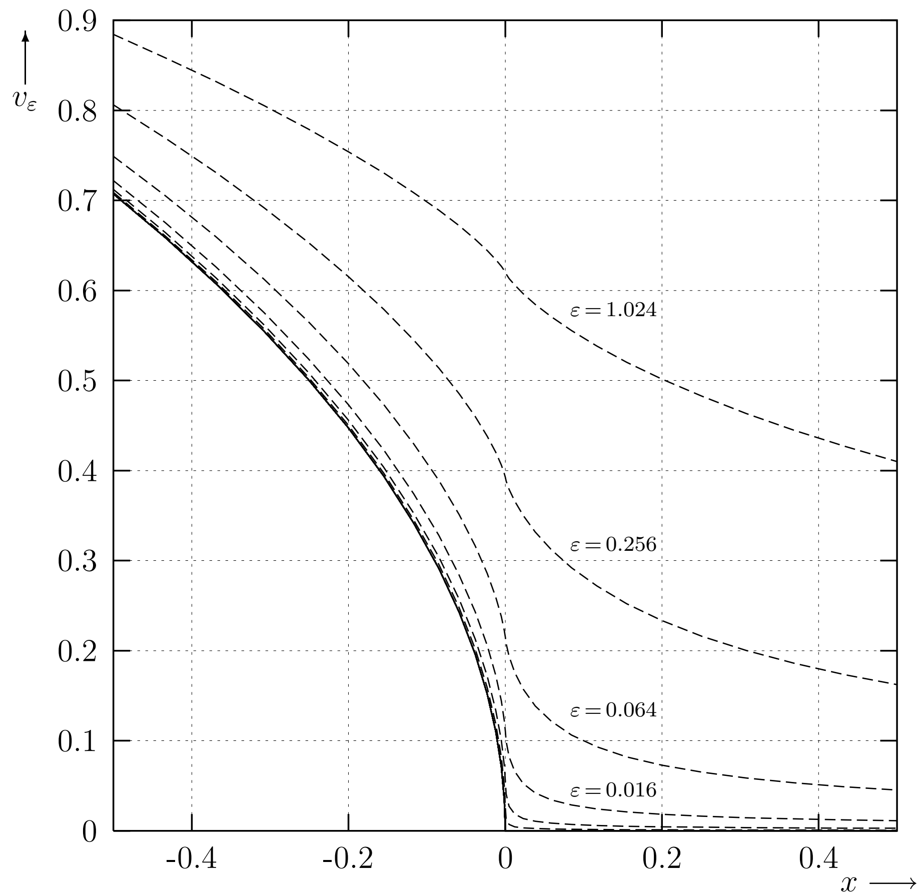

Figure 3. Solutions $\hat{v}_{\varepsilon,h}$

In Figure 3, we show (dotted lines) the computed solutions $\hat{v}_{\varepsilon,h}$ of (6.6) in a neighborhood of the singular point $(0,0)$ for a fixed value of h and various values of ε: 0.001, 0.004, 0.016, 0.064, 0.256, 1.024. The solid line is the trace v_0 of the singular function S^1. The number of nodes in Γ is set to 210.

