STABLE ASYMPTOTICS FOR ELLIPTIC SYSTEMS ON PLANE DOMAINS WITH CORNERS

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Abstract. *We consider boundary value problems for elliptic systems in the sense of Agmon-Douglis-Nirenberg on plane domains with corners, where the domain, the coefficients of the operators and the right hand sides all depend on a parameter. We construct corner singularities in such a way that the corresponding decomposition of the solution into regular and singular parts is* ***stable****, i. e. the regular part and the coefficients of the singular functions depend smoothly on the parameter. The construction of these singular functions continues the paper* [4] *and generalizes results known for second order scalar boundary value problems — see* [2, 3] [11].

Introduction

0.a Motivation. Parameter dependence in elliptic boundary value problems is important in a variety of applications, for example in the question of stability of numerical approximations, in shape optimization problems (optimal control of the domain), or in the case of quasi-cylindrical domains (the geometry is almost independent of one of the variables which can be considered as a parameter); an example of the latter are domains with edges which can be partly treated as corner domains depending on a parameter.

Unlike regular elliptic boundary value problems, for which the usual stability theory for invertible or Fredholm operators can be applied, boundary value problems with corner singularities may show phenomena of unboundedness or non-regularity in the dependence of the singular parts on a parameter. A well-known aspect of that instability is the occurrence of logarithmic terms in the asymptotics near a corner for certain isolated values of its opening angle ω, while such terms are absent for any other value of ω: for example, concerning the *first term* in the singular part of the solution of the Dirichlet problem, the set of exceptional openings is formed by the angles π/n with $n = 2, 3, \dots$ for the Laplace operator and this set reduces to $\omega_0 \simeq 0.813\pi$ for the biharmonic operator — see [6] for instance.

In the general situation, corner singularities are determined by eigenvalues and eigenfunctions of certain associated spectral problems. The multiplicity of these eigenvalues may change for some values of the parameter: then the differentiability with respect to the parameter may be lost (*"branching"*), or even if the eigenvalues remain smooth functions of the parameter, the eigenfunctions may become linearly dependent and their coefficients unbounded (*"crossing"*). The latter phenomenon of crossing is the only to appear in the case of second order scalar operators and was studied intensively in [2, 3], [10, 11].

There it was shown that certain linear combinations of the usual singular functions had to be constructed in order to obtain stability of the coefficients. These special linear combinations were given either by divided differences or, equivalently, in the form of complex contour integrals [2, §8]. The approach via contour integrals turns out to be successful also in the case of general boundary value problems elliptic in the sense of Agmon-Douglis-Nirenberg.

In the general case of higher order equations (order ≥ 4) or for systems, the first one of the above-mentioned phenomena, namely the "branching" of the eigenvalues may appear. This is well known for standard boundary values problems for the biharmonic operator or the Stokes system (the first branching occurs at the above-mentioned opening $\omega_0 \simeq 0.813\pi$). We show that also in this case the construction based on contour integrals leads to a form of the singular functions that generates smooth parameter dependence for the corresponding coefficients.

Our aim in this paper is twofold: We want to show that our construction of corner singularities, which continues the paper [4], provides on one hand stability and smoothness with respect to a parameter and on the other hand also rather explicit formulas for the singularities. Thus we think that these formulas, in spite of the intrinsic difficulties coming from the fact that we treat general ADN-elliptic systems and admit also corner angles π and 2π, are sufficiently explicit to be used, for example, in numerical approximations.

Our formulas can be compared to the general analytic functionals of Schulze [13, 14] and to the stable Keldyš chains of Schmutzler [12].

0.b Plan. In §1, we describe the class of parameter dependent corner problems that will be studied. We introduce a class of diffeomorphisms that does not exclude corner angles π and 2π. Thus corners may develop out of (or disappear into) smooth parts of the boundary and (inward) cusps may appear. We formulate the main results on the *existence* of corner asymptotics which depend in a stable and smooth way on the parameter (Theorems 1.1 and 1.3). The aim of the rest of the paper is the *description* of these stable asymptotics and the proof of the theorems.

In §2, corner singularities are constructed. The construction uses complex integrals on two levels: First for the construction of a basis of solutions of a certain associated system of ordinary differential equations without boundary conditions. Here the contour integral is only needed if the multiplicities of the (complex) characteristics of the principal part are changing; we illustrate this by several examples. In a second step, this solution basis is used to construct the eigenfunctions of a corresponding two-point boundary value problem. Here the above-mentioned phenomena of "crossing" and "branching" have to be treated. This section introduces the study of three types of spaces of singular functions associated to complex contours γ: $\mathscr{X}(\gamma)$ and $\mathscr{Y}(\gamma)$ for the principal parts of the asymptotics corresponding to flat and polynomial right hand sides, respectively (Theorems 2.7 and 2.12), and various spaces $\mathscr{Z}(\gamma)$ for the non-principal terms of the asymptotics (singular right hand sides, Theorem 2.21).

In §3, we formulate and prove the basic results of decomposition when the gain of regularity is ≤ 1 (here only the principal parts of the problem and of the asymptotics are involved).

In §4, we introduce the Taylor expansion of the boundary value problem at the corner (interior systems and boundary operators on the curved sides of the domain) and, using the spaces $\mathscr{X}(\gamma)$, $\mathscr{Y}(\gamma)$ and $\mathscr{Z}(\gamma)$, we give in Lemma 4.5 and Theorem 4.6 expressions in the form of finite sums of elementary terms, of the "abstract" stable asymptotics of §1 and prove the stability theorems 1.1 and 1.3. As an example, we calculate the first terms of these asymptotics for the Laplace operator on a domain limited by a portion of a parabola or of a circle (Examples 4.9 and 4.10).

In §5, starting from the formulas with contour integrals that we introduced in §2, we derive the radial and angular behavior of the terms in the asymptotics. For this we rely on a generalization of the Leibniz formula for divided differences to more general contour integrals which we present in the Appendix, §6.

0.c Outline of results. We explain roughly the structure of the stable asymptotics by a short discussion of their radial behavior. If r denotes the distance to the corner, the ordinary behavior that every one knows is $r^\mu \log^q r$ with μ describing a finite set of complex numbers. These μ appear as eigenvalues, and they are conveniently constructed as zeros of certain polynomials. Thus the ordinary asymptotics behave as

$$\frac{q!}{2i\pi}\int_\gamma \frac{r^\lambda}{(\lambda-\mu)^{q+1}}\,d\lambda\,.$$

where the contour γ surrounds μ. If the μ depend smoothly on a parameter, but q is not constant (for instance equal to 1 for some values of the parameter and to 0 everywhere else), we say that there are *crossings*. The stable asymptotics behave like divided differences of the function $r \rightarrow r^\lambda$:

$$S[\mu_1,\ldots,\mu_d;r] = \frac{1}{2i\pi}\int_\gamma \frac{r^\lambda}{(\lambda-\mu_1)\cdots(\lambda-\mu_d)}\,d\lambda\,,$$

where, in fact, $d = Q+1$ with Q the maximal value of q. But in general the μ_j themselves do not depend smoothly on the parameter; there are *branchings*. However there are polynomials a of degree $d \geq 2$ whose coefficients depend smoothly on the parameter which intervene as denominators for the radial functions:

$$\frac{1}{2i\pi} \int_\gamma \frac{r^\lambda}{a(\lambda)} \, d\lambda \, .$$

If there is a branching, it is impossible to factorize a into a product of stable monomials and all the functions giving the stable radial behavior are described by:

$$S[a, q_j; r] = \frac{1}{2i\pi} \int_\gamma \frac{r^\lambda \, q_j(\lambda)}{a(\lambda)} \, d\lambda \, ,$$

where $(q_j)_{j=1,\ldots,d}$ is a basis of the space $\mathbb{P}_{d-1}$ of polynomials of degree $< d$. In §5, we give examples for such formulas when $d = 2$ or 3.

1. Parameter dependent boundary value problems

1.a General setting. We consider a class of plane elliptic boundary value problems depending on a parameter

$$t \in \mathscr{T}, \quad \text{where} \quad \mathscr{T} \subset \mathbb{R} \quad \text{is a compact interval.}$$

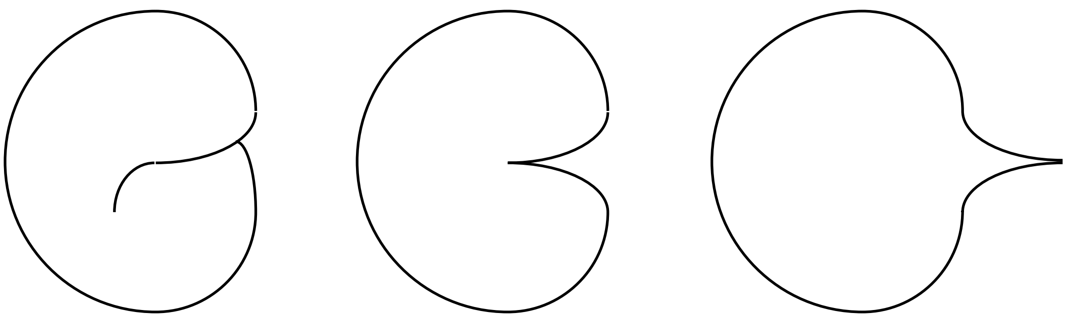
This means that we have:
- a family of bounded domains $\Omega_t \subset \mathbb{R}^2$,
- a family of elliptic systems of differential operators $\boldsymbol{L}_t$,
- a family of systems of boundary operators $\boldsymbol{B}_t$,

and we look for the solutions $\boldsymbol{u}_t$ of the problem

$$\begin{cases} \boldsymbol{L}_t \boldsymbol{u}_t = \boldsymbol{f}_t & \text{in} \quad \Omega_t \\ \boldsymbol{B}_t \boldsymbol{u}_t = \boldsymbol{g}_t & \text{on} \quad \partial\Omega_t \, . \end{cases}$$

Our results will be valid under the following global hypotheses — however, to simplify the notation, we will consider only a more restricted class of problems later on.

The domains Ω_t are piecewise smooth, i. e. their boundaries are composed of a finite number of $\mathscr{C}^\infty$ arcs $\partial_j\Omega_t$, $j = 1, \ldots, J$ whose dependence on t is also $\mathscr{C}^\infty$. The corner angles are $\mathscr{C}^\infty$ functions of t, and we assume that they are always contained in the interval $(0, 2\pi]$. Thus cracks and inward cusps are allowed (angle 2π) but not outward cusps (angle 0).



crack *inward cusp* *outward cusp*

Figure 1

A corner angle π corresponds to a point $\mathcal{O}$ where the boundary arcs meet tangentially. In Figure 2, we sketch an example of possible variations with respect to the parameter.

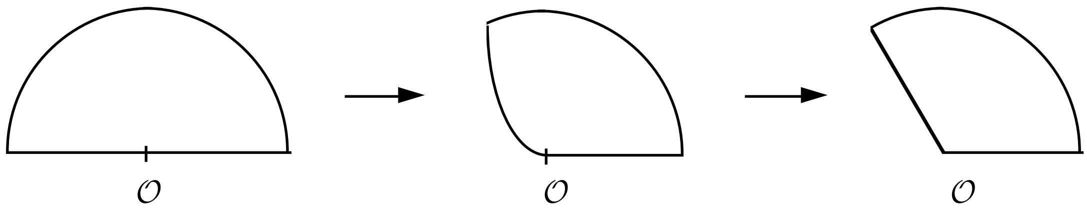


Figure 2

The operators $\boldsymbol{L}_t$ are systems properly elliptic in the sense of Agmon-Douglis-Nirenberg [1], their coefficients are $\mathscr{C}^\infty$ on $\overline{\Omega}_t$ and are $\mathscr{C}^\infty$ functions of the parameter $t \in \mathscr{T}$. On each arc $\partial_j\Omega_t$, the boundary operators $\boldsymbol{B}_t$ are defined by a system $\boldsymbol{B}_t^j$ which satisfies the usual covering condition and whose coefficients are $\mathscr{C}^\infty$ on $\overline{\partial_j\Omega_t}$ and are $\mathscr{C}^\infty$ functions of the parameter $t \in \mathscr{T}$ — so we can treat mixed boundary value problems.

For the right hand sides $\boldsymbol{f}_t$ and $\boldsymbol{g}_t$, we consider several levels of regularity with respect to the parameter t, namely mere boundedness, differentiability of order k, $\mathscr{C}^\infty$ or analyticity. Since differentiability with respect to t with values in some function space on Ω_t is, in general, not an intrinsic notion but requires the definition of diffeomorphisms between the Ω_t, by localization we will first reduce our class of domains to a more restricted class, where all domains Ω_t have only one corner and are the image of a single domain Ω by a family of (singular) diffeomorphisms χ_t.

1.b Localization. Let us denote $z = (z_1, z_2)$ the cartesian coordinates in $\mathbb{R}^2$. The first simplification is the localization at one "corner" $\mathcal{O}_t$ of Ω_t. $\mathcal{O}_t$ is the meeting point of two arcs $\partial_j\Omega_t$ and the function $t \mapsto \mathcal{O}_t$ is $\mathscr{C}^\infty$. We can always assume, via a global diffeomorphism on $\mathbb{R}^2$, $\mathscr{C}^\infty$ with respect to t, that the corner coincides with the origin $\mathcal{O}$ and that one of the two arcs meeting at $\mathcal{O}$ is a segment in the coordinate axis $z_2 = 0$. In a neighborhood of the origin $\mathcal{O}$, the domain Ω_t is described in polar coordinates (r, θ) by

$$0 \le \theta \le \omega_t(r)$$

where for a $r_0 > 0$

$$(t, r) \mapsto \omega_t(r) \text{ is } \mathscr{C}^\infty \text{ on } \mathscr{T} \times [0, r_0) \quad \text{and} \quad \forall t \in \mathscr{T},\ \ \omega_t(0) \in (0, 2\pi]\,.$$

As a convention, we denote by ω_t the opening of the tangent sector to Ω_t at $\mathcal{O}$, i. e.

$$\omega_t := \omega_t(0).$$

Note that, if there were no angle ω_t equal to π or 2π, we could even assume, after a global diffeomorphism, that $\omega_t(r)$ is a constant independent of r and t. But as we want to admit these somewhat delicate cases, we have to use a slightly more complicated setting.

In any case, there exists a family of diffeomorphisms

$$\chi_t \; : \; (r,\theta) \longmapsto (r_t,\theta_t)\,,$$

mapping a neighborhood of the corner $\mathcal{O}$ of a fixed domain Ω onto a neighborhood of the corner $\mathcal{O}_t$ of Ω_t. For Ω we can take any bounded domain whose boundary is $\mathscr{C}^\infty$ outside the origin and which coincides in the neighborhood of the origin with a sector

$$\Gamma = \{(r,\theta) \mid 0 < r < +\infty,\ 0 < \theta < \omega\}$$

of arbitrary opening $\omega \in (0, 2\pi)$. The mappings χ_t are smooth when expressed in polar coordinates:

$$(t,r,\theta) \mapsto r_t(r,\theta) \quad \text{and} \quad (t,r,\theta) \mapsto \theta_t(r,\theta) \quad \text{are } \mathscr{C}^\infty \text{ on } \mathscr{T} \times [0,r_0) \times [0,\omega]\,.$$

They satisfy

$$\begin{aligned} r_t(0,\theta) &= 0 \quad \text{for all } t,\theta\,, \\ \theta_t(r,0) &= 0, \quad \theta_t(r,\omega) = \omega_t(r_t(r,\omega)) \quad \text{for all } (t,r) \in \mathscr{T} \times [0,r_0)\,. \end{aligned}$$

For χ_t, we can always choose near the origin:

$$r_t = r, \qquad \theta_t(r,\theta) = \frac{\omega_t(r)}{\omega} \cdot \theta\,.$$

But such a diffeomorphism cannot be extended as a diffeomorphism (in cartesian coordinates) to the whole closure of Ω including the origin. Only if all angles ω_t are different from π and 2π, it is possible to choose ω and the χ_t so that they can be extended as $\mathscr{C}^\infty$ diffeomorphisms to all of $\mathbb{R}^2$ — namely, in this situation, either all angles ω_t are $< \pi$ and it suffices to take $\omega \in (0,\pi)$, or all angles ω_t are $> \pi$ and it suffices to take $\omega \in (\pi, 2\pi)$.

At this stage, we localize our problem in the neighborhood of the corner $\mathcal{O}_t$ and assume that the only corner of Ω_t is $\mathcal{O}_t$. To be completely specific, we assume that Ω is a subset of Γ, and we fix Ω_t globally by requiring

$$\chi_t \; : \; \Omega \longmapsto \Omega_t \quad \text{is a homeomorphism for all } t \in \mathscr{T}\,.$$

Since our stability results are of local nature, this is not really a restriction.

1.c ADN elliptic systems. We take as interior operators ADN-elliptic systems of multi-order $(m_1, \dots, m_N)$ as explained below. For the sake of simplicity, we consider Dirichlet boundary conditions only. We want to emphasize, however, that the corner singularities as constructed in the paper [4], are valid in the general situation described in Section 1.a and that the results of the present paper can be formulated for the general situation, too.

For the system $\boldsymbol{L}_t = (L_{t;kl})_{1\le k,l\le N}$ of partial differential operators, we make the following hypothesis: there exists $(m_1,\dots,m_N)\in\mathbb{N}^N$ such that $\boldsymbol{L}_t$ is ADN-elliptic of multi-degree $(m_k+m_l)_{1\le k,l\le N}$. This means that

$$\deg(L_{t;kl}) \le m_k+m_l, \quad 1\le k,l\le N$$

and if

$$\breve{\boldsymbol{L}}_t = (\breve{L}_{t;kl})_{1\le k,l\le N}$$

is the principal part of degree $(m_k+m_l)_{1\le k,l\le N}$ of $\boldsymbol{L}_t$, we assume that $\breve{\boldsymbol{L}}_t$ is properly elliptic in the sense of Agmon-Douglis-Nirenberg [1]. For such operators, the homogeneous Dirichlet conditions are written as

$$u_l \in \mathring{H}^{m_l}(\Omega_t), \quad 1\le l\le N$$

and this defines complementing boundary conditions.

Although, relying on [4], we could get the results of the present paper in the most general case, we are convinced that the simplifications in the notations and in the technical details of the proofs which are gained by making the symmetry assumption for the orders of the systems and by restricting to Dirichlet homogeneous conditions, improve the readability of our results sufficiently to justify this restriction. The formulation and proof of the corresponding results for the most general case require no essential new idea.

Let us denote by $\boldsymbol{m}$ the multi-order $(m_1,\dots,m_N)$. The Sobolev spaces naturally associated with the above problem are, for any $s\ge 0$:

$$\mathring{\boldsymbol{H}}^{\boldsymbol{m}} := \prod_{l=1}^{N} \mathring{H}^{m_l}, \qquad \boldsymbol{H}^{s+\boldsymbol{m}} := \prod_{l=1}^{N} H^{s+m_l}, \qquad \boldsymbol{H}^{s-\boldsymbol{m}} := \prod_{k=1}^{N} H^{s-m_k},$$

where the Sobolev spaces H^s are defined as usual (see [8]) and $\mathring{H}^s(\Omega)$ denotes the closure of $\mathscr{C}_0^\infty(\Omega)$ in $H^s(\Omega)$.

Thus the operator $\boldsymbol{L}_t$ is continuous from $\boldsymbol{H}^{s+\boldsymbol{m}}(\Omega_t)$ to $\boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t)$ for all $s\ge 0$, and the homogeneous Dirichlet conditions are defined by the condition $\boldsymbol{u}\in\mathring{\boldsymbol{H}}^{\boldsymbol{m}}(\Omega_t)$. We consider therefore the boundary value problem

$$\begin{cases} \boldsymbol{L}_t\boldsymbol{u}_t &=& \boldsymbol{f}_t \quad \text{in} \quad \Omega_t \\ \boldsymbol{u}_t &\in& \mathring{\boldsymbol{H}}^{\boldsymbol{m}}(\Omega_t)\,. \end{cases}$$

For the right hand side, we assume the (spatial) regularity

$$\boldsymbol{f}_t \in \boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t) \quad \text{for all} \quad t\in\mathscr{T}$$

with some $s>0$.

Assuming some initial regularity for $\boldsymbol{u}_t$:

$$\boldsymbol{u}_t \in \boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t) \quad \text{with} \quad 0 \le s_0 < s\,,$$

we have then for every $t \in \mathscr{T}$ the decomposition of $\boldsymbol{u}_t$ into corner singularities and a regular part corresponding to $\boldsymbol{H}^{s+\boldsymbol{m}}(\Omega_t)$ regularity. To be more precise, we have to introduce the spectral problem associated to the principal part $\boldsymbol{M}_t$ of $\boldsymbol{L}_t$ at the origin:

$$\boldsymbol{M}_t(\partial_z) = \breve{\boldsymbol{L}}_t(0, \partial_z)\,.$$

Let Λ_t denote the set of all complex numbers λ such that there exists a non-zero function, multi-homogeneous of degree $\lambda + \boldsymbol{m}$, of the form

$$\boldsymbol{u}(r,\theta) = r^{\lambda+\boldsymbol{m}}\boldsymbol{v}(\theta), \qquad \boldsymbol{v} \in \mathring{\boldsymbol{H}}^{\boldsymbol{m}}(0,\omega_t)$$

or in components ($l = 1, \dots, N$)

$$u_l(r,\theta) = r^{\lambda+m_l}v_l(\theta), \qquad v_l \in \mathring{H}^{m_l}(0,\omega_t)$$

that satisfies

$$\boldsymbol{M}_t\boldsymbol{u} = 0 \qquad \text{for} \quad r > 0, \quad \theta \in [0,\omega_t].$$

Λ_t is a discrete set in $\mathbb{C}$. It is the spectrum of a certain operator function $\mathring{\mathscr{A}}_t(\lambda)$ which we consider later on (see §2.a).

1.d Uniform estimates.

Here is our first stability result.

Theorem 1.1 *Let $s_0, s \in \mathbb{R}$ be given such that $0 \le s_0 < s$ and*

$$\operatorname{Re}\lambda \ne s-1 \quad \textit{for all} \quad \lambda \in \Lambda_t,\ t \in \mathscr{T} \qquad \textit{and} \qquad s \notin \mathbb{N}\,. \tag{1.1}$$

(i) *Then there exists a finite number of singular functions $\boldsymbol{S}_{\ell;t}$ ($\ell = 1, \dots, \mathcal{L}$) which are $\mathscr{C}^\infty$ functions of $t \in \mathscr{T}$, $r > 0$ (of course, not at $r = 0$!), and $\theta \in \mathbb{R}$ such that any solution $\boldsymbol{u}_t$ of the problem*

$$\begin{cases} \boldsymbol{L}_t\boldsymbol{u}_t &= \boldsymbol{f}_t \quad \textit{in} \quad \Omega_t \\ \boldsymbol{u}_t &\in \mathring{\boldsymbol{H}}^{\boldsymbol{m}}(\Omega_t)\,, \end{cases} \tag{1.2}$$

for which there holds

$$\boldsymbol{u}_t \in \boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)\,, \qquad \boldsymbol{f}_t \in \boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t)\,, \tag{1.3}$$

admits a decomposition

$$\boldsymbol{u}_t = \sum_{\ell=1}^{\mathcal{L}} c_{\ell;t}\,\boldsymbol{S}_{\ell;t} + \boldsymbol{u}_{\mathrm{reg};\,t} \tag{1.4}$$

with certain $c_{\ell;t} \in \mathbb{C}$ *and* $\boldsymbol{u}_{\mathrm{reg};\,t} \in \boldsymbol{H}^{s+\boldsymbol{m}}(\Omega_t)$*, with* $r^{-s-\boldsymbol{m}}\boldsymbol{u}_{\mathrm{reg};\,t} \in \boldsymbol{L}^2(\Omega_t)$*. Moreover, there is a constant* C*, independent of* t*,* $\boldsymbol{u}$ *and* $\boldsymbol{f}$*, such that for all* $t \in \mathscr{T}$ *there holds*

$$\sum_{\ell=1}^{\mathcal{L}} |c_{\ell;t}| + \|\boldsymbol{u}_{\mathrm{reg};\,t}\|_{\boldsymbol{H}^{s+\boldsymbol{m}}(\Omega_t)} + \|r^{-s-\boldsymbol{m}}\boldsymbol{u}_{\mathrm{reg};\,t}\|_{\boldsymbol{L}^2(\Omega_t)} \leq C\left(\|\boldsymbol{u}_t\|_{\boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)} + \|\boldsymbol{f}_t\|_{\boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t)}\right). \tag{1.5}$$

(ii) *If all angles* ω_t *are different from* π *and* 2π*, the condition* $s \notin \mathbb{N}$ *in* (1.1) *can be removed and all conclusions of* (i) *still hold, except that now* $r^{-s-\boldsymbol{m}+\varepsilon}\boldsymbol{u}_{\mathrm{reg};\,t} \in \boldsymbol{L}^2(\Omega_t)$ *and the estimate* (1.5) *must be replaced by*

$$\sum_{\ell=1}^{\mathcal{L}} |c_{\ell;t}| + \|\boldsymbol{u}_{\mathrm{reg};\,t}\|_{\boldsymbol{H}^{s+\boldsymbol{m}}(\Omega_t)} + \|r^{-s-\boldsymbol{m}+\varepsilon}\boldsymbol{u}_{\mathrm{reg};\,t}\|_{\boldsymbol{L}^2(\Omega_t)} \leq C\left(\|\boldsymbol{u}_t\|_{\boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)} + \|\boldsymbol{f}_t\|_{\boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t)}\right), \tag{1.6}$$

where $\varepsilon > 0$ *is an arbitrarily small fixed number.*

(iii) *If, in a neighborhood of* $r = 0$*, the diffeomorphisms* χ_t *and the operators* $\boldsymbol{L}_t$ *depend analytically on* t*, then, for any* $t_0 \in \mathscr{T}$*, there exists a decomposition as in* (1.4) *for* t *in a neighborhood of* t_0 *with singular functions* $\boldsymbol{S}_{\ell;t}$ *depending analytically on* t*.*

If the data depend regularly on the parameter t, the splitting (1.4) depends regularly on t too, as we are going to explain now.

