Weighted Analytic Regularity in Polyhedra

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Abstract

We explain a simple strategy to establish analytic regularity for solutions of second order linear elliptic boundary value problems. The abstract framework presented here helps to understand the proof of analytic regularity in polyhedral domains given in the authors' paper in *Math. Models Methods Appl. Sci.* **22 (8)** (2012). We illustrate this strategy by considering problems set in smooth domains, in corner domains and in polyhedra.

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1. Introduction

Solutions of elliptic boundary value problems with analytic data are analytic. This classical result has played an important role in the analysis of harmonic functions since Cauchy's time and in the analysis of more general elliptic problems since Hilbert formulated it as his 19th problem. Hilbert's problem for second order nonlinear problems in variational form in two variables was solved by Bernstein in 1904 [8]. After this, many techniques were developed for proving analyticity, culminating in the 1957 paper [26] by Morrey and Nirenberg on linear problems, where Agmon's elliptic regularity estimates in nested open sets were refined to get Cauchy-type analytic estimates, both in the interior of a domain and near analytic parts of its boundary.

Analyticity means exponentially fast approximation by polynomials, and therefore it plays an important role in numerical analysis, too. Analytic estimates have gained a renewed interest through the development of the p and h-p versions of the finite element method by Babuška and others [7]. In this context, applications often involve boundaries that are not globally analytic, but only piecewise analytic due to the presence of corners and edges, and therefore global elliptic regularity results cannot be used directly.

The way of proving such analytic regularity results (which we call here "type A" for short) is quite technical and often difficult to follow, as can be seen for example from papers by Babuška and Guo [5, 6, 18] devoted to corner domains. In order to tackle polyhedral domains with success, it was necessary to alleviate some difficulties as much as possible. In our paper [14] on type A results for polyhedral domains, we eventually completed a proof that had been missing for a long time. For doing this, we relied on already known basic regularity results of low order (called here "type B") and proved what remains after that. In this paper, we present this new approach in a more abstract and systematic way, which shows how to attain this aim with as little effort as possible. The program consists in dividing the proof into two fundamental steps.

The first step involves results of type B, namely basic regularity results that exist for many boundary value problems in domains with different regularity properties. Such results are often well known, in some cases since a long time (for example [1] in 1959 for smooth domains and [23] in 1967 corner domains), in others more recently (polyhedral domains [16, 25]).

The second step consists in proving "regularity shift" results, which we call "type S". In our context, they involve the proof of Cauchy type estimates for the derivatives of the solution at any order.

The abstract framework behind this approach can be summarized as follows:

$$\boxed{\text{Type B} + \text{Type S} \Longrightarrow \text{Type A}}$$

We illustrate this strategy in the context of linear elliptic boundary value problems. We first recall some results for the case of domains with analytic boundary that are well known but can be proved with the help of our program. Second we present some analytic regularity results for corner domains, extending results obtained by Babuška and others for polygonal domains [5, 6, 18, 21]. Finally we state recent results for polyhedral domains that we proved in [14], using anisotropic weighted Sobolev spaces introduced in [19, 20].

2. An abstract framework

We study the question of regularity of the solutions of elliptic boundary value problems. More precisely, consider a boundary value problem, written in compact form with a linear operator $\mathbb{P}$ as

$$\mathbb{P}\boldsymbol{u} = \boldsymbol{q}$$

where $\boldsymbol{q}$ may include interior, boundary, or interface data.

A regularity statement takes the form

$$\boxed{\boldsymbol{u} \in \mathbb{U}_{\mathsf{base}} \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Q}_{\mathsf{data}} \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{U}_{\mathsf{sol}}}$$

The ideal situation can be summarized as follows:

- $\mathbb{U}_{\mathsf{base}}$ is a space where existence of solutions is known,
- $\mathbb{U}_{\mathsf{sol}}$ is optimal in the sense that $\mathbb{P}$ is bounded from $\mathbb{U}_{\mathsf{sol}}$ into $\mathbb{Q}_{\mathsf{data}}$,
- if $\mathbb{Q}_{\mathsf{data}}$ is a space of piecewise analytic data, $\mathbb{U}_{\mathsf{sol}}$ is a space of piecewise analytic solutions.

In the literature, three types of relevant theorems can be found:

Type C *(Existence of solutions in a basic space $\mathbb{V}$) This is typically the consequence of a coercive variational formulation or, more generally, of a Fredholm alternative.*

Type B *(Basic regularity)*

$$\boxed{\boldsymbol{u} \in \mathbb{V} \quad \textit{and} \quad \boldsymbol{q} \in \mathbb{Q}^{\mathsf{B}}_{\mathsf{data}} \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{U}^{\mathsf{B}}_{\mathsf{sol}}}$$

for suitable $\mathbb{Q}^{\mathsf{B}}_{\mathsf{data}}$ and where $\mathbb{U}^{\mathsf{B}}_{\mathsf{sol}}$ is a space involving estimates on a finite number of derivatives (e.g. a space of strong solutions).

