A singularly perturbed mixed boundary value problem

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Abstract. *We study a mixed Neumann-Robin boundary value problem for the Laplace operator in a smooth domain in* $\mathbb{R}^2$ *. The Robin condition contains a parameter* ε *and tends to a Dirichlet condition as* $\varepsilon \to 0$ *. We give a complete asymptotic expansion of the solution in powers of* ε *. At the points where the boundary conditions change, there appear boundary layers of corner type of size* ε *. They describe how the singularities of the limit Dirichlet-Neumann problem are approximated. We give sharp estimates in various Sobolev norms and show in particular that there exist terms of order* $\mathscr{O}(\varepsilon \log \varepsilon)$ *.*

1 INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ . We consider the Laplace equation in Ω with mixed Robin-Neumann boundary condition. The Robin condition is imposed on the part Γ_R of the boundary and the Neumann condition on Γ_N (we assume that Γ is equal to $\overline{\Gamma_R} \cup \overline{\Gamma_N}$):

$$(\mathcal{P}_\varepsilon) \qquad \left\{ \begin{array}{rcll} \Delta u_\varepsilon & = & f & \text{in } \Omega \\ \partial_n u_\varepsilon & = & 0 & \text{on } \Gamma_N \\ \varepsilon \partial_n u_\varepsilon + u_\varepsilon & = & g & \text{on } \Gamma_R. \end{array} \right.$$

The two main features of this problem are

1. The intersection of $\overline{\Gamma_R}$ and $\overline{\Gamma_N}$ is not empty: we can suppose without restriction that $\overline{\Gamma_R} \cap \overline{\Gamma_N}$ is formed by two points $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, whence a singular behavior near these points has to be expected. In order to simplify the technical details of the description of this singular behavior, we make the assumption that in a neighborhood of the two points $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, the boundary Γ coincides with straight lines.

2. The Robin condition is $\varepsilon\partial_n + I$, where $\varepsilon > 0$ is a small parameter (∂_n denotes the derivative with respect to the exterior normal).

Thus, as $\varepsilon \to 0$, the problem changes its type, "degenerating" into the mixed Dirichlet-Neumann problem

$$(\mathcal{P}_0) \qquad \left\{ \begin{array}{rcll} \Delta u_0 & = & f & \text{in} \quad \Omega \\ \partial_n u_0 & = & 0 & \text{on} \quad \Gamma_N \\ u_0 & = & g & \text{on} \quad \Gamma_R. \end{array} \right.$$

This is a singular perturbation problem whose peculiarity lies in the question of the singular behavior at $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$. Indeed, for problem $(\mathcal{P}_\varepsilon)$, the first singular function behaves as $\mathscr{O}(r \log r)$, whereas, for problem $(\mathcal{P}_0)$, the first singular function behaves as $\mathscr{O}\left(r^{1/2}\right)$. Therefore the introduction of the small parameter ε can be considered as a regularization of the mixed Dirichlet-Neumann problem $(\mathcal{P}_0)$, see [2, 12]. If this regularization is done in view of a better numerical approximability of the solution, then one needs to understand how the singular functions for $\varepsilon = 0$ are approximated by "near-singular" functions for $\varepsilon > 0$.

If, on the other hand, one needs to approximate the problem $(\mathcal{P}_\varepsilon)$ for a small value of ε, then it might be profitable to use numerical methods that take the singularities of the limit problem $(\mathcal{P}_0)$ into account. This is in spite of the fact that these singularities are not present in the solution of the problem $(\mathcal{P}_\varepsilon)$. We shall see below in what sense they are "nearly present". Our estimates should be useful as a basis for error estimates for such numerical methods.

The problem $(\mathcal{P}_\varepsilon)$ can also be considered as the simplest one in a whole class of similar problems and it can serve as a starting point for various generalizations: Domains with corners, other elliptic boundary value problems of mathematical physics such as problems from fluid dynamics, electromagnetism or elasticity. One of the motivations for our paper were discussions with B. Szabo and I. Babuška on this subject.

Beyond its role of simple model for the above mentioned class of problems, the asymptotic behavior of solutions of $(\mathcal{P}_\varepsilon)$ is of particular interest for (at least) three reasons (see also [2, 12] for these and other motivations):

1. Problem $(\mathcal{P}_\varepsilon)$ can be considered as a penalization of the limit problem $(\mathcal{P}_0)$ — indeed such a method has been used to enforce Dirichlet conditions in FEM.
2. Problem $(\mathcal{P}_\varepsilon)$ in its actual form describes certain contact problems, where the reaction of the contact zone is modelized by a stiff spring.
3. This question of "sudden" transformation of a singular behavior into a completely different one at the limit is an old problem, whose correct solution requires modern techniques of multiscale asymptotics.

A problem like $(\mathcal{P}_\varepsilon)$, but with Γ_N empty, was considered by KIRSCH [7]. The mixed problem $(\mathcal{P}_\varepsilon)$ was studied by COLLI FRANZONE [1] who proved error estimates between u_ε and u_0 as $\varepsilon \to 0$ in various Sobolev norms. In these papers the Robin boundary condition is considered as a regularization of the Dirichlet condition, too, but in different sense from the regularization discussed above: The problem $(\mathcal{P}_\varepsilon)$ for $\varepsilon > 0$ is well-posed and has a nice variational form for $g \in L^2(\Gamma)$ which is, for instance, the standard regularity condition in boundary control problems, whereas for the weak solution of the Dirichlet problem, more regularity ($g \in H^{1/2}(\Gamma)$) is required. The mixed Dirichlet-Neumann problem $(\mathcal{P}_0)$ is even ill-posed for $g \in L^2(\Gamma)$!

The variational formulation of $(\mathcal{P}_\varepsilon)$ is: Find $u \in H^1(\Omega)$ such that

$$\forall v \in H^1(\Omega): \int_\Omega \nabla u_\varepsilon \cdot \nabla v \, dx + \frac{1}{\varepsilon} \int_{\Gamma_R} (u_\varepsilon - g)\, v \, ds = - \int_\Omega f\, v \, dx \,. \tag{1.1}$$

From this formulation one is immediately led to various interpretations of $(\mathcal{P}_\varepsilon)$ as Tikhonov regularization or Yosida approximation or penalization method for the limit problem $(\mathcal{P}_0)$, see the above references for applications of such interpretations.

Our aim is to construct an asymptotic expansion of u_ε in powers of ε and to deduce optimal error estimates. The main obstruction to the construction of a singular perturbation series of the form $u_\varepsilon \simeq u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots$, each term u^n being solution of problem $(\mathcal{P}_0)$ with $f = 0$ and $g = -\partial_n u^{n-1}$, is the lack of regularity of the solutions of the mixed problem $(\mathcal{P}_0)$.

Our technique is inspired by NAZAROV's paper [11]: it consists in correcting the singular part of the u^n by a sort of corner layer term. We are grateful for discussions with V. MAZ'YA and S. NAZAROV on this technique. In our problem, these corner layer terms are not exponentially decreasing as in the problems studied by NAZAROV [9, 10] and KELLOGG [6], where the limit operator as $\varepsilon \to 0$ has an order strictly lower than the operator for $\varepsilon > 0$.

In our situation, the boundary layer of thickness ε is situated near the two points $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$. Our method differs from the general strategy of "matching asymptotic expansions", explained by IL'IN [5]. Roughly speaking, matching asymptotic expansions consists in constructing two expansions:
- an outer expansion in variables x far from the boundary layer,
- an inner expansion in variables x/ε in the boundary layer,

and to determine the degrees of freedom in both expansions so that they coincide in an intermediate region. In contrast, we construct alternatively outer and inner terms in order to avoid a strongly singular behavior at the boundary layer of both the inner and outer terms (characteristic feature of a "bisingular problem"). As a result, we obtain for u_ε a composite expansion with all terms at least in $H^1(\Omega)$ and optimal estimates for the remainders, see Theorem 7.1.

In a neighborhood of the point $\boldsymbol{c}_i = 0$, one of the forms of this expansion reads:

$$u_\varepsilon \sim \sum_{n\in\mathbb{N}} \varepsilon^n u^n[\log\varepsilon](x) + \sum_{n\in\mathbb{N}} \varepsilon^{n+1/2} w^n[\log\varepsilon](\frac{x}{\varepsilon}), \tag{1.2}$$

the notation $[\log\varepsilon]$ indicates a polynomial behavior in $\log\varepsilon$. The exponents $n+1/2$ come from the singular functions of the limit problem $(\mathcal{P}_0)$. The singular behavior near 0 of u_ε comes from the boundary layer terms w^n only, see Theorem 8.6. An expansion as $x \to 0$ of the terms u^n added to the terms w^n yields the complete inner expansion of u_ε whereas an expansion as $x \to +\infty$ of the terms w^n added to the terms u^n yields the complete outer expansion of u_ε: this outer expansion necessarily contains powers of $\log\varepsilon$, too (see §9 the expression of the first logarithmic term in $\varepsilon\log\varepsilon$).

We conclude this introduction by an explanation about this apparent paradox of change in the nature of singularities when $\varepsilon \to 0$. In suitable polar coordinates, the most singular part of u_0 has the form $c^1(u^0)\, S^1$ where $S^1 = r^{1/2}\sin\theta/2$ and c^1 is a coefficient; the remaining part of u_0 is denoted $u_{0,\text{reg}}$ and belongs nearly to $H^{5/2}$. The most singular part of u_ε has the form $\gamma^1(u_\varepsilon)\,\Sigma^1$ where $\Sigma^1 = \frac{1}{\pi} r((\theta-\pi)\sin\theta - \log r\cos\theta)$ and γ_1 is a coefficient. The outcome of our constructions is an asymptotic expansion beginning as, *cf* (6.3)-(6.4):

$$u_\varepsilon = u_0 + \sqrt{\varepsilon}\, c^1(u^0)\, Y^1(\frac{r}{\varepsilon},\theta) + \mathscr{O}(\varepsilon\log\varepsilon) \tag{1.3a}$$

$$= u_{0,\text{reg}} + \sqrt{\varepsilon}\, c^1(u^0)\, K^1(\frac{r}{\varepsilon},\theta) + \mathscr{O}(\varepsilon\log\varepsilon) \tag{1.3b}$$

where $Y^1 = K^1 - S^1$ and the "profile" K^1 is the only solution of the Robin-Neumann problem with $\varepsilon = 1$ on a half-space, behaving like S^1 at infinity and belonging to H^1 in the neighborhood of 0, *see* Theorem 5.4. The boundary layer Y^1 decays like $r^{-1/2}\log r$ at infinity. The expansion (1.3a) is convenient to prove optimal estimates for $u_\varepsilon - u_0$, for example we find $\sqrt{\varepsilon}$ in the H^1-norm, *cf* Proposition 6.3, whereas the expansion (1.3b) yields a decomposition of u_ε in regular and singular parts relying on the decomposition of the profile K^1 itself in the neighborhood of 0. The most singular part of K^1 being $\gamma^1(K^1)\,\Sigma^1$, we obtain that

$$\gamma^1(u_\varepsilon) = \frac{1}{\sqrt{\varepsilon}}\, c^1(u^0)\,\gamma^1(K^1) + \mathscr{O}\left(\sqrt{\varepsilon}\log\varepsilon\right). \tag{1.4}$$

Indeed the behavior at infinity of K^1 like S^1 and its singular part in 0 like Σ^1 builds the link between the different behaviors of the solutions of problems $(\mathcal{P}_\varepsilon)$ and $(\mathcal{P}_0)$. In this sense, we can say that S^1 is "nearly present" in K^1.

It is somewhat instructive to compare this transformation in the nature of singularities with the regular perturbation occurring in the problem where we replace the Robin condition of problem $(\mathcal{P}_\varepsilon)$ by $\varepsilon\,\underline{r}\,\partial_n u_\varepsilon + u_\varepsilon$: in this case the singular functions associated with this new problem depend analytically on $\varepsilon \to 0$: the first exponent admits the expansion $\lambda(\varepsilon) = \frac{1}{2} + \frac{1}{2\pi}\,\varepsilon + \frac{1}{2\pi^2}\,\varepsilon^2 + \mathscr{O}(\varepsilon^3)$.