1.e Parameter regularity. In order to describe higher regularity with respect to the parameter, we need weighted Sobolev spaces $V_0^s(\Omega_t)$. Their definition is for $s \geq 0$ (see [9], [5], [2]):

$$v \in V_0^s(\Omega_t) \iff v \in H^s(\Omega_t) \quad \text{and} \quad r^{-s}v \in L^2(\Omega_t).$$

The norm is $\|v\|^2_{V_0^s(\Omega_t)} = \|v\|^2_{H^s(\Omega_t)} + \|r^{-s}v\|^2_{L^2(\Omega_t)}$. An equivalent norm would be

$$\|v\|_{H^s(\Omega_t)} + \sum_{|\beta|\leq s} \|r^{-s+|\beta|}\partial_z^\beta v\|_{L^2(\Omega_t)}.$$

For further reference we define for $\delta \in \mathbb{R}$

$$V_\delta^s(\Omega_t) = \{v \mid \ r^\delta v \in V_0^s(\Omega_t)\}.$$

The spaces V_0^s differ from H^s by the Taylor expansion at the origin. This is expressed by the following well-known lemma [5].

Lemma 1.2 *Let* $s > 0$*. For* $v \in H^s(\Omega_t)$ *there exist the traces at the origin* $\partial_z^\beta v(0)$*, for* $0 \leq |\beta| < s - 1$*. Let*

$$v_0(z) := v(z) - \sum_{|\beta|<s-1} \frac{z^\beta}{\beta!}\,\partial_z^\beta v(0).$$

(i) *If $s \notin \mathbb{N}$, v_0 belongs to $V_0^s(\Omega_t)$, and*

$$\|v_0\|_{V_0^s(\Omega_t)} + \sum_{|\beta|<s-1} |\partial_z^\beta v(0)| \le C \, \|v\|_{H^s(\Omega_t)} .$$

(ii) *If $s \in \mathbb{N}$, there (only) holds*

$$v_0 \in H^s(\Omega_t) \quad \textit{and} \quad r^{-s+\varepsilon} v_0 \in L^2(\Omega_t), \quad \forall \varepsilon > 0$$

(in other words, $v_0 \in H^s(\Omega_t) \cap V_\varepsilon^s(\Omega_t)$) and the corresponding norm estimate.

The importance of the spaces V_0^s lies in the fact that they are invariant with respect to the class of diffeomorphisms χ_t introduced in §1.b. Thus

$$\chi_t^* : v \mapsto v \circ \chi_t \qquad V_0^s(\Omega_t) \to V_0^s(\Omega)$$

is an isomorphism. This is an easy consequence of the representation of V_0^s in polar coordinates (see [5]). On the other hand, χ_t^* is not an isomorphism between $H^s(\Omega_t)$ and $H^s(\Omega)$, in general. Only if χ_t is a diffeomorphism of $\mathbb{R}^2$ in cartesian coordinates (which, once again, can be assumed if there are no angles π or 2π), the spaces H^s are invariant under χ_t^*.

We can now define the regularity with respect to $t \in \mathscr{T}$. Let $\kappa \in \{0, 1, , \ldots, +\infty\}$. Then the function $t \mapsto v_t$ is κ-times differentiable with values in $V_0^s(\Omega_t)$,

$$t \mapsto v_t \in \mathscr{C}^\kappa(\mathscr{T}, V_0^s(\Omega_t))$$

if the function $t \mapsto \chi_t^* v_t$ is in $\mathscr{C}^\kappa(\mathscr{T}, V_0^s(\Omega))$ in the usual sense, i. e.

$$t \mapsto \frac{d^j}{dt^j}(v_t \circ \chi_t) \quad \text{is continuous with values in} \quad V_0^s(\Omega), \quad j = 0, 1, \ldots, \kappa.$$

We abbreviate this as

$$t \mapsto v_t \in \mathscr{C}^\kappa(\mathscr{T}, V_0^s(\Omega_t)) \quad \Longleftrightarrow \quad t \mapsto \chi_t^* v_t \in \mathscr{C}^\kappa(\mathscr{T}, V_0^s(\Omega)). \tag{1.7}$$

We define analyticity with respect to t analogously:

$$t \mapsto v_t \in \mathscr{A}(\mathscr{T}, V_0^s(\Omega_t)) \quad \Longleftrightarrow \quad t \mapsto \chi_t^* v_t \in \mathscr{A}(\mathscr{T}, V_0^s(\Omega)). \tag{1.8}$$

If χ_t are diffeomorphisms in cartesian coordinates, we can use the corresponding definitions for $H^s(\Omega_t)$ instead of $V_0^s(\Omega_t)$:

$$t \mapsto v_t \in \mathscr{C}^\kappa(\mathscr{T}, H^s(\Omega_t)) \quad \Longleftrightarrow \quad t \mapsto \chi_t^* v_t \in \mathscr{C}^\kappa(\mathscr{T}, H^s(\Omega)). \tag{1.9}$$

In the general case for $s \notin \mathbb{N}$, we use the Taylor expansion

$$v_t(z) = v_{0;t}(z) + \sum_{|\beta|<s-1} \frac{z^\beta}{\beta!} \, \partial_z^\beta v_t(0) \tag{1.10}$$

and define

$$t \mapsto v_t \in \mathscr{C}^\kappa(\mathscr{T}, H^s(\Omega_t)) \Longleftrightarrow \begin{cases} \quad t \mapsto v_{0;t} \in \mathscr{C}^\kappa(\mathscr{T}, V_0^s(\Omega_t)) \\ \text{and} \\ \quad t \mapsto \partial_z^\beta v_t(0) \in \mathscr{C}^\kappa(\mathscr{T}) \quad (|\beta| < s-1), \end{cases} \tag{1.11}$$

respectively

$$t \mapsto v_t \in \mathscr{A}(\mathscr{T}, H^s(\Omega_t)) \Longleftrightarrow \begin{cases} \quad t \mapsto v_{0;t} \in \mathscr{A}(\mathscr{T}, V_0^s(\Omega_t)) \\ \text{and} \\ \quad t \mapsto \partial_z^\beta v_t(0) \in \mathscr{A}(\mathscr{T}) \quad (|\beta| < s-1). \end{cases} \tag{1.12}$$

In the case where the previous definition (1.9) of $\mathscr{C}^\kappa(\mathscr{T}, H^s(\Omega_t))$ is applicable, it is equivalent to the definition (1.11). One should note, however, that for $\kappa \geq 1$ these definitions are not "intrinsic". They depend on the choice of the diffeomorphisms χ_t, not only on the family of domains Ω_t.

Theorem 1.3 *Let $s_0, s \in \mathbb{R}$ be given such that $0 \leq s_0 < s$ and such that* (1.1) *holds. Let $\kappa \in \{0, 1, , \ldots, +\infty\}$ be given.*

(i) *Then any solution $\boldsymbol{u}_t$ of the problem* (1.2) *for which there holds*

$$t \mapsto \boldsymbol{u}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)), \qquad t \mapsto \boldsymbol{f}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t)), \tag{1.13}$$

admits the decomposition (1.4) *with coefficients $c_{\ell;t} \in \mathbb{C}$ and a regular part $\boldsymbol{u}_{\mathrm{reg};\,t} \in \boldsymbol{V}_0^{s+\boldsymbol{m}}(\Omega_t)$ which satisfy the estimate* (1.5) *and moreover*

$$t \mapsto c_{\ell;t} \in \mathscr{C}^\kappa(\mathscr{T}) \qquad t \mapsto \boldsymbol{u}_{\mathrm{reg};\,t} \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_0^{s+\boldsymbol{m}}(\Omega_t)). \tag{1.14}$$

(ii) *If all angles ω_t are different from π and 2π, the condition $s \notin \mathbb{N}$ in* (1.1) *can be removed and then any solution $\boldsymbol{u}_t$ of the problem* (1.2) *for which* (1.13) *holds admits the decomposition* (1.4) *with coefficients $c_{\ell;t} \in \mathbb{C}$ and a regular part $\boldsymbol{u}_{\mathrm{reg};\,t} \in \boldsymbol{H}^{s+\boldsymbol{m}}(\Omega_t)$ which satisfy the estimate* (1.6) *and moreover*

$$t \mapsto c_{\ell;t} \in \mathscr{C}^\kappa(\mathscr{T}) \qquad t \mapsto \boldsymbol{u}_{\mathrm{reg};\,t} \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{H}^{s+\boldsymbol{m}}(\Omega_t)). \tag{1.15}$$

(iii) *If, in a neighborhood $\mathscr{U}$ of $r = 0$, the diffeomorphisms χ_t and the operators $\boldsymbol{L}_t$ depend analytically on t, then to any $t_0 \in \mathscr{T}$ there exists a neighborhood $\mathscr{T}_0$ of t_0 in $\mathscr{T}$ such that for any solution $\boldsymbol{u}_t$ of the problem* (1.2) *with support in $\mathscr{U}$ for which there holds*

$$t \mapsto \boldsymbol{u}_t \in \mathscr{A}(\mathscr{T}_0, \boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)), \qquad t \mapsto \boldsymbol{f}_t \in \mathscr{A}(\mathscr{T}_0, \boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t)), \tag{1.16}$$

$\boldsymbol{u}_t$ admits the decomposition (1.4) *and moreover*

$$t \mapsto c_{\ell;t} \in \mathscr{A}(\mathscr{T}_0) \qquad t \mapsto \boldsymbol{u}_{\mathrm{reg};\,t} \in \mathscr{A}(\mathscr{T}_0, \boldsymbol{H}^{s+\boldsymbol{m}}(\Omega_t)). \tag{1.17}$$

Remark 1.4 If the operator $\boldsymbol{L}_t$ and the index s_0 are such that for the problem (1.2) there holds uniqueness of the solution $\boldsymbol{u}_t \in \boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)$, then the initial regularity hypothesis of $\boldsymbol{u}_t$ with respect to t, as required in (1.13), is not needed. It is then a consequence of the hypothesis on $\boldsymbol{f}_t$ in (1.13) (see also Remark 4.8*(ii)*).

■

2. Construction of stable singular functions

Let $\boldsymbol{M}_t(\partial_z)$ denote, as in §1, the principal part of the operator $\boldsymbol{L}_t(z;\partial_z)$ at the origin. As "usual" (see [3], [4]) the description of the singularities is performed in 3 steps:

- The singular functions of $\boldsymbol{M}_t(\partial_z)$ with zero right hand side;
- The singular functions of $\boldsymbol{M}_t(\partial_z)$ with polynomial right hand sides;
- The singular functions of $\boldsymbol{M}_t(z;\partial_z)$ with singular right hand sides.

Each step is performed in two stages: firstly solving an interior equation $\boldsymbol{M}_t\boldsymbol{u}_t^0 = \boldsymbol{f}_t$, secondly solving boundary conditions $\boldsymbol{C}_t\boldsymbol{u}_t = \boldsymbol{g}_t$ together with the equation $\boldsymbol{M}_t\boldsymbol{u}_t = 0$. For the first step, $\boldsymbol{f}_t$ is 0; for the second one $\boldsymbol{f}_t$ is polynomial; finally, for the third step $\boldsymbol{f}_t$ is generated by the solutions from the two previous steps and by an induction procedure.

2.a Singularities with zero right hand side. It is well known that any non-zero solution $\boldsymbol{u}$ of $\boldsymbol{M}_t\boldsymbol{u} = 0$ with homogeneous Dirichlet boundary conditions, which has the form

$$\boldsymbol{u}(r,\theta) = \sum_{q=0}^{Q} r^{\lambda+\boldsymbol{m}} \log^q r\, \boldsymbol{v}_q(\theta) \quad \text{with} \quad \boldsymbol{v}_q \in \mathring{\boldsymbol{H}}^{\boldsymbol{m}}(0,\omega_t) \tag{2.1}$$

with

$$s_0 - 1 < \operatorname{Re}\lambda < s - 1,$$

contributes to the singularities of the problem (1.2) with conditions (1.3). When $\boldsymbol{L}_t = \boldsymbol{M}_t$, i. e. when $\boldsymbol{L}_t$ is homogeneous with constant coefficients, and when Ω_t coincides with a sector near the origin, the above functions generate the whole space of singular functions of the problem (1.2) with $\boldsymbol{u}_t \in \boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)$ and $\boldsymbol{f}_t \in \boldsymbol{V}_0^{s-\boldsymbol{m}}(\Omega_t)$.

This is the reason for the introduction of the space $\mathscr{X}_t(s_0,s)$ as the space generated by the solutions $\boldsymbol{u}$ of $\boldsymbol{M}_t\boldsymbol{u} = 0$ of the form (2.1) with $s_0 - 1 < \operatorname{Re}\lambda < s - 1$. We are going to investigate the stability of the spaces $\mathscr{X}_t$ with respect to t and construct stable generators for them. To do that, we write $\boldsymbol{M}_t$ in polar coordinates as

$$\boldsymbol{M}_t(\partial_z) = \Big(\operatorname{diag}(r^{-\boldsymbol{m}})\Big) \times \mathscr{M}_t(\theta;\partial_\theta, r\partial_r) \times \Big(\operatorname{diag}(r^{-\boldsymbol{m}})\Big)$$

i. e. in components

$$\boldsymbol{M}_{t;kl}(\partial_z) = r^{-m_k}\, \mathscr{M}_{t;kl}(\theta;\partial_\theta, r\partial_r)\, r^{-m_l}$$

With this notation, any multi-homogeneous solution

$$\boldsymbol{u}(r,\theta) = r^{\lambda+\boldsymbol{m}}\boldsymbol{v}(\theta) \quad \text{of} \quad \boldsymbol{M}_t\boldsymbol{u} = 0$$

corresponds to a solution $\boldsymbol{v}(\theta)$ of the system of ordinary differential equations

$$\mathscr{M}_t(\theta;\partial_\theta,\lambda)\,\boldsymbol{v}(\theta) = 0. \tag{2.2}$$

Since we are considering homogeneous Dirichlet boundary conditions, we consider this differential operator on the space $\overset{\circ}{\boldsymbol{H}}{}^{\boldsymbol{m}}(0,\omega_t)$. Thus we define the family of operators

$$\begin{array}{rccc} \overset{\circ}{\mathscr{A}}_t(\lambda): & \overset{\circ}{\boldsymbol{H}}{}^{\boldsymbol{m}}(0,\omega_t) & \to & \boldsymbol{H}^{-\boldsymbol{m}}(0,\omega_t) \\ & \boldsymbol{v}(\theta) & \mapsto & \mathscr{M}_t(\theta;\partial_\theta,\lambda)\,\boldsymbol{v}(\theta) \end{array} \tag{2.3}$$

Later on, we shall introduce the corresponding operator family $\mathscr{A}_t(\lambda)$ with non-zero boundary conditions.

Due to the ellipticity of $\boldsymbol{M}_t$, $\lambda \mapsto \overset{\circ}{\mathscr{A}}_t(\lambda)$ has a meromorphic resolvent $\lambda \mapsto \overset{\circ}{\mathscr{A}}_t(\lambda)^{-1}$. The poles of this resolvent are given by the set Λ_t introduced above in §1.c, and its residues generate the spaces $\mathscr{X}_t$ of singularities (see [7] and [4, Lemma 4.1]):

Lemma 2.1 *Let γ be a simple closed contour in $\mathbb{C}$. Let $\mathscr{X}_t(\gamma)$ be the space generated by the solutions $\boldsymbol{u}$ of $\boldsymbol{M}_t\boldsymbol{u} = 0$ of the form* (2.1) *with $\lambda \in$ int γ. If $\gamma \cap \Lambda_t$ is empty, then*

$$\mathscr{X}_t(\gamma) = \{\int_\gamma r^{\lambda+\boldsymbol{m}} \overset{\circ}{\mathscr{A}}_t(\lambda)^{-1}\Psi(\lambda)\,d\lambda \mid \Psi(\lambda) \text{ holomorphic with values in } \boldsymbol{H}^{-\boldsymbol{m}}(0,\omega_t)\}$$

As a consequence of the ellipticity, the infinite strip $s_0 - 1 < \operatorname{Re}\lambda < s - 1$ contains only a finite number of poles in Λ_t and there exists a closed contour $\gamma(s_0, s)$ such that $\mathscr{X}_t(s_0,s) = \mathscr{X}_t(\gamma)$. Conversely, let $\gamma_t^1, \ldots, \gamma_t^J$ be a system of simple closed non-intersecting curves such that

$$\Lambda_t \cap \{\lambda \mid \operatorname{Re}\lambda \in (s_0-1, s-1)\} \subset \bigcup_{j=1}^{J} \mathit{int}\,\gamma_t^j.$$

Then

$$\mathscr{X}_t(s_0,s) = \bigoplus_{j=1}^{J} \mathscr{X}_t(\gamma_t^j). \tag{2.4}$$

Our first aim is the description of generators of $\mathscr{X}_t(\gamma)$ (see Corollary 2.8). To do this, we study the resolvent $\overset{\circ}{\mathscr{A}}_t(\lambda)^{-1}$: we construct first a basis of solutions of the homogeneous system (2.2) *without boundary conditions.* This construction was presented in detail in [4]. We describe here the results and illustrate them by some examples.

In the following construction, $t \in \mathscr{T}$ is a fixed value. We will consider the dependence (stability, regularity) on t later.

Let $\boldsymbol{M}_t(\xi_1,\xi_2)$ be the symbol of the operator $\boldsymbol{M}_t(\partial_{z_1},\partial_{z_2})$. We write it in Cayley-transformed coordinates:

$$\boldsymbol{M}_{t;+}(\alpha) := \boldsymbol{M}_t(\alpha+1, i(\alpha-1))\,; \quad \boldsymbol{M}_{t;-}(\alpha) := \boldsymbol{M}_t(1+\alpha, i(1-\alpha)) \quad (\alpha\in\mathbb{C}).$$

These are matrix polynomials in α whose determinants, due to the ellipticity of $\boldsymbol{M}_t$ do not vanish on the unit circle

$$\tilde{\gamma} = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}.$$

For any polynomial

$$A(\alpha) = \sum_{n=0}^{d} A_n\, \alpha^n,$$

we define the shifted polynomials $A^{\sharp\,\delta}(\alpha)$, $\delta = 0, \dots, d$ by

$$A^{\sharp\,\delta}(\alpha) := \sum_{n=\delta}^{d} A_n\, \alpha^{n-\delta}\,.$$

Finally, we need the following diagonal $N \times N$ matrices:

$$\begin{aligned} \boldsymbol{Z}^{+}(\lambda;\zeta,\zeta^*;\alpha) &= \left(\frac{(\alpha\zeta+\zeta^*)^{\lambda+m_l}}{(\lambda+m_l)(\lambda+m_l-1)\cdots(\lambda+1)}\,\delta_{kl}\right)_{1\le k,l\le N} \\ \boldsymbol{Z}^{-}(\lambda;\zeta,\zeta^*;\alpha) &= \left(\frac{(\zeta+\alpha\zeta^*)^{\lambda+m_l}}{(\lambda+m_l)(\lambda+m_l-1)\cdots(\lambda+1)}\,\delta_{kl}\right)_{1\le k,l\le N} \end{aligned} \tag{2.5}$$

If we identify $z \in \mathbb{R}^2$ with $\zeta = z_1 + i\, z_2 \in \mathbb{C}$, we have the relation

$$\boldsymbol{M}_t(\partial_{z_1}, \partial_{z_2})\, \boldsymbol{Z}^{\pm}(\lambda;\zeta,\overline{\zeta};\alpha) = \widetilde{\boldsymbol{Z}}^{\pm}(\lambda;\zeta,\overline{\zeta};\alpha)\, \boldsymbol{M}_{t,\pm}(\alpha) \tag{2.6}$$

for any λ, ζ, $\alpha \in \mathbb{C}$, where $\widetilde{\boldsymbol{Z}}^{\pm}$ is defined similarly to $\boldsymbol{Z}^{\pm}$ as

$$\begin{aligned} \widetilde{\boldsymbol{Z}}^{+}(\lambda;\zeta,\zeta^*;\alpha) &= \Big(\lambda(\lambda-1)\cdots(\lambda-m_k+1)\,(\alpha\zeta+\zeta^*)^{\lambda-m_k}\,\delta_{kl}\Big)_{1\le k,l\le N} \\ \widetilde{\boldsymbol{Z}}^{-}(\lambda;\zeta,\zeta^*;\alpha) &= \Big(\lambda(\lambda-1)\cdots(\lambda-m_k+1)\,(\zeta+\alpha\zeta^*)^{\lambda-m_k}\,\delta_{kl}\Big)_{1\le k,l\le N} \end{aligned} \tag{2.7}$$

We can now describe the space of solutions of the system (2.2). Let us denote

$$\mathfrak{W}_t(\lambda) := \{\boldsymbol{u}(r,\theta) = r^{\lambda+\boldsymbol{m}}\boldsymbol{v}(\theta) \mid \boldsymbol{M}_t(\partial_z)\,\boldsymbol{u} = 0 \quad \text{for} \quad \theta \in (0,\omega_t)\}$$

and its angular part

$$\mathscr{W}_t(\lambda) := \{\boldsymbol{v}(\theta) \in \mathscr{C}^\infty(\mathbb{R}) \otimes \mathbb{C}^N \mid \mathscr{M}_t(\theta;\partial_\theta,\lambda)\,\boldsymbol{v}(\theta) = 0 \quad \text{for} \quad \theta \in (0,\omega_t)\},$$

so that

$$\boldsymbol{u} = \Big(r^{\lambda+m_1}v_1(\theta),\dots,r^{\lambda+m_N}v_N(\theta)\Big) \in \mathfrak{W}_t(\lambda) \iff \boldsymbol{v} = (v_1,\dots,v_N) \in \mathscr{W}_t(\lambda)\,. \tag{2.8}$$

The following theorem was shown in [4, Theorem 2.1 and Lemma 4.2]:

Theorem 2.2 *Let λ be any complex number, and let d be the maximal degree of all the matrix elements of $\boldsymbol{M}_{t;\pm}$ as polynomials in α. We define:*

$$\begin{aligned} \mathfrak{W}_t^{\pm}(\lambda) := \Big\{\zeta \mapsto \boldsymbol{w}(\lambda,\zeta) \;\mid\; \exists \boldsymbol{q}^1,\dots,\boldsymbol{q}^d \in \mathbb{C}^N \\ \boldsymbol{w}(\lambda,\zeta) = \int_{\tilde{\gamma}} \boldsymbol{Z}^{\pm}(\lambda;\zeta,\overline{\zeta};\alpha)\,\boldsymbol{M}_{t;\pm}^{-1}(\alpha)\Big(\boldsymbol{M}_{t;\pm}^{\sharp\,d}(\alpha)\,\boldsymbol{q}^1 + \cdots + \boldsymbol{M}_{t;\pm}^{\sharp\,1}(\alpha)\,\boldsymbol{q}^d\Big)\,d\alpha\Big\}. \end{aligned} \tag{2.9}$$

(i) *For all* $\lambda \in \mathbb{C} \setminus \mathbb{N}$, *the dimension of* $\mathfrak{W}_t^{\pm}(\lambda)$ *is equal to* $|\boldsymbol{m}|$,

$$\mathfrak{W}_t^{+}(\lambda) \oplus \mathfrak{W}_t^{-}(\lambda) = \mathfrak{W}_t(\lambda)\,,$$

and there exists $\boldsymbol{q}^1_{\pm,h}, \dots, \boldsymbol{q}^d_{\pm,h}$ *in* $\mathbb{C}^N$ *for* $h = 1, \dots, |\boldsymbol{m}|$ *(independent of* λ *but possibly dependent on* t*, see Remark* 2.3 *below) such that the*

$$\boldsymbol{w}^{\pm}_{t;h}(\lambda,\zeta) := \int_{\tilde{\gamma}} \boldsymbol{Z}^{\pm}(\lambda;\zeta,\overline{\zeta};\alpha)\,\boldsymbol{M}^{-1}_{t;\pm}(\alpha)\left(\boldsymbol{M}^{\sharp\,d}_{t;\pm}(\alpha)\,\boldsymbol{q}^1_{\pm,h} + \cdots + \boldsymbol{M}^{\sharp\,1}_{t;\pm}(\alpha)\,\boldsymbol{q}^d_{\pm,h}\right) d\alpha$$

for $h = 1, \dots, |\boldsymbol{m}|$ *are a basis of* $\mathfrak{W}_t^{\pm}(\lambda)$.

(ii) *the functions* $\boldsymbol{v}_{t;h}(\lambda,\cdot)$ *defined for* $h = 1, \dots, 2|\boldsymbol{m}|$ *by*

$$\boldsymbol{v}_{t;h}(\lambda,\theta) := \begin{cases} \boldsymbol{w}^{+}_{t;h}(\lambda, e^{i\theta}) & \textit{for} \quad h = 1, \dots, |\boldsymbol{m}| \\ \boldsymbol{w}^{-}_{t;h-\boldsymbol{m}}(\lambda, e^{i\theta}) & \textit{for} \quad h = m+1, \dots, 2|\boldsymbol{m}| \end{cases}$$

are a basis of $\mathscr{W}_t(\lambda)$.

Remark 2.3 There is a lot of freedom in the choice of the vectors $\boldsymbol{q}^j_{\pm,h}$. It was shown in [4] that they can be chosen independently of $\lambda \in \mathbb{C} \setminus \mathbb{N}$. If a choice $\left(\boldsymbol{q}^1_{\pm,h}(t_0), \dots, \boldsymbol{q}^d_{\pm,h}(t_0)\right)_{h=1,\dots,|\boldsymbol{m}|}$ yields linearly independent functions $\boldsymbol{w}^{\pm}_{t_0;h}$ for one $t_0 \in \mathscr{T}$, this choice will work also for t in a neighborhood of t_0 due to the continuous dependence of $\boldsymbol{M}_{t;\pm}$ on t. Thus if $\mathscr{T}$ is sufficiently small, the vectors $\boldsymbol{q}^j_{\pm,h}$ can be chosen independently of t. In general, they will depend on t, but it is easy to see that they can be chosen as analytic (even polynomial) functions of $t \in \mathscr{T}$. In the sequel we assume that such a choice is fixed. In any case, the basis functions $\boldsymbol{w}^{\pm}_{t;h}$ and $\boldsymbol{v}_{t;h}$ are $\mathscr{C}^\infty$ with respect to t (or analytic, if $\boldsymbol{M}_t$ depends analytically on t). ■

Remark 2.4 From the representation (2.9) and the residue theorem, it is clear that the components of the elements of $\mathfrak{W}_t^{\pm}(\lambda)$ consist of linear combinations of the elements of $\boldsymbol{Z}^{\pm}$ and of their derivatives with respect to α, where α takes values α_0 in the zero set of $\det \boldsymbol{M}_{t;\pm}(\alpha)$ inside the unit circle.