Type A *(Analytic regularity)*

$$\boxed{\boldsymbol{u} \in \mathbb{V} \quad \textit{and} \quad \boldsymbol{q} \in \mathbb{Q}^{\mathsf{A}}_{\mathsf{data}} \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{U}^{\mathsf{A}}_{\mathsf{sol}}}$$

for suitable $\mathbb{Q}^{\mathsf{A}}_{\mathsf{data}}$ and where $\mathbb{U}^{\mathsf{A}}_{\mathsf{sol}}$ involves estimates on all derivatives with Cauchy-type growth.

As mentioned in the introduction, a fourth type of statement (Type S) plays a fundamental role in our strategy:

Type S *(Regularity Shift)*

$$\boxed{\boldsymbol{u} \in \mathbb{U}^{\mathsf{B}}_{\mathsf{sol}} \quad \textit{and} \quad \boldsymbol{q} \in \mathbb{Q}^{\mathsf{A}}_{\mathsf{data}} \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{U}^{\mathsf{A}}_{\mathsf{sol}}}$$

Our main strategy and idea is to use the scheme

$$\boxed{\text{Type B + Type S} \Longrightarrow \text{Type A}}$$

Hence our remaining task is to find suitable pairs $(\mathbb{U}^{\mathsf{B}}_{\mathsf{sol}}, \mathbb{U}^{\mathsf{A}}_{\mathsf{sol}})$ so that

1. A result of type B is known,
2. We are able to prove corresponding results of type S.

The spaces $\mathbb{U}^{\mathsf{B}}_{\mathsf{sol}}$ and $\mathbb{U}^{\mathsf{A}}_{\mathsf{sol}}$ will be built with the help of a countable set of semi-norms

$$|\cdot|_{\mathbb{X}^\ell}, \quad \ell \in \mathbb{N}_0 .$$

Typically, the semi-norm $|\cdot|_{\mathbb{X}^\ell}$ is a norm on derivatives ∂^α of order $|\alpha| = \ell$. With this sequence a full family of spaces can be associated in a natural way:

1. Finite regularity spaces for any natural number k

   $$\mathbb{X}^k = \Big\{\boldsymbol{u} : \ |\boldsymbol{u}|_{\mathbb{X}^\ell} < \infty, \ \forall \ell = 0, \dots, k\Big\}$$

   associated with the norm $\|\boldsymbol{u}\|_{\mathbb{X}^k} = \max_{\ell=0,\dots,k} |\boldsymbol{u}|_{\mathbb{X}^\ell}$,
2. Infinite regularity class

   $$\mathbb{X}^\infty = \Big\{\boldsymbol{u} : \ |\boldsymbol{u}|_{\mathbb{X}^\ell} < \infty, \ \forall \ell \in \mathbb{N}_0\Big\},$$

3. Analytic class

   $$\mathbb{X}^\varpi = \left\{\boldsymbol{u} \in \mathbb{X}^\infty : \ \sup_{\ell \in \mathbb{N}_1} \left(\frac{1}{\ell!} |\boldsymbol{u}|_{\mathbb{X}^\ell}\right)^{1/\ell} < \infty\right\}.$$

Note that Gevrey classes of order $s \geq 1$ could also be associated with the same semi-norms by replacing $\ell!$ by $(\ell!)^s$ in the definition of $\mathbb{X}^\varpi$.

Similar definitions for the spaces of the right hand sides $\boldsymbol{q}$ can be made. Denote the corresponding semi-norms by $|\cdot|_{\mathbb{Y}^\ell}$ and the corresponding spaces by $\mathbb{Y}^k$, $\mathbb{Y}^\infty$ and $\mathbb{Y}^\varpi$. Regularity results of types B and A associated with these families of semi-norms can be stated as follows:

Type B *(Basic regularity of order $m \in \mathbb{N}_1$)*

$$\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Y}^m \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^m}$$

with the estimate

$$\|\boldsymbol{u}\|_{\mathbb{X}^m} \le C\big(\|\mathbb{P}\boldsymbol{u}\|_{\mathbb{Y}^m} + \|\boldsymbol{u}\|_{\mathbb{V}}\big), \quad \forall \boldsymbol{u} \in \mathbb{X}^m$$

for some $C > 0$ independent of $\boldsymbol{u}$.

Type A *(Analytic regularity)*

$$\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Y}^\varpi \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^\varpi}$$

Results of type S associated with such families of semi-norms are of two sorts.