2 ASYMPTOTIC EXPANSION FOR A PROBLEM WITHOUT SINGULARITIES

In this section we consider the non-mixed problem with $\Gamma_N = \emptyset$

$$\begin{cases} \Delta u_\varepsilon &= 0 \quad \text{in} \quad \Omega \\ \varepsilon \partial_n u_\varepsilon + u_\varepsilon &= g \quad \text{on} \quad \Gamma. \end{cases} \tag{2.1}$$

We shall see that it is rather easy to construct the complete asymptotic expansion of u_ε as $\varepsilon \to 0$ in this case: there are no boundary layer or corner layer terms, so the expansion will have the form

$$u_\varepsilon = u^0 + \varepsilon u^1 + \cdots + \varepsilon^N u^N + r_\varepsilon^N. \tag{2.2}$$

By localization, this expansion will also describe the behavior of the solution u_ε of our original mixed problem $(\mathcal{P}_\varepsilon)$ away from the singular points, i. e. in the interior of Ω and near interior points of the boundary part Γ_R .

An analysis and applications of such a non-mixed singularly perturbed Robin problem have been described by KIRSCH [7] for the case of the Helmholtz equation.

The standard construction of the functions u^n , $n \geq 0$, uses the formal series $u_\varepsilon = \sum_n \varepsilon^n u^n$ which is inserted into the boundary value problem (2.1). Comparison of coefficients of ε^n gives the sequence of Dirichlet problems

$$\begin{cases} \Delta u^0 &= 0 \quad \text{in} \quad \Omega \\ u^0 &= g \quad \text{on} \quad \Gamma \end{cases} ;$$
$$\begin{cases} \Delta u^n &= 0 \quad \text{in} \quad \Omega \\ u^n &= -\partial_n u^{n-1} \quad \text{on} \quad \Gamma \end{cases} \quad (n = 1, 2, \dots). \tag{2.3}$$

If this sequence of Dirichlet problems is satisfied for $n = 0, \dots, N$, then the remainder r_ε^N satisfies the boundary value problem

$$\begin{cases} \Delta r_\varepsilon^N &= 0 \quad \text{in} \quad \Omega \\ \varepsilon \partial_n r_\varepsilon^N + r_\varepsilon^N &= -\varepsilon^{N+1} \partial_n u^N \quad \text{on} \quad \Gamma. \end{cases} \tag{2.4}$$

This is the same problem as the problem satisfied by u_ε , only with a different right hand side. Therefore all estimates for the remainders r_ε^N for any $N \geq 0$ are direct consequences of the basic estimates for u_ε (" $N = -1$ ").

The right hand side in (2.4) is obtained from g by repeated application of the Dirichlet-to-Neumann map (or Poincaré-Steklov operator)

$$T : u\Big|_\Gamma \mapsto \partial_n u\Big|_\Gamma , \qquad \text{where } u \text{ satisfies } \Delta u = 0 \text{ in } \Omega . \tag{2.5}$$

Namely, by (2.3) we have

$$u^n\Big|_\Gamma = (-T)^n g . \tag{2.6}$$

Now from the well-known properties of the operator T one can deduce all the relevant estimates. In fact, one can consider the whole problem treated in this paragraph as a variation on the theme that

$$(\varepsilon T + I)^{-1} = \sum_n (-\varepsilon T)^n$$

in the sense of asymptotic series. The fact that this is a singular perturbation problem and not a regular one corresponds to the fact that $\varepsilon T + I$ is a pseudodifferential operator of order one for $\varepsilon > 0$ and of order zero for $\varepsilon = 0$.

The following lemma contains the basic estimate for the problem (2.1). We use here the standard notation for the Sobolev spaces on Ω and on Γ .

Lemma 2.1 *Let u_ε , $\varepsilon > 0$, be the solution of the Robin problem* (2.1). *For any $s \in \mathbb{R}$ there exists a constant C independent of ε such that*

$$\|u_\varepsilon\|_{H^{1+s}(\Omega)} \le C \, \|g\|_{H^{\frac{1}{2}+s}(\Gamma)} . \tag{2.7}$$

Proof. We first estimate the trace $u_\varepsilon\Big|_\Gamma$ and then we conclude by using the regularity of the Dirichlet problem. We have

$$u_\varepsilon\Big|_\Gamma = (\varepsilon T + I)^{-1} g .$$

Now if $g \in H^{\frac{1}{2}}(\Gamma)$, we have the variational formulation of the Robin problem (2.1),

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla v \, dx + \frac{1}{\varepsilon} \int_\Gamma u_\varepsilon v \, ds = \frac{1}{\varepsilon} \int_\Gamma g v \, ds$$

for any $v \in H^1(\Omega)$. In particular, for $v = u_\varepsilon$ and $\varepsilon \ge 0$ we find

$$\|u_\varepsilon\Big|_\Gamma\|^2_{L^2(\Gamma)} \le \big\langle (\varepsilon T + I) u_\varepsilon\Big|_\Gamma , u_\varepsilon\Big|_\Gamma \big\rangle .$$

This shows that $\varepsilon T + I$ is a positive definite operator in $L^2(\Gamma)$ and that the operator norm of $(\varepsilon T + I)^{-1}$ in $L^2(\Gamma)$ is bounded uniformly with respect to $\varepsilon > 0$. Now the operator $T + I$ is a positive definite elliptic pseudodifferential operator of order one, and we can use its fractional powers, which commute with $\varepsilon T + I$, to transfer this uniform boundedness to Sobolev spaces of any order on Γ. (Note that we cannot use powers of T directly, because T is only semidefinite: it maps constants to zero.) Thus we have an estimate

$$\|u_\varepsilon\big|_\Gamma\|_{H^{\frac{1}{2}+s}(\Gamma)} \leq C\,\|g\|_{H^{\frac{1}{2}+s}(\Gamma)}$$

for any $s \in \mathbb{R}$ with C independent of $\varepsilon > 0$. If we combine this with the estimate for the Dirichlet problem

$$\|u_\varepsilon\|_{H^{1+s}(\Omega)} \leq C\,\|u_\varepsilon\big|_\Gamma\|_{H^{\frac{1}{2}+s}(\Gamma)}$$

we arrive at (2.7). ■

Corollary 2.2 *For any* $s \in \mathbb{R}$ *there exists a constant* C *independent of* $\varepsilon \in (0,1]$ *such that*

$$\|u_\varepsilon\|_{H^{1+s}(\Omega)} \leq C\,\varepsilon^{-1}\|g\|_{H^{-\frac{1}{2}+s}(\Gamma)}\,. \tag{2.8}$$

Proof. We rewrite the Robin problem (2.1) as follows

$$\begin{cases} \Delta u_\varepsilon &= 0 & \text{in} \quad \Omega \\ \partial_n u_\varepsilon + u_\varepsilon &= \varepsilon^{-1}(g - (1-\varepsilon)u_\varepsilon) & \text{on} \quad \Gamma. \end{cases}$$

This gives an estimate

$$\|u_\varepsilon\|_{H^{1+s}(\Omega)} \leq C\,\varepsilon^{-1}\|g - (1-\varepsilon)u_\varepsilon\|_{H^{-\frac{1}{2}+s}(\Gamma)} \leq C\,\varepsilon^{-1}\|g\|_{H^{-\frac{1}{2}+s}(\Gamma)}\,.$$

In the last inequality we used the result of Lemma 2.1. ■

By interpolation, we obtain also estimates for any

$$s \in \mathbb{R}, \quad s-1 \leq t \leq s\,,$$
$$\|u_\varepsilon\|_{H^{1+s}(\Omega)} \leq C\,\varepsilon^{t-s}\|g\|_{H^{\frac{1}{2}+t}(\Gamma)} \tag{2.9}$$

Theorem 2.3 *Let* $N \in \mathbb{N}$, $s,t \in \mathbb{R}$ *with* $s \leq t \leq s+1$. *For* $g \in H^{\frac{1}{2}+t+N}(\Gamma)$ *and* $0 < \varepsilon \leq 1$, *let* u_ε *be the solution of the Robin problem* (2.1). *For* $n = 0,\ldots,N$ *let* u^n *be constructed according to the sequence of Dirichlet problems* (2.3). *Then* u_ε *admits the asymptotic expansion* (2.2), *where the remainder* r_ε^N *satisfies the estimate*

$$\|r_\varepsilon^N\|_{H^{1+s}(\Omega)} \leq C\,\varepsilon^{N+t-s}\|g\|_{H^{\frac{1}{2}+t+N}(\Gamma)}\,. \tag{2.10}$$

Here the constant C *does not depend on* ε *and* g.

Proof. It suffices to apply the previous estimates (2.9) of u_ε to the problem (2.4) satisfied by r_ε^N and to note that

$$\|\partial_n u^n\|_{H^{-\frac{1}{2}+t}(\Gamma)} = \|T^{n+1}g\|_{H^{-\frac{1}{2}+t}(\Gamma)} \leq C\,\|g\|_{H^{\frac{1}{2}+t+N}(\Gamma)}\,.$$

■

3 REGULARITY AND SINGULARITIES OF THE MIXED PROBLEM $(\mathcal{P}_0)$

We return to our initial problem, where the two boundary parts Γ_R and Γ_N are non-empty and have at least two common points $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$. From KONDRAT'EV's work [8], we know that the solutions of problem $(\mathcal{P}_0)$ has singular solutions which behave like powers $j-\frac{1}{2}$ of the distance r_i to $\boldsymbol{c}_i$. The first consequence is the limited regularity in the scale of the Sobolev spaces H^s, *cf* ESKIN [4], DAUGE [3]:

Theorem 3.1 *For all* $s \in (-\frac{1}{2}, \frac{1}{2})$*, for all* $g \in H^{\frac{1}{2}+s}(\Gamma_R)$*, problem* $(\mathcal{P}_0)$ *with* $f=0$ *has a unique solution* u *in the space* $H^{1+s}(\Omega)$*, and the following estimate holds*

$$\|u\|_{H^{1+s}(\Omega)} \leq C\,\|g\|_{H^{\frac{1}{2}+s}(\Gamma_R)}. \tag{3.1}$$

As a complement of information, let us recall that for $s = \pm\frac{1}{2}$, problem $(\mathcal{P}_0)$ is not of closed range, whereas for $s < -\frac{1}{2}$ solutions are not unique and for $s > \frac{1}{2}$ solutions in $H^{1+s}(\Omega)$ do not always exist.

More precisely, for $i = 1, 2$, let (r_i, θ_i) denote the polar coordinates centered at $\boldsymbol{c}_i$ and oriented such that, in a neighborhood of $\boldsymbol{c}_i$, the domain Ω coincides with the set $r_i > 0$, $\theta_i \in (0, \pi)$, and Γ_R coincides with the set $r_i > 0$, $\theta_i = 0$.

A basis of the singular functions of the mixed Dirichlet-Neumann problem is given by

$$S^j(r,\theta) = r^{j-\frac{1}{2}} \sin(j-\tfrac{1}{2})\theta, \tag{3.2}$$

for $j = 1, 2, \ldots$, and the localized singular functions of our problem can be written as

$$S_i^j = \chi(r_i)\, S^j(r_i, \theta_i), \tag{3.3}$$

where χ is a smooth cut-off function which is equal to 1 in a neighborhood of 0.

Theorem 3.2 *Let* $M \geq 1$ *be an integer. For all* $s \in (-\frac{1}{2}, \frac{1}{2})$*, for all* $g \in H^{\frac{1}{2}+s+M}(\Gamma_R)$ *the unique solution* $u \in H^1(\Omega)$ *of problem* $(\mathcal{P}_0)$ *with* $f=0$ *admits the splitting*

$$u = u_{\mathrm{reg}(M)} + \sum_{i=1}^{2} \sum_{j=1}^{M} c_i^j(u)\, S_i^j \tag{3.4}$$

where $u_{\mathrm{reg}(M)}$ *belongs to* $H^{1+s+M}(\Omega)$ *and the* $c_i^j(u)$ *are coefficients; moreover we have the estimate*

$$\|u_{\mathrm{reg}(M)}\|_{H^{1+s+M}(\Omega)} + \sum_{i=1}^{2}\sum_{j=1}^{M} |c_i^j(u)| \ \leq\ C\,\|g\|_{H^{\frac12+s+M}(\Gamma_R)}\,. \tag{3.5}$$

As we have seen in the previous section, the terms u^n of the formal asymptotic expansion of the solution u_ε of problem $(\mathcal{P}_\varepsilon)$ would be the solution of problem $(\mathcal{P}_0)$ with $f = 0$ and $g = -\partial_n u^{n-1}$. Thus we also have to consider the expansion of solutions with ***singular*** data, e.g. solutions of the mixed Dirichlet-Neumann problem with right-hand side $f = 0$ and $g = -\partial_n S^j$.