More precisely, if $\boldsymbol{w} \in \mathfrak{W}_t^{+}(\lambda)$, then $\boldsymbol{w} = (w_1, \dots, w_N)$, and $w_l(z)$ is given in terms of the complex variable ζ as a linear combination of functions of the form

$$\zeta^n(\alpha_0\zeta + \overline{\zeta})^{\lambda+m_l-n} = \Big(\frac{\zeta}{\overline{\zeta}}\Big)^n \overline{\zeta}^{\lambda+m_l}\Big(1 + \alpha_0\,\frac{\zeta}{\overline{\zeta}}\Big)^{\lambda+m_l-n}$$

where $n \in \mathbb{N}$ and $\det \boldsymbol{M}_{t;+}(\alpha_0) = 0$, $|\alpha_0| < 1$ ($n = 0$ if α_0 is a simple pole of $\boldsymbol{M}^{-1}_{t;\pm}$). Similarly, if $\boldsymbol{w} \in \mathfrak{W}_t^{-}(\lambda)$, then its component $w_l(z)$ is a linear combination of functions of the form

$$\overline{\zeta}^n(\alpha_0\overline{\zeta} + \zeta)^{\lambda+m_l-n} = \Big(\frac{\overline{\zeta}}{\zeta}\Big)^n \zeta^{\lambda+m_l}\Big(1 + \alpha_0\,\frac{\overline{\zeta}}{\zeta}\Big)^{\lambda+m_l-n}$$

where $n \in \mathbb{N}$ and $\det \boldsymbol{M}_{t;-}(\alpha_0) = 0$, $|\alpha_0| < 1$.

Note that these functions are well defined for $|\alpha| < 1$ and ζ in a sector of opening $\omega \leq 2\pi$. ■

Example 2.5

(i) For the scalar case $N = 1$, we denote by $d_\pm$ the degrees of the polynomials $M_\pm(\alpha)$. For the sake of simplicity, we omit the index t. The representation (2.9) then simplifies to

$$\mathfrak{W}^+(\lambda) = \Big\{ \int_{\tilde\gamma} (\alpha\zeta + \overline{\zeta})^{\lambda+\boldsymbol{m}}\, M_+^{-1}(\alpha)\, f(\alpha)\, d\alpha \;\mid\; f \in \mathbb{P}_{d_+-1}[\alpha] \Big\};$$

$$\mathfrak{W}^-(\lambda) = \Big\{ \int_{\tilde\gamma} (\zeta + \alpha\overline{\zeta})^{\lambda+\boldsymbol{m}}\, M_-^{-1}(\alpha)\, f(\alpha)\, d\alpha \;\mid\; f \in \mathbb{P}_{d_--1}[\alpha] \Big\}.$$

(ii) Note that even in the scalar case, the multiplicities of the roots α_0 may depend on the parameter t. In this case, the functions $\zeta^n(\alpha_0\zeta + \overline{\zeta})^{\lambda+m_l-n}$ are not regular with respect to t. The linear combinations defined by the contour integrals in Theorem 2.2, however, depend regularly on t. Consider the following example of a scalar operator of order $2m = 4$:

$$M_t(\partial_z) = \frac{1}{16}\Big[(1-t)^2\Delta^2 - 8t\partial_{z_1}^2\partial_{z_2}^2\Big] = (\partial_\zeta^2 - t\partial_{\overline{\zeta}}^2)(\partial_{\overline{\zeta}}^2 - t\partial_\zeta^2).$$

M_t is elliptic for $|t| < 1$. We find

$$M_{t;\pm}(\alpha) = (\alpha^2 - t)(1 - t\alpha^2)\,.$$

There is a branching point $t = 0$. For $t \neq 0$ the functions

$$(\pm\sqrt{t}\,\zeta + \overline{\zeta})^{\lambda+2} \quad \text{and} \quad (\pm\sqrt{t}\,\overline{\zeta} + \zeta)^{\lambda+2}$$

are a basis of $\mathfrak{W}_t(\lambda)$, and for $t = 0$ the double root at $\alpha = 0$ gives the basis functions $\overline{\zeta}^{\lambda+2}$, $\zeta\overline{\zeta}^{\lambda+1}$, $\zeta^{\lambda+2}$, and $\overline{\zeta}\zeta^{\lambda+1}$.
This does not depend smoothly on t. If we use the above formulas in *(i)*, however, we can choose $2\pi i f(\alpha) = 1 - t\alpha^2$ and $2\pi i f(\alpha) = 2\alpha(1 - t\alpha^2)$ and obtain basis functions that are analytic with respect to t:

$$\begin{aligned}
\boldsymbol{w}_{t;1}^+ &= \frac{1}{\sqrt{t}}\Big[(\overline{\zeta} + \sqrt{t}\,\zeta)^{\lambda+2} - (\overline{\zeta} - \sqrt{t}\,\zeta)^{\lambda+2}\Big]\\
\boldsymbol{w}_{t;2}^+ &= (\overline{\zeta} + \sqrt{t}\,\zeta)^{\lambda+2} + (\overline{\zeta} - \sqrt{t}\,\zeta)^{\lambda+2}\\
\boldsymbol{w}_{t;1}^- &= \frac{1}{\sqrt{t}}\Big[(\zeta + \sqrt{t}\,\overline{\zeta})^{\lambda+2} - (\zeta - \sqrt{t}\,\overline{\zeta})^{\lambda+2}\Big]\\
\boldsymbol{w}_{t;2}^- &= (\zeta + \sqrt{t}\,\overline{\zeta})^{\lambda+2} + (\zeta - \sqrt{t}\,\overline{\zeta})^{\lambda+2}\,.
\end{aligned}$$

(iii) The two-dimensional Stokes system in *complex form* associates to (u, u^*, p) the triple (f, f^*, g) by

$$\begin{cases}
2\partial_\zeta\partial_{\overline{\zeta}}u \qquad\qquad + \partial_{\overline{\zeta}}p = \frac{1}{2}f\\
\qquad\qquad 2\partial_\zeta\partial_{\overline{\zeta}}u^* + \partial_\zeta p = \frac{1}{2}f^*\\
\partial_\zeta u + \partial_{\overline{\zeta}}u^* \qquad\qquad = \frac{1}{2}g\,.
\end{cases}$$

Here

$$\partial_\zeta = \frac{1}{2}(\partial_{z_1} - i\,\partial_{z_2}) \qquad \partial_{\overline{\zeta}} = \frac{1}{2}(\partial_{z_1} + i\,\partial_{z_2})$$

and the complex velocity components are

$$u := u_1 + i\,u_2 \qquad u^* := u_1 - i\,u_2$$

where (u_1, u_2) is the usual velocity field. For this system we find

$$\boldsymbol{M}_+(\alpha) = \begin{pmatrix} 2\alpha & 0 & 1 \\ 0 & 2\alpha & \alpha \\ \alpha & 1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{M}_-(\alpha) = \begin{pmatrix} 2\alpha & 0 & \alpha \\ 0 & 2\alpha & 1 \\ 1 & \alpha & 0 \end{pmatrix}.$$

From the representation in Theorem 2.2, we find that the following triples $\boldsymbol{w}_j^{\pm}(\lambda,\cdot)$ for $j = 1, 2$ are bases of $\mathfrak{W}^{\pm}(\lambda)$:

$$\boldsymbol{w}_1^+(\lambda;\zeta) = (\overline{\zeta}^{\lambda+1}, 0, 0)\,; \qquad \boldsymbol{w}_2^+(\lambda;\zeta) = (-(\lambda+1)\overline{\zeta}^{\lambda}\zeta,\ \overline{\zeta}^{\lambda+1},\ 2(\lambda+1)\overline{\zeta}^{\lambda})\,;$$

$$\boldsymbol{w}_1^-(\lambda;\zeta) = (0, \zeta^{\lambda+1}, 0)\,; \qquad \boldsymbol{w}_2^-(\lambda;\zeta) = (\zeta^{\lambda+1},\ -(\lambda+1)\zeta^{\lambda}\overline{\zeta},\ 2(\lambda+1)\zeta^{\lambda})\,.$$

■

In order to construct solutions for the homogeneous Dirichlet problem

$$\mathring{\mathscr{A}}_t(\lambda)\,\boldsymbol{v} = 0$$

(see (2.3)), we define the following objects.

For a function $\boldsymbol{v} = (v_1, \ldots, v_N) \in \boldsymbol{H}^m(0, \omega_t)$, we introduce the vector of traces corresponding to the Dirichlet conditions on $\theta = 0$ and $\theta = \omega_t$: The column vector $\mathscr{C}_t\,\boldsymbol{v}$ of length $2|\boldsymbol{m}|$ contains first the $|\boldsymbol{m}|$ terms

$$v_1(0),\ \partial_n v_1(0),\ \ldots,\ \partial_n^{m_1-1} v_1(0),\ v_2(0),\ \partial_n v_2(0),\ \ldots,\ \partial_n^{m_N-1} v_N(0);$$

and then the corresponding $|\boldsymbol{m}|$ terms

$$v_1(\omega_t),\ \partial_n v_1(\omega_t),\ \ldots,\ \partial_n^{m_1-1} v_1(\omega_t),\ v_2(\omega_t),\ \partial_n v_2(\omega_t),\ \ldots,\ \partial_n^{m_N-1} v_N(\omega_t).$$

The normal derivative ∂_n corresponds here to $-\partial_\theta$ for $\theta = 0$ and to ∂_θ for $\theta = \omega_t$.

With the solution basis $\boldsymbol{v}_{t;h}(\lambda,\cdot)$ defined in Theorem 2.2, we define now

• $\mathscr{N}_t(\lambda)$: the $2|\boldsymbol{m}| \times 2|\boldsymbol{m}|$ matrix of complex numbers whose $2|\boldsymbol{m}|$ columns are the $\mathscr{C}_t\,\boldsymbol{v}_{t;h}(\lambda,\cdot)$ for $h = 1, \ldots, 2|\boldsymbol{m}|$ and

$$D_t(\lambda) := \det \mathscr{N}_t(\lambda);$$

• $\mathscr{F}_t(\lambda)$: the $N \times 2|\boldsymbol{m}|$ matrix of $\mathscr{C}^\infty$ functions on $[0, \omega_t]$ whose $2|\boldsymbol{m}|$ columns are the $\boldsymbol{v}_{t;h}(\lambda,\cdot)$ for $h = 1, \ldots, 2|\boldsymbol{m}|$.

Thus

$$\mathscr{N}_t(\lambda) = \mathscr{C}_t \circ \mathscr{F}_t(\lambda).$$

The operator $\mathscr{M}_t(\lambda)$, when acting from $\boldsymbol{H}^{s+\boldsymbol{m}}(0,\omega_t)$ to $\boldsymbol{H}^{s-\boldsymbol{m}}(0,\omega_t)$ has a right inverse $\mathscr{R}_t(\lambda)$ which is a holomorphic function of $\lambda \in \mathbb{C}$ (see [4, Theorem 3.3 and §4.b]).
Note that, according to Remark 2.3, $\mathscr{N}_t$, D_t, $\mathscr{C}_t$, and $\mathscr{R}_t$ all depend smoothly on t.

From the description of the Dirichlet problem

$$\mathring{\mathscr{A}}_t(\lambda)\, \boldsymbol{v} = \boldsymbol{f}$$

as the system

$$\begin{cases} \mathscr{M}_t(\lambda)\, \boldsymbol{v} &= \boldsymbol{f} \\ \mathscr{C}_t\, \boldsymbol{v} &= 0\,, \end{cases}$$

one obtains with these definitions immediately the following result.

Proposition 2.6 (i) *The set $\Lambda_t \setminus \mathbb{Z}$ of non-integer poles of the resolvent $\mathring{\mathscr{A}}_t(\lambda)^{-1}$ is the set of non-integer roots of the equation*

$$D_t(\lambda) = 0.$$

(ii) *For any $\lambda \in \mathbb{C} \setminus \Lambda_t$ there holds*

$$\mathring{\mathscr{A}}_t(\lambda)^{-1} = \mathscr{R}_t(\lambda) - \mathscr{F}_t(\lambda)\, \mathscr{N}_t(\lambda)^{-1} \mathscr{C}_t\, \mathscr{R}_t(\lambda).$$

As a corollary, we obtain descriptions of the spaces of singular functions $\mathscr{X}_t(\gamma)$ in terms of the above defined finite dimensional objects.

Theorem 2.7 *Let γ be a simple closed contour such that $\Lambda_t \cap \gamma = \emptyset$.*
(i) *Then*

$$\mathscr{X}_t(\gamma) = \Big\{(r,\theta) \mapsto \int_\gamma r^{\lambda+\boldsymbol{m}} \mathscr{F}_t(\lambda,\theta)\, \mathscr{N}_t(\lambda)^{-1} \mathscr{G}(\lambda)\, d\lambda \ \mid\ \mathscr{G} : \mathbb{C} \mapsto \mathbb{C}^{2|\boldsymbol{m}|} \textit{ holomorphic}\Big\}$$

(ii) *Let $a_{t,\gamma}$ be the polynomial in λ with leading coefficient 1 that has as its roots precisely the roots of $D_t(\lambda)$ inside γ with the same multiplicities: it is given for $\lambda' \notin \overline{\mathit{int}\,\gamma_t}$ by*

$$a_{t,\gamma}(\lambda') = \exp\Big(\frac{1}{2i\pi} \int_\gamma \frac{d}{d\lambda} D_t(\lambda)\, \frac{\log(\lambda-\lambda')}{D_t(\lambda)}\, d\lambda\Big).$$

Let $d(\gamma)$ be the degree of $a_{t,\gamma}$. Then

$$\mathscr{X}_t(\gamma) = \Big\{(r,\theta) \mapsto \int_\gamma r^{\lambda+\boldsymbol{m}} \mathscr{F}_t(\lambda,\theta)\, \mathscr{N}_t(\lambda)^{-1} \mathscr{G}(\lambda)\, d\lambda \ \mid\ \mathscr{G} \in \mathbb{P}_{d(\gamma)-1}[\lambda] \otimes \mathbb{C}^{2|\boldsymbol{m}|}\Big\}.$$

We also have the inclusion

$$\begin{aligned} \mathscr{X}_t(\gamma) &\subset \Big\{(r,\theta) \mapsto \int_\gamma r^{\lambda+\boldsymbol{m}} \mathscr{F}_t(\lambda,\theta)\, \frac{\mathscr{G}(\lambda)}{D_t(\lambda)}\, d\lambda \ \mid\ \mathscr{G} \in \mathbb{P}_{d(\gamma)-1}[\lambda] \otimes \mathbb{C}^{2|\boldsymbol{m}|}\Big\} \\ &= \Big\{(r,\theta) \mapsto \int_\gamma r^{\lambda+\boldsymbol{m}} \mathscr{F}_t(\lambda,\theta)\, \frac{\mathscr{G}(\lambda)}{a_{t,\gamma}(\lambda)}\, d\lambda \ \mid\ \mathscr{G} \in \mathbb{P}_{d(\gamma)-1}[\lambda] \otimes \mathbb{C}^{2|\boldsymbol{m}|}\Big\}. \end{aligned}$$

Corollary 2.8 *Let $(\gamma_t)_{t\in\mathscr{T}}$ be a family of simple closed contours depending continuously on $t \in \mathscr{T}$ such that*

$$\forall t \in \mathscr{T}: \quad \Lambda_t \cap \gamma_t = \emptyset.$$

(i) *Then $d(\gamma_t) =: d$ is independent of $t \in \mathscr{T}$.*

(ii) *Let furthermore $(\mathscr{G}_\ell)_{\ell=1,\dots,2|\boldsymbol{m}|d}$ be a basis of $\mathbb{P}_{d-1}[\lambda] \otimes \mathbb{C}^{2|\boldsymbol{m}|}$. Then the functions*

$$\boldsymbol{X}_{\ell;t}(r,\theta) = \int_{\gamma_t} r^{\lambda+\boldsymbol{m}} \mathscr{F}_t(\lambda,\theta)\, \mathscr{N}_t(\lambda)^{-1} \mathscr{G}_\ell(\lambda)\, d\lambda$$

are $\mathscr{C}^\infty$ functions of $t \in \mathscr{T}$, $r > 0$, and $\theta \in \mathbb{R}$ (analytic if the data are analytic) and they generate the space $\mathscr{X}_t(\gamma_t)$ for all $t \in \mathscr{T}$.

(iii) *Let $(t,\lambda) \mapsto \mathscr{G}(t,\lambda) \in \mathbb{C}^{2|\boldsymbol{m}|}$ be a function of class $\mathscr{C}^\kappa(\mathscr{T})$, holomorphic in a neighborhood of $\overline{\operatorname{int}\gamma_t}$ for each t, and let $\boldsymbol{v}_t \in \mathscr{X}_t(\gamma_t)$ be defined by*

$$\boldsymbol{v}_t(r,\theta) = \int_{\gamma_t} r^{\lambda+\boldsymbol{m}} \mathscr{F}_t(\lambda,\theta)\, \mathscr{N}_t(\lambda)^{-1} \mathscr{G}(t,\lambda)\, d\lambda\,. \tag{2.10}$$

Then there are coefficients $c_\ell \in \mathscr{C}^\kappa(\mathscr{T})$, $\ell = 1,\dots,2|\boldsymbol{m}|d$ such that

$$\boldsymbol{v}_t(r,\theta) = \sum_\ell c_\ell(t)\, \boldsymbol{X}_{\ell;t}(r,\theta)\,. \tag{2.11}$$

Furthermore, there is an estimate with a constant C independent of $\mathscr{G}$:

$$\sum_\ell \|c_\ell\|_{\mathscr{C}^\kappa(\mathscr{T})} \le C \sup_{t\in\mathscr{T},\, 0\le\kappa'\le\kappa} \|\partial_t^{\kappa'} \mathscr{G}(t,\cdot)\|_{L^1(\gamma_t)}\,. \tag{2.12}$$

Proof. Let

$$\mathscr{G}(t,\lambda) = a_{t,\gamma}(\lambda)\, \mathscr{H}(t,\lambda) + \sum_{\ell=1}^{2|\boldsymbol{m}|d} c_\ell(t)\, \mathscr{G}_\ell(\lambda)$$

be the Euclidean division of $\mathscr{G}(t,\cdot)$ by $a_{t,\gamma}$ (see Proposition 6.5 in the Appendix). Here $\mathscr{H}(t,\cdot)$ is holomorphic, so it does not contribute to the integral (2.10). The coefficients c_ℓ are given by the formulas

$$c_\ell(t) = \frac{1}{2i\pi} \int_{\gamma_t} \frac{\mathscr{G}(t,\lambda)\cdot Q^*_{\ell,t}(\lambda)}{a_{t,\gamma}(\lambda)}\, d\lambda\,,$$

where $Q^*_{\ell,t}$, $\ell = 1,\dots,2|\boldsymbol{m}|d$ is a certain basis of $\mathbb{P}_{d-1}[\lambda] \otimes \mathbb{C}^{2|\boldsymbol{m}|}$ depending regularly on $t \in \mathscr{T}$. The formula (2.11) and the estimate (2.12) follow immediately. ■

Remark 2.9 The family of spaces $(\mathscr{X}_t(\gamma_t))_{t\in\mathscr{T}}$ is a ($\mathscr{C}^\infty$ or analytic) vector bundle of constant fiber dimension $d_0 \le d$. ■

Remark 2.10 The generating functions of the space $\mathscr{X}_t(\gamma)$ are in a simple and natural way functions of the complex variable $\zeta = re^{i\theta}$, because the matrix function $r^{\lambda+\boldsymbol{m}}\mathscr{F}_t$ appearing in their definition is the $N \times 2|\boldsymbol{m}|$ matrix of functions

$$\boldsymbol{W}_t(\lambda) \;:\; \zeta \longmapsto \boldsymbol{W}_t(\lambda,\zeta) = r^{\lambda+\boldsymbol{m}} \mathscr{F}_t(\lambda,\theta)$$

whose first $|\boldsymbol{m}|$ columns are the basis vectors $\boldsymbol{w}^+_{t;h}(\lambda,\cdot)$ as defined in Theorem 2.2, while the second $|\boldsymbol{m}|$ columns are the vector functions $\boldsymbol{w}^-_{t;h}(\lambda,\cdot)$, $h = 1,\dots,|\boldsymbol{m}|$. ■

2.b Singularities with polynomial right hand sides.

The second step in the description of the general form of the singular functions is the case of polynomial right hand sides. We are thus looking for solutions of the form (2.1) of the system $\boldsymbol{M}_t\boldsymbol{u} = \boldsymbol{f}$ where $\boldsymbol{f} \in \mathbb{P}[z_1, z_2] \otimes \mathbb{C}^N$, i. e. the components of the vector function $\boldsymbol{f}$ are polynomials in the cartesian variables (z_1, z_2).

We use, as above, the notation $\mathbb{P}_n$ for the space of polynomials of degree at most n and $\mathbb{P}_{(n)}$ for the space of homogeneous polynomials of degree n. In order to obtain a convenient description, we decompose the space of vector polynomials into subspaces of (multi-)homogeneous polynomials according to the multi-order of our system $\boldsymbol{L}$: for any $\lambda_0 \in \mathbb{N}$,

$$\mathbb{P}_{(\lambda_0+\boldsymbol{m})} = \prod_{l=1}^{N} \mathbb{P}_{(\lambda_0+m_l)} \quad \text{and} \quad \mathbb{P}_{(\lambda_0-\boldsymbol{m})} = \prod_{k=1}^{N} \mathbb{P}_{(\lambda_0-m_k)} \,.$$

Thus the components of $\boldsymbol{f} \in \mathbb{P}_{(\lambda_0-\boldsymbol{m})}$ have the form $f_k(r,\theta) = r^{\lambda_0-m_k}\varphi_k(\theta)$, where φ_k belongs to a space of trigonometric polynomials in θ of degree $\lambda_0 - m_k$; we denote the space of these trigonometric polynomials $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N)$ such that $r^{\lambda_0-\boldsymbol{m}}\varphi$ is a polynomial, by $\mathbb{T}_{\lambda_0-\boldsymbol{m}}$.

Let then $\mathscr{Y}_t(\gamma)$ be the space generated by all the solutions $\boldsymbol{u}$ of $\boldsymbol{M}_t\boldsymbol{u} = \boldsymbol{f}$ of the form (2.1) with $\lambda \in \mathit{int}\,\gamma$ and $\boldsymbol{f} \in \bigoplus_{\lambda_0 \in (\mathit{int}\,\gamma)\cap\mathbb{N}} \mathbb{P}_{(\lambda_0-\boldsymbol{m})}$.
If $(\mathit{int}\,\gamma)\cap\mathbb{N}$ is empty, $\mathscr{Y}_t(\gamma)$ coincides with $\mathscr{X}_t(\gamma)$. We can without restriction assume that $\mathit{int}\,\gamma$ contains only one integer element λ_0 — cf. the decomposition (2.4).

The Mellin transforms of the polynomials in $\mathbb{P}_{(\lambda_0-\boldsymbol{m})}$ are meromorphic vector functions Ψ with a simple pole in $\mathit{int}\,\gamma$ of the form

$$\Psi(\lambda) = \Phi(\lambda) + (\lambda - \lambda_0)^{-1}\boldsymbol{\varphi} \quad \text{with} \quad \boldsymbol{\varphi} \in \mathbb{T}_{\lambda_0-\boldsymbol{m}} \,,$$

where Φ is holomorphic with values in $\boldsymbol{H}^{-\boldsymbol{m}}(0, \omega_t)$. In analogy to Lemma 2.1 we have

Lemma 2.11 *Let γ be a simple closed contour in $\mathbb{C}$, $\mathit{int}\,\gamma$ containing the only integer λ_0. If $\gamma \cap \Lambda_t$ is empty, then*

$$\mathscr{Y}_t(\gamma) = \{\int_\gamma r^{\lambda+\boldsymbol{m}}\, \mathring{\mathscr{A}}_t(\lambda)^{-1}\Psi(\lambda)\, d\lambda \mid \Psi(\lambda) \textit{ meromorphic with a simple pole in } \lambda_0$$
$$\textit{and a polar part in } \mathbb{T}_{\lambda_0-\boldsymbol{m}}\}.$$

Due to the presence of the term $\mathscr{R}_t(\lambda)$ in the formula for $\mathring{\mathscr{A}}_t(\lambda)^{-1}$ in Proposition 2.6, the preceding lemma cannot be used directly to give a description of $\mathscr{Y}_t(\gamma)$ in the spirit of Theorem 2.7. In [4], this description was obtained by writing the polynomial right hand side as a special case of a very general form of singular functions. Here we shall give a slightly simpler and more precise formulation.