Type S standard *(Standard regularity shift, basic order $m \in \mathbb{N}_1$) For all $k > m$*

$$\boxed{\boldsymbol{u} \in \mathbb{X}^m \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Y}^k \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^k}$$

with the estimates

$$\|\boldsymbol{u}\|_{\mathbb{X}^k} \le C\big(\|\mathbb{P}\boldsymbol{u}\|_{\mathbb{Y}^k} + \|\boldsymbol{u}\|_{\mathbb{X}^m}\big), \quad \forall \boldsymbol{u} \in \mathbb{X}^m,$$

for some $C > 0$ independent of $\boldsymbol{u}$ but that can depend on k.

Type S with Cauchy estimates *(Regularity shift with Cauchy estimates, basic order $m \in \mathbb{N}_1$) For all $k > m$*

$$\boxed{\boldsymbol{u} \in \mathbb{X}^m \quad \text{and} \quad \boldsymbol{q} \in \mathbb{Y}^k \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^k}$$

with the estimates

$$\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k} \le A^{k+1}\Big(\sum_{\ell=0}^{k} \frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + \|\boldsymbol{u}\|_{\mathbb{X}^m}\Big), \quad \forall \boldsymbol{u} \in \mathbb{X}^m \tag{1}$$

with a constant $A > 0$ independent of k and of $\boldsymbol{u}$.

A variant of Type S—later used with anisotropic norms—allows a loss of two derivatives: For all $k > m$

$$\boxed{\boldsymbol{u} \in \mathbb{X}^m \quad \textit{and} \quad \boldsymbol{q} \in \mathbb{Y}^{k+2} \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^k}$$

with the estimates

$$\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k} \le A^{k+1}\Big(\sum_{\ell=0}^{k+2}\frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + \|\boldsymbol{u}\|_{\mathbb{X}^m}\Big), \quad \forall \boldsymbol{u} \in \mathbb{X}^m \tag{2}$$

with a constant $A > 0$ independent of k and of $\boldsymbol{u}$.

Let us finish this section by a theorem summarizing our strategy in this abstract framework.

Theorem 2.1. *If there exists $m \in \mathbb{N}_1$ such that*

1. *Type B holds for the order m,*
2. *Type S with Cauchy estimates holds with the basic order m,*

then Type A holds, namely

$$\boxed{\boldsymbol{u} \in \mathbb{V} \quad \textit{and} \quad \boldsymbol{q} \in \mathbb{Y}^\varpi \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^\varpi}$$

Proof. Choose an arbitrary $k \in \mathbb{N}_1$ such that $k > m$. Since for all $Y = (y_0, \dots, y_k) \in \mathbb{R}^{k+1}$ we have

$$\sum_{\ell=0}^{k}|y_\ell| \le (k+1)\max_{\ell=0,\dots,k}|y_\ell|,$$

and $k+1 \le e^{k+1}$, the Cauchy estimate (1) implies with $A_1 = eA$

$$\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k} \le A_1^{k+1}\Big(\max_{\ell=0,\dots,k}\frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + \|\boldsymbol{u}\|_{\mathbb{X}^m}\Big).$$

By taking the kth root of this estimate, we obtain

$$\begin{aligned}\Big(\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k}\Big)^{1/k} &\le A_1^{1+1/k}\Big(\max_{\ell=0,\dots,k}\frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + \|\boldsymbol{u}\|_{\mathbb{X}^m}\Big)^{1/k}\\ &\le A_2\Big\{\max_{\ell=0,\dots,k}\Big(\frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell}\Big)^{1/k} + \|\boldsymbol{u}\|_{\mathbb{X}^m}^{1/k}\Big\},\end{aligned}$$

with $A_2 = \max\{A_1, A_1^2\}$, using $(a+b)^{\frac{1}{k}} \le a^{\frac{1}{k}} + b^{\frac{1}{k}}$ for all positive real numbers a, b.

Now by assumption on the data, there exists $C \ge 0$ such that

$$|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^0} \le C \quad \text{and} \quad \frac{1}{\ell!} |\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} \le C^\ell, \quad \forall \ell \in \mathbb{N}_1.$$

We can assume that $C \ge 1$ and hence $C^{\frac{\ell}{k}} \le C$ for $\ell \le k$. We arrive at

$$\left(\frac{1}{k!} |\boldsymbol{u}|_{\mathbb{X}^k}\right)^{1/k} \le A_2 \left(C + \|\boldsymbol{u}\|_{\mathbb{X}^m}^{1/k}\right) \le A_2 \left(C + \max\{1, \|\boldsymbol{u}\|_{\mathbb{X}^m}\}\right).$$

This proves that $\boldsymbol{u} \in \mathbb{X}^\varpi$.