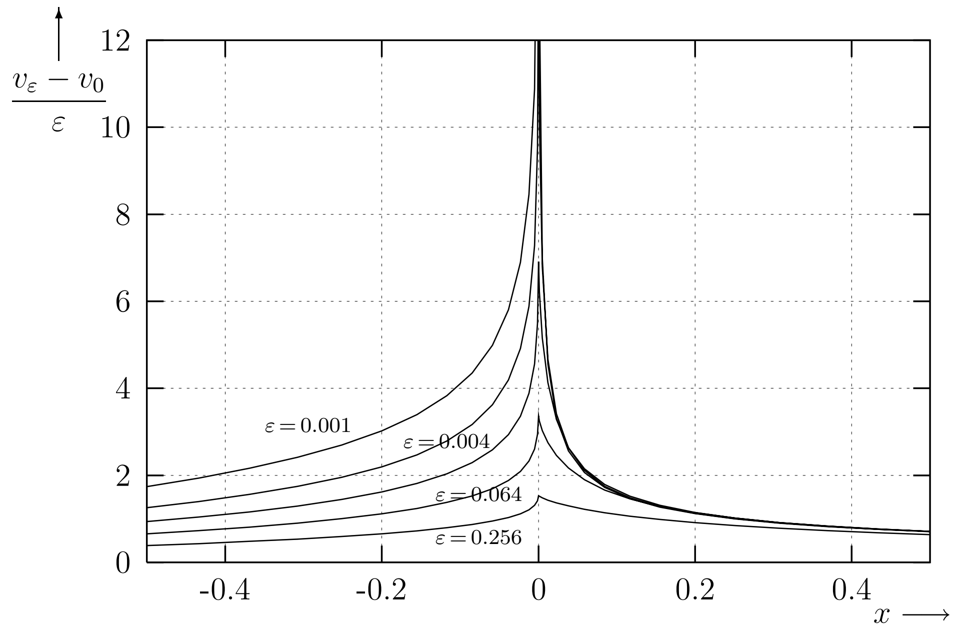

Figure 4. Difference $(\hat{v}_{\varepsilon,h} - v_0)/\varepsilon$

One can see the essentially self-similar behavior of $v_\varepsilon \sim c\,\varepsilon^{1/2}\,K^1(r/\varepsilon)$. Also visible are the $\mathscr{O}\,(\varepsilon)$ behavior of v_ε in the interior of Γ_R and the $\mathscr{O}\left(\varepsilon^{1/2}\right)$ behavior of v_ε at the singular point 0.

In Figure 4, for the same number of nodes and a similar set of values for ε, we plot $\varepsilon^{-1}(\hat{v}_{\varepsilon,h} - v_0)$. The function v_0 is the trace of S^1 and the exact solution of problem $(\mathcal{P}_0)$. According to (3.4), this function behaves like $w^1_\varepsilon \sim \varepsilon^{-1/2}\,W^1(r/\varepsilon)$. We see very clearly the behavior independent of ε on Γ_R, and we can guess a behavior like $|\log\varepsilon|$ on Γ_N, corresponding to what can be predicted from the asymptotics of K^1 Lemma 3.1.

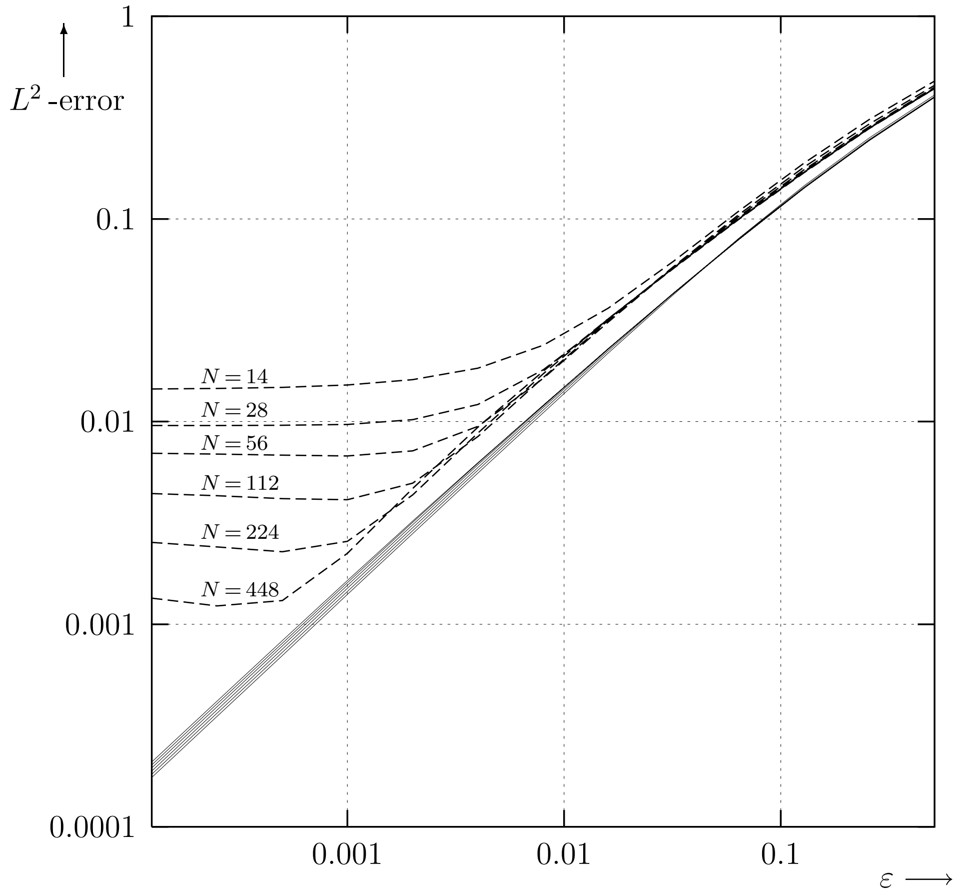

Figure 5. L^2-errors

In Figure 5, we plot the L^2 errors between $\hat{v}_{\varepsilon,h}$ and v_0 on Γ (broken lines) and Γ_R (solid lines). On Γ_R, as function of ε for various values of the total number N of nodes on Γ, the error $\|\hat{v}_{\varepsilon,h} - v_0\|_{L^2}$ is (for this range of N and ε) essentially proportional to ε and varies very little with N. On the whole boundary Γ, we see a $\mathscr{O}(\varepsilon + h)$ behavior.