Theorem 2.12 *Let γ be a simple closed contour such that $\Lambda_t \cap \gamma = \emptyset$ and such that $(\mathit{int}\,\gamma) \cap \mathbb{N} = \{\lambda_0\}$. Then, with $a_{t,\gamma}$ the polynomial introduced in Theorem 2.7, we* *have the inclusion*

$$\mathscr{Y}_t(\gamma) \;\subset\; \Big\{(r,\theta)\mapsto \int_\gamma r^{\lambda+\boldsymbol{m}}\mathscr{F}_t(\lambda,\theta)\,\frac{\mathscr{G}(\lambda)}{(\lambda-\lambda_0)\,a_{t,\gamma}(\lambda)}\,d\lambda \;\mid\; \mathscr{G}\in\mathbb{P}_{d(\gamma)}[\lambda]\otimes\mathbb{C}^{2|\boldsymbol{m}|}\Big\}$$
$$+\;\mathbb{P}_{(\lambda_0+\boldsymbol{m})}.$$

Proof. The theorem is a consequence of the following more precise statement and of Theorem 2.7. ■

Lemma 2.13 *Let $\boldsymbol{f}_t\in\mathbb{P}_{(\lambda_0-\boldsymbol{m})}$ be such that its coefficients are $\mathscr{C}^\kappa(\mathscr{T})$ functions (or analytic). Suppose that also the coefficients of $\boldsymbol{M}_t$ have this regularity with respect to $t\in\mathscr{T}$ and that γ_t depends continuously on t with $\Lambda_t\cap\gamma_t=\emptyset$ and $(int\,\gamma)\cap\mathbb{N}=\{\lambda_0\}$ for all t. Then the system*

$$\begin{cases} \boldsymbol{M}_t\,\boldsymbol{u}_t &=& \boldsymbol{f}_t \\ \mathscr{C}_t\,\boldsymbol{u}_t(r,\cdot) &=& 0 \quad (r>0), \end{cases}$$

has a solution of the form

$$\boldsymbol{u}_t=\boldsymbol{u}_{\mathrm{pol};\,t}+\boldsymbol{u}_{\mathrm{sing};\,t}$$

where $\boldsymbol{u}_{\mathrm{pol};\,t}\in\mathbb{P}_{(\lambda_0+\boldsymbol{m})}$, the coefficients of $\boldsymbol{u}_{\mathrm{pol};\,t}$ are $\mathscr{C}^\kappa(\mathscr{T})$ (or analytic) and there exists $\mathscr{G}_t\in\mathbb{P}_{d(\gamma_t)}[\lambda]\otimes\mathbb{C}^{2|\boldsymbol{m}|}$ whose coefficients are $\mathscr{C}^\kappa(\mathscr{T})$ (or analytic) such that

$$\boldsymbol{u}_{\mathrm{sing};\,t}(r,\theta)=\int_{\gamma_t} r^{\lambda+\boldsymbol{m}}\mathscr{F}_t(\lambda,\theta)\,\frac{\mathscr{G}_t(\lambda)}{(\lambda-\lambda_0)\,a_{t,\gamma_t}(\lambda)}\,d\lambda\,. \tag{2.13}$$

Proof. We show first that

$$\boldsymbol{M}_t\;:\;\mathbb{P}_{(\lambda_0+\boldsymbol{m})}\longmapsto\mathbb{P}_{(\lambda_0-\boldsymbol{m})}$$

is surjective. Let $f\in\mathbb{P}_{(\lambda_0-\boldsymbol{m})}$. Consider the Dirichlet problem on B_2, the disk of radius 2:

$$\boldsymbol{M}_t\,\boldsymbol{w}=\boldsymbol{f},\qquad w\in\mathring{\boldsymbol{H}}^{\boldsymbol{m}}(B_2).$$

This is an elliptic boundary value problem which is therefore solvable provided $\boldsymbol{f}$ satisfies a finite number of solvability conditions. These can be satisfied by modifying $\boldsymbol{f}$ on $B_2\setminus B_1$ (here we use that $\boldsymbol{M}_t$ has constant coefficients and hence the kernel of the adjoint problem is generated by analytic functions). There exists therefore a solution $\boldsymbol{w}\in\mathring{\boldsymbol{H}}^{\boldsymbol{m}}(B_2)$ of $\boldsymbol{M}_t\boldsymbol{w}=\widetilde{\boldsymbol{f}}$ with $\widetilde{\boldsymbol{f}}=\boldsymbol{f}$ on B_1.

For $r\le 1$ we have therefore $\boldsymbol{M}_t\boldsymbol{w}=\boldsymbol{f}$, and $\boldsymbol{w}$ is analytic in (z_1,z_2) for $r\le 1$. Let $\boldsymbol{u}$ be the Taylor polynomial of $\boldsymbol{w}$ at 0 of (multi-)degree $\lambda_0+\boldsymbol{m}$. Then the l-th component of $\boldsymbol{M}_t(\boldsymbol{w}-\boldsymbol{u})$ vanishes to the order $\mathcal{O}(r^{\lambda_0-m_l+1})$ at the origin. On the other hand, $\boldsymbol{M}_t(\boldsymbol{w}-\boldsymbol{u})$ is a polynomial of (multi-)degree $\le\lambda_0-\boldsymbol{m}$. Hence $\boldsymbol{M}_t(\boldsymbol{w}-\boldsymbol{u})\equiv 0$ for $r\le 1$. Let $\boldsymbol{u}_0\in\mathbb{P}_{(\lambda_0+\boldsymbol{m})}$ be the part of $\boldsymbol{u}$ homogeneous of degree $\lambda_0+\boldsymbol{m}$. We find again $\boldsymbol{M}_t(\boldsymbol{u}-\boldsymbol{u}_0)=0$, so that finally $\boldsymbol{M}_t\boldsymbol{u}_0=\boldsymbol{f}$ holds.

As a surjective mapping between two finite-dimensional spaces, $\boldsymbol{M}_t$ has a right inverse $\mathscr{R}_{t,\,\mathrm{pol}}(\lambda_0)$ which depends regularly on t.

We can therefore define

$$\boldsymbol{u}_{\mathrm{pol};\,t} = \mathscr{R}_{t,\,\mathrm{pol}}(\lambda_0)\, \boldsymbol{f}_t\,.$$

We have to find now a function $\boldsymbol{u}_{\mathrm{sing};\,t}$ satisfying

$$\boldsymbol{M}_t\, \boldsymbol{u}_{\mathrm{sing};\,t} = 0, \qquad \mathscr{C}_t\, \boldsymbol{u}_{\mathrm{sing};\,t} = \mathscr{C}_t\, \boldsymbol{u}_{\mathrm{pol};\,t}\,.$$

Using Mellin transformation as in [4, (5.6)], we find a solution of the form

$$\boldsymbol{u}_{\mathrm{sing};\,t}(r,\theta) = \int_{\gamma_t} r^{\lambda+\boldsymbol{m}}\, \mathscr{F}_t(\lambda,\theta)\, \mathscr{N}_t(\lambda)^{-1}\, \frac{\mathscr{G}_t(\lambda)}{(\lambda-\lambda_0)}\, d\lambda$$

with $\lambda \mapsto \mathscr{G}_t(\lambda)$ holomorphic. This implies (2.13). ■

Remark 2.14 In some cases, the space $\mathscr{Y}_t(\gamma)$ coincides with the space of polynomials $\mathbb{P}_{(\lambda_0+\boldsymbol{m})}$; for example:
(i) If the angle ω is different from π or 2π, and we assume in addition to the hypotheses of Theorem 2.12 that $\Lambda_t \cap (int\,\gamma) = \emptyset$, then $\mathscr{Y}_t(\gamma) = \mathbb{P}_{(\lambda_0+\boldsymbol{m})}$.
(ii) If ω is π or 2π and the only possible element of Λ_t in $\overline{int\,\gamma}$ is $\lambda_0 \in \mathbb{N}$ then $\mathscr{Y}_t(\gamma) = \mathbb{P}_{(\lambda_0+\boldsymbol{m})}$. ■

Proof. *(i)* Since $\lambda_0 \notin \Lambda_t$, the operator

$$\begin{array}{cccccc} (\boldsymbol{M}_t, \mathscr{C}_t) & : & \mathbb{P}_{(\lambda_0+\boldsymbol{m})} & \longmapsto & \mathbb{P}_{(\lambda_0-\boldsymbol{m})} & \times & \mathbb{C}^{2|\boldsymbol{m}|} \\ & & \boldsymbol{w} & \longrightarrow & (\boldsymbol{M}_t\boldsymbol{w} & , & \mathscr{C}_t\boldsymbol{w}\Big|_{r=1}) \end{array}$$

is bijective: indeed, since

$$\dim \mathbb{P}_{(\lambda_0+\boldsymbol{m})} = \dim \mathbb{P}_{(\lambda_0-\boldsymbol{m})} \times \mathbb{C}^{2|\boldsymbol{m}|} = (\lambda_0+1)N + |\boldsymbol{m}|\,,$$

it suffices to prove that the operator is injective; if $\boldsymbol{w} \in \mathbb{P}_{(\lambda_0+\boldsymbol{m})}$, $\boldsymbol{w} = r^{\lambda_0+\boldsymbol{m}}\boldsymbol{\varphi}$, satisfies $\boldsymbol{M}_t\boldsymbol{w} = 0$ and $\mathscr{C}_t\boldsymbol{w}|_{r=1} = 0$ then $\mathring{\mathscr{A}}_t(\lambda_0)\boldsymbol{\varphi} = 0$ and the assumption $\boldsymbol{\varphi} \neq 0$ would lead to the contradiction $\lambda_0 \in \Lambda_t$.
Therefore to any $\boldsymbol{w} \in \mathscr{Y}_t(\gamma)$, there exists $\boldsymbol{u} \in \mathbb{P}_{(\lambda_0+\boldsymbol{m})}$ such that $\boldsymbol{M}_t\boldsymbol{u} = \boldsymbol{M}_t\boldsymbol{w}$ and $\mathscr{C}_t\boldsymbol{u}(r,\cdot) = \mathscr{C}_t\boldsymbol{w}(r,\cdot)$. As $\Lambda_t \cap (int\,\gamma) = \emptyset$, $\mathscr{X}_t(\gamma) = \{0\}$. Hence $\boldsymbol{u} = \boldsymbol{w}$.

For the proof of *(ii)* we refer to the paper [?]. There this is shown for the case of general boundary conditions and it is furthermore shown that for $\omega = 2\pi$, Λ_t contains only integers and half integers. ■

2.c Singular right hand sides. The third step in the description of the singular functions is the construction of classes of singularities which are "closed" under the solution of the model boundary value problem. That means

If $\boldsymbol{f}$ and $\boldsymbol{g}$ have these forms, then both problems

$$\boldsymbol{M}_t\, \boldsymbol{u} = \boldsymbol{f} \quad \textit{(no boundary conditions)}$$

and

$$\boldsymbol{M}_t\, \boldsymbol{u} = 0, \qquad \mathscr{C}_t\, \boldsymbol{u}(r,\cdot) = \boldsymbol{g}(r)$$

have solutions with a similar form.

We shall, in fact, define two classes of spaces $\mathscr{Z}_t(\gamma; a; \nu)$ and $\widetilde{\mathscr{Z}}_t(\gamma; a; \nu)$ depending on a simple closed contour $\gamma \subset \mathbb{C}$, a polynomial a and an integer ν. The difference between $\mathscr{Z}_t$ and $\widetilde{\mathscr{Z}}_t$ corresponds to the difference between $\boldsymbol{Z}^{\pm}$ and $\widetilde{\boldsymbol{Z}}^{\pm}$ in (2.5) and (2.7) respectively.

Definition 2.15 Let $\gamma \cap \mathbb{N} = \emptyset$ and $a \neq 0$ on γ. Then $\mathscr{Z}_t(\gamma; a; \nu)$ is the space generated by all functions $\boldsymbol{u}^{+}$ and $\boldsymbol{u}^{-}$ where

$$\boldsymbol{u}^{\pm}(\zeta) = \int_{\gamma} \frac{\boldsymbol{V}^{\pm}(\lambda, \zeta)\, \mathscr{H}(\lambda)}{a(\lambda)}\, d\lambda \quad \text{with} \quad \mathscr{H} \in \mathbb{P}_{\deg(a)-1}[\lambda] \otimes \mathbb{C}^N \tag{2.14}$$

where $\boldsymbol{V}^{\pm}(\lambda, \zeta)$ is the diagonal matrix $\operatorname{diag}(v_l^{\pm}(\lambda, \zeta))$ with $(v_1^{\pm}, \ldots, v_N^{\pm}) =: \boldsymbol{v}^{\pm}$ defined by

$$\boldsymbol{v}^{\pm}(\lambda, \zeta) = \int_{\tilde{\gamma}} \frac{\boldsymbol{Z}^{\pm}(\lambda; \zeta, \overline{\zeta}; \alpha)\, \boldsymbol{h}(\alpha)}{\det \boldsymbol{M}_{t;\pm}(\alpha)^{\nu}}\, d\alpha \quad \text{with} \quad \boldsymbol{h} \in \mathbb{P}_{|\boldsymbol{m}|\nu-1}[\alpha] \otimes \mathbb{C}^N. \tag{2.15}$$

The space $\widetilde{\mathscr{Z}}_t(\gamma; a; \nu)$ is defined analogously, where $\boldsymbol{Z}^{\pm}$ in (2.15) is replaced by $\widetilde{\boldsymbol{Z}}^{\pm}$.

■

It is clear that

$$\mathscr{Z}_t(\gamma; a; \nu) \subset \mathscr{Z}_t(\gamma'; a'; \nu'), \quad \text{and} \quad \widetilde{\mathscr{Z}}_t(\gamma; a; \nu) \subset \widetilde{\mathscr{Z}}_t(\gamma'; a'; \nu')$$

provided that $\mathit{int}\,\gamma \subset \mathit{int}\,\gamma'$, a divides a' and $\nu \leq \nu'$.

Note that, in polar coordinates, the functions in $\mathscr{Z}_t(\gamma; a; \nu)$ are of the form

$$\sum_{j,q} r^{\lambda_j + \boldsymbol{m}} \log^q r\, \boldsymbol{v}_{j,q}(\theta)$$

where λ_j are the roots of a in $\mathit{int}\,\gamma$, whereas the elements of $\widetilde{\mathscr{Z}}_t(\gamma; a; \nu)$ are of the form

$$\sum_{j,q} r^{\lambda_j - \boldsymbol{m}} \log^q r\, \boldsymbol{v}_{j,q}(\theta).$$

A simple calculation shows that the operator $\boldsymbol{M}_t$ maps $\mathscr{Z}_t(\gamma; a; \nu)$ into $\widetilde{\mathscr{Z}}_t(\gamma; a; \nu)$. Lower order terms in the differential operator and terms coming from the Taylor expansion of the coefficients give rise to a shift in the contour γ:

Lemma 2.16 *Let $\gamma \subset \mathbb{C}$ be a simple closed contour. Let $p \in \mathbb{N}$ and $\boldsymbol{L}_p$ be a $N \times N$ matrix of multi-homogeneous differential operators with polynomial coefficients of the following form, for $k, l = 1, \ldots, N$:*

$$L_{p,kl} = \zeta^{\beta_{1kl}}\, \overline{\zeta}^{\beta_{2kl}}\, \partial_{\zeta}^{\,i_{1kl}}\, \partial_{\overline{\zeta}}^{\,i_{2kl}} \quad \textit{with} \quad -\beta_{1kl} - \beta_{2kl} + i_{1kl} + i_{2kl} = m_k + m_l - p\,.$$

Then there exists a polynomial b_0 with integer roots and $\nu_0 \in \mathbb{N}$ such that for any polynomial a without roots on γ and any $\nu \in \mathbb{N}$, the operator

$$\boldsymbol{L}_p \quad \textit{maps} \quad \mathscr{Z}_t(\gamma; a; \nu) \quad \textit{into} \quad \widetilde{\mathscr{Z}}_t(\gamma'; a'; \nu')$$

with

$$\gamma' = \gamma + p = \{\lambda \in \mathbb{C} \mid \lambda - p \in \gamma\}, \quad a'(\lambda) = a(\lambda - p) \cdot b_0(\lambda), \quad \nu' = \nu + \nu_0\,.$$

Proof. We consider one component u_l of an element $\boldsymbol{u} \in \mathscr{Z}_t(\gamma; a; \nu)$ and

$$w_k = \zeta^{\beta_{1kl}} \overline{\zeta}^{\beta_{2kl}} \partial_\zeta^{\,i_{1kl}} \partial_{\overline{\zeta}}^{\,i_{2kl}} u_l(\zeta) \quad \text{with} \quad -\beta_{1kl} - \beta_{2kl} + i_{1kl} + i_{2kl} = m_k + m_l - p\,.$$

We have to show that w_k is the k-th component of an element $\boldsymbol{w} \in \widetilde{\mathscr{Z}}_t(\gamma'; a'; \nu')$. By definition, $u_l = u_l^+ + u_l^-$, where

$$u_l^\pm = \int_\gamma \left(\int_{\tilde\gamma} Z_l^\pm(\lambda; \zeta, \overline{\zeta}; \alpha)\, \frac{h_l(\alpha)}{b_\pm^\nu(\alpha)}\, d\alpha \right) \frac{\mathscr{H}_l(\lambda)}{a(\lambda)}\, d\lambda. \tag{2.16}$$

Here $\mathscr{H}_l(\lambda)$ and $h_l(\alpha)$ are (scalar) polynomials, $Z_l^\pm$ is the l-th entry of the diagonal matrix defined in (2.5), and $b_\pm^\nu$ is the polynomial $b_\pm^\nu(\alpha) = \det \boldsymbol{M}_{t;\pm}(\alpha)^\nu$.

We will give the details only for u_l^+. By definition (2.5) of Z_l^+, (2.16) is of the form

$$u_l^+(\zeta) = \int_\gamma \left(\int_{\tilde\gamma} (\alpha\zeta + \overline{\zeta})^{\lambda + m_l}\, \frac{h_l(\alpha)}{b_+^\nu(\alpha)}\, d\alpha \right) \frac{\mathscr{H}_l(\lambda)}{a(\lambda)\, b_1(\lambda)}\, d\lambda\,, \tag{2.17}$$

where $b_1(\lambda)$ is a polynomial with integer roots. Using the binomial formula for

$$\overline{\zeta}^{\beta_2} = (\alpha\zeta + \overline{\zeta} - \alpha\zeta)^{\beta_2},$$

we can write

$$\zeta^{\beta_1} \overline{\zeta}^{\beta_2} \partial_\zeta^{\,i_1} \partial_{\overline{\zeta}}^{\,i_2} (\alpha\zeta + \overline{\zeta})^{\lambda + m_l}$$

as a sum of terms of the form

$$\frac{b_2(\lambda)}{b_3(\lambda)}\, \alpha^{i_1 + \beta_3} \partial_\alpha^{\beta_1 + \beta_3} (\alpha\zeta + \overline{\zeta})^{\lambda + m_l + \beta_1 + \beta_2 - i_1 - i_2}$$

where $0 \le \beta_3 \le \beta_2$, and b_2 and b_3 are polynomials with integer roots. Note that $\lambda + m_l + \beta_1 + \beta_2 - i_1 - i_2 = \lambda + p - m_k$. Now we use partial integration in the integral over $\tilde\gamma$ with respect to α, which gives a denominator $b_+^{\nu'}(\alpha)$ with $\nu' = \nu + \beta_1 + \beta_2$. We find that w_k is a sum of terms of the form

$$\int_\gamma \left(\int_{\tilde\gamma} \widetilde{Z}_k^+(\lambda + p; \zeta, \overline{\zeta}; \alpha)\, \frac{h_l(\alpha)}{b_+^{\nu'}(\alpha)}\, d\alpha \right) \frac{\mathscr{H}_l'(\lambda)}{a(\lambda)\, b_4(\lambda)}\, d\lambda$$

with $\widetilde{Z}_k^+$ as defined in (2.7). According to Definition 2.15, this is indeed the form of the k-th component of a function in $\widetilde{\mathscr{Z}}_t(\gamma + p; a'; \nu')$. ∎

Lemma 2.17 *Let γ be a simple closed curve, $\nu \in \mathbb{N}$ and a be a polynomial without zeros on γ. Then to any $\boldsymbol{f} \in \widetilde{\mathscr{Z}}_t(\gamma; a; \nu)$, there exists $\boldsymbol{u} \in \mathscr{Z}_t(\gamma; a; \nu + 1)$, solution of*

$$\boldsymbol{M}_t(\partial_z)\, \boldsymbol{u} = \boldsymbol{f}.$$

Proof. Let

$$\boldsymbol{f} = \int_\gamma \left(\int_{\tilde\gamma} \widetilde{\boldsymbol{Z}}^\pm(\lambda; \zeta, \overline{\zeta}; \alpha)\, \frac{\boldsymbol{h}(\alpha)}{\det \boldsymbol{M}_{t;\pm}(\alpha)^\nu}\, d\alpha \right) \frac{\mathscr{H}(\lambda)}{a(\lambda)}\, d\lambda. \tag{2.18}$$

Then with relation (2.6), we see immediately that

$$\boldsymbol{u} = \int_\gamma \left(\int_{\tilde\gamma} \boldsymbol{Z}^\pm(\lambda;\zeta,\overline{\zeta};\alpha)\, \frac{\boldsymbol{M}_{t;\pm}(\alpha)^{-1}\,\boldsymbol{h}(\alpha)}{\det \boldsymbol{M}_{t;\pm}(\alpha)^\nu}\, d\alpha \right) \frac{\mathscr{H}(\lambda)}{a(\lambda)}\, d\lambda \tag{2.19}$$

is the desired solution. ■

Let us now study inhomogeneous boundary conditions corresponding to singular functions. Let $\boldsymbol{C}_t$ be the boundary trace operator corresponding to Dirichlet boundary conditions on Γ_t: $\boldsymbol{C}_t\boldsymbol{u}$ is a function of $r > 0$ defined by

$$\Big(\boldsymbol{C}_t\boldsymbol{u}\Big)(r) = \Big(u_1(r,0),\dots,\partial_n^{m_N-1}u_N(r,0);\ u_1(r,\omega_t),\dots,\partial_n^{m_N-1}u_N(r,\omega_t)\Big). \tag{2.20}$$

The relation with the operator $\mathscr{C}_t$ as defined in §2.a is given by

$$\Big(\boldsymbol{C}_t\, r^{\boldsymbol{m}}\boldsymbol{u}\Big)(r) = r^{\boldsymbol{\mu}}\,\mathscr{C}_t\boldsymbol{u}(r,\cdot)\,, \tag{2.21}$$

where

$$\boldsymbol{\mu} = (\sigma_1,\dots,\sigma_{|\boldsymbol{m}|},\sigma_1,\dots,\sigma_{|\boldsymbol{m}|}) \tag{2.22}$$

and

$$\boldsymbol{\sigma} = (m_1, m_1-1,\dots,1,m_2,\dots,1,\dots,m_N,\dots,1). \tag{2.23}$$

The spaces of traces corresponding to the spaces $\mathscr{Z}_t$ are defined as follows.

Definition 2.18 $\mathscr{S}(\gamma;a)$ is the space generated by all functions $\boldsymbol{g} = (g_1,\dots,g_{2|\boldsymbol{m}|})$ of the form

$$\boldsymbol{g}(r) = \int_\gamma r^{\lambda+\boldsymbol{\mu}}\,\frac{\boldsymbol{\psi}(\lambda)}{a(\lambda)}\,d\lambda \quad \text{with} \quad \boldsymbol{\psi} \in \mathbb{P}_{\deg(a)-1}[\lambda]\otimes\mathbb{C}^{2|\boldsymbol{m}|}\,. \tag{2.24}$$

■

The trace result is the following:

Lemma 2.19 *Let $\gamma \in \mathbb{C}$ be a simple closed contour. Then there exists a polynomial b_0 with integer roots such that for any polynomial a without roots on γ and any $\nu \in \mathbb{N}$, the operator*

$$\boldsymbol{C}_t \quad \textit{maps} \quad \mathscr{Z}_t(\gamma;a;\nu) \quad \textit{into} \quad \mathscr{S}(\gamma;a\cdot b_0)\,.$$

We leave the simple proof to the reader. We will prove a more general statement, analogue of Lemma 2.16 for the trace operators, in Lemma 4.1 below. Here, we prove the fundamental statements about the solution of the boundary value problem $(\boldsymbol{M}_t,\boldsymbol{C}_t)$ with singular right hand side.

Lemma 2.20 *Let $\gamma \subset \mathbb{C}$ be a simple closed contour with $\Lambda_t \cap \gamma = \emptyset$. Let $a_{t,\gamma}$ be the characteristic polynomial defined in Theorem 2.7. Let a be a polynomial without zeros on γ. Then to any $\boldsymbol{g} \in \mathscr{S}(\gamma;a)$ there exists a solution $\boldsymbol{u} \in \mathscr{Z}_t(\gamma;a\cdot a_{t,\gamma};1)$ of the system*

$$\begin{cases} \boldsymbol{M}_t\,\boldsymbol{u} &= 0 \\ \boldsymbol{C}_t\,\boldsymbol{u} &= \boldsymbol{g}\,. \end{cases}$$

Proof. Let $d = \deg(a)$. Let $\mathscr{G}$ be the unique polynomial in $\mathbb{P}_{d+d(\gamma)}[\lambda] \otimes \mathbb{C}^{2|\boldsymbol{m}|}$ such that

$$\frac{1}{a(\lambda)\cdot a_{t,\gamma}(\lambda)}\,\mathscr{G}(\lambda) - \mathscr{N}_t^{-1}(\lambda)\,\frac{\boldsymbol{\psi}(\lambda)}{a(\lambda)}$$

has no poles in $int\,\gamma$ (see the Appendix). We define $\boldsymbol{u}$ by

$$\boldsymbol{u}(r,\theta) = \int_\gamma r^{\lambda+\boldsymbol{m}}\,\mathscr{F}_t(\lambda,\theta)\,\frac{\mathscr{G}(\lambda)}{a(\lambda)\cdot a_{t,\gamma_t}(\lambda)}\,d\lambda\,.$$

From the definition of $\mathscr{F}_t$ and Theorem 2.2, we see that $\boldsymbol{u}$ belongs to $\mathscr{Z}_t(\gamma; a\cdot a_{t,\gamma}; 1)$ and satisfies $\boldsymbol{M}_t\boldsymbol{u} = 0$. By the definition of $\mathscr{G}$, we can write $\boldsymbol{u}$ as

$$\boldsymbol{u}(r,\theta) = \int_\gamma r^{\lambda+\boldsymbol{m}}\,\mathscr{F}_t(\lambda,\theta)\,\mathscr{N}_t(\lambda)^{-1}\,\frac{\boldsymbol{\psi}(\lambda)}{a(\lambda)}\,d\lambda\,.$$

The operator $\mathscr{C}_t = \mathscr{C}_t(\theta,\partial_\theta)$ commutes with the integral and with the multiplication by powers of r. Using $\mathscr{C}_t\mathscr{F}_t = \mathscr{N}_t$, we find with (2.21)

$$\boldsymbol{C}_t\,\boldsymbol{u}(r) = \int_\gamma r^{\lambda+\boldsymbol{\mu}}\,\frac{\boldsymbol{\psi}(\lambda)}{a(\lambda)}\,d\lambda = \boldsymbol{g}(r)\,.$$

■

Theorem 2.21 *Consider the boundary value problem on Γ_t:*

$$\begin{cases} \boldsymbol{M}_t\,\boldsymbol{u}_t &= \boldsymbol{f}_t\,, \\ \boldsymbol{C}_t\,\boldsymbol{u}_t &= \boldsymbol{g}_t\,, \end{cases} \tag{2.25}$$

where, for any $t \in \mathscr{T}$, $\boldsymbol{f}_t \in \widetilde{\mathscr{Z}}_t(\gamma_t; a_t; \nu)$ and $\boldsymbol{g}_t \in \mathscr{S}(\gamma_t, a_t)$. Suppose that $\boldsymbol{f}_t$ and $\boldsymbol{g}_t$ depend regularly on $t \in \mathscr{T}$, i.e. all the polynomials in their definitions are $\mathscr{C}^\kappa(\mathscr{T})$ functions (or analytic), that γ_t depends continuously on t, and that for every t the hypotheses of lemmas 2.17 and 2.20 are satisfied. Then there exists a polynomial b_0 with integer roots such that the problem (2.25) has a solution

$$\boldsymbol{u}_t \in \mathscr{Z}_t(\gamma_t; a_t\cdot b_0\cdot a_{t,\gamma_t}; \nu+1)$$

which depends regularly on $t \in \mathscr{T}$, too.