The precise formulation of this problem requires the definition of the mixed problem on the half-space (which is the "model" domain in the neighborhood of the boundary points of Ω). Let

$$\Pi = \{(x,y)\in\mathbb{R}^2 \mid \quad y>0\}$$

be the half-space. $\mathbb{R}_+$ and $\mathbb{R}_-$ are the two parts of its boundary. The relevant mixed problem is

$$\begin{cases} \Delta u &=& 0 & \text{in } \ \Pi \\ \partial_n u &=& 0 & \text{on } \ \mathbb{R}_- \\ u &=& g & \text{on } \ \mathbb{R}_+. \end{cases} \tag{3.6}$$

As well-known in the theory of corner problems, the spaces in which such a problem with singular right-hand side can be solved are generated by functions of the form $r^\lambda \log^q r\,\psi_q(\theta)$. In our particular situation, we only use those corresponding to $\lambda = j - 1/2$:

$$\mathscr{S}^j = \Big\{u = r^{j-\frac12}\sum_{q=0}^{Q}\log^q r\,\psi_q(\theta) \mid \quad \psi_q\in\mathscr{C}^\infty([0,\pi])\Big\} \tag{3.7}$$

We have:

Lemma 3.3 *Let* j *be an integer. We set* $S^{j(0)} = S^j$ *defined in* (3.2)*. For each integer* $\ell\geq 1$*, there exists a solution* $S^{j(\ell)}\in\mathscr{S}^{j-\ell}$ *to problem* (3.6) *with datum* $g = -\partial_n S^{j(\ell-1)}$*.*

The proof of this lemma is classical and is already essentially contained in [8], see also [3]. The degree in $\log r$ increases by one at each step because of a resonance (this fact is particular for the angle π, and disappears if one has, instead of Π, a cone of opening ω such that π/ω is not rational).

The above sequences $S^{j(1)}, S^{j(2)},\ldots,$ generated by S^j are not unique since $S^{j(\ell)}$ is defined up to the addition of any $c\,S^{j-\ell}$.

Definition 3.4 For any integers j and $\ell \geq 1$, we define the "associated" singular functions $S^{j,\ell}$ as a particular solution of the system

$$\begin{cases} S^{j,0} = S^j \\ S^{j,\ell} \in \mathscr{S}^{j-\ell} \text{ solution of problem (3.6) with datum } g = -\partial_n S^{j,\ell-1}. \end{cases}$$

■

Then for any sequence $S^{j(1)}, S^{j(2)}, \ldots$ as in Lemma 3.3, there exist coefficients $c^{(p)}$ such that

$$\begin{aligned} S^{j(1)} &= S^{j,1} + c^{(1)}\, S^{j-1} \\ S^{j(2)} &= S^{j,2} + c^{(1)}\, S^{j-1,1} + c^{(2)}\, S^{j-2} \\ \ldots \end{aligned} \tag{3.8}$$

Using the complex form of the coordinates $\zeta = r\, e^{i\theta}$, we can prove that

$$S^{j,\ell} = \operatorname{Re}\Big((-\zeta)^{j-\ell-\frac{1}{2}}\, P^{j,\ell}[\log(-\zeta)] \Big)\,, \tag{3.9}$$

where $P^{j,\ell}$ is a polynomial of degree ℓ with real coefficients that can be determined recursively from the coefficients of $P^{j,\ell-1}$.

We can choose the following formula for the first associated function $S^{j+1,1}$:

$$S^{j+1,1}(r,\theta) = \frac{1}{\pi}(j+\tfrac{1}{2})\, r^{j-\frac{1}{2}} \Big((\pi-\theta)\cos(j-\tfrac{1}{2})\theta - \log r \sin(j-\tfrac{1}{2})\theta \Big)\,. \tag{3.10}$$

Returning to the mixed problem $(\mathcal{P}_0)$ on Ω, we define $S_i^{j,\ell}$ like in (3.3) by

$$S_i^{j,\ell} = \chi(r_i)\, S^{j,\ell}(r_i,\theta_i).$$

For any $1 \leq \ell < k$, we obtain the following splitting (3.11) for the solution u with data $f = 0$ and $g = -\sum_i C_i\, \partial_n S_i^{k,\ell-1}$: for any $M \geq k-\ell$

$$u \;=\; u_{\mathrm{reg}(M)} \;+\; \sum_{i=1}^{2} \Big(C_i\, S_i^{k,\ell} \;+\; \sum_{j=1}^{M} c_i^j(u)\, S_i^j \Big). \tag{3.11}$$

Remark 3.5 The splitting (3.11) holds for instance with $g = -\partial_n S_i^2$, whereas with $g = -\partial_n S_i^1$ which does not belong to $L^2(\Gamma_R)$, problem $(\mathcal{P}_0)$ is ***not well-posed***: the uniqueness of solution is lost. ■

4 BASIC ESTIMATES

Let us recall that u_ε is the solution of problem $(\mathcal{P}_\varepsilon)$ for a small $\varepsilon > 0$ and u_0 is the solution of problem $(\mathcal{P}_0)$ with the same data. Let r_ε be the remainder

$$r_\varepsilon = u_\varepsilon - u_0.$$

Our aim is to obtain estimates for u_ε and r_ε. For simplicity, we assume (without restriction) that $f = 0$. The arguments that we present now are essentially contained in [1], but we write them in a more systematic way and improve slightly the results of [1].

Reducing the problem to the boundary, we want to estimate in Sobolev norms the traces γu_ε and γr_ε of u_ε and r_ε on the part Γ_R of the boundary. It is now important to give a precise definition for the Sobolev spaces on Γ_R. For any $s \in \mathbb{R}$, the space $H^s(\Gamma)$ on the whole boundary of Ω is obviously defined like the H^s space on the unit circle $\mathbb{T}$. The space $H^s(\Gamma_R)$ is the space of the restrictions to Γ_R of the distributions belonging to $H^s(\Gamma)$. And we also need the space $\widetilde{H}^s(\Gamma_R)$ which is the dual space of $H^{-s}(\Gamma_R)$. Then we set

$$\mathcal{H}^s(\Gamma_R) = \begin{cases} H^s(\Gamma_R) & \text{if } s \geq 0, \\ \widetilde{H}^s(\Gamma_R) & \text{if } s \leq 0. \end{cases} \tag{4.1}$$

The scale of the spaces $\mathcal{H}^s$ is an interpolation scale: for any $\theta \in [0,1]$

$$\left[\mathcal{H}^s, \mathcal{H}^t\right]_\theta = \mathcal{H}^{(1-\theta)s+\theta t}.$$

We start from the variational formulation (1.1) of problem $(\mathcal{P}_\varepsilon)$ and obtain

$$|u_\varepsilon|^2_{H^1(\Omega)} + \frac{1}{\varepsilon}\,\|\gamma u_\varepsilon\|^2_{L^2(\Gamma_R)} \;=\; \frac{1}{\varepsilon}\,\langle g, \gamma u_\varepsilon\rangle, \tag{4.2}$$

where $|v|_{H^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)^2}$ is the H^1-seminorm. As a first consequence, we obtain that

$$\frac{1}{\varepsilon}\,\|\gamma u_\varepsilon\|^2_{L^2(\Gamma_R)} \;\leq\; \frac{1}{\varepsilon}\,\|g\|_{L^2(\Gamma_R)}\,\|\gamma u_\varepsilon\|_{L^2(\Gamma_R)}\,.$$

Whence

$$\|\gamma u_\varepsilon\|_{L^2(\Gamma_R)} \;\leq\; \|g\|_{L^2(\Gamma_R)}\,. \tag{4.3}$$

We can also start from the inequalities

$$\langle g, \gamma u_\varepsilon\rangle \;\leq\; \|g\|_{\mathcal{H}^{-1/2}(\Gamma_R)}\,\|\gamma u_\varepsilon\|_{\mathcal{H}^{1/2}(\Gamma_R)}$$

and for $0 < \varepsilon$ bounded

$$|u_\varepsilon|^2_{H^1(\Omega)} + \frac{1}{\varepsilon}\,\|\gamma u_\varepsilon\|^2_{L^2(\Gamma_R)} \geq c\,\|u_\varepsilon\|^2_{H^1(\Omega)} \geq c'\,\|\gamma u_\varepsilon\|^2_{\mathcal{H}^{1/2}(\Gamma_R)}\,.$$

Whence from (4.2):

$$\|\gamma u_\varepsilon\|_{\mathcal{H}^{1/2}(\Gamma_R)} \ \le\ C\,\varepsilon^{-1}\|g\|_{\mathcal{H}^{-1/2}(\Gamma_R)}\,. \tag{4.4}$$

In order to write the reduction to the boundary, we introduce the Dirichlet-to-Neumann operator T of the Dirichlet-Neumann mixed problem $(\mathcal{P}_0)$: for $g \in \mathcal{H}^{1/2}(\Gamma_R)$, Tg is the Neumann trace on Γ_R $\gamma\partial_n u_0$ in $\mathcal{H}^{-1/2}(\Gamma_R)$ of the solution u_0 . As a consequence of Theorem 3.1:

$$T\ :\ \mathcal{H}^s(\Gamma_R) \longrightarrow \mathcal{H}^{s-1}(\Gamma_R) \quad \text{is continuous} \quad \forall s \in (0,1). \tag{4.5}$$

As u_ε is the solution of problem $(\mathcal{P}_0)$ with Dirichlet datum γu_ε , we obviously have

$$(\varepsilon T + I)(\gamma u_\varepsilon) = g,$$

whence the invertibility of $\varepsilon T + I$ and

$$\gamma u_\varepsilon = (\varepsilon T + I)^{-1} g. \tag{4.6}$$

Since the trace of r_ε on Γ_R is equal to $\gamma u_\varepsilon - g$, we obtain

$$\gamma r_\varepsilon = \Big((\varepsilon T + I)^{-1} - I\Big)\, g. \tag{4.7}$$

Thus we have to study the continuity properties of the operators $(\varepsilon T + I)^{-1}$ and $(\varepsilon T + I)^{-1} - I$. Note that we cannot use the perturbation series for $(\varepsilon T + I)^{-1}$ as in §2, since powers T^n , with $n \ge 2$ are not well-defined, see (4.5). The beginning of the perturbation series (" $N = 0$ ") appears in formulas (4.8) and (4.10) below.

From (4.3) and (4.4), we obtain that the norm of $(\varepsilon T + I)^{-1}$ from $\mathcal{H}^t(\Gamma_R)$ into $\mathcal{H}^s(\Gamma_R)$ is $\mathscr{O}(\varepsilon^{t-s})$ for the two pairs (s,t) equal to $(0,0)$ and $(\frac12, -\frac12)$. Then the following definitions are natural.

Let $\mathscr{U}$ be the set of pairs (s,t) such that the following estimate holds

$$\|\gamma u_\varepsilon\|_{\mathcal{H}^s(\Gamma_R)} \ \le\ C\,\varepsilon^{t-s}\,\|g\|_{\mathcal{H}^t(\Gamma_R)}$$

for any $g \in \mathcal{H}^s(\Gamma_R)$ and let $\mathscr{R}$ be the set of pairs (s,t) such that the following estimate holds

$$\|\gamma r_\varepsilon\|_{\mathcal{H}^s(\Gamma_R)} \ \le\ C\,\varepsilon^{t-s}\,\|g\|_{\mathcal{H}^t(\Gamma_R)}$$

for any $g \in \mathcal{H}^s(\Gamma_R)$. We already know that $(0,0)$ and $(\frac12, -\frac12)$ belong to $\mathscr{U}$. In order to extend the knowledge of these sets $\mathscr{U}$ and $\mathscr{R}$ we use that, with the help of a standard interpolation argument, they are convex sets of $\mathbb{R}^2$, and we rely on the two following lemmas.

Lemma 4.1 *For all* $s \in (0,1)$ *and for all* $t \leq s$ *,*

$$(s,t) \in \mathscr{U} \Longleftrightarrow (s-1,t) \in \mathscr{R}.$$

Proof. We start from the formula

$$(\varepsilon T + I)^{-1} - I = -\varepsilon T \, (\varepsilon T + I)^{-1}. \tag{4.8}$$

The direction $\Rightarrow$ is then a direct consequence of the assumption and of the continuity property (4.5).