N	ε	$L^2(\Gamma_R)$	$H^1(\Gamma_R)$	$L^2(\Gamma)$	$H^1(\Gamma)$	$H^{1/2}(\Gamma)$
14	0.064	0.908	0.042	0.901	0.035	0.468
28	0.032	0.909	0.039	0.917	0.020	0.469
56	0.016	0.911	0.022	0.930	0.006	0.468
112	0.008	0.911	0.012	0.939	0.002	0.470
224	0.004	0.908	0.006	0.946	0.001	0.473

Table 2. Numerical convergence rates for $h \sim \varepsilon$

Note that we can prove that the theoretical convergence rates of $\hat{v}_{\varepsilon,h} - v_0$ in the norms $L^2(\Gamma)$ and $L^2(\Gamma_R)$ are the same as in Table 1 if $h \geq \varepsilon$, and are in $\varepsilon \log^{1/2} \varepsilon$ if $h \leq \varepsilon$ (due to the difference $v_\varepsilon - v_0$).

As shown in Table 2, we get a very good match between theoretical convergence rates and numerical ones when ε and N vary together so that their product remains constant.

7 ENRICHED SUBSPACES

In Section 6, we showed that essentially optimal $\mathscr{O}\left(h^{1-\delta}\right)$ convergence in $L^2(\Gamma)$ can be achieved, uniformly in ε. If a higher rate of convergence is desired, then increasing the polynomial degree (e.g. to quadratic) will not improve this rate, due to the singular components. In this section, we briefly describe how the rate of approximation of such singular components can be improved by enriching the finite element space with appropriate singular functions. This idea was introduced and studied computationally in the context of the finite element method in [16], [17]. By using the p version rather than the h version, and adding enough singular functions, we can in practice achieve an *exponential* rate. This can be understood to be the p version of the method of enriched subspaces, which is classical for the h version (see [15]).

The method is as follows. First, a fixed mesh is introduced on Γ, such that the points $\boldsymbol{c}_i$ are nodal points. Typically, only a few elements are needed. Let I_i^N and I_i^R be the elements on the two sides of $\boldsymbol{c}_i$ — we assume the length of each of these is $\mathscr{O}(1)$, so that the support of the cut-off function χ appearing in various decompositions above (such as (2.2), (2.4), etc.) lies within these elements.

Now we use the p version, i.e. our subspace $\mathcal{V}_p$ contains all continuous functions whose restrictions to each element are in $\mathbb{P}_p$. These will give a high algebraic rate, uniform in ε, while approximating the smooth components of the solution. To approximate the unsmooth components, we add appropriate singular functions to the elements I_i^N and I_i^R.

The results of Sections 2-4 show that $u_{\mathrm{sing}}^{[N]}(\varepsilon)$ have a rather complicated behavior, which can be summarized by the estimate (4.15), which is uniform in ε. We therefore need to augment the spaces $\mathcal{V}_p$ so that any function that satisfies (4.15) is well-approximated. It is no longer sufficient to just add the limiting singularities of $r^{1/2}$ and $r \log r$ from Section 2, since these are not enough to approximate these functions. Instead, we add $n = n(p)$ singular functions over each of the four elements I_i^N, I_i^R. If $\chi(x)$ is the cut-off function on one of these elements, say I_1^N, where x is the distance from $\boldsymbol{c}_1$ along Γ, then on I_1^N these are defined as follows.

Let $\alpha_1 \leq \frac{1}{2}$ and let $\alpha_2 > \alpha_1$. Define $s = (\alpha_2 - \alpha_1)/n(p)$. Then we add the singular functions

$$z_j = x^{\alpha_1 + js}\chi(x), \quad j = 0 \dots n(p) \tag{7.1}$$

to the set of polynomials already defined on I_1^N.

We have the following result (stated for the case that the length of I_1^N is 1).