If we use the variant (2) instead of (1), we have to take the maximum over $\ell \in \{0, \ldots, k+2\}$ in the right hand sides, and we conclude with the argument that $C^{\frac{\ell}{k}} \le C^3$ for $\ell \le k+2$. ■

In conclusion, it suffices to realize the program Type B + Type S to obtain results of Type A for any desired situation.

3. Smooth domains

We consider the following situation: Ω is a smooth bounded domain with an analytic boundary. We denote by $\partial_{\boldsymbol{s}}\Omega$ with $\boldsymbol{s} \in \mathscr{S}$, the connected components of $\partial\Omega$ (obviously $\mathscr{S}$ is a finite set). The system $\mathbb{P}$ corresponds to a second order linear elliptic system with boundary conditions of Dirichlet or Neumann type, see [13] for the details. More precisely $\mathbb{P} = (L, T_{\boldsymbol{s}}, D_{\boldsymbol{s}})$ are operators with analytic coefficients such that

- L is the interior operator (can be a square system),
- $T_{\boldsymbol{s}}$ is a boundary operator of order 1, for $\boldsymbol{s} \in \mathscr{S}_N$,
- $D_{\boldsymbol{s}}$ is boundary operator of order 0, for $\boldsymbol{s} \in \mathscr{S}_D$,

when $\mathscr{S} = \mathscr{S}_N \cup \mathscr{S}_D$ is a fixed splitting of $\mathscr{S}$. So the equation $\mathbb{P}\boldsymbol{u} = \boldsymbol{q}$ takes the form of the mixed boundary value problem

$$\begin{cases} L\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ T_{\boldsymbol{s}}\boldsymbol{u} = \boldsymbol{g}_{\boldsymbol{s}} & \text{on } \partial_{\boldsymbol{s}}\Omega, \ \boldsymbol{s} \in \mathscr{S}_N, \\ D_{\boldsymbol{s}}\boldsymbol{u} = \boldsymbol{h}_{\boldsymbol{s}} & \text{on } \partial_{\boldsymbol{s}}\Omega, \ \boldsymbol{s} \in \mathscr{S}_D. \end{cases}$$

Theorems of Type C, B, and A have been known for more than 50 years in the framework of Sobolev spaces: Here the norms of solutions and right hand sides are defined as

$$|\boldsymbol{u}|_{\mathbb{X}^\ell} = \sum_{|\alpha|=\ell} \|\partial_{\boldsymbol{x}}^\alpha \boldsymbol{u}\|_{L^2(\Omega)},$$

$$|\boldsymbol{q}|_{\mathbb{Y}^\ell} = \sum_{|\alpha|=\ell-2} \|\partial_{\boldsymbol{x}}^\alpha \boldsymbol{f}\|_{L^2(\Omega)} + \sum_{\substack{\boldsymbol{s}\in\mathscr{S}_N \\ |\alpha|=\ell-2}} \|\partial_{\boldsymbol{x}}^\alpha \boldsymbol{g}_{\boldsymbol{s}}\|_{H^{\frac12}(\partial_{\boldsymbol{s}}\Omega)} + \sum_{\substack{\boldsymbol{s}\in\mathscr{S}_D \\ |\alpha|=\ell-1}} \|\partial_{\boldsymbol{x}}^\alpha \boldsymbol{h}_{\boldsymbol{s}}\|_{H^{\frac12}(\partial_{\boldsymbol{s}}\Omega)}.$$

We refer to the papers [26, 1, 2] for the following results.

Theorem 3.1. *(Type C)* [1, 2] $\mathbb{P} : \mathbb{X}^m \to \mathbb{Y}^m$ *is a Fredholm operator for any* $m \geq 2$*.*

Theorem 3.2. *(Type B)* [1, 2] *For any* $k > 2$*, one has*

$$\boxed{\boldsymbol{u} \in \mathbb{X}^2 \quad \textit{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^k \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^k}$$

Theorem 3.3. *(Type A)* [26]

$$\boxed{\boldsymbol{u} \in \mathbb{X}^2 \quad \textit{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^\varpi \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^\varpi}$$

With respect to these three classical theorems, it seems that nothing else could be added by our strategy. Nevertheless it is not easy to find regularity shift results with Cauchy-type estimates in the literature. We have proved such estimates in smooth domains by using nested open sets on model problems, like in the Morrey–Nirenberg proof [26] of analytic regularity, and a Faà di Bruno formula for local coordinate transformations. This proof, which clarifies some older proofs, is given in detail in [13] (Theorem 2.7.1).