Proof. We solve first the interior equation with the help of Lemma 2.17, correct the traces with Lemma 2.19, then solve the boundary value problem with Lemma 2.20 and observe that all operations in the proofs of the lemmas conserve the regularity in t. ■

3. Regularity in weighted Sobolev spaces

Following the by now classical Kondratev method [7] (see also [5], [9]), we consider now regularity in weighted Sobolev spaces. There are two kinds of regularity results: The first one follows from the Agmon-Douglis-Nirenberg estimates on smooth domains, applied to a dyadic partition on a neighborhood of a corner. Here the regularity with respect to a parameter is well known, and we present the result (Proposition 3.1) therefore without proof. The second result describes the asymptotic behavior at the corner and its proof uses Mellin transformation. In Theorem 3.2, we present a version where the gain in regularity at the corner is described in terms of the weight, whereas the Sobolev exponent remains fixed. By combining Proposition 3.1 and Theorem 3.2 one can obtain versions where the gain in regularity is described by changing the weight and the Sobolev index at the same time.

In addition to the weighted Sobolev spaces (see §1.e)

$$\boldsymbol{V}_\delta^{s+\boldsymbol{m}}(\Omega_t) = \prod_{l=1}^{N} V_\delta^{s+m_l}(\Omega_t) \quad \text{and} \quad \boldsymbol{V}_\delta^{s-\boldsymbol{m}}(\Omega_t) = \prod_{k=1}^{N} V_\delta^{s-m_k}(\Omega_t),$$

we consider the following spaces of (Dirichlet) traces on the boundary $\partial\Omega_t$:

$$\boldsymbol{V}_\delta^{s+\boldsymbol{\sigma}}(\partial\Omega_t) = \prod_{h=1}^{|\boldsymbol{m}|} V_\delta^{s+\sigma_h}(\partial\Omega_t)\,.$$

If we define the trace operator $\boldsymbol{B}_t(\partial_z)$ by

$$\boldsymbol{B}_t(\partial_z)\boldsymbol{u} = \Big(u_1,\ \partial_n u_1, \ldots, \partial_n^{m_1-1}u_1,\ u_2, \ldots, \partial_n^{m_2-1}u_2, \ldots, \partial_n^{m_N-1}u_N\Big)\Big|_{\partial\Omega_t},$$

we have then the continuous trace mapping (for $s > -\frac{1}{2}$)

$$\boldsymbol{B}_t(\partial_z) : \boldsymbol{V}_\delta^{s+\boldsymbol{m}}(\Omega_t) \longrightarrow \boldsymbol{V}_\delta^{s+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t)$$

where $\boldsymbol{\sigma} = (m_1, m_1-1, \ldots, 1, m_2, \ldots, 1, \ldots, m_N, \ldots, 1)$.

Proposition 3.1 *Let $\delta_0,\ \delta_1,\ s_0,\ s_1 \in \mathbb{R}$ be such that*

$$-\tfrac{1}{2} < s_0 < s_1 \quad \textit{and} \quad s_0 - \delta_0 = s_1 - \delta_1.$$

Let $\kappa \in \mathbb{N}$. Let $\boldsymbol{u}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_0}^{s_0+\boldsymbol{m}}(\Omega_t))$ be a solution of

$$\left\{\begin{array}{rcl} \boldsymbol{L}_t\boldsymbol{u}_t &=& \boldsymbol{f}_t, \\ \boldsymbol{B}_t\boldsymbol{u}_t &=& \boldsymbol{g}_t, \end{array}\right. \tag{3.1}$$

with

$$\boldsymbol{f}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_1}^{s_1-\boldsymbol{m}}(\Omega_t)) \quad \textit{and} \quad \boldsymbol{g}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_1}^{s_1+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t)).$$

Then $\boldsymbol{u}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_1}^{s_1+\boldsymbol{m}}(\Omega_t))$, and there is an estimate

$$\begin{aligned}\|\boldsymbol{u}_t\|_{\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_1}^{s_1+\boldsymbol{m}}(\Omega_t))} \ \le\ C\Big(&\|\boldsymbol{u}_t\|_{\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_0}^{s_0+\boldsymbol{m}}(\Omega_t))} + \\ &\|\boldsymbol{f}_t\|_{\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_1}^{s_1-\boldsymbol{m}}(\Omega_t))} + \|\boldsymbol{g}_t\|_{\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_1}^{s_1+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t))}\Big).\end{aligned}$$

Theorem 3.2 *Let δ_0, $\delta_1 \in \mathbb{R}$, $s > -\frac{1}{2}$ be such that*

$$\delta_0 > \delta_1 \geq \delta_0 - 1$$

and

$$\forall \lambda \in \Lambda_t \ : \ \operatorname{Re}\lambda \neq s - 1 - \delta_0, \ \ \operatorname{Re}\lambda \neq s - 1 - \delta_1.$$

Let $\kappa \in \mathbb{N}$. Let $\boldsymbol{u}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}^{s+\boldsymbol{m}}_{\delta_0}(\Omega_t))$ be a solution of the boundary value problem (3.1) *with*

$$\boldsymbol{f}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}^{s-\boldsymbol{m}}_{\delta_1}(\Omega_t)) \quad \textit{and} \quad \boldsymbol{g}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}^{s+\boldsymbol{\sigma}-\frac{1}{2}}_{\delta_1}(\partial\Omega_t)).$$

Then there exists $\boldsymbol{u}_{\mathrm{reg};\,t} \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}^{s+\boldsymbol{m}}_{\delta_1}(\Omega_t))$ and $c_{\ell;t} \in \mathscr{C}^\kappa(\mathscr{T})$ such that

$$\boldsymbol{u}_t = \boldsymbol{u}_{\mathrm{reg};\,t} + \sum_{\ell=1}^{\mathcal{L}} c_{\ell;t}\, \boldsymbol{X}_{\ell;t}\,.$$

The singular functions $\boldsymbol{X}_{\ell;t}$ belong to the space $\mathscr{X}_t(\gamma_t)$, where γ_t is a curve surrounding the finite set

$$\Lambda_t \cap \{\lambda \in \mathbb{C} \mid s - 1 - \delta_0 < \operatorname{Re}\lambda < s - 1 - \delta_1\}.$$

They are $\mathscr{C}^\infty$ functions in $(t, r, \theta) \in \mathscr{T} \times (0, \infty) \times [0, \omega_t]$ independent of $\boldsymbol{u}_t$, $\boldsymbol{f}_t$, $\boldsymbol{g}_t$. There are estimates

$$\sum_{\ell=1}^{\mathcal{L}} \|c_{\ell;t}\|_{\mathscr{C}^\kappa(\mathscr{T})} + \|\boldsymbol{u}_{\mathrm{reg};\,t}\|_{\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}^{s+\boldsymbol{m}}_{\delta_1}(\Omega_t))} \ \leq \ C\Big(\|\boldsymbol{u}_t\|_{\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}^{s+\boldsymbol{m}}_{\delta_0}(\Omega_t))} + \|\boldsymbol{f}_t\|_{\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}^{s-\boldsymbol{m}}_{\delta_1}(\Omega_t))} + \|\boldsymbol{g}_t\|_{\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}^{s+\boldsymbol{\sigma}-\frac{1}{2}}_{\delta_1}(\partial\Omega_t))}\Big).$$

Proof. We begin by transforming the boundary value problem (3.1) with the help of the diffeomorphism $\chi_t : \Omega \to \Omega_t$ (see §1.b). In the following we will reserve the notation with t as a subscript for objects defined on Ω_t, whereas for those defined on Ω, we write t as a variable. For the function

$$u(t, r, \theta) = (\chi_t^* u_t)(r, \theta) = (u_t \circ \chi_t)(r, \theta)\,,$$

we obtain a family of boundary value problems on the fixed domain Ω which we write as

$$\left\{ \begin{array}{rcll} \boldsymbol{L}(t, r, \theta; r\partial_r, \partial_\theta)\, \boldsymbol{u}(t, z) & = & \boldsymbol{f}(t, z) & (t, z) \in \mathscr{T} \times \Omega, \\ \boldsymbol{C}(\partial_z)\, \boldsymbol{u}(t, z) & = & \boldsymbol{g}(t, z) & (t, z) \in \mathscr{T} \times \partial\Omega. \end{array} \right. \tag{3.2}$$

The hypotheses on $\boldsymbol{u}$, $\boldsymbol{f}$, $\boldsymbol{g}$ are conserved:

$$\boldsymbol{u} \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}^{s+\boldsymbol{m}}_{\delta_0}(\Omega)), \quad \boldsymbol{f} \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}^{s-\boldsymbol{m}}_{\delta_1}(\Omega)), \quad \boldsymbol{g} \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}^{s+\boldsymbol{\sigma}-\frac{1}{2}}_{\delta_1}(\partial\Omega)).$$

Note that the differential operator $\boldsymbol{L}(t, \cdot)$ does not, in general, have regular coefficients when written in cartesian coordinates. Due to our hypotheses on the diffeomorphism χ_t, however, the operator

$$\mathscr{L}(t, r, \theta; r\partial_r, \partial_\theta) = r^{\boldsymbol{m}} \boldsymbol{L}(t, r, \theta; r\partial_r, \partial_\theta)\, r^{\boldsymbol{m}}$$

has coefficients regular in $(r,\theta) \in [0,r_0) \times [0,\omega]$. Let

$$\mathscr{M}(t,\theta; r\partial_r, \partial_\theta) = \mathscr{L}(t,0,\theta; r\partial_r, \partial_\theta)$$

and

$$\boldsymbol{M}(t,r,\theta; r\partial_r, \partial_\theta) = r^{-\boldsymbol{m}} \mathscr{M}(t,\theta; r\partial_r, \partial_\theta)\, r^{-\boldsymbol{m}}.$$

Then we have the following continuous mapping

$$\boldsymbol{L}(t,r,\theta; r\partial_r, \partial_\theta) - \boldsymbol{M}(t,r,\theta; r\partial_r, \partial_\theta) \; : \; \boldsymbol{V}^{s+\boldsymbol{m}}_{\delta_0}(\Omega) \longrightarrow \boldsymbol{V}^{s-\boldsymbol{m}}_{\delta_0-1}(\Omega) \subset \boldsymbol{V}^{s-\boldsymbol{m}}_{\delta_1}(\Omega)$$

which is also continuous

$$\mathscr{C}^{\kappa}(\mathscr{T}, \boldsymbol{V}^{s+\boldsymbol{m}}_{\delta_0}(\Omega)) \longrightarrow \mathscr{C}^{\kappa}(\mathscr{T}, \boldsymbol{V}^{s-\boldsymbol{m}}_{\delta_1}(\Omega)).$$

This principal part $\boldsymbol{M}(t,\cdot)$ is related to the principal part $\boldsymbol{M}_t$ as defined in §1.c by transformation with the principal part $\breve{\chi}_t$ of the diffeomorphism χ_t: Let

$$\breve{\chi}_t(r,\theta) = (\breve{r}_t, \breve{\theta}_t) \quad \text{with} \quad \breve{r}_t(r,\theta) = r \cdot \lim_{\rho\to 0+} \frac{r_t(\rho,\theta)}{\rho}, \quad \breve{\theta}_t(r,\theta) = \theta_t(0,\theta).$$

Thus for the choice described in §1.b, one finds $\breve{\chi}_t(r,\theta) = (r, \theta\, \dfrac{\omega_t}{\omega})$.

Let $\breve{\chi}^*_t$ be the corresponding transformation operator, i.e.

$$\breve{\chi}^*_t u = u \circ \breve{\chi}_t\,.$$

Then we have

$$\boldsymbol{M}(t,r,\theta; r\partial_r, \partial_\theta) = \breve{\chi}^*_t\, \boldsymbol{M}_t(\partial_z) \left(\breve{\chi}^*_t\right)^{-1}. \tag{3.3}$$

This can easily be seen if we introduce the dilation operator T_ρ by $T_\rho u(z) = u(\rho z)$ and use the relations

$$\boldsymbol{M}_t = \lim_{\rho\to 0+} \rho^{\boldsymbol{m}}\, T_\rho\, \boldsymbol{L}_t\, T_\rho^{-1} \rho^{\boldsymbol{m}}\,; \qquad \boldsymbol{M}(t,\cdot) = \lim_{\rho\to 0+} \rho^{\boldsymbol{m}}\, T_\rho\, \boldsymbol{L}(t,\cdot)\, T_\rho^{-1} \rho^{\boldsymbol{m}}\,;$$

and

$$\breve{\chi}^*_t = \lim_{\rho\to 0+} T_\rho\, \chi^*_t\, T_\rho^{-1}\,; \qquad \boldsymbol{L}(t,\cdot) = \chi^*_t\, \boldsymbol{L}_t \left(\chi^*_t\right)^{-1}.$$

Let φ be a cut-off function near the corner point, i.e. $\varphi \in C^\infty_0(\mathbb{R}^2)$, $\varphi(r,\theta) = 1$ for $r < R_0$, $\varphi(r,\theta) = 0$ for $r > 2R_0$, where $R_0 > 0$ is chosen such that for $r \le 2R_0$, Ω coincides with the sector Γ (see §1.b). Let $\tilde{\boldsymbol{u}} = \varphi \boldsymbol{u}$. Then $\tilde{\boldsymbol{u}}$ satisfies a boundary value problem

$$\begin{cases} \boldsymbol{M}(t,r,\theta; r\partial_r, \partial_\theta)\, \tilde{\boldsymbol{u}}(t,z) &= \tilde{\boldsymbol{f}}(t,z), \\ \boldsymbol{C}(\partial_z)\, \tilde{\boldsymbol{u}}(t,z) &= \tilde{\boldsymbol{g}}(t,z), \end{cases} \tag{3.4}$$

where the initial regularity hypotheses on $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{f}}$, $\tilde{\boldsymbol{g}}$ are still the same as before. The problem (3.4) is (multi-) homogeneous in r and can be considered as defined on Γ. It can therefore be subject to Mellin transformation: Let

$$\begin{aligned} \hat{\boldsymbol{u}}(t,\lambda,\theta) &:= \int_0^\infty r^{-\lambda-\boldsymbol{m}} \tilde{\boldsymbol{u}}(t,r,\theta)\, \frac{dr}{r}\,, \\ \hat{\boldsymbol{f}}(t,\lambda,\theta) &:= \int_0^\infty r^{-\lambda+\boldsymbol{m}} \tilde{\boldsymbol{f}}(t,r,\theta)\, \frac{dr}{r}\,, \end{aligned}$$

$$\widehat{\boldsymbol{g}}^j(t,\lambda) \;:=\; \int_0^\infty r^{-\lambda-\boldsymbol{\sigma}}\widetilde{\boldsymbol{g}}(t,r,\theta)\,\frac{dr}{r} \quad (\theta = j\cdot\omega,\ \ j=0,1)$$

Then $\widehat{\boldsymbol{u}}$ is holomorphic for $\operatorname{Re}\lambda < s-\delta_0-1$, while $\widehat{\boldsymbol{f}}$ and $\widehat{\boldsymbol{g}}^j$ are holomorphic for $\operatorname{Re}\lambda < s-\delta_1-1$, and for $\operatorname{Re}\lambda < s-\delta_0-1$, one has the boundary value problem on $[0,\omega]$:

$$\left\{\begin{array}{rcl} \mathscr{M}(t,\theta;\lambda,\partial_\theta)\,\widehat{\boldsymbol{u}}(t,\lambda,\theta) &=& \widehat{\boldsymbol{f}}(t,\lambda,\theta), \\ \mathscr{C}^j(\partial_\theta)\,\widehat{\boldsymbol{u}}(t,\lambda,\cdot) &=& \widetilde{\boldsymbol{g}}^j(t,\lambda) \end{array}\right. \tag{3.5}$$

where we set for $\theta = j\omega,\ \ j=0,1$:

$$\mathscr{C}^j(\partial_\theta)\boldsymbol{v} = \Big(v_1(\theta),\ldots,\partial_\theta^{m_1-1}v_1(\theta),\ v_2(\theta),\ldots,\partial_\theta^{m_2-1}v_2(\theta),\ldots,\partial_\theta^{m_N-1}v_N(\theta)\Big).$$

We abbreviate this as

$$\mathscr{A}(t,\lambda)\,\widehat{\boldsymbol{u}}(t,\lambda) = \begin{pmatrix}\widehat{\boldsymbol{f}}(t,\lambda)\\ \widehat{\boldsymbol{g}}(t,\lambda)\end{pmatrix} = \widehat{\boldsymbol{F}}(t,\lambda)\,.$$

Inverse Mellin transform gives

$$\widetilde{\boldsymbol{u}}(t,r,\cdot) = \frac{1}{2i\pi}\int_{\operatorname{Re}\lambda = s-\delta_0-1} r^{\lambda+\boldsymbol{m}}\mathscr{A}(t,\lambda)^{-1}\,\widehat{\boldsymbol{F}}(t,\lambda)\;d\lambda\,.$$

If γ_t is a contour around the (finitely many) poles of $\mathscr{A}(t,\lambda)^{-1}$ between the lines $\operatorname{Re}\lambda = s-\delta_0-1$ and $\operatorname{Re}\lambda = s-\delta_1-1$, then we can write

$$\begin{array}{rcl} \widetilde{\boldsymbol{u}}(t,r,\cdot) &=& \dfrac{1}{2i\pi}\displaystyle\int_{\operatorname{Re}\lambda=s-\delta_1-1} r^{\lambda+\boldsymbol{m}}\mathscr{A}(t,\lambda)^{-1}\,\widehat{\boldsymbol{F}}(t,\lambda)\;d\lambda \\ && +\displaystyle\int_{\gamma_t} r^{\lambda+\boldsymbol{m}}\mathscr{A}(t,\lambda)^{-1}\,\frac{1}{2i\pi}\,\widehat{\boldsymbol{F}}(t,\lambda)\;d\lambda \\ &=:& \widetilde{\boldsymbol{u}}_{\mathrm{reg}} + \widetilde{\boldsymbol{u}}_{\mathrm{sing}}. \end{array} \tag{3.6}$$

Here the first term on the right hand side, $\widetilde{\boldsymbol{u}}_{\mathrm{reg}}$, is a function in $\boldsymbol{V}_{\delta_1}^{s+\boldsymbol{m}}(\Gamma)$ for each t which is easily seen to belong to $\mathscr{C}^\kappa(\mathscr{T},\boldsymbol{V}_{\delta_1}^{s+\boldsymbol{m}}(\Gamma))$ and to satisfy the estimate

$$\|\widetilde{\boldsymbol{u}}_{\mathrm{reg}}\|_{\mathscr{C}^\kappa(\mathscr{T},\boldsymbol{V}_{\delta_1}^{s+\boldsymbol{m}}(\Gamma))} \le\; C\Big(\|\widetilde{\boldsymbol{f}}\|_{\mathscr{C}^\kappa(\mathscr{T},\boldsymbol{V}_{\delta_1}^{s-\boldsymbol{m}}(\Gamma))} + \|\widetilde{\boldsymbol{g}}\|_{\mathscr{C}^\kappa(\mathscr{T},\boldsymbol{V}_{\delta_1}^{s+\boldsymbol{\sigma}-\frac12}(\partial\Gamma))}\Big),$$

so it contributes to the regular part $\boldsymbol{u}_{\mathrm{reg};\,t}$.

For the description of the second term, $\widetilde{\boldsymbol{u}}_{\mathrm{sing}}$, we would like to use Corollary 2.8. We cannot apply it directly to the integral involving the resolvent $\mathscr{A}(t,\lambda)^{-1}$, because, unlike the operator $\mathring{\mathscr{A}}_t(\lambda)$, the operator $\mathscr{A}(t,\lambda)$ is not derived from a differential operator with constant coefficients with respect to cartesian coordinates. We do not, therefore, have the explicit description of a basis of the kernel of $\mathscr{M}(t,\lambda)$ as we have it for $\mathscr{M}_t(\lambda)$ (see Theorem 2.2). We can, however, use the description given in Proposition 2.6 for the resolvent of $\mathring{\mathscr{A}}_t$, together with the isomorphism induced by the principal part $\breve{\chi}_t$ of the diffeomorphism χ_t between the solutions of $\mathscr{M}_t\boldsymbol{v}_t = 0$ and $\mathscr{M}(t,\cdot)\boldsymbol{v}(t) = 0$.

Namely, from the definitions, from the identities

$$\begin{array}{rcl} \mathscr{M}(t,\theta;\partial_\theta,\lambda)\,\boldsymbol{v}(\theta) &=& r^{-\lambda}\mathscr{M}(t,\theta;r\partial_r,\partial_\theta)\;r^\lambda\,\boldsymbol{v}(\theta)\,, \\ \mathscr{M}_t(\theta;\partial_\theta,\lambda)\,\boldsymbol{w}(\theta) &=& r^{-\lambda}\mathscr{M}_t(\theta;r\partial_r,\partial_\theta)\;r^\lambda\,\boldsymbol{w}(\theta)\,, \end{array}$$

and from (3.3), we find

$$\mathscr{M}(t,\lambda)\,\boldsymbol{v} = \Big(r^{-\lambda+\boldsymbol{m}}\ \check{\chi}_t^*\ r^{\lambda-\boldsymbol{m}}\Big)\mathscr{M}_t(\lambda)\Big(r^{-\lambda-\boldsymbol{m}}\big(\check{\chi}_t^*\big)^{-1} r^{\lambda+\boldsymbol{m}}\Big)\boldsymbol{v}\,. \tag{3.7}$$

Note that the mapping which associates to any function $\boldsymbol{w}$ defined on $(0,\omega_t)$ the function on $(0,\omega)$:

$$\theta \longmapsto \Big(\partial_r r_t(0,\theta)\Big)^{\lambda-\boldsymbol{m}}\boldsymbol{w}(\theta_t(0,\theta)) \quad \Big(= (r^{-\lambda+\boldsymbol{m}}\ \check{\chi}_t^*\ r^{\lambda-\boldsymbol{m}})\,\boldsymbol{w}\ \ \forall r>0\Big),$$

is an isomorphism of $\mathscr{C}^\infty([0,\omega_t])\otimes\mathbb{C}^N$ onto $\mathscr{C}^\infty([0,\omega])\otimes\mathbb{C}^N$, depending smoothly on t and λ.

From (3.7), we see that the solution of (3.5) is given by

$$\begin{aligned}\widehat{\boldsymbol{u}}(t,\lambda) &= \mathscr{A}(t,\lambda)^{-1}\,\widehat{\boldsymbol{F}}(t,\lambda)\\ &= \mathscr{R}(t,\lambda)\widehat{\boldsymbol{f}}(t,\lambda) - \mathscr{F}(t,\lambda)\mathscr{N}(t,\lambda)^{-1}\Big(\mathscr{C}(t,\lambda)\mathscr{R}(t,\lambda)\widehat{\boldsymbol{f}}(t,\lambda) - \widehat{\boldsymbol{g}}(t,\lambda)\Big)\end{aligned}$$

where the operators and matrices $\mathscr{R}(t,\lambda)$, $\mathscr{F}(t,\lambda)$, $\mathscr{C}(t,\lambda)$, $\mathscr{N}(t,\lambda)$ are related to $\mathscr{R}_t(\lambda)$, $\mathscr{F}_t(\lambda)$, $\mathscr{C}_t(\lambda)$, $\mathscr{N}_t(\lambda)$ in Proposition 2.6 via the isomorphisms of equation (3.7).

It follows that $\widetilde{\boldsymbol{u}}_{\rm sing}$ has the representation

$$\widetilde{\boldsymbol{u}}_{\rm sing}(t,r,\cdot) = \int_{\gamma_t} r^{\lambda+\boldsymbol{m}}\mathscr{F}(t,\lambda)\mathscr{N}(t,\lambda)^{-1}\mathscr{G}(t,\lambda)\,d\lambda,$$

with a $\mathscr{G}(t,\lambda)$ holomorphic in $\lambda\in\overline{int\,\gamma_t}$, regular in t, and estimated by

$$\sup_{\lambda\in\gamma_t}\|\mathscr{G}(\cdot,\lambda)\|_{\mathscr{C}^\kappa(\mathscr{T})} \le C\Big(\|\widetilde{\boldsymbol{f}}\|_{\mathscr{C}^\kappa(\mathscr{T},\boldsymbol{V}^{s-\boldsymbol{m}}_{\delta_1}(\Gamma))} + \|\widetilde{\boldsymbol{g}}\|_{\mathscr{C}^\kappa(\mathscr{T},\boldsymbol{V}^{s+\boldsymbol{\sigma}-\frac12}_{\delta_1}(\partial\Gamma))}\Big).$$

The analogue of Corollary 2.8 shows now

$$\widetilde{\boldsymbol{u}}_{\rm sing}(t,r,\theta) = \sum_\ell c_\ell(t)\,\boldsymbol{X}_\ell(t,r,\theta)$$

where the functions $(r,\theta)\mapsto\boldsymbol{X}_\ell(t,r,\theta)$ span the space $\mathscr{X}(t,\gamma_t)$ of all solutions $\boldsymbol{v}$ of $\boldsymbol{M}(t)\,\boldsymbol{v}=0$, $\boldsymbol{C}\boldsymbol{v}=0$, where $\boldsymbol{v}$ is quasi-homogeneous as in (2.1) with exponents $\lambda\in int\,\gamma_t$. This space $\mathscr{X}(t,\gamma_t)$ is mapped isomorphically onto the space $\mathscr{X}_t(\gamma_t)$ studied in §2 by the mapping $\big(\check{\chi}_t^*\big)^{-1}$. The functions

$$\boldsymbol{X}_{\ell;t} = \big(\check{\chi}_t^*\big)^{-1}\boldsymbol{X}_\ell(t)$$

are therefore a generating set of singular functions in $\mathscr{X}_t(\gamma_t)$ depending smoothly on t. The function

$$\boldsymbol{u}_{{\rm sing};\,t} = \big(\check{\chi}_t^*\big)^{-1}\widetilde{\boldsymbol{u}}_{\rm sing}(t)$$

has therefore the properties of the singular part as stated in the theorem.