Proposition 7.1 *Let $f(x) = \phi(x)\chi(x)$, defined on $I = [0,1]$, where*

$$|D^\alpha \phi(x)| \leq M^{|\alpha|+1}\alpha!\, x^{\delta - |\alpha|} \tag{7.2}$$

for some $\delta > -\frac{1}{2}$ and some $M > 0$. Then, for all $n \in \mathbb{N}$, there exists a function

$$f_n(x) = \sum_{j=0}^{n} a_j z_j$$

such that for any integer $0 \leq l < \min\{\delta, \alpha_1\} + \frac{1}{2}$,

$$\|f - f_n\|_{H^l(I)} \leq C\beta^{n^{1/3}} \tag{7.3}$$

where $0 < C$ and $0 < \beta < 1$ are constants independent of n.

Note that $f(x) = f_n(x) = 0$ at $x = 0$ and 1.

Proposition 7.1 shows that the functions (7.1) approximate singular functions satisfying (7.2) at an *exponential* rate in n. The proof of this proposition is rather technical, and may be found in [5]. Here, we give a heuristic argument for the main idea.

Suppose we consider the function

$$f(x) = x^\delta, \quad \delta > 0. \tag{7.4}$$

Then it is easy to see that f satisfies (7.2). Let us choose an $\alpha_1 < \delta$ and form the functions z_j (using some $\alpha_2 > \alpha_1$). Then we consider

$$\begin{aligned} \inf_{a_j} |f - \sum a_j z_j| &= x^{\alpha_1} \inf_{a_j} |x^{\delta-\alpha_1} - \sum a_j x^{js}| \\ &= x^{\alpha_1} \inf_{a_j} |y^{(\delta-\alpha_1)/s} - \sum a_j y^j| \end{aligned} \tag{7.5}$$

where $y = x^s$.

Looking at equation (7.5), we see that f is approximated at a rate proportional to the rate at which the function y^γ is approximated by polynomials in y , where

$$\gamma = \frac{\delta - \alpha_1}{s} = Cn \to \infty \text{ as } n \to \infty.$$

It can be shown that this rate of approximation is superexponential, since y^γ is an increasingly smooth function, and approximation of smooth functions by polynomials y^j is superexponential.

Note that the above heuristic does not involve an explicit knowledge of the exponent δ — all we need is the value of an α_1 that is less than δ , for it to work. In particular, if we are approximating a linear combination of x^δ functions (as happens in the case of the singularities in our contact problem), then these are well-approximated as well, by choosing any such single value of α_1 that is smaller than all the values of δ .

Moreover, as shown in [5], the argument above can be modified when the function to be approximated satisfies the more general criterion of Proposition 7.1. The error estimate we get now is somewhat degraded, but is still exponential.

Using Proposition 4.6 and Proposition 7.1, we can now obtain a function $u_{\mathrm{sing},p} \in \mathcal{V}_p$ satisfying

$$\|u_{\mathrm{sing}}^{[N]}(\varepsilon) - u_{\mathrm{sing},p}\|_{\mathcal{V},\varepsilon} \le C\varepsilon^{-\delta-1/2}\beta^{(n(p))^{1/3}}. \tag{7.6}$$

We remark that the negative power of ε in estimate (7.6) appears due to a use of the crude bound

$$\|v\|_{\mathcal{V},\varepsilon} \le \varepsilon^{-1}\|v\|_{H^{1/2}(\Gamma)}$$

and can probably be eliminated by a more careful analysis. In any case, note that taking

$$n(p) = \left(p^{1/3} + (\delta + 1/2)\log_\beta \varepsilon\right)^3 \tag{7.7}$$

eliminates the dependency on ε and gives the uniform estimate,

$$\|u_{\mathrm{sing}}^{[N]}(\varepsilon) - u_{\mathrm{sing},p}\|_{\mathcal{V},\varepsilon} \le C\beta^{p^{1/3}}. \tag{7.8}$$

Hence the rate of convergence is no longer blocked due to the singular components.

Let us remark that the heuristic presented above and the proof presented in [5] also suggest a theoretical basis of why the enriched subspace finite element method described in [16], [17] gives near-exponential convergence. One of the limitations discussed in that reference was the fact that the matrices one gets due to the addition of the singular functions (which are far from being orthogonal) can be rather poor. However, due to the exponential convergence, very few such singular functions are needed in practice (this was observed experimentally in [16], [17]), and the condition numbers one might obtain would still be reasonable. An alternative remedy would be to orthogonalize the functions z^j before using them.

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Authors'adresses

M. Costabel and M. Dauge — *U.R.A. 305 du C.N.R.S. – IRMAR*
Université de Rennes 1 – Campus de Beaulieu
35042 Rennes Cedex 03, FRANCE

M. Suri — *Department of Mathematics and Statistics*
University of Maryland Baltimore County
Baltimore, MD 21250, USA.