Theorem 3.4. *(Type S)* [13] *With the assumptions of the present section, there exists* $A > 0$ *such that for any* $k \geq 2$ *and* $\boldsymbol{u} \in \mathbb{X}^2$*, we have*

$$\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k} \leq A^{k+1}\Big(\sum_{\ell=0}^{k} \frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + \|\boldsymbol{u}\|_{\mathbb{X}^2}\Big).$$

Let us notice that if the problem associated with $\mathbb{P}$ admits a coercive variational formulation on a subspace $\mathbb{V}$ of $H^1(\Omega)$, the corresponding theorems exist and have a more canonical form:

C Existence in $\mathbb{V}$ is direct,

B Basic regularity for $m = 2$: If $\mathbb{P}\boldsymbol{u} \in \mathbb{Y}^2$, then $\boldsymbol{u} \in \mathbb{X}^2$, which means that the variational solution is a strong solution. The proof is based on the well-known method of tangential difference quotients of Nirenberg,

A, S All estimates involve the norm $\|\boldsymbol{u}\|_{H^1(\Omega)}$ in the right-hand side.

4. Corner domains

Here we consider an analytic corner domain Ω with corner set $\mathscr{C}$, in the sense that it is a bounded domain of $\mathbb{R}^d$, $d \geq 2$, such that its boundary is analytic except at a finite number of points (the corner set), and for each corner $\boldsymbol{c}$ there exist neighborhoods $\mathcal{U}$ of $\boldsymbol{c}$ and $\mathcal{U}'$ of $\boldsymbol{0}$, a regular cone $K_{\boldsymbol{c}}$ and an analytic bijective map F from $\Omega \cap \mathcal{U}$ to $K_{\boldsymbol{c}} \cap \mathcal{U}'$. In 2D, such domains are simply piecewise analytic and are called polygonal domains, and if the boundary is piecewise flat, the domain is a polygon.

For the results in this section, we consider a second order elliptic boundary value system $\mathbb{P} = (L, T_{\boldsymbol{s}}, D_{\boldsymbol{s}})$ with analytic coefficients. To simplify our presentation, we assume that it corresponds to a coercive problem on a closed subspace $\mathbb{V}$ of $H^1(\Omega)$ and that the boundary data are zero (in other words $\boldsymbol{q} \equiv \boldsymbol{f}$).

Theorems of Type C are directly based on the Lax-Milgram lemma.

Theorems of Type B and "S standard" are well known, starting with the pioneering paper of Kondrat'ev [23]. Such results use weighted Sobolev spaces of the following type:

$$|\boldsymbol{u}|_{\mathbb{X}^\ell} = \sum_{|\alpha|=\ell} \|w_\ell\, \partial_{\boldsymbol{x}}^\alpha \boldsymbol{u}\|_{L^2(\Omega)} \quad \text{and} \quad |\boldsymbol{f}|_{\mathbb{Y}^\ell} = \sum_{|\alpha|=\ell-2} \|w_\ell\, \partial_{\boldsymbol{x}}^\alpha \boldsymbol{f}\|_{L^2(\Omega)}\,,$$

where $w_\ell(\boldsymbol{x})$, $\ell \in \mathbb{N}_0$ is the family of weights of general type

$$w_\ell(\boldsymbol{x}) = r(\boldsymbol{x})^{\ell+\beta}, \quad r(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \mathscr{C}),$$

with a real parameter $\beta \in \mathbb{R}$. So in this situation, the norm of the space $\mathbb{X}^m$ is equivalent to

$$\sum_{|\alpha|\leq m} \|r(\boldsymbol{x})^{|\alpha|+\beta}\, \partial_{\boldsymbol{x}}^\alpha \boldsymbol{u}\|_{L^2(\Omega)}\,.$$

4.1. An example: Results of type B for the Laplacian in a polygon

For the presentation of detailed examples of results of type B, let us start with the Laplace equation with Dirichlet conditions in a polygon. Let $\omega = \max_{\boldsymbol{c}\in\mathscr{C}} \omega_{\boldsymbol{c}}$ be the largest opening angle of the polygon Ω ($\omega_{\boldsymbol{c}}$ being the interior opening of Ω at the corner $\boldsymbol{c}$). Let $(r_{\boldsymbol{c}}, \theta_{\boldsymbol{c}})$ be polar coordinates centered at $\boldsymbol{c}$.

Theorem 4.1. *(Type B)* [23] *Let $\boldsymbol{u} \in H^1_0(\Omega)$ be the variational solution of $\Delta\boldsymbol{u} = \boldsymbol{f}$. Let $m \geq 2$ and let β be such that $0 < -\beta - 1 < \frac{\pi}{\omega}$ and $w_\ell = r^{\ell+\beta}$, $0 \leq \ell \leq m$. Then*

$$\boldsymbol{f} \in \mathbb{Y}^m \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^m. \tag{3}$$

Remark 4.2. 1. The condition $0 < -\beta - 1$, i.e. $\beta < -1$ implies that w_1 is unbounded and guarantees that $\mathbb{X}^2$ is compactly embedded in $H^1(\Omega)$.