For the direction $\Leftarrow$, we cannot simply invert T in the above formula (4.8) (T is not invertible), but we can use the inverse of $T + I$:

$$\frac{1}{\varepsilon}\,(T+I)^{-1}\,\Big((1-\varepsilon)(\varepsilon T + I)^{-1} - I\Big) \;=\; -(\varepsilon T + I)^{-1}. \tag{4.9}$$

If $t \leq s$ the identity is continuous from $\mathcal{H}^t(\Gamma_R)$ into $\mathcal{H}^s(\Gamma_R)$. Thus, if (s,t) belongs to $\mathscr{R}$, the operator $((1-\varepsilon)(\varepsilon T + I)^{-1} - I)$ has the same properties of continuity as $(\varepsilon T + I)^{-1} - I$. The above formula (4.9) and the continuity of $(T+I)^{-1}$ from $\mathcal{H}^{s-1}(\Gamma_R)$ into $\mathcal{H}^s(\Gamma_R)$ ends the proof. ∎

Lemma 4.2 *For all* $t \in (-1,0)$ *and for all* $t \leq s$ *,*

$$(s,t) \in \mathscr{U} \Longleftrightarrow (s,t+1) \in \mathscr{R}.$$

The proof is very similar to the previous one: instead of (4.8), we use

$$(\varepsilon T + I)^{-1} - I = -\varepsilon(\varepsilon T + I)^{-1} T \tag{4.10}$$

and instead of (4.9)

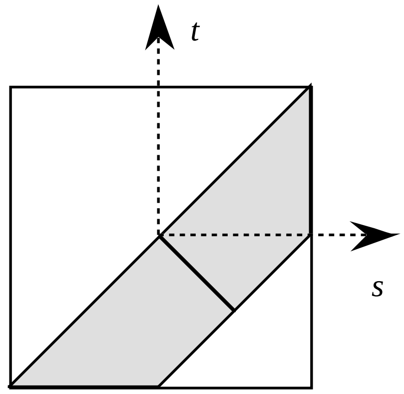
$$\frac{1}{\varepsilon}\,\Big((1-\varepsilon)(\varepsilon T + I)^{-1} - I\Big)\,(T+I)^{-1} \;=\; -(\varepsilon T + I)^{-1}. \tag{4.11}$$

Theorem 4.3 *The set* $\mathscr{U}$ *satisfies*

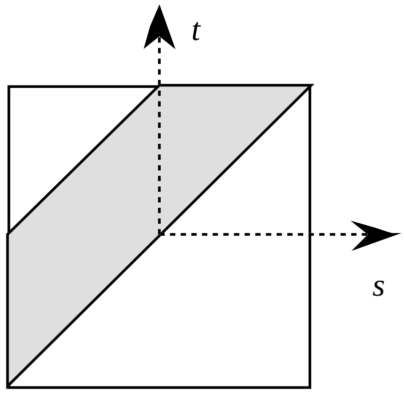
$$\{(s,t) \in (-1,1) \times (-1,1) \mid \quad s-1 \leq t \leq s\} \;\subset\; \mathscr{U} \tag{4.12}$$

and the set $\mathscr{R}$ *satisfies*

$$\{(s,t) \in (-1,1) \times (-1,1) \mid \quad s \leq t \leq s+1\} \;\subset\; \mathscr{R}. \tag{4.13}$$



$\mathscr{U}$



$\mathscr{R}$

The proof uses the fact that $(0,0)$ and $(\frac{1}{2},-\frac{1}{2})$ belong to $\mathscr{U}$, thus the segment linking these points is contained in $\mathscr{U}$. Then we transport this set into $\mathscr{R}$ by lemmas 4.1 and 4.2 and take the convex hull of the set thus obtained. Then we return to $\mathscr{U}$, next to $\mathscr{R}$. We end by transporting the diagonal of $\mathscr{R}$ into $\mathscr{U}$.

Combining the above results with Theorem 3.1, we obtain

Corollary 4.4 *For the solution u_ε of problem $(\mathcal{P}_\varepsilon)$ with $f=0$ and the remainder $r_\varepsilon = u_\varepsilon - u_0$, where u_0 is the solution of problem $(\mathcal{P}_0)$, the following estimates hold*

$$\forall s \in \Big(-\tfrac{1}{2},\tfrac{1}{2}\Big), \quad s-1 \le t \le s \ : \quad \|u_\varepsilon\|_{H^{1+s}(\Omega)} \ \le \ C\,\varepsilon^{t-s}\,\|g\|_{\mathcal{H}^{\frac{1}{2}+t}(\Gamma_R)} \tag{4.14}$$

$$\forall s \in \Big(-\tfrac{1}{2},\tfrac{1}{2}\Big), \quad s \le t < \tfrac{1}{2} \ : \quad \|r_\varepsilon\|_{H^{1+s}(\Omega)} \ \le \ C\,\varepsilon^{t-s}\,\|g\|_{H^{\frac{1}{2}+t}(\Gamma_R)}\,. \tag{4.15}$$

Remark 4.5 If f is any function in $L^2(\Omega)$, the above estimates still hold with the part $\|g\|_{H^{\frac{1}{2}+t}(\Gamma_R)}$ in the right hand sides replaced by $\Big(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}+t}(\Gamma_R)}\Big)$. This can easily be proved by subtracting from u_ε the solution of a Neumann problem on Ω with data f in Ω and a smooth function φ with support inside Γ_R on Γ . ∎

5 THE MODEL PROBLEM $(\mathcal{P}_1)$. SINGULARITIES AT INFINITY

To go further in the construction of the asymptotic expansion of u_ε , the first idea is to solve the mixed Dirichlet-Neumann problem $(\mathcal{P}_0)$ with $f=0$ and $g=-\partial_n u_0$. But, in general, the coefficients $c_i^1(u_0)$ of the first singular functions S_i^1 are not zero. Thus this mixed problem is not well-posed.

The idea is to solve directly the problem $(\mathcal{P}_\varepsilon)$ for these singular parts in a model situation, i.e. on the half-plane Π : using the homogeneity of such a geometry, all problems $(\mathcal{P}_\varepsilon)$ are equivalent by the change of variables $r \mapsto r/\varepsilon$ to the problem $(\mathcal{P}_1)$ with $\varepsilon = 1$:

$$(\mathcal{P}_1) \qquad \left\{ \begin{array}{rcll} \Delta u &=& f & \text{in} \quad \Pi \\ \partial_n u &=& 0 & \text{on} \quad \mathbb{R}_- \\ \partial_n u + u &=& g & \text{on} \quad \mathbb{R}_+, \end{array} \right.$$

with $f = 0$ and $g = -\partial_n S^j$. We expect the solutions to define "corner layer terms" with structure $w(\frac{r}{\varepsilon}, \theta)$.

As will be seen below, the solution of this problem is closely linked with the solutions of the homogeneous problem $(\mathcal{P}_1)$ (with $f = 0$ and $g = 0$):

$$\left\{ \begin{array}{rcll} \Delta K &=& 0 & \text{in} \quad \Pi \\ \partial_n K &=& 0 & \text{on} \quad \mathbb{R}_- \\ \partial_n K + K &=& 0 & \text{on} \quad \mathbb{R}_+. \end{array} \right. \tag{5.1}$$

Roughly speaking, to each S^j corresponds a solution K^j of the homogeneous problem (5.1) which behaves like S^j at infinity (when $r \to +\infty$) and $K^j - S^j$ solves problem $(\mathcal{P}_1)$ with $f = 0$ and $g = -\partial_n S^j$.

Theorem 5.1 *For any integer $j \geq 0$, let $\mathscr{K}^j$ be the space of solutions of problem (5.1) which are $\mathscr{O}\left(r^{j-1/2}\right)$ as $r \to +\infty$ and H^1 on any bounded subdomain of Π. Then*

$$\forall j \geq 1, \quad \dim \mathscr{K}^j = j,$$

and for all $K \in \mathscr{K}^j$, there exists a constant $c(K)$ such that

$$K = c(K)\, S^j + \mathscr{o}\left(r^{j-1/2}\right).$$

An essential argument for the proof is the existence of a variational space V on Π where problem $(\mathcal{P}_1)$ is uniquely solvable. Let V be the space

$$V = \left\{ u \mid \quad \nabla u \in L^2(\Pi)^2\,; \quad u\Big|_{\mathbb{R}_+} \in L^2(\mathbb{R}_+) \right\}, \tag{5.2}$$

endowed with the norm $\|u\|_V^2 = \int_\Pi |\nabla u|^2 + \int_{\mathbb{R}_+} |u|^2$. With the help of the Lax-Milgram theorem, we obtain immediately:

Lemma 5.2 *For any f in the dual space $H^1(\Pi)^*$ of $H^1(\Pi)$ and any $g \in L^2(\mathbb{R}_+)$, the problem $(\mathcal{P}_1)$ on Π whose variational formulation reads*

$$\forall v \in V, \quad \int_\Pi \nabla u \cdot \nabla v + \int_{\mathbb{R}_+} (u - g)\, v \;=\; -\int_\Pi f\, v,$$

has a unique solution $u \in V$.

Another key argument is the asymptotic expansion of any element K of $\mathscr{K}^j$ at infinity (when $r \to +\infty$). By Mellin transformation, we obtain:

Lemma 5.3 *Let $K \in \mathscr{K}^j$. For any $P \in \mathbb{N}$, we can expand K as a sum*

$$K = \sum_{p=0}^{P} K^{(p)} + \mathscr{O}\left(r^{j-P-1/2}\right), \quad K^{(p)} \in \mathscr{S}^{j-p}, \tag{5.3}$$

where the space $\mathscr{S}^j$ is defined in (3.7) *and $K^{(p)}$ is solution of the mixed problem* (3.6) *on Π with datum $g = -\partial_n K^{(p-1)}$; for $p = 0$, $K^{(0)}$ is solution of the totally homogeneous problem* (3.6)*.*

Proof of Theorem 5.1. Let χ be a smooth cut-off function equal to 1 in a neighborhood of 0 and 0 in a neighborhood of $+\infty$. We remark that $(1-\chi)S^0$ belongs to V.

Let $K \in \mathscr{K}^j$. In the expansion (5.3) of K, the term $K^{(0)}$ is a solution in $\mathscr{S}^j$ of the totally homogeneous problem (3.6). Thus it has the form $c^{(0)}\, S^j$.

Let $K \in \mathscr{K}^0$. Writing its expansion (5.3) with $P = 1$ and cutting it by $(1-\chi)$, we see that K belongs to V. Thus Lemma 5.2 yields that $K = 0$.

For $j \geq 1$, let $K \in \mathscr{K}^j$. For the expansion (5.3) of K, there exist coefficients $c^{(\ell)}$ such that (compare with (3.8))

$$\begin{aligned}
&K^{(0)} = c^{(0)}\, S^j \\
&K^{(1)} = c^{(0)}\, S^{j,1} + c^{(1)}\, S^{j-1} \\
&K^{(2)} = c^{(0)}\, S^{j,2} + c^{(1)}\, S^{j-1,1} + c^{(2)}\, S^{j-2} \\
&\ldots
\end{aligned}$$

Like above, we obtain that

$$K - (1-\chi) \sum_{\ell=0}^{j-1} c^{(\ell)} \sum_{p=0}^{j-\ell} S^{j-\ell,p}$$

belongs to V. This proves that $\dim \mathscr{K}^j \leq j$. Conversely, we note that for any $k \geq 1$

$$\left(\Delta\Big((1-\chi)\sum_{p=0}^{k} S^{k,p}\Big),\ (1-\chi)\sum_{p=0}^{k} S^{k,p}\Big|_{\mathbb{R}_+} \right)$$

belongs to $H^1(\Pi)^* \times L^2(\mathbb{R}_+)$. Lemma 5.2 yields that there exists a unique $v \in V$ such that

$$v + (1-\chi)\sum_{p=0}^{k} S^{k,p} \in \mathscr{K}^k.$$

This ends the proof of Theorem 5.1 and proves also the following theorem. ■

Theorem 5.4 *For any integer* $j \geq 1$ *, there exists a unique element* K^j *in* $\mathscr{K}^j$ *satisfying*

$$K^j = S^j + S^{j,1} + \cdots + S^{j,j-1} + \mathscr{O}\left(r^{-1/2}\log^j r\right) \quad \textit{as } r \to +\infty. \tag{5.4}$$

Remark 5.5 Relying on (5.3), it is possible to push forward the asymptotic expansion of K^j : we have

$$K^j \;=\; S^j + S^{j,1} + \cdots + S^{j,j-1} + S^{j,j} + \mathscr{O}\left(r^{-1/2}\right),$$

where this $\mathscr{O}\left(r^{-1/2}\right)$ cannot be replaced by $\mathscr{o}\left(r^{-1/2}\right)$ because of the possible presence of a term $c\,S^0$ in the asymptotics. Relying on (3.8) and (5.3), we obtain that there exist (unique) coefficients $c^{j(p)}$ such that for any $P \geq j$

$$K^j \;=\; \sum_{\ell=0}^{P} S^{j,\ell} \;+\; \sum_{p=0}^{P-j} c^{j(p)} \sum_{\ell=0}^{P-p} S^{-p,\ell} + \mathscr{o}\left(r^{j-P-1/2}\right) \quad \text{as } r \to +\infty. \tag{5.5}$$

■

6 THE FIRST TERMS IN THE EXPANSION. SHARP ESTIMATES

We suppose now that the right hand side is more regular than required for the basic estimates in Section 4. This allows us to construct further terms in the expansion of u_ε and to get eventually remainder terms that behave like arbitrary high powers of ε as $\varepsilon \to 0$. It will also allow sharp estimates on the first terms in the expansion. For example, we shall see that the powers of ε given in Corollary 4.4 are sharp in the sense that there is no $\mathscr{O}(\varepsilon)$ estimate for r_ε in any Sobolev norm on Ω .