It remains to show that

$$\boldsymbol{u}_{{\rm reg};\,t} = \boldsymbol{u}_t - \boldsymbol{u}_{{\rm sing};\,t}$$

is in $\mathscr{C}^\kappa(\mathscr{T},\boldsymbol{V}^{s+\boldsymbol{m}}_{\delta_1}(\Omega_t))$ and satisfies the right estimates. Now

$$\boldsymbol{u}_{{\rm reg};\,t} = \big(\chi_t^*\big)^{-1}(1-\varphi)\boldsymbol{u}(t) + \big(\chi_t^*\big)^{-1}\widetilde{\boldsymbol{u}}_{\rm reg}(t) + \Big(\big(\chi_t^*\big)^{-1} - \big(\check{\chi}_t^*\big)^{-1}\Big)\widetilde{\boldsymbol{u}}_{\rm sing}(t).$$

We have to consider only the last term. Hence we have to show that $\chi_t^* - \breve{\chi}_t^*$ maps the functions $\boldsymbol{X}_{\ell;t}$ into $\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_1}^{s+\boldsymbol{m}}(\Omega))$. But from the assumptions on χ_t it follows by Taylor expansion that $\chi_t^* - \breve{\chi}_t^*$ maps $\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_0}^{s+\boldsymbol{m}}(\Omega))$ into $\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_0-1}^{s+\boldsymbol{m}}(\Omega))$.

■

Remark 3.3 Singular functions in $\mathscr{X}_t(\gamma_t)$ and, more generally in $\mathscr{Z}_t(\gamma_t; a; \nu)$ (see Definition 2.15), belong to $\boldsymbol{V}_\delta^{s+\boldsymbol{m}}(\Omega_t)$ if $s - \delta - 1 < \inf\{\operatorname{Re}\lambda \mid \lambda \in \gamma_t\}$. ■

Theorem 3.2 can be extended to the case of right hand sides in ordinary Sobolev spaces. The singular functions $\boldsymbol{X}_{\ell;t} \in \mathscr{X}_t(\gamma_t)$ are then replaced by $\boldsymbol{Y}_{\ell;t} \in \mathscr{Y}_t(\gamma_t)$, where $\mathscr{Y}_t(\gamma_t)$ is the space studied in §2.b.

Corollary 3.4 *Let $s > 0$, let $0 < \delta_0 \le s$ and $\delta_1 = \max\{0, \delta_0 - 1\}$ with the additional condition that $s \notin \mathbb{N}$ if $\delta_1 = 0$. We assume that*

$$\forall \lambda \in \Lambda_t \ : \ \operatorname{Re}\lambda \neq s - 1 - \delta_0, \ \ \operatorname{Re}\lambda \neq s - 1 - \delta_1.$$

Let $\kappa \in \mathbb{N}$. Let $\boldsymbol{u}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_0}^{s+\boldsymbol{m}}(\Omega_t))$ be a solution of the boundary value problem (3.1) *with*

$$\boldsymbol{f}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t)) \quad \textit{and} \quad \boldsymbol{g}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_1}^{s+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t)).$$

Then there exist $\boldsymbol{u}_{\mathrm{reg};\,t} \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_1}^{s+\boldsymbol{m}}(\Omega_t))$ and $c_{\ell;t} \in \mathscr{C}^\kappa(\mathscr{T})$ such that

$$\boldsymbol{u}_t = \boldsymbol{u}_{\mathrm{reg};\,t} + \sum_{\ell=1}^{\mathcal{L}} c_{\ell;t}\, \boldsymbol{Y}_{\ell;t}\,.$$

The singular functions $\boldsymbol{Y}_{\ell;t}$ belong to the space $\mathscr{Y}_t(\gamma_t)$, where γ_t is a curve surrounding the finite set

$$(\Lambda_t \cup \mathbb{N}) \ \cap \ \{\lambda \in \mathbb{C} \mid s - 1 - \delta_0 < \operatorname{Re}\lambda < s - 1 - \delta_1\}.$$

They are $\mathscr{C}^\infty$ functions in $(t, r, \theta) \in \mathscr{T} \times (0, \infty) \times [0, \omega_t]$ independent of $\boldsymbol{u}_t$, $\boldsymbol{f}_t$, $\boldsymbol{g}_t$. There are estimates

$$\sum_{\ell=1}^{\mathcal{L}} \|c_{\ell;t}\|_{\mathscr{C}^\kappa(\mathscr{T})} + \|\boldsymbol{u}_{\mathrm{reg};\,t}\|_{\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_1}^{s+\boldsymbol{m}}(\Omega_t))} \ \le \ C\Big(\|\boldsymbol{u}_t\|_{\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_0}^{s+\boldsymbol{m}}(\Omega_t))} + \|\boldsymbol{f}_t\|_{\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t))} + \|\boldsymbol{g}_t\|_{\mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_1}^{s+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t))}\Big).$$

Proof. By §1.e, $\boldsymbol{f}_t$ has a Taylor expansion

$$\boldsymbol{f}_t = \boldsymbol{f}_{0;t} + \boldsymbol{f}_{\mathrm{pol};\,t}$$

with $\boldsymbol{f}_{0;t} \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta_1}^{s-\boldsymbol{m}}(\Omega_t))$ and $\boldsymbol{f}_{\mathrm{pol};\,t} \in \mathscr{C}^\kappa(\mathscr{T}, \mathbb{P}_{(\lambda_0-\boldsymbol{m})})$. Here $\lambda_0 = [s - \delta_0] \in \mathbb{N}$. The proof follows then immediately from Lemma 2.13 and Theorem 3.2. ■

Remark 3.5 If the diffeomorphisms χ_t are smooth in cartesian coordinates, then $s \in \mathbb{N}$ is allowed for $\delta_1 = 0$ (cf [5]). ■

Remark 3.6 For the case of analytic parameter dependence, there hold analogues of the statements of Proposition 3.1, Theorem 3.2 and Corollary 3.4. ■

4. Complete asymptotics

In this section, we finish the proof of Theorem 1.3 and we describe the complete stable asymptotics. We do not give a separate proof of Theorem 1.1, because this follows from the same arguments by just omitting the regularity with respect to the parameter $t \in \mathscr{T}$. For the same reason, we do not repeat the proof for the case of analytic parameter dependence.

4.a Taylor expansion of the operators. We begin by considering the Taylor expansion at $r = 0$ of the interior operator $\boldsymbol{L}_t$ and the boundary operator $\boldsymbol{B}_t$. An explicit description of this expansion was given in [4, §1.d]. Here we describe it in a shorter way by using the dilation operator T_ρ defined by

$$T_\rho u(z) = u(\rho z)$$

as in the proof of Theorem 3.2. Consider the differential-operator-valued function

$$\rho \longmapsto \boldsymbol{L}(\rho) := \rho^{\boldsymbol{m}}\, T_\rho\, \boldsymbol{L}_t\, T_\rho^{-1} \rho^{\boldsymbol{m}} \qquad (\rho > 0)\,. \tag{4.1}$$

Let

$$\boldsymbol{L}(\rho) \sim \sum_{p \geq 0} \rho^p\, \boldsymbol{L}_t^{(p)}$$

be its Taylor expansion at $\rho = 0$, i.e.

$$\boldsymbol{L}_t^{(p)} = \frac{1}{p!}\, \frac{d^p}{d\rho^p}\, \boldsymbol{L}(\rho)\Big|_{\rho=0}\,.$$

Then $\boldsymbol{L}_t^{(0)} = \boldsymbol{M}_t$, and $\boldsymbol{L}_t^{(p)}$ is a multi-homogeneous differential operator with polynomial coefficients, linear combination of operators $\boldsymbol{L}_p$ as considered in Lemma 2.16. If we define the remainder $R^P(\boldsymbol{L}_t)$ by

$$\boldsymbol{L}_t = \sum_{p=0}^{P-1} \boldsymbol{L}_t^{(p)} + R^P(\boldsymbol{L}_t)\,,$$

then for any s and δ we have continuous mappings

$$R^P(\boldsymbol{L}_t) \;:\; \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_\delta^{s+\boldsymbol{m}}(\Omega_t)) \longrightarrow \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{V}_{\delta-P}^{s-\boldsymbol{m}}(\Omega_t)). \tag{4.2}$$

For the Taylor expansion of the boundary operator, we consider the family of operators

$$\rho \longmapsto \boldsymbol{B}(\rho) := \rho^{-\boldsymbol{\mu}}\, T_\rho\, \boldsymbol{B}_t\, T_\rho^{-1} \rho^{\boldsymbol{m}} \qquad (\rho > 0)\,. \tag{4.3}$$

In this formula, we interpret the boundary operator $\boldsymbol{B}_t$ in a neighborhood of the origin as the $(2|\boldsymbol{m}| \times N)$ matrix valued operator $\boldsymbol{B}_t$ defined by

$$\Big(\boldsymbol{B}_t\, \boldsymbol{u}\Big)(r) = \Big(u_1(r,0), \ldots, \partial_n^{m_N-1} u_N(r,0);\; u_1(r,\omega_t(r)), \ldots, \partial_n^{m_N-1} u_N(r,\omega_t(r))\Big). \tag{4.4}$$

Thus for $\rho > 0$, the operator $\boldsymbol{B}(\rho)$ maps a N-component vector function of (r,θ) defined in a neighborhood of $(0,r_0]\times[0,\omega_t]$ to a $2|\boldsymbol{m}|$-component vector function of r, where the first $|\boldsymbol{m}|$ components correspond to the traces for $\theta=0$ and the second $|\boldsymbol{m}|$ components to those for $\theta=\omega_t(\rho r)$. For example, the $(|\boldsymbol{m}|+1)$-st component of $\boldsymbol{B}(\rho)\boldsymbol{u}(r)$ is $u_1(r,\omega_t(\rho r))$. Let

$$\boldsymbol{B}(\rho)\sim\sum_{p\geq 0}\rho^p\,\boldsymbol{B}_t^{(p)}$$

be the Taylor expansion of $\boldsymbol{B}(\rho)$ at $\rho=0$. For $p=0$, we find $\boldsymbol{B}_t^{(0)}=\boldsymbol{C}_t$ according to (2.20). We define the remainder $R^P(\boldsymbol{B}_t)$ by

$$\boldsymbol{B}_t=\sum_{p=0}^{P-1}\boldsymbol{B}_t^{(p)}+R^P(\boldsymbol{B}_t)\,.$$

We use this Taylor expansion to describe the action of $\boldsymbol{B}_t$ on singular functions in the spaces $\mathscr{Z}_t(\gamma;a;\nu)$ of Definition 2.15 and we obtain a statement corresponding to Lemma 2.16.

Lemma 4.1 *Let γ, a, ν, p satisfy the hypotheses of Lemma 2.16. Then there exists a polynomial b_0 with integer roots such that the operator*

$$\boldsymbol{B}_t^{(p)}\quad\textit{maps}\quad\mathscr{Z}_t(\gamma;a;\nu)\quad\textit{into}\quad\mathscr{S}(\gamma';a')$$

with

$$\gamma'=\gamma+p=\{\lambda\in\mathbb{C}\mid\lambda-p\in\gamma\},\quad a'(\lambda)=a(\lambda-p)\cdot b_0(\lambda)\,.$$

Proof. We write the proof for $\boldsymbol{u}^+$ as defined in (2.14)–(2.15), the case of $\boldsymbol{u}^-$ being similar. From (2.14), we have

$$\boldsymbol{B}_t^{(p)}\boldsymbol{u}^+(r)=\int_\gamma\frac{\boldsymbol{B}_t^{(p)}\,\boldsymbol{V}^+(\lambda,\zeta)\,\mathscr{H}(\lambda)}{a(\lambda)}\,d\lambda\quad\text{with}\quad\mathscr{H}\in\mathbb{P}_{\deg(a)-1}[\lambda]\otimes\mathbb{C}^N\tag{4.5}$$

and from (2.15), we obtain

$$\boldsymbol{B}_t^{(p)}\boldsymbol{v}^+(\lambda,\zeta)=\int_{\tilde\gamma}\frac{\boldsymbol{B}_t^{(p)}\,\boldsymbol{Z}^+(\lambda;\zeta,\overline{\zeta};\alpha)\,\boldsymbol{h}(\alpha)}{\det\boldsymbol{M}_{t;+}(\alpha)^\nu}\,d\alpha\quad\text{with}\quad\boldsymbol{h}\in\mathbb{P}_{|\boldsymbol{m}|\nu-1}[\alpha]\otimes\mathbb{C}^N.\tag{4.6}$$

From the definition of $\boldsymbol{B}_t^{(p)}$, we see that it satisfies

$$\rho^{-\boldsymbol{\mu}}\,T_\rho\,\boldsymbol{B}_t^{(p)}\,T_\rho^{-1}\rho^{\boldsymbol{m}}=\rho^p\,\boldsymbol{B}_t^{(p)}.$$

Together with the homogeneity of the functions $\boldsymbol{Z}^+$

$$T_\rho\,\boldsymbol{Z}^+(\lambda;\zeta,\overline{\zeta};\alpha)=\boldsymbol{Z}^+(\lambda;\,\rho\,\zeta\,,\,\rho\,\overline{\zeta}\,;\alpha)=\rho^{\lambda+\boldsymbol{m}}\,\boldsymbol{Z}^+(\lambda;\zeta,\overline{\zeta};\alpha)\,,$$

this shows that the function $\boldsymbol{G}^+(\lambda;r;\alpha):=\boldsymbol{B}_t^{(p)}\,\boldsymbol{Z}^+(\lambda;\zeta,\overline{\zeta};\alpha)(r)$ satisfies

$$T_\rho\,\boldsymbol{G}^+(\lambda;r;\alpha)=\rho^{\lambda+p+\boldsymbol{\mu}}\,\boldsymbol{G}^+(\lambda;r;\alpha)\,.$$

Setting $b(\lambda):=(\lambda+1)\cdots(\lambda+\max_l m_l)$ and defining

$$\boldsymbol{g}^+(\lambda;\alpha):=\boldsymbol{G}^+(\lambda;1;\alpha)\cdot b(\lambda),$$

we therefore obtain a function $\boldsymbol{g}^+$ holomorphic in λ and α such that

$$\boldsymbol{G}^+(\lambda; r; \alpha) = r^{\lambda+p+\boldsymbol{\mu}} \, \frac{\boldsymbol{g}^+(\lambda;\alpha)}{b(\lambda)} \, .$$

Let

$$\boldsymbol{\psi}_1(\lambda) = \int_{\tilde\gamma} \frac{\boldsymbol{g}^+(\lambda;\alpha)\, \boldsymbol{h}(\alpha)}{\det \boldsymbol{M}_{t;+}(\alpha)^\nu} \; d\alpha \, .$$

With (4.5)–(4.6), we find

$$\boldsymbol{B}_t^{(p)} \boldsymbol{u}^+(r) = \int_\gamma r^{\lambda+p+\boldsymbol{\mu}} \, \frac{\boldsymbol{\Psi}_1(\lambda)\, \mathscr{H}(\lambda)}{a(\lambda)} \; d\lambda \, ,$$

which, after Euclidean division modulo a, reduces to the required form. Finally we obtain

$$\boldsymbol{B}_t^{(p)} \boldsymbol{u}(r) = \int_{\gamma+p} r^{\lambda+\boldsymbol{\mu}} \, \frac{\boldsymbol{\psi}(\lambda)}{a(\lambda-p)\cdot b_0(\lambda)} \; d\lambda \, . \tag{4.7}$$

■

Lemma 4.2 *Let γ, a, ν be as in the preceding lemma. Then there is $r_0 > 0$ such that for any $P \in \mathbb{N}$*

$$R^P(\boldsymbol{B}_t) \quad \textit{maps} \quad \mathscr{Z}_t(\gamma; a; \nu) \quad \textit{into} \quad \boldsymbol{V}_{\delta-P}^{s+\boldsymbol{\mu}-\frac12}(0, r_0)$$

for any s, δ satisfying (compare with Remark 3.3)

$$s - \delta - 1 < \inf\{\operatorname{Re}\lambda \mid \lambda \in \gamma_t\} \, .$$

Proof. We consider again a function $\boldsymbol{u}^+$ as in the previous proof. From the definition 4.3 of $\boldsymbol{B}(\rho)$ and the definition of $R^P(\boldsymbol{B}_t)$, we find

$$\rho^{-\boldsymbol{\mu}}\, T_\rho\, R^P(\boldsymbol{B}_t)\, T_\rho^{-1} \rho^{\boldsymbol{m}} = \rho^P \widetilde{R}(\rho) \, ,$$

where $\widetilde{R}(\rho)$ is $\mathscr{C}^\infty$ at $\rho = 0$ and $\widetilde{R}(0) = R^P(\boldsymbol{B}_t)$.
Setting $\boldsymbol{G}^+(\lambda; r; \alpha) := \big(R^P(\boldsymbol{B}_t)\, \boldsymbol{Z}^+(\lambda; \zeta, \overline{\zeta}; \alpha)\big)(r)$, we find

$$T_\rho\, \boldsymbol{G}^+(\lambda; r; \alpha) = \rho^{\lambda+p+\boldsymbol{\mu}}\, \widetilde{\boldsymbol{G}}^+(\lambda; r; \alpha; \rho) \, .$$

Here $\widetilde{\boldsymbol{G}}^+(\lambda; r; \alpha; \rho) = \widetilde{R}(\rho)\, \boldsymbol{Z}^+(\lambda; \zeta, \overline{\zeta}; \alpha)$ and we see that, with the polynomial $b(\lambda)$ as above, the function

$$(\lambda, r, \alpha, \rho) \longmapsto \widetilde{\boldsymbol{G}}^+(\lambda; r; \alpha; \rho) \cdot b(\lambda)$$

is holomorphic in λ and α and $\mathscr{C}^\infty$ in $r > 0$ and $\rho \ge 0$. Thus writing

$$\boldsymbol{G}^+(\lambda; r; \alpha) = T_r\, \boldsymbol{G}^+(\lambda; 1; \alpha) = r^{\lambda+p+\boldsymbol{\mu}}\, \widetilde{\boldsymbol{G}}^+(\lambda; 1; \alpha; r)$$

we see that $r \mapsto \boldsymbol{G}^+(\lambda; r; \alpha)$ belongs to $\boldsymbol{V}_{\delta-P}^{s+\boldsymbol{\mu}-\frac12}(0, r_0)$ for any α and any λ with $\operatorname{Re}\lambda > s - \delta - 1$ such that $b(\lambda) \neq 0$. This is uniform in $\alpha \in \tilde\gamma$ and $\lambda \in \gamma_t$. Integrating over $\tilde\gamma$ and γ_t according to (2.14)–(2.15), we find $\boldsymbol{u}^+ \in \boldsymbol{V}_{\delta-P}^{s+\boldsymbol{\mu}-\frac12}(0, r_0)$. ■

Whereas the preceding two lemmas were formulated for a fixed $t \in \mathscr{T}$, their proofs show that the regularity with respect to t is conserved.

Proposition 4.3 *Let the hypotheses of Lemmas* 4.1 *and* 4.2 *be satisfied for every* $t \in \mathscr{T}$. *Let* γ_t *depend continuously on* t. *Suppose that* $\boldsymbol{u}_t \in \mathscr{Z}_t(\gamma_t; a_t; \nu)$ *is smooth with respect to* t *in the sense that all the polynomials in* (2.14)–(2.15) *are* $\mathscr{C}^\infty$ *in the variable* t. *Then also the polynomial* a' *and the polynomial* $\boldsymbol{\psi}$ *in the formula* (4.7) *for* $\boldsymbol{B}_t^{(p)}\boldsymbol{u}_t$ *are* $\mathscr{C}^\infty$ *with respect to* $t \in \mathscr{T}$. *Furthermore,*

$$R^P(\boldsymbol{B}_t)\, \boldsymbol{u}_t \in \mathscr{C}^\infty(\mathscr{T}, \boldsymbol{V}_{\delta-P}^{s+\boldsymbol{\mu}-\frac{1}{2}}(0, r_0)) .$$

4.b Expansion of the singular functions.

In the following we denote by $\mathscr{C}^\infty(\mathscr{T}, \mathscr{Z}_t(\gamma_t; a_t; \nu))$ the space of functions

$$t \longmapsto \boldsymbol{u}_t \in \mathscr{Z}_t(\gamma_t; a_t; \nu), \qquad t \in \mathscr{T}$$

such that the polynomials in their definition (2.14)–(2.15) are $\mathscr{C}^\infty$ functions of $t \in \mathscr{T}$. Analogously, we define $\mathscr{C}^\infty(\mathscr{T}, \mathscr{X}_t(\gamma_t))$ and $\mathscr{C}^\infty(\mathscr{T}, \mathscr{Y}_t(\gamma_t))$ — we do not know whether this is equivalent to the requirement that the functions themselves are $\mathscr{C}^\infty$ functions of $\mathscr{T}$, but this is not important for the following.

We will now describe the construction of the complete singular functions of the operator $(\boldsymbol{L}_t, \boldsymbol{B}_t)$. By $\partial\Omega_t^0$ we denote a neighborhood of the corner in Ω_t which can be identified with a neighborhood of the origin in $\partial\Gamma_t$.

Lemma 4.4 *Let* $\gamma_t \subset \mathbb{C}$ *be a family of simple closed curves depending continuously on* $t \in \mathscr{T}$. *Suppose that for an integer* $P \geq 2$ *and all* $t \in \mathscr{T}$ *there holds*

$$(\gamma_t + p) \cap \mathbb{N} = \emptyset, \quad (\gamma_t + p) \cap \Lambda_t = \emptyset \quad \textit{for } p = 0, \dots, P-1.$$

Then for any $\boldsymbol{X}_t \in \mathscr{C}^\infty(\mathscr{T}, \mathscr{X}_t(\gamma_t))$ *there exist* $\boldsymbol{X}_t^{(p)}$, $p = 0, \dots, P-1$, *with*

$$\begin{array}{rcl} \boldsymbol{X}_t^{(0)} & = & \boldsymbol{X}_t, \\ \boldsymbol{X}_t^{(p)} & \in & \mathscr{C}^\infty\Big(\mathscr{T},\ \mathscr{Z}_t(\gamma_t + p;\ a_{t,\gamma_t}^{(p)} \cdot b_0^{(p)}\ ;\ 2p+1)\Big), \quad p = 1, \dots, P-1, \end{array}$$

such that the function

$$S^P(\boldsymbol{X}_t) := \sum_{p=0}^{P-1} \boldsymbol{X}_t^{(p)}$$

satisfies

$$\begin{array}{rcl} \boldsymbol{L}_t\, S^P(\boldsymbol{X}_t) & \in & \mathscr{C}^\infty(\mathscr{T}, \boldsymbol{V}_{\delta-P}^{s-\boldsymbol{m}}(\Omega_t)), \\ \boldsymbol{B}_t\, S^P(\boldsymbol{X}_t) & \in & \mathscr{C}^\infty(\mathscr{T}, \boldsymbol{V}_{\delta-P}^{s+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t^0)) \end{array}$$

for any s *and* δ *with* $s - \delta - 1 < \inf\{\operatorname{Re}\lambda \mid \lambda \in \gamma_t,\ t \in \mathscr{T}\}$. *The polynomials* $a_{t,\gamma_t}^{(p)}$ *are given by*

$$a_{t,\gamma_t}^{(p)}(\lambda) = a_{t,\gamma_t+p}(\lambda) \cdot a_{t,\gamma_t+p-1}(\lambda - 1) \cdots a_{t,\gamma_t}(\lambda - p) \tag{4.8}$$

and $b_0^{(p)}$ *has integer roots.*

Proof. We construct the $\boldsymbol{X}_t^{(p)}$ inductively as solutions of the boundary value problems

$$\Big(\boldsymbol{M}_t, \boldsymbol{C}_t\Big)\boldsymbol{X}_t^{(p)} = -\sum_{q=0}^{p-1}\Big(\boldsymbol{L}_t^{(p-q)}, \boldsymbol{B}_t^{(p-q)}\Big)\boldsymbol{X}_t^{(q)}. \tag{4.9}$$

Note that this holds also for $p = 0$, namely $(\boldsymbol{M}_t, \boldsymbol{C}_t)\boldsymbol{X}_t^{(0)} = 0$ due to the definition of the space $\mathscr{X}_t(\gamma_t)$. It is clear that $\boldsymbol{X}_t^{(0)}$ belongs to $\mathscr{C}^\infty(\mathscr{T}, \mathscr{Z}_t(\gamma_t;\, a_{t,\gamma_t}; 1))$. Suppose that for $q = 1, \ldots, p-1$, $\boldsymbol{X}_t^{(q)} \in \mathscr{C}^\infty(\mathscr{T}, \mathscr{Z}_t(\gamma_t;\, a_{t,\gamma_t}^{(q)} \cdot b_0^{(q)};\, 2q+1))$ have been constructed. Then according to Lemma 2.16 and Lemma 4.1

$$\Big(\boldsymbol{L}_t^{(p-q)}, \boldsymbol{B}_t^{(p-q)}\Big)\boldsymbol{X}_t^{(q)} = (\boldsymbol{f}_t^{(q)}, \boldsymbol{g}_t^{(q)})$$

with

$$\begin{aligned} \boldsymbol{f}_t^{(q)} &\in \widetilde{\mathscr{Z}}_t(\gamma_t + p\,;\; a_{t,\gamma_t}^{(q)}(\lambda - p + q)\cdot b_1^{(q)}(\lambda)\,;\; 2q+1+\nu_0^{(q)}) \\ \boldsymbol{g}_t^{(q)} &\in \mathscr{S}(\gamma_t + p\,;\; a_{t,\gamma_t}^{(q)}(\lambda - p + q)\cdot b_1^{(q)}(\lambda)), \end{aligned}$$

with a polynomial $b_1^{(q)}$ with integer roots. Considering the degrees of the differential operators in $\boldsymbol{L}_t^{(p-q)}$ and the proof of Lemma 2.16, we see that $\nu_0^{(q)} \le p - q$, hence $2q + 1 + \nu_0^{(q)} \le 2p$.