2. Since the strongest singularity present in the solution $\boldsymbol{u}$

$$\boldsymbol{x} \longmapsto r_{\boldsymbol{c}}^{\pi/\omega} \sin \frac{\pi\theta_{\boldsymbol{c}}}{\omega_{\boldsymbol{c}}}$$

belongs to $\mathbb{X}^m$ for all m if and only if the condition $-\beta - 1 < \frac{\pi}{\omega}$ holds, we directly see that this condition is a necessary condition for (3).

For the Laplace equation with Neumann condition in a polygon, the previous functional setting is unsuitable. The main reason is that for each corner $\boldsymbol{c}$, radial functions $\eta_{\boldsymbol{c}}(r_{\boldsymbol{c}})$ with $\eta_{\boldsymbol{c}} \equiv 1$ near 0 and a sufficiently small support are solution of the Neumann problem in Ω with smooth right hand sides, while they do not belong to $\mathbb{X}^2$ if $0 < -\beta - 1$ because $w_0 = r^\beta$. But this problem disappears as soon as we consider derivatives. Therefore the remedy is to modify the first weights and take

$$w_\ell = r^{\max\{0,\ell+\beta\}} \simeq \min\{1, r^{\ell+\beta}\}, \quad \ell \in \mathbb{N}_0\,.$$

For instance if we choose[1] $\beta = -\frac{3}{2}$, then $w_0 = w_1 = 1$, and $w_\ell = r^{\ell+\beta}$ as before if $\ell \geq 2$. This defines what is called "non-homogeneous norms".

[1]The choice $\beta = -\frac{3}{2}$ satisfies the condition $0 < -\beta - 1 < \frac{\pi}{\omega}$ for any polygon without crack, i.e. when $\omega < 2\pi$.

Theorem 4.3. *(Type B)* [24] *Let* $\boldsymbol{u} \in H^1(\Omega)$ *be the variational solution of the Neumann problem* $\Delta \boldsymbol{u} = \boldsymbol{f}$ *with* $\partial_n \boldsymbol{u} = 0$. *Let* $m \geq 2$ *and let* β *be such that* $0 < -\beta - 1 < \frac{\pi}{\omega}$ *and* $w_\ell = r^{\max\{0,\ell+\beta\}}$, $0 \leq \ell \leq m$. *Then*

$$\boldsymbol{f} \in \mathbb{Y}^m \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^m.$$

4.2. General results of type B for corner domains

In the general setting described at the beginning of this section, that means problems in variational form on corner domains in any dimension, results of type B are also well known. In our terminology, such a result can be written as follows.

Theorem 4.4. *(Type B)* [23, 24] *Let* Ω *be a corner domain in* $\mathbb{R}^d$. *There exists a real number* $b^*(\Omega, \mathbb{P}) > 1 - \frac{d}{2}$ *such that the following holds. Let* $m \geq 2$ *and let* $\beta < -1$ *be such that* $-\beta - \frac{d}{2} < b^*(\Omega, \mathbb{P})$. *Choose the weights* $w_\ell = r^{\max\{0,\ell+\beta\}}$ *(non-homogeneous norms). Then*

$$\boldsymbol{u} \in \mathbb{V} \quad \textit{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^m \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^m$$

Remark 4.5. The weights $w_\ell = r^{\ell+\beta}$ (homogeneous norms) are suitable if $\boldsymbol{u} \in \mathbb{V}$ implies $\frac{\boldsymbol{u}}{r} \in L^2(\Omega)$ (like it was the case for the Dirichlet problem in a polygon). In this case, an analogous statement holds involving a positive number $b(\Omega, \mathbb{P})$, whose optimal value is determined by the Mellin corner spectra $\sigma(\mathfrak{P}_{\boldsymbol{c}})$ of the operator $\mathbb{P}$, and is, in general, different from $b^*(\Omega, \mathbb{P})$. The latter quantity also takes polynomial functions into account, namely via the condition of "injectivity modulo polynomials" of [16].

4.3. First results of type A in polygonal domains

Weighted analytic regularity has first been studied by Babuška and Guo, and proofs were given for some standard boundary value problems in polygonal domains. In particular, they gave complete proofs, based on Morrey's results for smooth domains, for the Laplace equation or the Lamé system with Dirichlet or Neumann conditions. In our language, the result can be written as follows.