The first term after $u_0 = u^0$ in the expansion is a correction term w^0 to u^0 of corner layer type. The idea is to use the splitting of u^0 , solution of $(\mathcal{P}_0)$, into regular and singular parts as in Theorem 3.2, and then to replace the singular functions S^j by their "near-singular" counterparts K^j that were constructed in the previous section.

More precisely, in the first step we assume that $g \in H^{1+\delta}$ for some $\delta > 0$ and $f = 0$. We can then apply Theorem 3.2 with $M = 1$ to u^0 :

$$u^0 = u^0_{\mathrm{reg}(1)} + \sum_{i=1}^{2} c^1_i(u^0)\, S^1_i\,. \tag{6.1}$$

Recall that the singular function S^1_i was defined in (3.3) from the model function S^1 via a cut-off function χ and that it is homogeneous of degree $1/2$. We have for any $\varepsilon > 0$

$$S^1_i = \chi(r_i)\, S^1(r_i,\theta_i) = \sqrt{\varepsilon}\,\chi(r_i)\, S^1(\frac{r_i}{\varepsilon},\theta_i)\,. \tag{6.2}$$

We define

$$w^0(\varepsilon) = \sum_{i=1}^{2} c_i^1(u^0)\, \chi(r_i)\, (K^1 - S^1)(\frac{r_i}{\varepsilon}, \theta_i) \tag{6.3}$$

and

$$\begin{aligned} \tilde{u}^0(\varepsilon) &= u^0 + \sqrt{\varepsilon}\, w^0(\varepsilon) \\ &= u^0_{\mathrm{reg}(1)} + \sqrt{\varepsilon} \sum_{i=1}^{2} c_i^1(u^0)\, \chi(r_i)\, K^1(\frac{r_i}{\varepsilon}, \theta_i)\,. \end{aligned} \tag{6.4}$$

If the support of χ is chosen sufficiently small, then ∂_n commutes with the multiplication by $\chi(r_i)$, and the function $\chi(r_i)\, K^1(\frac{r_i}{\varepsilon}, \theta_i)$ satisfies the homogeneous boundary conditions of the problem $(\mathcal{P}_\varepsilon)$.

The remainder $\tilde{r}_\varepsilon = u_\varepsilon - \tilde{u}^0(\varepsilon)$ satisfies therefore the boundary value problem

$$\left\{ \begin{array}{rclll} \Delta \tilde{r}_\varepsilon &=& -\sqrt{\varepsilon}\, \Delta w^0 & \text{in} & \Omega \\ \partial_n \tilde{r}_\varepsilon &=& 0 & \text{on} & \Gamma_N \\ \varepsilon \partial_n \tilde{r}_\varepsilon + \tilde{r}_\varepsilon &=& -\,\varepsilon\, \partial_n u^0_{\mathrm{reg}(1)} & \text{on} & \Gamma_R\,. \end{array} \right. \tag{6.5}$$

Thus the boundary conditions are already in a shape for the application of the basic estimates with a gain of a power of ε, but due to the cut-off function, the Laplace equation is now inhomogeneous, and we have to expand its right hand side into powers of ε. This is done in the following lemma.

Lemma 6.1 *For any* $P \geq 1$*, we have*

$$-\sqrt{\varepsilon}\, \Delta w^0(\varepsilon) = \sum_{p=1}^{P} \varepsilon^p\, f^{0p}[\log \varepsilon] + f^{0(P)}(\varepsilon)\,. \tag{6.6}$$

Here $f^{0p}[\log \varepsilon]$ *are polynomials of degree* p *in* $\log \varepsilon$*:*

$$f^{0p}[\log \varepsilon] = \sum_{q=0}^{p} f^{0pq} \log^q \varepsilon$$

with coefficients $f^{0pq} \in \mathscr{C}^\infty(\overline{\Omega})$*. The remainder satisfies*

$$\|f^{0(P)}(\varepsilon)\|_{L^2(\Omega)} = \mathscr{O}\left(\varepsilon^P\right) \quad \textit{as } \varepsilon \to 0\,. \tag{6.7}$$

Proof. From the definition (6.3) of w^0 we find

$$\Delta w^0(\varepsilon) = \sum_{i=1}^{2} c_i^1(u^0) \left(2\nabla\chi(r_i) \cdot \nabla(K^1 - S^1)(\tfrac{r_i}{\varepsilon}, \theta_i) + \Delta\chi(r_i)\, (K^1 - S^1)(\tfrac{r_i}{\varepsilon}, \theta_i)\right) \tag{6.8}$$

Now we use the decomposition of K^1 given in Lemma 5.3 and in (5.4), (5.5) which can obviously be differentiated. Let us first consider a term $K^{(p)} \in \mathscr{S}^{1-p}$. We have

$$K^{(p)} = r^{\frac{1}{2}-p}\psi(\theta)[\log r]\,,$$

hence

$$K^{(p)}(\frac{r}{\varepsilon},\theta) = \varepsilon^{p-\frac{1}{2}} f^{(p)}(r,\theta)[\log\varepsilon]\,, \tag{6.9}$$

where the coefficients of the polynomial $f^{(p)}(r,\theta)[\log\varepsilon]$ in $\log\varepsilon$ are $\mathscr{C}^\infty$ for $r \neq 0$. The same form (6.9) with the same power of ε holds for the gradient $\nabla\left(K^{(p)}(\frac{r}{\varepsilon},\theta)\right)$. In (6.8) only derivatives of the cut-off function χ appear which vanish in a neighborhood of the singularities at $r_i = 0$. Therefore these terms have the form $\varepsilon^p f^{0p}[\log\varepsilon]$ as stated in the lemma. Note that in $K^1 - S^1$ only terms $K^{(p)}$ with $p \geq 1$ appear.

It remains to prove the norm estimate (6.7).

Assume that the support of $\nabla\chi$ is contained in the annulus $\{r \in (a,b)\}$ for some $0 < a < b$. Then for a function of the form

$$f(\varepsilon) = \Delta\chi(r_i)F(\frac{r_i}{\varepsilon},\theta_i)$$

with

$$F(r,\theta) = \mathscr{O}\left(r^{\frac{1}{2}-P}\right) \qquad \text{as } r \to \infty\,,$$

we have

$$\|f(\varepsilon)\|^2_{L^2(\Omega)} \leq \mathscr{O}(1)\cdot\int_a^b \left|\frac{r}{\varepsilon}\right|^{1-2P} r\,dr = \mathscr{O}\left(\varepsilon^{2P-1}\right)\,.$$

The same estimate is obtained for a term

$$\nabla\chi(r_i)\cdot\nabla F(\frac{r_i}{\varepsilon},\theta_i)\,.$$

Together with the factor $\sqrt{\varepsilon}$ we obtain (6.7). ∎

We now use this expansion for $P = 1$ in order to define

$$f^1[\log\varepsilon] = f^{01}[\log\varepsilon] = f^{010} + f^{011}\log\varepsilon. \tag{6.10}$$

With this right hand side, we can now define the term u^1 in our expansion of u_ε as solution of the mixed boundary value problem $(\mathcal{P}_0)$

$$\left\{\begin{array}{rcll} \Delta u^1[\log\varepsilon] &=& f^1[\log\varepsilon] & \text{in } \Omega \\ \partial_n u^1[\log\varepsilon] &=& 0 & \text{on } \Gamma_N \\ u^1[\log\varepsilon] &=& -\,\partial_n u^0_{\mathrm{reg}(1)} & \text{on } \Gamma_R\,. \end{array}\right. \tag{6.11}$$

This notation of polynomials in $\log\varepsilon$ which we shall use now systematically means here that we have

$$u^1[\log\varepsilon] = u^{10} + u^{11}\log\varepsilon\,,$$

where u^{10} and u^{11} are solutions of the following two mixed problems independent of ε :

$$\left\{\begin{array}{rclll} \Delta u^{10} &=& f^{010} & \text{in} & \Omega \\ \partial_n u^{10} &=& 0 & \text{on} & \Gamma_N \\ u^{10} &=& -\,\partial_n u^0_{\text{reg}(1)} & \text{on} & \Gamma_R \end{array}\right. \tag{6.12}$$

and

$$\left\{\begin{array}{rclll} \Delta u^{11} &=& f^{011} & \text{in} & \Omega \\ \partial_n u^{11} &=& 0 & \text{on} & \Gamma_N \\ u^{11} &=& 0 & \text{on} & \Gamma_R\,. \end{array}\right. \tag{6.13}$$

We can now define the expansion up to $\mathscr{O}(\varepsilon)$.

Theorem 6.2 *For the solution u_ε of $(\mathcal{P}_\varepsilon)$, with $f \in H^1(\Omega)$ and $g \in H^{1+\delta}(\Gamma_R)$ for some $\delta > 0$, we have an expansion*

$$u_\varepsilon = u^0 + \varepsilon\, u^1[\log\varepsilon] + \sqrt{\varepsilon}\, w^0(\varepsilon) + r^1_\varepsilon\,, \tag{6.14}$$

where r^1_ε satisfies estimates $\quad\forall s \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad s \le t < \frac{1}{2}$:

$$\|r^1_\varepsilon\|_{H^{1+s}(\Omega)} \le C\,\varepsilon^{1+t-s}|\log\varepsilon|\,\Big(\|f\|_{H^1(\Omega)} + \|g\|_{H^{\frac{3}{2}+t}(\Gamma_R)}\Big)\,. \tag{6.15}$$

Proof. We consider the boundary value problem satisfied by r^1_ε :

$$\left\{\begin{array}{rclll} \Delta r^1_\varepsilon &=& f^{0(1)}(\varepsilon) & \text{in} & \Omega \\ \partial_n r^1_\varepsilon &=& 0 & \text{on} & \Gamma_N \\ (\varepsilon\partial_n + 1)\, r^1_\varepsilon &=& -\,\varepsilon^2\partial_n u^1[\log\varepsilon] & \text{on} & \Gamma_R\,. \end{array}\right. \tag{6.16}$$

We compare this with the problem satisfied by u_ε and use the basic estimate (4.14) and Remark 4.5. Replacing there t by $t-1$, we obtain for s and t as in the theorem

$$\|r^1_\varepsilon\|_{H^{1+s}(\Omega)} \le C\left(\|f^{0(1)}(\varepsilon)\|_{L^2(\Omega)} + \varepsilon^{t-1-s}\|\varepsilon^2\partial_n u^1[\log\varepsilon]\|_{\mathcal{H}^{-\frac{1}{2}+t}(\Gamma_R)}\right) \tag{6.17}$$

The right hand side $f^{0(1)}(\varepsilon)$ has, according to Lemma 6.1 with $P = 1$, an L^2 norm of order $\mathscr{O}(\varepsilon)$. If we use the lemma with $P = 2$, we even find that it is $\mathscr{O}\left(\varepsilon^2\log^2\varepsilon\right)$. Its dependence on f and g is via the constants $c^1_i(u^0)$, see the definition (6.3) of w^0 . Therefore this contribution to r_ε^1 allows estimates as in (6.15) with any power of ε less than 2.