Hence we conclude from Theorem 2.21 that the boundary value problem

$$\Big(\boldsymbol{M}_t, \boldsymbol{C}_t\Big)\boldsymbol{X}_t^{(p)} = -\sum_{q=0}^{p-1}(\boldsymbol{f}_t^{(q)}, \boldsymbol{g}_t^{(q)})$$

has a solution $\boldsymbol{X}_t^{(p)} \in \mathscr{Z}_t(\gamma_t + p\,;\; a_t \cdot b_0^{(p)} \cdot a_{t,\gamma_t+p}\,;\; 2p+1)$ where $b_0^{(p)}$ has the required form and a_t is the least common multiple of

$$a_{t,\gamma_t}^{(q)}(\lambda - p + q) = a_{t,\gamma_t+q}(\lambda - p + q)\cdots a_{t,\gamma_t+1}(\lambda - p + 1)\cdot a_{t,\gamma_t}(\lambda - p) \quad \text{for } q = 0, \ldots, p-1.$$

Thus $\boldsymbol{X}_t^{(p)}$ is in the right space.

Now we compute $(\boldsymbol{L}_t, \boldsymbol{B}_t)S^P(\boldsymbol{X}_t)$ by using their Taylor expansion

$$\begin{aligned} \Big(\boldsymbol{L}_t, \boldsymbol{B}_t\Big)S^P(\boldsymbol{X}_t) &= \sum_{p=0}^{P-1}\Big(\boldsymbol{L}_t^{(p)}, \boldsymbol{B}_t^{(p)}\Big)S^P(\boldsymbol{X}_t) \qquad + R^P\Big(\boldsymbol{L}_t, \boldsymbol{B}_t\Big)S^P(\boldsymbol{X}_t) \\ &= \sum_{p=0}^{P-1}\sum_{q=0}^{p}\Big(\boldsymbol{L}_t^{(p-q)}, \boldsymbol{B}_t^{(p-q)}\Big)\boldsymbol{X}_t^{(q)} + \sum_{\substack{p+q\ge P\\ p,q\le P-1}}\Big(\boldsymbol{L}_t^{(p)}, \boldsymbol{B}_t^{(p)}\Big)\boldsymbol{X}_t^{(q)} \\ &\qquad + R^P\Big(\boldsymbol{L}_t, \boldsymbol{B}_t\Big)S^P(\boldsymbol{X}_t). \end{aligned}$$

According to the construction, the first sum on the right hand side vanishes.

For the second sum, we note that for $p + q \ge P$ we have

$$\boldsymbol{L}_t^{(p)}\boldsymbol{X}_t^{(q)} \in \widetilde{\mathscr{Z}}_t(\gamma_t + p + q;\, a';\, \nu') \subset \boldsymbol{V}_{\delta-P}^{s-\boldsymbol{m}}(\Omega_t)$$

and

$$\boldsymbol{B}_t^{(p)}\boldsymbol{X}_t^{(q)} \in \mathscr{S}(\gamma_t + p + q;\, a') \subset \boldsymbol{V}_{\delta-P}^{s+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t^0),$$

because $s - (\delta - P) - 1 < \inf\{\operatorname{Re}\lambda \mid \lambda \in \gamma_t + p + q\}$.

Finally, for the remainders we use the fact that

$$\boldsymbol{X}_t^{(p)} \in \mathscr{Z}_t(\gamma_t + p;\ a_{t,\gamma_t}^{(p)} \cdot b_0^{(p)};\ 2p+1)$$

and hence, with (4.2),

$$R^P(\boldsymbol{L}_t)\boldsymbol{X}_t^{(p)} \in \boldsymbol{V}_{\delta-P}^{s-\boldsymbol{m}}(\Omega_t)$$

and, with Lemma 4.2

$$R^P(\boldsymbol{B}_t)\boldsymbol{X}_t^{(p)} \in \boldsymbol{V}_{\delta-P}^{s+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t^0).$$

For the traces, we identified $|\boldsymbol{m}|$-component vector functions on $\partial\Omega_t^0$ with $2|\boldsymbol{m}|$-component vector functions on $(0, r_0)$, and thus $\boldsymbol{V}_{\delta-P}^{s+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t^0)$ with $\boldsymbol{V}_{\delta-P}^{s+\boldsymbol{\mu}-\frac{1}{2}}(0, r_0)$. The proof is complete if we remark that all above constructions conserve the regularity with respect to $t \in \mathscr{T}$. ■

If we admit polynomial right hand sides, we obtain the following generalisation.

Lemma 4.5 *In addition to the hypotheses of Lemma* 4.4*, suppose that int* γ_t *contains exactly one integer element* λ_0.
Then for any $\boldsymbol{Y}_t \in \mathscr{C}^\infty(\mathscr{T}, \mathscr{Y}_t(\gamma_t))$ *there exist* $\boldsymbol{Y}_t^{(p)}$, $p = 0, \ldots, P-1$, *with*

$$\begin{aligned}\boldsymbol{Y}_t^{(0)} &= \boldsymbol{Y}_t,\\ \boldsymbol{Y}_t^{(p)} &\in \mathscr{C}^\infty\Big(\mathscr{T},\ \mathscr{Z}_t(\gamma_t + p\,;\ a_{t,\gamma_t}^{(p)} \cdot b_1^{(p)}\,;\ 2p+1) + \mathbb{P}_{(\lambda_0+p+\boldsymbol{m})}\Big), \quad p = 1, \ldots, P-1,\end{aligned}$$

such that the function

$$S^P(\boldsymbol{Y}_t) = \sum_{p=0}^{P-1} \boldsymbol{Y}_t^{(p)}$$

satisfies

$$\begin{aligned}\boldsymbol{L}_t\, S^P(\boldsymbol{Y}_t) &\in \mathscr{C}^\infty(\mathscr{T}, \boldsymbol{V}_{\delta-P}^{s-\boldsymbol{m}}(\Omega_t) + \mathbb{P}_{(\lambda_0-\boldsymbol{m})}),\\ \boldsymbol{B}_t\, S^P(\boldsymbol{Y}_t) &\in \mathscr{C}^\infty(\mathscr{T}, \boldsymbol{V}_{\delta-P}^{s+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t^0)).\end{aligned}$$

The polynomials $a_{t,\gamma_t}^{(p)}$ *are given in* (4.8) *and* $b_1^{(p)}$ *are polynomials with integer roots.*

Proof. The proof follows the same scheme as above. The functions $\boldsymbol{Y}_t^{(p)}$ are solutions of the boundary value problems

$$\Big(\boldsymbol{M}_t, \boldsymbol{C}_t\Big)\boldsymbol{Y}_t^{(p)} = -\sum_{q=0}^{p-1}\Big(\boldsymbol{L}_t^{(p-q)}, \boldsymbol{B}_t^{(p-q)}\Big)\boldsymbol{Y}_t^{(q)}. \tag{4.10}$$

The main difference is the situation for $p = 0$: as $\boldsymbol{Y}_t \in \mathscr{Y}_t(\gamma_t)$, we have

$$(\boldsymbol{M}_t, \boldsymbol{C}_t)\boldsymbol{Y}_t^{(0)} = (\boldsymbol{f}_{\text{pol}}, 0), \quad \text{with} \quad \boldsymbol{f}_{\text{pol}} \in \mathbb{P}_{(\lambda_0-\boldsymbol{m})}.$$

The construction of the $\boldsymbol{Y}_t^{(q)}$ is based on Lemma 2.13 and Theorem 2.21. With similar calculations as in the previous lemma, we obtain

$$\Big(\boldsymbol{L}_t, \boldsymbol{B}_t\Big)S^P(\boldsymbol{Y}_t) = (\boldsymbol{f}_{\text{pol}}, 0) + \sum_{\substack{p+q\geq P\\ p,\,q\leq P-1}} \Big(\boldsymbol{L}_t^{(p)}, \boldsymbol{B}_t^{(p)}\Big)\boldsymbol{Y}_t^{(q)} + R^P\Big(\boldsymbol{L}_t, \boldsymbol{B}_t\Big)S^P(\boldsymbol{Y}_t).$$

Hence the lemma. ■

4.c Proof of Theorem 1.3. We will show now that the singular functions $S^P(\boldsymbol{Y}_t)$ are the correct objects to use in the decomposition Theorems 1.1 and 1.3.

Thus let s be given as in (1.1). Without restriction (see, however, Remark 4.7) we assume $s_0 = 0$. We choose a sequence $\xi_0, \dots, \xi_I \in \mathbb{R}$ with the properties

$$\begin{gathered} -2 < \xi_0 \le -1 < \xi_1 \dots < \xi_I = s-1, \\ \xi_i \notin \mathbb{N};\ \ |\xi_i - \xi_{i-1}| < 1;\ \ \operatorname{Re}\lambda \ne \xi_i \text{ for all } \lambda \in \Lambda_t,\ t \in \mathscr{T} \quad (i = 1, \dots, I). \end{gathered} \tag{4.11}$$

Since this restricts the variation of the Λ_t, the condition (4.11) might not be satisfied for the whole interval $\mathscr{T}$. In this case we can find a covering of $\mathscr{T}$ by open subintervals for which (4.11) is possible. We will then get the results of Theorems 1.1 and 1.3 for these subintervals and we can patch them together with the help of a smooth partition of unity. For the case of analytic dependence on the parameter, we only obtain a local result. So we assume now that (4.11) is valid. We can then cover the set

$$(\Lambda_t \cup \mathbb{N}) \cap \{\lambda \in \mathbb{C} \mid \xi_0 < \operatorname{Re}\lambda < s-1\}$$

with a finite number of open sets of the form $\mathit{int}\,\gamma_t^j$, $j = 1, \dots, J$, where γ_t^j are simple closed curves depending continuously on $t \in \mathscr{T}$ and satisfying

$$(\gamma_t^j + p) \ \cap\ (\Lambda_t \cup \mathbb{N}) \ \cap\ \{\lambda \in \mathbb{C} \mid \xi_0 < \operatorname{Re}\lambda < s-1\} = \emptyset$$

for all $j = 1, \dots, J$, $t \in \mathscr{T}$ and $p \in \mathbb{N}$, and

$$\operatorname{Re}\lambda \ne \xi_i \quad \text{for all } \lambda \in \gamma_t^j,\ \ t \in \mathscr{T},\ \ j = 1, \dots, J;\ \ i = 1, \dots, I.$$

According to Theorem 2.12, each of the spaces $\mathscr{Y}_t(\gamma_t^j)$ is generated by a finite number of functions described there.

The following theorem then holds.

Theorem 4.6 *Let* $\{\boldsymbol{Y}_{\ell;t} \mid \ell = 1, \dots, \mathcal{L}\}$ *be a generating set of the space* $\oplus_{j=1,\dots,J}\mathscr{Y}_t(\gamma_t^j)$ *such that each* $\boldsymbol{Y}_{\ell;t}$ *belongs to* $\mathscr{C}^\infty(\mathscr{T}, \mathscr{Y}_t(\gamma_t^{j_\ell}))$ *for some* j_ℓ*. For each* $\ell = 1, \dots, \mathcal{L}$*, let* i *and* j *be such that*

$$\boldsymbol{Y}_{\ell;t} \in \mathscr{Y}_t(\gamma_t^j) \quad \text{and} \quad \xi_i < \operatorname{Re}\lambda < \xi_{i+1} \text{ for all } \lambda \in \gamma_t^j,$$

and define $P_\ell = [s - \xi_i]$*. Then with the singular functions* $\boldsymbol{S}_{\ell;t}$ *defined by*

$$\boldsymbol{S}_{\ell;t} = S^{P_\ell}(\boldsymbol{Y}_{\ell;t})\,,$$

Theorems 1.1 *and* 1.3 *hold.*

Proof. Let $\boldsymbol{f}_t \in \boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t)$ be given and $\boldsymbol{u}_t \in \mathring{\boldsymbol{H}}^{\boldsymbol{m}}(\Omega_t)$ be a solution of the boundary value problem (1.2). We begin by using the interior regularity theorem, Proposition 3.1:

$$\boldsymbol{u}_t \in \mathring{\boldsymbol{H}}^{\boldsymbol{m}}(\Omega_t) \subset \boldsymbol{V}_0^{\boldsymbol{m}}(\Omega_t), \quad \boldsymbol{L}_t\,\boldsymbol{u}_t \in \boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t) \subset \boldsymbol{V}_s^{s-\boldsymbol{m}}(\Omega_t), \quad \boldsymbol{B}_t\,\boldsymbol{u}_t = 0$$

implies $\boldsymbol{u}_t \in \boldsymbol{V}_s^{s+\boldsymbol{m}}(\Omega_t) \subset \boldsymbol{V}_{\delta_0}^{s+\boldsymbol{m}}(\Omega_t)$ with $\delta_0 = s - \xi_0 - 1$. We define

$$\delta_i = s - \xi_i - 1 = \xi_I - \xi_i.$$

With $\boldsymbol{u}_t^{(0)} = \boldsymbol{u}_t$, we have therefore the beginning $i = 0$ of the following situation:

$$\boldsymbol{u}_t^{(i)} \in \boldsymbol{V}_{\delta_i}^{s+\boldsymbol{m}}(\Omega_t); \quad \boldsymbol{L}_t\, \boldsymbol{u}_t^{(i)} \in \boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t); \quad \boldsymbol{B}_t\, \boldsymbol{u}_t^{(i)} \in \boldsymbol{V}_0^{s+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t). \tag{4.12}$$

We will show that by splitting off suitable singular functions, we can pass from i to $i+1$. At the end, we shall have $\boldsymbol{u}_{\mathrm{reg};\,t} = \boldsymbol{u}_t^{(I)}$.

In (4.12), we can apply Corollary 3.4 and obtain

$$\boldsymbol{u}_t^{(i)} = \boldsymbol{u}_{\mathrm{reg};\,t}^{(i)} + \sum_\ell c_{\ell;t}\, \boldsymbol{Y}_{\ell;t}$$

with

$$\boldsymbol{u}_{\mathrm{reg};\,t}^{(i)} \in \boldsymbol{V}_{\max(0,\delta_i-1)}^{s+\boldsymbol{m}}(\Omega_t) \subset \boldsymbol{V}_{\delta_{i+1}}^{s+\boldsymbol{m}}(\Omega_t)$$

and $\boldsymbol{Y}_{\ell;t} \in \mathscr{Y}_t(\gamma_t^j)$ for a γ_t^j such that

$$\lambda \in \gamma_t^j \implies \operatorname{Re}\lambda \in (s-1-\delta_i,\ s-1-\delta_{i+1}) = (\xi_i,\, \xi_{i+1}).$$

We take a smooth cut-off function $\eta(r)$ supported in $[0, r_0)$ and define:

$$\boldsymbol{u}_t^{(i+1)} = \boldsymbol{u}_{\mathrm{reg};\,t}^{(i)} + \sum_\ell c_{\ell;t}\Big(\boldsymbol{Y}_{\ell;t} - \eta\cdot S^{P_\ell}(\boldsymbol{Y}_{\ell;t})\Big).$$

Each term in the sum is in $\boldsymbol{V}_\delta^{s+\boldsymbol{m}}(\Omega_t)$ for any $\delta > s-1-\inf\{\operatorname{Re}\lambda \mid \lambda\in\gamma_t^j\}-1$ (see Remark 3.3) and thus in $\boldsymbol{V}_{\delta_{i+1}}^{s+\boldsymbol{m}}(\Omega_t)$, since

$$\delta_{i+1} > \delta_i - 1 = s - \xi_i - 2 > s - 2 - \inf\{\operatorname{Re}\lambda \mid \lambda\in\gamma_t^j\}.$$

Hence $\boldsymbol{u}_t^{(i+1)} \in \boldsymbol{V}_{\delta_{i+1}}^{s+\boldsymbol{m}}(\Omega_t)$. Furthermore we have

$$\boldsymbol{u}_t^{(i+1)} = \boldsymbol{u}_t^{(i)} - \eta\sum_\ell c_{\ell;t}\, S^{P_\ell}(\boldsymbol{Y}_{\ell;t}).$$

Lemma 4.5 gives us

$$\begin{aligned}
\boldsymbol{L}_t\, S^{P_\ell}(\boldsymbol{Y}_{\ell;t}) &\in \boldsymbol{V}_{\delta_i-P_\ell}^{s-\boldsymbol{m}}(\Omega_t) + \mathbb{P}_{(\lambda_j-\boldsymbol{m})},\\
\boldsymbol{B}_t\, S^{P_\ell}(\boldsymbol{Y}_{\ell;t}) &\in \boldsymbol{V}_{\delta_i-P_\ell}^{s+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t^0),
\end{aligned}$$

where λ_j is the only integer in *int* γ_t^j. If there is no integer in *int* γ_t^j, the polynomial part in $\boldsymbol{L}_t\, S^{P_\ell}(\boldsymbol{Y}_{\ell;t})$ is absent. Also, $\delta_i - P_\ell \le \delta_i - (s-\xi_i-1) = 0$. Hence

$$\begin{aligned}
\boldsymbol{L}_t\, S^{P_\ell}(\boldsymbol{Y}_{\ell;t}) &\in \boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t),\\
\boldsymbol{B}_t\, S^{P_\ell}(\boldsymbol{Y}_{\ell;t}) &\in \boldsymbol{V}_0^{s+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t^0),
\end{aligned}$$

and with (4.12) we find

$$\boldsymbol{L}_t\, \boldsymbol{u}_t^{(i+1)} \in \boldsymbol{H}^{s-\boldsymbol{m}}(\Omega_t) \quad\text{and}\quad \boldsymbol{B}_t\, \boldsymbol{u}_t^{(i+1)} \in \boldsymbol{V}_0^{s+\boldsymbol{\sigma}-\frac{1}{2}}(\partial\Omega_t).$$

This shows the induction step, and the proof of the theorem is complete if we note once again that all the tools used here conserve the regularity in $t \in \mathscr{T}$. ■

Remark 4.7

(i) If $\boldsymbol{u}_t \in \boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)$ is given as in Theorems 1.1 and 1.3 with $s_0 > 0$, then, due to the obvious inclusion $\boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t) \subset \boldsymbol{H}^{\boldsymbol{m}}(\Omega_t)$, we can apply the above theorem for $s_0 = 0$. Note that, however, we then obtain for $\boldsymbol{u}_t$ a decomposition which uses

as singular functions the $S^{P_\ell}(\boldsymbol{Y}_{\ell;t})$ where $\boldsymbol{Y}_{\ell;t} \in \mathscr{Y}_t(\gamma_t^{j_\ell})$, and the γ_t^j cover all the singular exponents with real parts in $(\xi_0, s-1)$, not only those in $(s_0-1, s-1)$. Now functions in $\mathscr{Y}_t(\gamma_t)$ with $\operatorname{Re}\gamma_t \subset (-\infty, s_0-1)$ can only belong to $\boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)$ if they are polynomials. Thus the singular functions $\boldsymbol{S}_{\ell;t}$ which actually appear in the decomposition of a $\boldsymbol{u}_t \in \boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)$ will have the form $S^{P_\ell}(\boldsymbol{Y}_{\ell;t})$
(a) with $\boldsymbol{Y}_{\ell;t} \in \mathscr{Y}_t(\gamma_t^j)$ and $\operatorname{Re}\gamma_t^j \subset (s_0-1, s-1)$ — so the $\boldsymbol{Y}_{\ell;t}$ themselves belong to $\boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)$ without belonging to $\boldsymbol{H}^{s+\boldsymbol{m}}(\Omega_t)$,
(b) or with a polynomial $\boldsymbol{Y}_{\ell;t}$.

(ii) We consider the case *(b)*.
• If an opening ω_t is equal to π or 2π, the following situation may occur: A polynomial $\boldsymbol{Y}_{\ell;t}$ whose exponents do not correspond to the interval $(s_0-1, s-1)$, generates terms $\boldsymbol{Y}_{\ell;t}^{(p)}$ which belong to $\boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)$ without belonging to $\boldsymbol{H}^{s+\boldsymbol{m}}(\Omega_t)$, although they cannot be represented as combinations of other functions $\boldsymbol{Y}_{\ell';t}^{(p')}$ coming from a $\boldsymbol{Y}_{\ell';t}$ whose exponents correspond to the interval $(s_0-1, s-1)$.
• On the other hand, if the openings ω_t are never equal to π or 2π, this situation never occurs. See also the examples in the next subsection (§4.d). ■

Remark 4.8
(i) If certain eigenvalues μ_t in Λ_t cross the line $\operatorname{Re}\lambda = s_0-1$, i.e. $\operatorname{Re}\mu_t < s_0-1$ for $t \in \mathscr{T}_0$ and $\operatorname{Re}\mu_t > s_0-1$ for $t \in \mathscr{T}_1$ with $\mathscr{T}_0$ and $\mathscr{T}_1$ two open subsets of $\mathscr{T}$, then the corresponding functions $\boldsymbol{S}_{\ell;t}$ do not belong to $\boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)$ for $t \in \mathscr{T}_0$, but they belong to $\boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)$ for $t \in \mathscr{T}_1$. In the $\mathscr{C}^\infty$ case, this situation is treated by localization in t. In the analytic case, the assumption $\boldsymbol{u}_t \in \mathscr{A}(\mathscr{T}, \boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t))$ implies that the corresponding coefficients $c_{\ell;t}$ are identically 0 on $\mathscr{T}$.

(ii) If problem (1.2) has a unique solution in $\boldsymbol{u} \in \boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)$ for any $\boldsymbol{f} \in \boldsymbol{H}^{s_0-\boldsymbol{m}}(\Omega_t)$ and any $t \in \mathscr{T}$, then

$$\boldsymbol{f}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{H}^{s_0-\boldsymbol{m}}(\Omega_t)) \quad \Longrightarrow \quad \boldsymbol{u}_t \in \mathscr{C}^\kappa(\mathscr{T}, \boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t))$$

and

$$\boldsymbol{f}_t \in \mathscr{A}(\mathscr{T}, \boldsymbol{H}^{s_0-\boldsymbol{m}}(\Omega_t)) \quad \Longrightarrow \quad \boldsymbol{u}_t \in \mathscr{A}(\mathscr{T}, \boldsymbol{H}^{s_0+\boldsymbol{m}}(\Omega_t)).$$

Moreover, the eigenvalues in Λ_t never cross the line $\operatorname{Re}\lambda = s_0-1$. This situation is well known for $s_0 = 0$ when $\boldsymbol{L}$ is a strongly coercive system. ■

4.d Examples. We take $N=1$ and $L=\Delta$ and give a few examples of singular functions for domains with curved sides. Here we describe the first terms in the asymptotics of a solution of the Dirichlet problem for a fixed opening ω at $\mathcal{O}$. Next we discuss stable expressions for these terms in the neighborhood of the angles $\omega=\pi$ and $\omega=2\pi$.

We are going to consider two types of curved sides:
(i) the parabola $(\mathcal{P}) : y = x^2$, whose branch $x>0$, $y>0$ is described in polar coordinates by

$$r \longrightarrow \omega_{\mathcal{P}}(r) = r - \frac{5r^3}{6} + \mathcal{O}(r^5) \quad \text{as} \quad r \to 0;$$

(ii) the circle $(\mathcal{C})\,:\; y = x^2 + y^2$, whose branch $x > 0,\ y > 0$ is described in polar coordinates by

$$r \;\longrightarrow\; \omega_{\mathcal{C}}(r) = r + \frac{r^3}{6} + \mathcal{O}(r^5) \quad \text{as} \quad r \to 0.$$

Example 4.9 We consider for $\omega \in (\frac{\pi}{2}, 2\pi]$ the domain

$$\{(x,y) \mid 0 < \theta < \omega - \omega_{\mathcal{P}}(r)\}.$$

• For ω different from π and 2π the generating term of the first singularity is an element of a space $\mathscr{X}(\gamma)$:

$$X^{(0)}_{\omega,1}(x,y) = \operatorname{Im} \zeta^{\frac{\pi}{\omega}}.$$

From the Taylor expansion of its trace on $\theta = \omega - \omega_{\mathcal{P}}(r)$, we see that the first generation of derived terms has to satisfy Dirichlet conditions:

$$X^{(1)}_{\omega,1}\Big|_{\theta=0} = 0; \quad X^{(1)}_{\omega,1}\Big|_{\theta=\omega} = -\frac{\pi}{\omega}\, r^{\frac{\pi}{\omega}+1}.$$

This gives

$$X^{(1)}_{\omega,1}(x,y) = \frac{\pi}{\omega}\,\frac{1}{\sin\omega}\;\operatorname{Im} \zeta^{\frac{\pi}{\omega}+1}.$$

The generating terms of the second and the third singularities are

$$X^{(0)}_{\omega,2}(x,y) = \operatorname{Im} \zeta^{\frac{2\pi}{\omega}}; \quad X^{(0)}_{\omega,3}(x,y) = \operatorname{Im} \zeta^{\frac{3\pi}{\omega}}.$$

The first polynomial generating the space $\mathscr{Y}(\gamma)$ where γ surrounds 1, is

$$Y^{(0)}_{\omega,1}(x,y) = y\,(x\sin\omega - y\cos\omega).$$

These singular functions depend regularly on ω for $\omega \in (\frac{\pi}{2}, \pi) \cup (\pi, 2\pi)$. In order to obtain stable expressions for ω near π or 2π, we have to change $X^{(1)}_{\omega,1}$ into

$$X^{(1),\pi}_{\omega,1} = \frac{\pi}{\omega}\,\frac{1}{\sin\omega}\,\Big(\operatorname{Im} \zeta^{\frac{\pi}{\omega}+1} - \operatorname{Im} \zeta^{\frac{2\pi}{\omega}}\Big)$$

or

$$X^{(1),2\pi}_{\omega,1} = \frac{\pi}{\omega}\,\frac{1}{\sin\omega}\,\Big(\operatorname{Im} \zeta^{\frac{\pi}{\omega}+1} - \operatorname{Im} \zeta^{\frac{3\pi}{\omega}}\Big).$$

These stable expressions for π and 2π are obtained as integrals according to the proof of Lemma 2.20 where the contour γ goes around $\{\frac{\pi}{\omega}, \frac{2\pi}{\omega} - 1\}$ and $\{\frac{\pi}{\omega}, \frac{3\pi}{\omega} - 1\}$ respectively. Here $r^{\lambda+\boldsymbol{m}}\mathscr{F}(\lambda,\theta)$ is the vector function $(\zeta^{\lambda+1}, \bar\zeta^{\lambda+1})$ and $a(\lambda) = \lambda - \frac{\pi}{\omega}$.