Theorem 4.6. *(Type A)* [5, 6, 18] *There exists $\beta \in (-2,-1)$ such that with the weights $w_\ell = r^{\max\{0,\ell+\beta\}}$ there holds*

$$\boxed{\boldsymbol{u} \in \mathbb{V} \quad \textit{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^\varpi \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^\varpi}$$

Remark 4.7. (Exponential convergence) [5, 6, 18] This weighted analytic regularity allows to prove the exponential convergence of the h-p version of the finite element method.

4.4. General results of type S in corner domains

In the general situation described at the beginning of this section, results of type S are available.

Theorem 4.8. *(Type S standard)* [23] *With the homogeneous weights $w_\ell = r^{\ell+\beta}$, we have for all $k \geq 2$ and all $\beta \in \mathbb{R}$,*

$$\boxed{\boldsymbol{u} \in \mathbb{X}^2 \quad \textit{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^k \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^k}$$

with the estimates

$$\|\boldsymbol{u}\|_{\mathbb{X}^k} \leq C\big(\|\mathbb{P}\boldsymbol{u}\|_{\mathbb{Y}^k} + \|\boldsymbol{u}\|_{\mathbb{X}^2}\big),$$

where C is a positive constant that depends on β and k.

In other words for any β, if $r^{|\alpha|+\beta}\boldsymbol{u} \in L^2(\Omega)$ for $|\alpha| \leq 2$, then $r^{|\alpha|+\beta}\boldsymbol{u} \in L^2(\Omega)$ for all $|\alpha| \leq k$ if the right-hand side has the corresponding regularity. This is an unconditional elliptic regularity shift result for corner domains.

In [14] we have proved Type S results with Cauchy-type estimates for homogeneous operators $\mathbb{P}$ with constant coefficients in corner domains.

Theorem 4.9. *(Type S with Cauchy estimates)* [14]

1. *If we use the homogeneous weights $w_\ell = r^{\ell+\beta}$, then for all $\beta \in \mathbb{R}$ there exists $A > 0$ such that for any $k \geq 2$ and $\boldsymbol{u} \in \mathbb{X}^2$*

$$\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k} \leq A^{k+1}\Big(\sum_{\ell=0}^{k}\frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + \|\boldsymbol{u}\|_{\mathbb{X}^1}\Big).$$

2. *If we use the non-homogeneous weights* $w_\ell = r^{\max\{0,\ell+\beta\}}$*, then for all* $\beta \in \mathbb{R}$ *and* $m \geq \max\{-\beta, 2\}$ *there exists* $A > 0$ *such that for any* $k \geq m$ *and* $\boldsymbol{u} \in \mathbb{X}^m$

$$\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k} \leq A^{k+1}\Big(\sum_{\ell=m+1}^{k}\frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + |\boldsymbol{u}|_{\mathbb{X}^m}\Big).$$

Proof. The proof is based on the following three steps

1. Use unweighted estimates (of Cauchy type) in a fixed annulus far from the corner $\boldsymbol{c}$.
2. Use scaling arguments to get scaled estimates on annuli closer to the corner; the weight appears in a natural way.
3. Sum over a dyadic partition of a neighborhood of $\boldsymbol{c}$.

■

As a corollary of the previous results of type B and S, results of type A in corner domains follow in the general case of homogeneous operators $\mathbb{P}$ with constant coefficients *sine dolore* (see [12] for more general situations of operators with lower order terms).

Theorem 4.10. *(Type A)*[14] *Let* Ω *be a corner domain in* $\mathbb{R}^d$*. With the same optimal number* $b^*(\Omega,\mathbb{P})$ *as in Theorem 4.4, if* $\beta < -1$ *is such that* $-\beta - \frac{d}{2} < b^*(\Omega,\mathbb{P})$ *and if we choose the weights* $w_\ell = r^{\max\{0,\ell+\beta\}}$, $\ell \in \mathbb{N}$*, then*

$$\boldsymbol{u} \in \mathbb{V} \quad \textit{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^\varpi \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^\varpi. \tag{4}$$

Remark 4.11. The homogeneous weights $w_\ell = r^{\ell+\beta}$ can be used instead if the implication

$$\boldsymbol{u} \in \mathbb{V} \Longrightarrow \frac{\boldsymbol{u}}{r} \in L^2(\Omega)$$

holds. In this case, (4) holds if $\beta < -1$ and $-\beta - \frac{d}{2} < b(\Omega,\mathbb{P})$.

5. Polyhedral domains

For several reasons, polyhedral domains are the main aim here. One reason is, of course, their importance for the applications. The second reason is that for the proof of analytic regularity they necessitate a clear strategy like the one described here. For polygonal domains or, more generally domains with isolated conical points, and even for domains with smooth edges, it is possible to arrive at a proof of analytic regularity taking Morrey's analytic regularity result for smooth domains as a starting point, see [5, 6], for example. The complexity of the proofs then makes it practically impossible to take the next step and treat polyhedral domains as well. With our strategy, the proof for polyhedral domains became possible [14], and it will also be feasible to extend it to a class of elliptic systems including lower order terms and variable coefficients. This has yet to be published, however [12].