For the last term in (6.17) we find

$$\begin{aligned} \|\partial_n u^1[\log\varepsilon]\|_{\mathcal{H}^{-\frac12+t}(\Gamma_R)} &\le C\left(\|f^1[\log\varepsilon]\|_{L^2(\Omega)} + \|\partial_n u^0_{\mathrm{reg}(1)}\|_{\mathcal{H}^{\frac12+t}(\Gamma_R)}\right) \\ &\le C\,|\log\varepsilon|\left(\|f\|_{H^1(\Omega)} + \|g\|_{H^{\frac32+t}(\Gamma_R)}\right) \end{aligned} \tag{6.18}$$

Here we used the regularity estimate (3.5) of Theorem 3.2 for the problem $(\mathcal{P}_0)$. ∎

With the help of the expansion (6.14), we can now prove sharp estimates for some of the limit cases in Section 4, in particular in (4.13) and (4.15). It suffices to compute explicit norms of the terms $\varepsilon u^1[\log\varepsilon]$ and $\sqrt{\varepsilon}\,w^0(\varepsilon)$, but we have to pay attention to possible cancellations between these terms.

Proposition 6.3 *Assume that* $f=0$ *and* $g\in H^{1+\delta}(\Gamma_R)$ *for some* $\delta>0$ *. Then for* $r_\varepsilon = u_\varepsilon - u_0$ *we have the estimates*

$$\|r_\varepsilon\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon\log\varepsilon) \tag{6.19}$$

$$\|r_\varepsilon\|_{H^{1+s}(\Omega)} = \mathcal{O}\left(\varepsilon^{\frac12-s}\right) \quad \textit{for } s\in\left(-\tfrac12,\tfrac12\right) \tag{6.20}$$

$$\|r_\varepsilon\big|_{\Gamma_R}\|_{L^2(\Gamma_R)} = \mathcal{O}\left(\varepsilon\sqrt{|\log\varepsilon|}\right) \tag{6.21}$$

These estimates cannot be improved, in general.

Proof. Recall from (3.10) and (5.5) the expansion as $r\to\infty$

$$\begin{aligned} (K^1-S^1)(r,\theta) &= \frac{1}{2\pi}\,r^{-1/2}\Big((\pi-\theta)\cos\tfrac{\theta}{2} + \log r\,\sin\tfrac{\theta}{2}\Big) \\ &\qquad + c^{1(0)}\,r^{-1/2}\,\sin\tfrac{\theta}{2} + \mathcal{O}\left(r^{-3/2}\log^2 r\right). \end{aligned} \tag{6.22}$$

This gives (locally near $\boldsymbol{c}_i$, we omit the index i for brevity)

$$\sqrt{\varepsilon}\,w^0(\varepsilon) = c\,\varepsilon\log\varepsilon\,\chi(r)\,(1-\chi(\tfrac{r}{\varepsilon}))\,r^{-1/2}\,\sin\tfrac{\theta}{2} + \tilde{w}(\varepsilon)\,, \tag{6.23}$$

where

$$\|\tilde{w}(\varepsilon)\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon) \qquad \text{as } \varepsilon\to0\,.$$

This proves (6.19). In order to show that (6.19) is sharp, we have to show that the two terms in $\varepsilon\log\varepsilon$ cannot cancel each other. Suppose the contrary,

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon\log\varepsilon}\|\varepsilon u^1[\log\varepsilon] + \sqrt{\varepsilon}\,w^0(\varepsilon)\|_{L^2(\Omega)} = 0\,.$$

Hence

$$\|\chi(r)\Big(u^{11} - c\,(1-\chi(\tfrac{r}{\varepsilon}))r^{-1/2}\sin\tfrac{\theta}{2}\Big)\|_{L^2(\Omega)} \to 0\,.$$

This contradicts the $H^1(\Omega)$ regularity of the solution u^{11} of the mixed boundary value problem (6.13), unless $u^{11} = w^0 = 0$, that is $c_i^1(u^0) = 0$.

In order to show (6.21), we note that on Γ_R we have

$$u^{11} = 0 \quad \text{and} \quad (K^1 - S^1)(r, 0) = \tfrac{1}{2}\, r^{-1/2} + \mathscr{O}\left(r^{-3/2}\log^2 r\right).$$

Thus there are no terms in $\varepsilon \log \varepsilon$, and the dominant term in the $L^2(\Omega)$ norm comes from

$$\|\sqrt{\varepsilon}\chi(r)(1 - \chi(\tfrac{r}{\varepsilon}))(\tfrac{r}{\varepsilon})^{-1/2}\|^2_{L^2(\mathbb{R}_+)} = \mathscr{O}\left(\int_{a\varepsilon}^{b} \left|\sqrt{\varepsilon}(\tfrac{r}{\varepsilon})^{-1/2}\right|^2 dr\right) = \mathscr{O}\left(\varepsilon^2 \log \varepsilon\right). \tag{6.24}$$

All other terms are of order ε. This proves (6.21). Once again, the dominant term can only be absent if $c_i^1(u^0) = 0$.

If, in (6.24), we take Besov semi-norms with positive index, we find without much difficulty for the $H^s(\Gamma_R)$ norm with $s > 0$ a behavior $\mathscr{O}(\varepsilon^{1-s})$. The same power of ε is obtained for $H^s(\Gamma_R)$ norms of terms of the form $\sqrt{\varepsilon}\chi(r)v(r/\varepsilon)$, where $v \in H^s(\mathbb{R}_+)$. We obtain

$$\|r_\varepsilon\big|_{\Gamma_R}\|_{H^s(\Gamma_R)} = \mathscr{O}\left(\varepsilon^{1-s}\right) \qquad \text{for any } s \in (0,1). \tag{6.25}$$

From this we arrive at (6.20) by using the basic regularity of the problem $(\mathcal{P}_0)$ for r_ε. ■

7 THE COMPLETE CONSTRUCTION

If the data are more regular, we can extend the above construction of the first terms u^0, w^0 and u^1 in the asymptotics of the solution u_ε of problem $(\mathcal{P}_\varepsilon)$. In w^0 there appears the first corner layer term $(K^1 - S^1)(r/\varepsilon)$. In the next terms, we have other corner layer terms corresponding to the remainder $\mathscr{O}\left(r^{-1/2}\log^j r\right)$ of the asymptotic expansion at infinity of K^j, *cf* (5.4). We introduce the notation

$$Y^j := K^j - \left(S^j + S^{j,1} + \cdots + S^{j,j-1}\right). \tag{7.1}$$

As already used in the previous section, we denote by $[\log \varepsilon]$ the polynomial dependence with respect to $\log \varepsilon$, excluding any other dependence in ε.

Here follows our result of complete asymptotic expansion for u_ε in two kinds of terms:

1. The "outer expansion" formed by terms u^n which have the standard regularity of variational solutions of the limit mixed Dirichlet-Neumann problem $(\mathcal{P}_0)$,

2. The "inner expansion" formed by corner layer terms w^n whose profiles decay as $\mathscr{O}\left(r^{-1/2}\right)$ only.

Theorem 7.1 *Let* $N \geq 1$. *For the solution* u_ε *of* $(\mathcal{P}_\varepsilon)$, *with* $f \in H^N(\Omega)$ *and* $g \in H^{N+\delta}(\Gamma_R)$ *for some* $\delta > 0$, *we have an expansion*

$$u_\varepsilon = u^0 + \cdots + \varepsilon^N u^N[\log\varepsilon] + \varepsilon^{1/2}\, w^0(\varepsilon) + \cdots + \varepsilon^{N-1/2}\, w^{N-1}(\varepsilon) + r_\varepsilon^N\,, \tag{7.2}$$

where u^n *and* w^n *satisfy the recurrence relations* (7.4) *below and where* r_ε^N *satisfies estimates* $\forall s \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad s \leq t < \frac{1}{2}$:

$$\|r_\varepsilon^N\|_{H^{1+s}(\Omega)} \leq C\, \varepsilon^{N+t-s} |\log\varepsilon|^N \left(\|f\|_{H^N(\Omega)} + \|g\|_{H^{N+\frac{1}{2}+t}(\Gamma_R)}\right). \tag{7.3}$$

The outer expansion terms $u^n[\log\varepsilon]$ *are of degree* n *in* $\log\varepsilon$ *and are the solutions of the mixed problems:*

$$\left\{\begin{array}{rcll} \Delta u^n[\log\varepsilon] & = & f^n[\log\varepsilon] & \textit{in}\ \ \Omega \\ \partial_n u^n[\log\varepsilon] & = & 0 & \textit{on}\ \ \Gamma_N \\ u^n[\log\varepsilon] & = & -\,\partial_n u^{n-1}_{\mathrm{reg}(1)}[\log\varepsilon] & \textit{on}\ \ \Gamma_R\,, \end{array}\right. \tag{7.4a}$$

for all $1 \leq n \leq N$ *(for* $n = 0$, *the last right hand side is* g) *; the interior right hand sides* $f^n[\log\varepsilon]$ *come from the corner layer terms by equations* (7.4c) *and* (7.4d) *below. The corner layer terms have the expressions, for* $0 \leq n \leq N-1$

$$w^n(\varepsilon) = \sum_{i=1}^{2} \chi(r_i) \left[\sum_{j=1}^{n+1} c_i^j(u^0, \ldots, u^{n+1-j})[\log\varepsilon]\, Y^j(\frac{r_i}{\varepsilon}, \theta_i)\right], \tag{7.4b}$$

where the coefficients $c_i^j(u^0, \ldots, u^{n+1-j})[\log\varepsilon]$ *are linear combinations with coefficients polynomial in* $\log\varepsilon$ *of the coefficients* $c_i^{n+1}(u^0),\ c_i^n(u^1), \ldots, c_i^j(u^{n+1-j})$, *see* (7.9). *The interior right hand sides* f^n *of* (7.4a) *are the sum of the contributions corresponding to* ε^n

$$f^n[\log\varepsilon] = \sum_{\ell=0}^{n-1} f^{\ell\, n}[\log\varepsilon], \quad \textit{for}\ n \geq 1 \quad (f^0 = f\ \ \textit{for}\ n = 0) \tag{7.4c}$$

which are issued from the w^ℓ *for* $0 \leq \ell < n$ *by*

$$-\varepsilon^{\ell+1/2}\, \Delta w^\ell(\varepsilon) = \sum_{n=\ell+1}^{P} \varepsilon^n f^{\ell n}[\log\varepsilon] + f^{\ell(P)}(\varepsilon) \tag{7.4d}$$

for any $P \geq \ell + 1$ *and where the remainder satisfies* $\|f^{\ell(P)}(\varepsilon)\|_{L^2(\Omega)} = \mathscr{O}\left(\varepsilon^P\right)$ *as* $\varepsilon \to 0$.

PROOF. The proof works by induction over N. We recall that the remainder r_ε^N is defined by

$$r_\varepsilon^N \;=\; u_\varepsilon - \left(\sum_{n=0}^{N} \varepsilon^n u^n + \sum_{n=0}^{N-1} \varepsilon^{n+1/2} w^n \right).$$

To the relations (7.4a)-(7.4d), we add the boundary value problem for r_ε^N

$$\left\{ \begin{array}{rcccc} \Delta r_\varepsilon^N & = & f^{(N)}(\varepsilon) & \text{in} & \Omega \\ \partial_n r_\varepsilon^N & = & 0 & \text{on} & \Gamma_N \\ (\varepsilon\partial_n + 1)\, r_\varepsilon^N & = & -\,\varepsilon^{N+1}\partial_n u^N[\log\varepsilon] & \text{on} & \Gamma_R\,, \end{array} \right. \tag{7.4e}$$

where

$$f^{(N)} = \sum_{\ell=0}^{N-1} f^{\ell(N)}(\varepsilon). \tag{7.5}$$

For $N = 1$, all relations (7.4) are proved in the previous section. We assume that they hold for $N \geq 1$ and we have to prove them for $N+1$.