• When $\omega = \pi$, the first term in a space $\mathscr{X}(\gamma)$ is a polynomial:

$$X^{(0)}_{\omega,1}(x,y) = \operatorname{Im} \zeta.$$

and the corresponding generated term is

$$X^{(1)}_{\pi,1}(x,y) = -\frac{1}{\pi}\;\operatorname{Im} \zeta^2 \log\zeta$$

which is the limit as $\omega \to \pi$ of $X^{(1),\pi}_{\omega,1}$. This function is the first non-polynomial term in the asymptotics for $\omega = \pi$. Nevertheless, it does not correspond to the polar part of the resolvent of $\mathring{\mathscr{A}}(\lambda)$ at $\lambda = 1$. For $u \in H^{s_0+1}(\Omega) \cap \mathring{H}^1(\Omega)$ with $\Delta u = f \in H^{s-1}(\Omega)$ and $1 < s_0 < 2 < s$, if we consider the singularities of u with respect to H^{s+1}, we have to include the term $X^{(1)}_{\pi,1}$, although its "parent" term $X^{(0)}_{\pi,1}$ has an exponent outside the relevant interval (s_0, s), cf. Remark 4.7.

The polynomial terms of degree 2 are

$$X^{(0)}_{\omega,2}(x,y) = 2xy \quad \text{and} \quad Y^{(0)}_{\omega,1}(x,y) = y^2\,.$$

- When $\omega = 2\pi$, the first term is

$$X^{(0)}_{\omega,1}(x,y) = \operatorname{Im}\zeta^{\frac{1}{2}}.$$

and the corresponding generated term is

$$X^{(1)}_{\omega,1}(x,y) = \frac{1}{4\pi}\ \operatorname{Im}\zeta^{\frac{3}{2}}\log\zeta.$$

The same considerations as for the case $\omega = \pi$ hold for this term.

The first polynomial term is an element of a space $\mathscr{X}(\gamma)$:

$$X^{(0)}_{\omega,2}(x,y) = \operatorname{Im}\zeta$$

and the corresponding generated term is

$$X^{(1)}_{\omega,2}(x,y) = -\frac{1}{2\pi}\ \operatorname{Im}\zeta^2\log\zeta.$$

The next term is

$$X^{(0)}_{\omega,3}(x,y) = \operatorname{Im}\zeta^{\frac{3}{2}}.$$

■

Example 4.10 We consider the $\mathscr{C}^3$ domain

$$\{(x,y) \mid \omega_{\mathcal{C}}(r) < \theta < \pi - \omega_{\mathcal{P}}(r)\}.$$

The first polynomial is

$$X^{(0)}_1(x,y) = \operatorname{Im}\zeta$$

and we find for the first three derived terms

$$X^{(1)}_1(x,y) = -\operatorname{Re}\zeta^2, \quad X^{(2)}_1(x,y) = 0, \quad X^{(3)}_1(x,y) = \frac{1}{\pi}\operatorname{Im}\zeta^4\log\zeta.$$

Here again, for H^{s+1}-regularity with $s > 4$, we have to consider the term $X^{(3)}_1$ whose "parent" $X^{(0)}_1$ has an exponent 1 which may be outside the interval (s_0, s). ■

5. Structure of the stable asymptotics

The principal parts of the singular functions are elements of spaces $\mathscr{X}_t(\gamma_t)$ or $\mathscr{Y}_t(\gamma_t)$, whereas the other terms are described as elements of spaces $\mathscr{Z}_t$. We shall give additional information on the behavior of such functions in the polar coordinates (r,θ).

5.a Radial behavior.

It suffices to consider the elements of the spaces $\mathscr{Z}_t$, since the radial behavior is similar for the spaces $\mathscr{X}_t(\gamma_t)$ or $\mathscr{Y}_t(\gamma_t)$.

Lemma 5.1 *Assume that the coefficients of a family a_t of polynomials of degree d belong to $\mathscr{C}^\kappa(\mathscr{T})$ and that the contours γ_t are such that int γ_t contains all the roots of a_t. Let $\nu \in \mathbb{N}$ and let $\boldsymbol{u}_t$ belong to $\mathscr{C}^\kappa(\mathscr{T}, \mathscr{Z}_t(\gamma_t; a_t; \nu))$. If $q_{t,1}, \dots, q_{t,d} \in \mathscr{C}^\kappa(\mathscr{T}, \mathbb{P}_{d-1})$ are for each t a basis of $\mathbb{P}_{d-1}$, then there exist d functions of θ:*

$$\boldsymbol{\varphi}_{t,1}, \dots, \boldsymbol{\varphi}_{t,d} \in \mathscr{C}^\kappa(\mathscr{T}, \mathscr{C}^\infty([0, 2\pi]) \otimes \mathbb{C}^N)$$

such that

$$\boldsymbol{u}_t(r,\theta) = \sum_{j=1}^{d} \left(\int_{\gamma_t} \frac{r^{\lambda+\boldsymbol{m}}\, q_{t,j}(\lambda)}{a_t(\lambda)}\, d\lambda \right) \cdot \boldsymbol{\varphi}_{t,j}(\theta)\,.$$

Proof. By Definition 2.15, we have

$$\boldsymbol{u}_t(\zeta) = \int_\gamma \frac{\boldsymbol{V}_t^+(\lambda,\zeta)\, \mathscr{H}_t^+(\lambda) + \boldsymbol{V}_t^-(\lambda,\zeta)\, \mathscr{H}_t^-(\lambda)}{a_t(\lambda)}\, d\lambda\,.$$

From the formulas (2.15), we see that $\boldsymbol{V}_t^\pm(\lambda,\zeta)$ has the form $r^{\lambda+\boldsymbol{m}} \boldsymbol{\Psi}_t^\pm(\lambda,\theta)$. Setting

$$\boldsymbol{\Psi}_t(\lambda,\theta) = \boldsymbol{\Psi}_t^+(\lambda,\theta)\, \mathscr{H}_t^+(\lambda) + \boldsymbol{\Psi}_t^-(\lambda,\theta)\, \mathscr{H}_t^-(\lambda),$$

we obtain an holomorphic function of λ, $\mathscr{C}^\infty$ in θ and $\mathscr{C}^\kappa(\mathscr{T})$. Then

$$\boldsymbol{u}_t(\zeta) = \int_\gamma \frac{r^{\lambda+\boldsymbol{m}}\, \boldsymbol{\Psi}_t(\lambda,\theta)}{a_t(\lambda)}\, d\lambda\,,$$

and the use of the "Leibniz formula" (6.4) yields the lemma with

$$\boldsymbol{\varphi}_{t,j}(\theta) = \frac{1}{2i\pi} \int_\gamma \frac{\boldsymbol{\Psi}_t(\lambda,\theta)\, q^*_{t,j}(\lambda)}{a_t(\lambda)}\, d\lambda\,.$$

■

Remark 5.2 The relation of this lemma with the notion of *"crossing"* or *"branching"* is the following: when the roots of the polynomials a_t all depend smoothly on t, and when some of them coincide for a certain $t_0 \in \mathscr{T}$, we say that there is a *crossing* in t_0. When the roots of the polynomials a_t do not depend smoothly on t, there is a change of multiplicity in the points where the roots are not regular and we say that there is a *branching* in such points.

In the asymptotics $S^P(\boldsymbol{Y})$ of singular functions, the polynomials a_t are products of characteristic polynomials a_{t,γ_t} whose coefficients are smooth. When the polynomials

a_{t,γ_t} have the degree 1, as it is the case for second order scalar problems, we only have crossings. ■

We can give expressions of the radial functions with respect to the divided differences of the function $\lambda \to r^\lambda$. We recall the notation used in [2]: for $\mu_1, \ldots, \mu_d$ in $int\,\gamma$,

$$S[\mu_1, \ldots, \mu_d; r] = \frac{1}{2i\pi} \int_\gamma \frac{r^\lambda}{(\lambda - \mu_1) \cdots (\lambda - \mu_d)}\, d\lambda\,. \tag{5.1}$$

Here, for a polynomial $a \in \mathbb{P}_d$ with leading coefficient equal to 1 and a polynomial $q \in \mathbb{P}_{d-1}$, we set

$$S[a, q; r] = \frac{1}{2i\pi} \int_\gamma \frac{r^\lambda\, q(\lambda)}{a(\lambda)}\, d\lambda\,. \tag{5.2}$$

If we denote by $\mu_1, \ldots, \mu_d$ the roots of a, there holds

$$S[a, q; r] = q(r\partial_r) \Big(S[\mu_1, \ldots, \mu_d; r]\Big),$$

and also, by formula (6.4):

$$S[a, q; r] = \sum_{j=1}^{d} S[\mu_1, \ldots, \mu_j; r] \cdot q[\mu_j, \ldots, \mu_d]\,.$$

By special choices of q, we obtain

$$S[a, \frac{d^k a}{d\lambda^k}; r] = k! \sum_{j_1 < \ldots < j_k} S[\mu_{j_1}, \ldots, \mu_{j_k}; r]\,, \tag{5.3}$$

and, if the μ_j are distinct,

$$S[a, \lambda^k; r] = \sum_{j=1}^{d} \frac{\lambda_j^k\, r^{\lambda_j}}{\prod_{i \neq j} (\lambda_j - \lambda_i)}\,. \tag{5.4}$$

Remark 5.3 When $\kappa = 0$, i.e. when we only need a continuous dependence in t, we can always use the ordinary divided differences (and not necessarily the "symmetrized" ones as above) because the roots $\mu_{t,j}$ depend continuously on t. ■

Let us give detailed expressions of stable bases of the form $S[a_t, q_{t,j}; r]_{j=1,\ldots,d}$ in a few particular cases ($d = 2$, 3).

• **Case of 2 roots.**

– If the roots $\mu_{t,1}$ and $\mu_{t,2}$ depend smoothly on t and are never equal to each other ($t \in \overline{\mathscr{T}}$), we can choose for instance $q_{t,1}(\lambda) = \lambda - \mu_{t,2}$ and $q_{t,2}(\lambda) = \lambda - \mu_{t,1}$, which gives as a stable basis $\{r^{\mu_{t,1}},\, r^{\mu_{t,2}}\}$.

– If the roots $\mu_{t,1}$ and $\mu_{t,2}$ depend smoothly on t and can be equal to each other, we choose for instance $q_{t,1}(\lambda) = \lambda - \mu_{t,2}$ and $q_{t,2}(\lambda) = 1$, which gives as a stable basis

$$\Big\{r^{\mu_{t,1}},\ \ \frac{r^{\mu_{t,1}} - r^{\mu_{t,2}}}{\mu_{t,1} - \mu_{t,2}}\Big\}.$$

– If the roots $\mu_{t,1}$ and $\mu_{t,2}$ do not depend smoothly on t, we choose for instance $q_{t,1}(\lambda) = 2\lambda - \mu_{t,1} - \mu_{t,2}$ and $q_{t,2}(\lambda) = 1$, which gives as a stable basis

$$\Big\{r^{\mu_{t,1}} + r^{\mu_{t,2}},\ \ \frac{r^{\mu_{t,1}} - r^{\mu_{t,2}}}{\mu_{t,1} - \mu_{t,2}}\Big\}.$$

• **Case of 3 roots.**

– If the roots $\mu_{t,1}$, $\mu_{t,2}$ and $\mu_{t,3}$ depend smoothly on t and can be equal to each other, we choose for instance $q_{t,1}(\lambda) = (\lambda - \mu_{t,2})(\lambda - \mu_{t,3})$, $q_{t,2}(\lambda) = \lambda - \mu_{t,3}$ and $q_{t,3}(\lambda) = 1$, which gives as a stable basis

$$\Big\{r^{\mu_{t,1}},\ \ S[\mu_{t,1}, \mu_{t,2}; r],\ \ S[\mu_{t,1}, \mu_{t,2}, \mu_{t,3}; r]\Big\}.$$

– If only the root $\mu_{t,3}$ depends smoothly on t, we can choose for instance $q_{t,1}(\lambda) = (2\lambda - \mu_{t,1} - \mu_{t,2})(\lambda - \mu_{t,3})$, $q_{t,2}(\lambda) = \lambda - \mu_{t,3}$ and $q_{t,3}(\lambda) = 1$, which gives as a stable basis

$$\Big\{r^{\mu_{t,1}} + r^{\mu_{t,2}},\ \ S[\mu_{t,1}, \mu_{t,2}; r],\ \ S[\mu_{t,1}, \mu_{t,2}, \mu_{t,3}; r]\Big\}.$$

– If the roots $\mu_{t,1}$, $\mu_{t,2}$ and $\mu_{t,3}$ do not depend smoothly on t, we can choose for instance $q_{t,1}(\lambda) = 3\lambda^2 - 2\lambda(\mu_{t,1} + \mu_{t,2} + \mu_{t,3}) + \mu_{t,1}\mu_{t,2} + \mu_{t,2}\mu_{t,3} + \mu_{t,3}\mu_{t,1}$ and $q_{t,2}(\lambda) = 3\lambda - \mu_{t,1} - \mu_{t,2} - \mu_{t,3}$, $q_{t,3}(\lambda) = 1$, which gives as a stable basis

$$\Big\{r^{\mu_{t,1}} + r^{\mu_{t,2}} + r^{\mu_{t,3}},\ \ S[\mu_{t,1}, \mu_{t,2}; r] + S[\mu_{t,2}, \mu_{t,3}; r] + S[\mu_{t,3}, \mu_{t,1}; r],\ \ S[\mu_{t,1}, \mu_{t,2}, \mu_{t,3}; r]\Big\}.$$

5.b Angular behavior. In the most general situation, the angular behavior is determined by the functions $\boldsymbol{\varphi}_{t,j}(\theta)$ as defined in the proof of Lemma 5.1. In the cases where we know $\det \boldsymbol{M}_{t;\pm}(\alpha)$ sufficiently well to compute the integrals in formula (2.15) explicitly, we can obtain explicit expressions for $\boldsymbol{u}_t$ in terms of the variables ζ and $\overline{\zeta}$. For example, we proved in [4] that for systems "invariant by rotation" the roots of $\det \boldsymbol{M}_{t;\pm}(\alpha) = 0$ are all equal to 0. Thus the following result, whose proof is similar to that of Lemma 5.1, is interesting in numerous cases:

Lemma 5.4 *We suppose that the roots of* $\det \boldsymbol{M}_{t;\pm}(\alpha) = 0$ *are all equal to* 0. *Assume that the coefficients of a family* a_t *of polynomials of degree* d *belong to* $\mathscr{C}^\kappa(\mathscr{T})$ *and that the contours* γ_t *are such that int* γ_t *contains all the roots of* a_t. *Let* $\nu \in \mathbb{N}$ *and let* $\boldsymbol{u}_t$ *belong to* $\mathscr{C}^\kappa(\mathscr{T}, \mathscr{X}_t(\gamma_t; a_t; \nu))$. *If* $q_{t,1}, \ldots, q_{t,d} \in \mathscr{C}^\kappa(\mathscr{T}, \mathbb{P}_{d-1})$ *are for each* t *a basis of* $\mathbb{P}_{d-1}$, *then there exist* $2d$ *polynomials* $\boldsymbol{U}^+_{t,1}, \ldots, \boldsymbol{U}^+_{t,d}$ *and* $\boldsymbol{U}^-_{t,1}, \ldots, \boldsymbol{U}^-_{t,d}$ *in* $\mathbb{P}[X] \otimes \mathbb{C}^N$ *with* $\mathscr{C}^\kappa(\mathscr{T})$ *coefficients such that — with the notation* (5.2)*:*

$$\boldsymbol{u}_t(r,\theta) = \sum_{j=1}^{d} S[a_t, q_{t,j}; \overline{\zeta}]\ \operatorname{diag}(\overline{\zeta}^{\boldsymbol{m}}) \cdot \boldsymbol{U}^+_{t,j}\left(\frac{\zeta}{\overline{\zeta}}\right) + S[a_t, q_{t,j}; \zeta]\ \operatorname{diag}(\zeta^{\boldsymbol{m}}) \cdot \boldsymbol{U}^-_{t,j}\left(\frac{\overline{\zeta}}{\zeta}\right).$$

6. Appendix: Stable divided differences and Leibniz formulas

All the expressions of our singular functions are based on complex contour integrals $\int_\gamma q_t(\lambda)\, a_t(\lambda)^{-1}\, d\lambda$ with, as denominator, a polynomial a_t whose coefficients depend smoothly on t and a numerator which is holomorphic, or is more specifically written as a product of two holomorphic functions which also depend smoothly on t. The divided differences which we used in [2, 3] could also be used here if the roots of the polynomials a_t depend smoothly on t too. But although the continuity of the coefficients of the polynomials is transmitted to their roots, higher regularity ($\kappa \geq 1$) is not transmitted in general.

Proposition 6.1
(i) *Let $\gamma \subset \mathbb{C}$ be a simple contour and let a be a polynomial of degree $d \geq 1$, whose d roots belong to int γ. Let $q_1, \dots, q_d$ be a basis of $\mathbb{P}_{d-1}$. Then there exists a (dual) basis $q_1^*, \dots, q_d^*$ of $\mathbb{P}_{d-1}$ such that*

$$\mathfrak{B}_a(q_j, q_k^*) := \frac{1}{2i\pi} \int_\gamma \frac{q_j(\lambda)\, q_k^*(\lambda)}{a(\lambda)}\, d\lambda = \delta_{jk}. \tag{6.1}$$

(ii) *Assume that the coefficients of a family a_t of polynomials of degree d belong to $\mathscr{C}^\kappa(\mathscr{T})$ and that the contours γ_t are such that int γ_t contain all the roots of a_t. If $q_{t,1}, \dots, q_{t,d} \in \mathscr{C}^\kappa(\mathscr{T}, \mathbb{P}_{d-1})$ are for each t a basis of $\mathbb{P}_{d-1}$, then the polynomials $q_{t,1}^*, \dots, q_{t,d}^*$ of the dual basis have also $\mathscr{C}^\kappa(\mathscr{T})$ coefficients.*

The proof of the proposition is based on two lemmas.

Lemma 6.2 *If a and γ satisfy the hypotheses of Proposition* 6.1(i) *and if $q \in \mathbb{P}_{d-2}$, then*

$$\int_\gamma \frac{q(\lambda)}{a(\lambda)}\, d\lambda = 0.$$

Lemma 6.3 [1] *Let $a(\lambda) = \sum_{j=0}^d c_j \lambda^j$ and γ be as in Proposition* 6.1(i)*. Let, for $k = 1, \dots, d$, $a^{\sharp k}(\lambda) = \sum_{j=k}^d c_j \lambda^{j-k}$ be the shifted polynomials associated to a. Then, for $j, k = 1, \dots, d$*

$$\frac{1}{2i\pi} \int_\gamma \frac{\lambda^{j-1}\, a^{\sharp k}(\lambda)}{a(\lambda)}\, d\lambda = \delta_{jk}. \tag{6.2}$$

Proof. One can choose γ such that $0 \in$ *int* γ, without changing the above integral.

- If $j \leq k-1$, the integral (6.2) is 0 by Lemma 6.2.
- If $j \geq k$, $\lambda^j a^{\sharp k}(\lambda) = \lambda^{j-k} a(\lambda) + \lambda^{j-k} r_k$, with $r_k \in \mathbb{P}_{k-1}$.
  - If $j = k$, the integral (6.2) is equal to

$$\underbrace{\frac{1}{2i\pi} \int_\gamma \frac{1}{\lambda}\, d\lambda}_{=1} \quad + \quad \underbrace{\frac{1}{2i\pi} \int_\gamma \frac{r_k(\lambda)}{\lambda\, a(\lambda)}\, d\lambda}_{=0 \text{ by Lemma 6.2}}$$

– If $j > k$, the integral (6.2) is equal to

$$\underbrace{\frac{1}{2i\pi}\int_\gamma \lambda^{j-k-1}\,d\lambda}_{=0} \quad + \quad \underbrace{\frac{1}{2i\pi}\int_\gamma \frac{\lambda^{j-k-1}r_k(\lambda)}{a(\lambda)}\,d\lambda}_{=0 \text{ by Lemma 6.2}}$$

■

Proof of Proposition 6.1. We have just exhibited a basis and a dual basis, which are also stable with respect to the parameter regularity. We generalize to any basis q_j by the introduction of the coefficients c_{jl} such that

$$q_j(\lambda) = \sum_{l=1}^{d} c_{jl}\lambda^{j-1}.$$

The matrix (c_{jl}) is invertible. Let $(\tilde{c}_{jl})$ be its inverse. With

$$q_k^*(\lambda) = \sum_{l=1}^{d} \tilde{c}_{lk} a^{\sharp l},$$

we obtain the relations (6.1). ■

Corollary 6.4 *We assume that γ and a are as in Proposition* 6.1 *and that a is a product $a_1 \cdots a_P$. Let for each p, $q_{p,j}$ and $q_{p,k}^*$ be biorthogonal bases for the form $\mathfrak{B}_{a_p}$. Then the basis*

$$\{q_{1,j}\} \cup \{a_1 \cdot q_{2,j}\} \cup \{a_1 \cdot a_2 \cdot q_{3,j}\} \cup \cdots \cup \{a_1 \cdots a_{P-1} \cdot q_{P,j}\}$$

and the dual basis

$$\{q_{1,k}^* \cdot a_2 \cdots a_P\} \cup \cdots \cup \{q_{P-2,k}^* \cdot a_{P-1} \cdot a_P\} \cup \{q_{P-1,k}^* \cdot a_P\} \cup \{q_{P,k}^*\}$$

are biorthogonal for the form $\mathfrak{B}_a$.

Biorthogonal bases of $\mathfrak{B}_a$ provide expressions for the (Hermite) interpolation polynomial on the roots of a:

Proposition 6.5 *We assume that γ and a are as in Proposition* 6.1. *Let u be a holomorphic function in a neighborhood of $\overline{\operatorname{int}\gamma}$. Then there exists a unique polynomial $u[a] \in \mathbb{P}_{d-1}$ such that*

$$\forall q \in \mathbb{P}_{d-1} \qquad \mathfrak{B}_a(q, u - u[a]) = 0.$$

For instance, if the roots $\mu_1, \ldots, \mu_d$ of a are all simple, $u[a]$ is the Lagrange interpolant of u at $\mu_1, \ldots, \mu_d$. More generally, $u[a]$ appears as the remainder of the Euclidean division of u by a:

$$\exists\, \varphi \text{ holomorphic}, \qquad u = a\,\varphi + u[a].$$

For any biorthogonal bases q_j and q_k^* of $\mathfrak{B}_a$, we have

$$u[a] = \sum_{k=1}^{d} \mathfrak{B}_a(q_k, u)\, q_k^*\,. \tag{6.3}$$

In particular, if a and u depend smoothly on a parameter t, so does $u[a]$.

We deduce “Leibniz formulas”:

Proposition 6.6 *We assume that γ and a are as in Proposition* 6.1. *Let u and v be holomorphic functions in a neighborhood of $\overline{\operatorname{int}\gamma}$. For any biorthogonal bases q_j and q_k^* of $\mathfrak{B}_a$, there holds $\mathfrak{B}_a(u,v) = \sum_{k=1}^d \mathfrak{B}_a(q_k,u)\,\mathfrak{B}_a(q_k^*,v)$, i.e.*

$$\frac{1}{2i\pi}\int_\gamma \frac{u\,v}{a}\,d\lambda = \sum_{k=1}^d \left(\frac{1}{2i\pi}\int_\gamma \frac{u\,q_k}{a}\,d\lambda\right)\left(\frac{1}{2i\pi}\int_\gamma \frac{v\,q_k^*}{a}\,d\lambda\right). \tag{6.4}$$

The proof is straightforward, using that $\mathfrak{B}_a(u,v) = \mathfrak{B}_a(u[a],v[a])$ and the equalities $u[a] = \sum_{k=1}^d \mathfrak{B}_a(q_k,u)\,q_k^*$ and $v[a] = \sum_{j=1}^d \mathfrak{B}_a(q_j^*,v)\,q_j$.

Example 6.7
(i) If $a(\lambda) = \prod_{l=1}^d(\lambda-\mu_l)$, a possible choice is (cf Corollary 6.4)

$$q_1 = 1,\quad q_j(\lambda) = \prod_{l=1}^{j-1}(\lambda-\mu_l)\ \text{ for } j \geq 2\,;\quad q_d^* = 1,\quad q_k(\lambda) = \prod_{l=k+1}^{d}(\lambda-\mu_l)\ \text{ for } k \leq d-1.$$

One recovers the ordinary divided differences and the corresponding product formula (cf [2, §8]). In the special case $\mu_1 = \ldots = \mu_d$, the formula (6.4) is just the ordinary Leibniz formula for the derivative of order $d-1$ of the product uv.

(ii) In connection with Lemma 6.2, for any fixed λ_0, we can take $q_j(\lambda) = (\lambda-\lambda_0)^{j-1}$. The dual basis is then given by $q_k^* = a_{\lambda_0}^{\sharp k}$, which denotes the quotient of the Euclidean division of a by $(\lambda-\lambda_0)^k$.

(iii) One can choose the derivatives of a: $q_j(\lambda) = \dfrac{d^j a}{d\lambda^j}$ (see (5.3)). ■

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