Since $u^0, \ldots, u^N$ are already determined, we can define w^N by formula (7.4b): we only have to give the precise definition of the coefficients $c_i^j(u^0, \ldots, u^{N+1-j})$ for $j = 1, \ldots, N+1$, which will become clear hereafter. Anyway, we can prove like for Lemma 6.1 that $-\varepsilon^{N+1/2}\,\Delta w^N(\varepsilon)$ has an expansion according to (7.4d). Let us set, like for $N = 1$:

$$\tilde r_\varepsilon^N \;=\; u_\varepsilon - \left(\sum_{n=0}^{N} \varepsilon^n u^n + \sum_{n=0}^{N} \varepsilon^{n+1/2} w^n \right).$$

Then

$$\tilde r_\varepsilon^N = r_\varepsilon^N - \varepsilon^{N+1/2} w^N.$$

We are going to define the coefficients $c_i^j(u^0, \ldots, u^{N+1-j})$ so that there holds

$$\left\{ \begin{array}{rcccc} \Delta \tilde r_\varepsilon^N & = & \varepsilon^{N+1} f^{N+1}[\log\varepsilon] + f^{(N+1)}(\varepsilon) & \text{in} & \Omega \\ \partial_n \tilde r_\varepsilon^N & = & 0 & \text{on} & \Gamma_N \\ (\varepsilon\partial_n + 1)\, \tilde r_\varepsilon^N & = & -\,\varepsilon^{N+1}\partial_n u^N_{\mathrm{reg}(1)}[\log\varepsilon] & \text{on} & \Gamma_R\,. \end{array} \right. \tag{7.6}$$

We have $\Delta \tilde r_\varepsilon^N = \Delta r_\varepsilon^N - \varepsilon^{N+1/2}\Delta w^N$. Since

$$-\varepsilon^{N+1/2}\,\Delta w^N(\varepsilon) = \varepsilon^{N+1} f^{N\,N+1}[\log\varepsilon] + f^{N(N+1)}(\varepsilon)$$

the combination with (7.4c) and (7.5) yields the interior equation of (7.6).

Concerning the boundary term on Γ_R, we have near each corner $\boldsymbol{c}_i$ (we write

r for r_i)

$$(\varepsilon\partial_n + 1)(-\varepsilon^{N+1/2}\, w^N) \;=\; \varepsilon^{N+3/2} \sum_{j=1}^{N+1} c^j \partial_n S^{j,j-1}(\frac{r}{\varepsilon}). \tag{7.7}$$

Now, from relations (7.4a) and splittings of the type (3.11) for u^n with $M = N+1-n$, $n = 0, \ldots, N$, we obtain that

$$u^N = u^N_{\mathrm{reg}(1)} + \sum_{j=1}^{N+1} c_i^j(u^{N+1-j})\, S^{j,j-1}. \tag{7.8}$$

Thus

$$\begin{aligned}(\varepsilon\partial_n + 1)\,\tilde r^N_\varepsilon \;=\; & -\varepsilon^{N+1}\partial_n u^N_{\mathrm{reg}(1)}[\log\varepsilon] \\ & + \varepsilon^{N+1}\partial_n \sum_{j=1}^{N+1} \Big(-c_i^j(u^{N+1-j})\, S^{j,j-1}(r) + c^j \sqrt{\varepsilon}\, S^{j,j-1}(\frac{r}{\varepsilon}) \Big).\end{aligned}$$

Thus, we have (7.6) if we choose the coefficients c^j so that the above sum with respect to j is zero: with the help of the homogeneity relations between the $S^{j,j-1}$ (which are of "degree" $1/2$) we obtain that there exist polynomials $P_{j,k}$ of degree $j-k$ such that there holds

$$\frac{1}{\sqrt{\varepsilon}}\, S^{j,j-1}(\varepsilon R) \;=\; \sum_{k=1}^{j} P_{j,k}[\log\varepsilon]\, S^{k,k-1}(R), \quad \forall\varepsilon > 0,\; R > 0$$

and we set

$$c_i^j(u^0, \ldots, u^{N+1-j})[\log\varepsilon] \;=\; \sum_{\ell=j}^{N+1} P_{\ell,j}[\log\varepsilon]\, c_i^\ell(u^{N+1-\ell}), \tag{7.9}$$

so that we have

$$\sum_{j=1}^{N+1} c_i^j(u^{N+1-j})\, S^{j,j-1}(r) \;=\; \sum_{j=1}^{N+1} c_i^j(u^0, \ldots, u^{N+1-j}) \sqrt{\varepsilon}\, S^{j,j-1}(\frac{r}{\varepsilon}). \tag{7.10}$$

The proof of (7.6) is now complete.

It is clear that, if we define u^{N+1} by (7.4a) for $n = N+1$, the new remainder r^{N+1}_ε satisfies (7.4e) with $N+1$ instead N . The recurrence is proved. The estimates for r^N_ε are deduced from the boundary value problem that it satisfies, like for $N = 1$. ■

Like for $N = 0$ (Proposition 6.3), we can deduce sharp error estimates for r^N_ε , for example, if the data are smooth enough

$$\|r^N_\varepsilon\|_{L^2(\Omega)} \;=\; \mathscr{O}\left(\varepsilon^{N+1}\log\varepsilon\right) \tag{7.11}$$

$$\|r_\varepsilon^N\|_{H^{1+s}(\Omega)} \quad = \quad \mathscr{O}\left(\varepsilon^{N+\frac{1}{2}-s}\right) \qquad \text{for } s \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \tag{7.12}$$

(7.13)

8 HIGHER NORM ESTIMATES

The expansion obtained in the previous section can be used to get estimates in higher order Sobolev norms. For this, however, the terms in the expansion will have to be rearranged in different ways depending on the desired norm. As the regularity increases, one encounters two different kinds of thresholds: The first one is associated with the singularities of the mixed Dirichlet-Neumann problem $(\mathcal{P}_0)$ which contain half-integer powers of r. This occurs at half-integer Sobolev indices. The second one corresponds to the singularities of the mixed Robin-Neumann problem $(\mathcal{P}_\varepsilon)$ which contain integer powers and logarithms of r. This occurs at integer Sobolev indices.

Thus in a first step, we consider Sobolev spaces H^{1+s} with $s \in (\frac{1}{2}, 1)$. For $\varepsilon > 0$, u_ε belongs to this space, whereas u^0 and therefore $r_\varepsilon = u_\varepsilon - u^0$ contain the first singular function S^1 and have to be decomposed into regular and singular parts. Here as throughout this whole section, we assume that the right sides f and g are sufficiently smooth. For simplicity and without restriction of generality, we assume that $f = 0$.

We first extend the basic estimate (4.14) in Corollary 4.4 to this range.

Proposition 8.1

$$\forall s \in \left(-\tfrac{1}{2}, 1\right),\ t < \tfrac{1}{2},\ s-1 \le t \le s: \quad \|u_\varepsilon\|_{H^{1+s}(\Omega)} \ \le\ C\,\varepsilon^{t-s}\,\|g\|_{\mathcal{H}^{\frac{1}{2}+t}(\Gamma_R)}\,. \tag{8.1}$$

Proof. Let $s \in (\frac{1}{2}, 1)$. We write the Robin boundary condition of $(\mathcal{P}_\varepsilon)$ as

$$\partial_n u_\varepsilon = \varepsilon^{-1}(g - u_\varepsilon) \quad \text{on } \Gamma_R. \tag{8.2}$$

The regularity of the Neumann problem gives an estimate

$$\|u_\varepsilon\|_{H^{1+s}(\Omega)} \ \le\ C\left(\varepsilon^{-1}\|r_\varepsilon\big|_{\Gamma_R}\|_{\mathcal{H}^{-\frac{1}{2}+s}(\Gamma_R)} + \|u_\varepsilon\|_{\mathcal{H}^{-\frac{1}{2}+s}(\Gamma_R)}\right).$$

If we combine this with (4.13):

$$\|r_\varepsilon\big|_{\Gamma_R}\|_{\mathcal{H}^{-\frac{1}{2}+s}(\Gamma_R)} \ \le\ C\,\varepsilon^{t-s+1}\,\|g\|_{\mathcal{H}^{\frac{1}{2}+t}(\Gamma_R)}\,,$$

we arrive at (8.1). ■

Let us recall the definition of $\tilde{r}_\varepsilon$ from (6.4) :

$$\tilde{r}_\varepsilon = u_\varepsilon - \sqrt{\varepsilon}\,w^0(\varepsilon) - u^0 = u_\varepsilon - \sqrt{\varepsilon}\sum_{i=1}^{2} c_i^1(u^0)\,\chi(r_i)\,K^1(\tfrac{r_i}{\varepsilon}, \theta_i) - u^0_{\mathrm{reg}(1)}\,. \tag{8.3}$$

Here $u^0_{\mathrm{reg}(1)}$ has $H^{5/2-\delta}$ regularity for any $\delta > 0$, whereas the other terms in the second decomposition have the regularity of the solutions of $(\mathcal{P}_\varepsilon)$. Thus the following estimate can be interpreted as an explanation of how the regular part $u^0_{\mathrm{reg}(1)}$ of u^0 is approximated by $u_\varepsilon - \sqrt{\varepsilon}\sum_{i=1}^2 c_i^1(u^0)\,\chi(r_i)\,K^1(\frac{r_i}{\varepsilon},\theta_i)$.

Proposition 8.2

$$\forall s \in \left(\tfrac{1}{2}, 1\right),\ s \le t < \tfrac{3}{2}: \quad \|\tilde{r}_\varepsilon\|_{H^{1+s}(\Omega)} \ \le\ C\,\varepsilon^{t-s}\,\|g\|_{H^{\frac{1}{2}+t}(\Gamma_R)} \tag{8.4}$$

Proof. From the boundary value problem (6.5) satisfied by $\tilde{r}_\varepsilon$ we see that $\tilde{r}_\varepsilon$ is composed of a term of order $\mathscr{O}(\varepsilon\log\varepsilon)$, due to Lemma 6.1, and a solution of $(\mathcal{P}_\varepsilon)$ with g replaced by $-\varepsilon\partial_n u^0_{\mathrm{reg}(1)}\Big|_{\Gamma_R}$. We can therefore use Proposition 8.1 to obtain

$$\|\tilde{r}_\varepsilon\|_{H^{1+s}(\Omega)} \ \le\ C\,\varepsilon^{(t-1)-s}\,\|\varepsilon\partial_n u^0_{\mathrm{reg}(1)}\Big|_{\Gamma_R}\|_{\mathcal{H}^{-\frac{1}{2}+t}(\Gamma_R)}\,.$$

If we combine this with the regularity of the problem $(\mathcal{P}_0)$ as given in (3.5) (with $M = 1$, $s = t-1$):

$$\|\partial_n u^0_{\mathrm{reg}(1)}\Big|_{\Gamma_R}\|_{\mathcal{H}^{-\frac{1}{2}+t}(\Gamma_R)} \ \le\ C\,\|g\|_{H^{\frac{1}{2}+t}(\Gamma_R)}\,,$$

we arrive at (8.4) ■

In order to obtain an approximation with higher powers of ε but still in the same range of regularity, we use the complete expansion (7.2) and recall the definition for $N \in \mathbb{N}$

$$\tilde{r}^N_\varepsilon = u_\varepsilon - \sum_{n=0}^N \varepsilon^n\,u^n[\log\varepsilon] - \sqrt{\varepsilon}\sum_{n=0}^N \varepsilon^n\,w^n(\varepsilon)\,. \tag{8.5}$$

Note that this differs from r^N_ε only by the last term $\varepsilon^{N+\frac{1}{2}}w^N$.

Proposition 8.3

$$\forall s \in \left(\tfrac{1}{2}, 1\right),\ s \le t < \tfrac{3}{2}: \quad \|\tilde{r}^N_\varepsilon\|_{H^{1+s}(\Omega)} \ \le\ C\,\varepsilon^{N+t-s}\,\|g\|_{H^{\frac{1}{2}+t+N}(\Gamma_R)} \tag{8.6}$$

Proof. The boundary value problem (7.6) satisfied by $\tilde{r}^N_\varepsilon$ differs from the problem (7.4e) satisfied by r^N_ε by the right hand sides: In the interior, a term of order $\mathscr{O}\left(\varepsilon^{N+1}\log^{N+1}\varepsilon\right)$ is added, and on Γ_R, $\partial_n u^n$ is replaced by $\partial_n u^n_{\mathrm{reg}(1)}$. The proof then proceeds as for Proposition 8.2. ■

Remark 8.4 The definition of $\tilde{r}^N_\varepsilon$ can be written as

$$\tilde{r}^N_\varepsilon = u_\varepsilon - \sum_{n=0}^N \varepsilon^n\,u^n_{\mathrm{reg}(1)}[\log\varepsilon] - \sqrt{\varepsilon}\sum_{n=0}^N \varepsilon^n\,\tilde{w}^n(\varepsilon)\,, \tag{8.7}$$

where now the corner layer terms $\tilde{w}^n(\varepsilon)$ are no longer decaying as $(r/\varepsilon)^{-1/2}$, but growing as $(r/\varepsilon)^{1/2}$. They give the approximation of the singular parts $u^n - u^n_{\mathrm{reg}(1)}$. We have the explicit formula, compare (7.10), (7.4b) and (7.1),

$$\tilde{w}^n(\varepsilon) = \sum_{i=1}^{2} \chi(r_i) \left[\sum_{j=1}^{n+1} c_i^j(u^0, \dots, u^{n+1-j})[\log \varepsilon]\, \tilde{Y}^j(\frac{r_i}{\varepsilon}, \theta_i) \right] \tag{8.8}$$

with

$$\tilde{Y}^j := K^j - \sum_{l=0}^{j-2} S^{j,l} \tag{8.9}$$

Thus, compared to Y^j, one uses one term less in the asymptotic expansion of K^j at infinity. ∎

The second threshold appears at H^2 regularity due to the singularities of solutions of problem $(\mathcal{P}_\varepsilon)$. We will have to decompose u_ε into a regular and singular part and obtain estimates for the convergence of both. This decomposition is based on the well-known regularity of the Neumann problem with discontinuous data ("mixed Neumann-Neumann problem", so to say).

Lemma 8.5 *Let $M \geq 1$, $s \in (0,1)$ and $g \in H^{-\frac{1}{2}+M+s}(\Gamma_R)$ be given. Let $u \in H^1(\Omega)$ be solution of the boundary value problem*

$$\begin{cases} \Delta u &= 0 \quad \text{in } \Omega \\ \partial_n u &= 0 \quad \text{on } \Gamma_N \\ \partial_n u &= g \quad \text{on } \Gamma_R \,. \end{cases} \tag{8.10}$$

Then u admits a splitting

$$u = u_{\mathrm{reg,Neu}(M)} + \sum_{i=1}^{2} \chi(r_i) \sum_{j=1}^{M} \gamma_i^j(u)\, \Sigma^j(r_i, \theta_i)\,, \tag{8.11}$$

where the singular functions have the form

$$\Sigma^j(r,\theta) = \frac{1}{\pi}\, r^j \Big((\theta - \pi)\sin(j\theta) - \log r \cos(j\theta) \Big)\,. \tag{8.12}$$

The coefficients $\gamma_i^j(u)$ are local: They depend only on the Taylor expansion of g at $\boldsymbol{c}_i$. In particular,

$$\gamma_i^1(u) = g(\boldsymbol{c}_i).$$

There is an estimate

$$\|u_{\mathrm{reg,Neu}(M)}\|_{H^{1+s+M}(\Omega)} + \sum_{i=1}^{2} \sum_{j=1}^{M} |\gamma_i^j(u)| \;\leq\; C \left(\|g\|_{H^{-\frac{1}{2}+s+M}(\Gamma_R)} + \|u\|_{L^2(\Omega)} \right). \tag{8.13}$$

If we decompose u_ε according to this lemma:

$$u_\varepsilon = (u_\varepsilon)_{\mathrm{reg,Neu}(M)} + \sum_{i=1}^{2} \chi(r_i) \sum_{j=1}^{M} \gamma_i^j(u_\varepsilon)\, \Sigma^j(r_i,\theta_i)\,, \tag{8.14}$$

our expansion allows us to describe the asymptotics as $\varepsilon \to 0$ both for the regular part $(u_\varepsilon)_{\mathrm{reg,Neu}(M)}$ and for the coefficients $\gamma_i^j(u_\varepsilon)$. We shall restrict ourselves to $M = 1$ and describe the asymptotics of the leading coefficient $\gamma_i^1(u_\varepsilon)$.

Theorem 8.6 *Let $N \in \mathbb{N}$ and $g \in H^{N+1+\delta}(\Gamma_R)$ for some $\delta > 0$. Then $\gamma_i^1(u_\varepsilon)$ has an expansion as $\varepsilon \to 0$*

$$\gamma_i^1(u_\varepsilon) = \varepsilon^{-\frac12} \sum_{n=0}^{N} \varepsilon^n \gamma_i^{1,n}[\log\varepsilon] + \mathscr{O}\left(\varepsilon^N\right)\,. \tag{8.15}$$

The coefficients $\gamma_i^{1,n}[\log\varepsilon]$ in this expansion have the explicit expression, compare (7.4b)*,*

$$\gamma_i^{1,n}[\log\varepsilon] = -w^n(\varepsilon)(\boldsymbol{c}_i) = -\sum_{j=1}^{n+1} c_i^j(u^0,\dots,u^{n+1-j})[\log\varepsilon]\, K^j(0)\,. \tag{8.16}$$

In particular, the leading term in $\gamma_i^1(u_\varepsilon)$ is

$$-\,\varepsilon^{-\frac12}\, c_i^1(u^0)\, K^1(0)\,.$$

Proof. Writing the problem $(\mathcal{P}_\varepsilon)$ once more as Neumann problem as in (8.2), we obtain

$$\gamma_i^1(u_\varepsilon) = \partial_n u_\varepsilon(\boldsymbol{c}_i) = \varepsilon^{-1}(g-u_\varepsilon)(\boldsymbol{c}_i)\,.$$

Now we use the expansion (7.2) of Theorem 7.1 up to $N+1$ and note that $u^0(\boldsymbol{c}_i) = g(\boldsymbol{c}_i)$, whereas $u^n(\boldsymbol{c}_i) = 0$ for $n \geq 1$. In fact, $u^n = -\partial_n u^{n-1}_{\mathrm{reg}(1)}$ on Γ_R. This function is continuous on Γ and zero on Γ_N, hence its value in $\boldsymbol{c}_i$ is zero. Thus we find the expansion (8.15) with a remainder term estimated by

$$|\varepsilon^{-1} r_\varepsilon^{N+1}(\boldsymbol{c}_i)| \;\leq\; C\,\varepsilon^{-1} \|r_\varepsilon^{N+1}\|_{H^{1+\delta}(\Omega)} = \varepsilon^{-1}\mathscr{O}\left(\varepsilon^{N+1}\right) = \mathscr{O}\left(\varepsilon^N\right)\,.$$

The formula (8.16) is obtained from the definition (7.1), (7.4b) of w^n by noting that $S^{j,\ell}(0) = 0$ there. ∎

Along the lines of these last propositions, the interested reader should now be able to obtain estimates in Sobolev norms of any desired regularity. For example, for $H^s(\Omega)$ with $2 < s < 5/2$, the object to estimate is $(\tilde r_\varepsilon)_{\mathrm{reg,Neu}(1)}$, whereas for estimates with $5/2 < s < 3$, new corner layers derived from the functions K^j and their asymptotics at infinity, similarly to $\tilde w^n$, but with a behavior in $(r/\varepsilon)^{3/2}$, have to be introduced, and functions $u^n_{\mathrm{reg}(2)}$ instead of $u^n_{\mathrm{reg}(1)}$ have to be used.

9 CONCLUDING REMARKS

When considering the asymptotics (7.2), we can see that the different objects are not canonical (because of the cut-off functions $\chi(r_i)$). In contrast, the complete inner and outer expansions, although not valid one the whole domain Ω , are uniquely determined, whereas some parts of the terms of the composite expansion can be attached to either of the two scales. In particular, the outer expansion has the form

$$\sum_{n\in\mathbb{N}}\sum_{p=0}^{n} \varepsilon^n \log^p \varepsilon \, v^{n,p}(x). \tag{9.1}$$

In particular, $v^{0,0} = u^0$ and $v^{1,1}$ is the sum of u^{11} and the first term of the asymptotics of $\sqrt{\varepsilon}\, w^0$ at infinity, i.e. $-\sum_i \chi(r_i)\, r_i^{-1/2} \sin\theta_i/2$. Considering equation (6.13) and noting that $f^{011} = \Delta\Big(\sum_i \chi(r_i)\, r_i^{-1/2} \sin\theta_i/2\Big)$, we obtain the "canonical" expression

$$v^{1,1} = \sum_{i=1}^{2} c_i^1\, X_i^1; \tag{9.2}$$

the coefficients c_i^1 are the coefficients of the first singularities S_i^1 of u^0 , like in (6.1) and the functions X_i^1 are the (unique) solutions of the homogeneous mixed problem $(\mathcal{P}_0)$ which are equal to $\chi(r_i)\, r_i^{-1/2} \sin\theta_i/2$ modulo $H^1(\Omega)$ respectively.

It is also interesting to know how the results of Theorem 7.1 can be extended to other geometries of the domain Ω .

If Ω is still smooth, but if we relax the hypothesis about the geometry in the neighborhood of the transition points $\boldsymbol{c}_i$ for the boundary conditions, allowing that the boundary is any regular curve in the neighborhood of these points, then the asymptotics of u_ε has a similar form, involving integral powers of ε and $\log\varepsilon$ for the outer expansion terms and half-integers $n+1/2$ for the inner expansion terms (with possibly higher degrees in $\log\varepsilon$).

The proof is still more technical. For example, one can use a change of variables to flatten the boundary in the neighborhood of $\boldsymbol{c}_i$, and one obtains a new operator $L(x,\partial_x) = \Delta + A$, instead of Δ , with $A = \sum_{|\alpha|=1} a_\alpha \partial_\alpha$ a first order operator with smooth coefficients. The construction of the corner layer model terms K^j and Y^j involves the operator $\varepsilon^{-2} L(\varepsilon x, \varepsilon^{-1}\partial_x)$ on the half plane Π . The corresponding boundary value problem is a *regular perturbation* of problem $(\mathcal{P}_1)$ introduced at the beginning of section 5.

Another interesting generalization is the case when the domain Ω is polygonal. For a corner situated in the Neumann region Γ_N , there is no special effect due to the singular perturbation.

If we have a corner at a transition point $\boldsymbol{c}$, let us suppose that Ω coincides with a sector Π_ω of opening ω in the neighborhood of $\boldsymbol{c}$. Then we can still construct an asymptotics for u_ε in a similar way, but taking account of the following changes (for the sake of simplicity, we assume that π/ω is not rational):

1. The set of the singular exponents λ of the mixed Dirichlet-Neumann problem is
$$\Lambda = \{\lambda \mid \quad \lambda = \frac{k\pi}{\omega} + \frac{\pi}{2\omega}, \quad k \in \mathbb{Z}\}$$
and the singular functions $S^\lambda = r^\lambda \sin \lambda\theta$ are associated to functions $S^{\lambda,p}$ which are homogeneous of degree $\lambda - p$, like in Lemma 3.3 (if π/ω is rational, there are some logarithmic terms);
2. The homogeneous solutions K^λ of problem $(\mathcal{P}_1)$ on Π_ω can be defined so that for any $\lambda \in \Lambda$, $\lambda > 0$:
$$Y^\lambda := K^\lambda - \sum_{p=0}^{[\lambda]} S^{\lambda,p} = \mathscr{O}\left(r^{\lambda-1-[\lambda]}\right)$$
with $[\lambda]$ the integral part of λ.
3. The regular part of order 1, $u_{\mathrm{reg}(1)}$ is obtained by subtracting the singular parts with exponent $\nu \in (0,1]$.

Then, instead of (7.4b), we obtain in a neighborhood of $\boldsymbol{c}$, an asymptotics of the form
$$u_\varepsilon(x) \sim \sum_{n\in\mathbb{N}} \varepsilon^n u^n(x) + \sum_{\substack{\lambda\in\Lambda \\ \lambda>0}} \sum_{p\in\mathbb{N}} \varepsilon^{\lambda+p} c^{\lambda,p}\, Y^\lambda(\frac{x}{\varepsilon}). \tag{9.3}$$

If we have a corner inside the Robin region Γ_N, we have a similar analysis: the model profiles K^λ are now constructed from the homogeneous Dirichlet problem on the sector Π_ω and the exponents λ are those of the Dirichlet problem $k\pi/\omega$.

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