5.1. Corners, edges, distance functions and weights

Let Ω be a polyhedron in $\mathbb{R}^3$. Its boundary is a finite union of plane polygons, the faces. The segments forming the boundaries of the faces are the edges $\boldsymbol{e} \in \mathscr{E}$ of Ω, and the ends of the edges are the corners $\boldsymbol{c} \in \mathscr{C}$ of Ω. Edge openings may be equal to 2π, allowing domains with crack surfaces. For defining weighted Sobolev spaces, we use weights depending on the distance to the singular parts of the boundary. Several types of distances are used: the distance r to the singular points, the distance $r_{\boldsymbol{c}}$ to a corner $\boldsymbol{c}$, the distance $r_{\mathscr{C}}$ to the corner set, and finally the distance $r_{\boldsymbol{e}}$ to an edge $\boldsymbol{e}$. Note that $r_{\mathscr{C}}$ is equivalent to $\prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}$ and r is equivalent to $\prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \big(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\big)$.

One can then consider two ways of generating weights:

1. A simple way by choosing $\beta \in \mathbb{R}$ and using powers of r:

$$\boxed{w_\ell = r^{\ell+\beta}} \quad \text{or} \quad \boxed{w_\ell = r^{\max\{0,\ell+\beta\}}}$$

2. A more refined method by choosing different weights for each corner and each edge, expressed by a weight multi-index $\underline{\beta} = (\beta_{\boldsymbol{c}}, \beta_{\boldsymbol{e}})_{\boldsymbol{c}\in\mathscr{C},\boldsymbol{e}\in\mathscr{E}}$ and taking

$$\boxed{w_\ell = \prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\ell+\beta_{\boldsymbol{c}}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \Big(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\Big)^{\ell+\beta_{\boldsymbol{e}}}} \text{ or } \boxed{w_\ell = \prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\max\{0,\ell+\beta_{\boldsymbol{c}}\}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \Big(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\Big)^{\max\{0,\ell+\beta_{\boldsymbol{e}}\}}}$$

Note that if $\beta_{\boldsymbol{c}} \equiv \beta_{\boldsymbol{e}} \equiv \beta$, then $\prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\ell+\beta_{\boldsymbol{c}}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \big(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\big)^{\ell+\beta_{\boldsymbol{e}}} \simeq r^{\ell+\beta}$.

5.2. Type B in polyhedral domains

We state results of type B for problems $\mathbb{P}$ that admit a $\mathbb{V}$-coercive variational formulation in a closed subspace $\mathbb{V}$ of $H^1(\Omega)$ determined by essential boundary conditions and for operators with smooth coefficients.

Theorem 5.1. *(Type B)* [25, 12] *For numbers $b^*_{\boldsymbol{c}}(\Omega,\mathbb{P}) > -\frac{1}{2}$ and $b_{\boldsymbol{e}}(\Omega,\mathbb{P}) > 0$ that can be determined in an optimal way from Mellin corner and edge spectra $\sigma(\mathfrak{P}_{\boldsymbol{c}})$ and $\sigma(\mathfrak{P}_{\boldsymbol{e}})$ combined with the condition of injectivity modulo polynomials, the following holds. Let $m \geq 2$ and let $\underline{\beta} < -1$ be such that $-\beta_{\boldsymbol{c}} - \frac{3}{2} < b^*_{\boldsymbol{c}}(\Omega,\mathbb{P})$ and $-\beta_{\boldsymbol{e}} - 1 < b_{\boldsymbol{e}}(\Omega,\mathbb{P})$. Then by choosing the weights $w_\ell = \prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\max\{0,\ell+\beta_{\boldsymbol{c}}\}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \big(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\big)^{\max\{0,\ell+\beta_{\boldsymbol{e}}\}}$, we have*

$$\boldsymbol{u} \in \mathbb{V} \quad \textit{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^m \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^m.$$

Example 5.2. *Let $\omega_{\boldsymbol{e}}$ be the interior opening angle at the edge $\boldsymbol{e}$, and let $\Theta_{\boldsymbol{c}}$ be the intersection of the unit sphere centered at $\boldsymbol{c}$ with the cone that coincides with Ω near the corner $\boldsymbol{c}$. One finds, cf* [16, 12]
(i) *for the Laplace equation with zero Dirichlet boundary conditions,*

$$b_{\boldsymbol{e}}(\Omega,\mathbb{P}) = \frac{\pi}{\omega_{\boldsymbol{e}}}, \quad b_{\boldsymbol{c}}(\Omega,\mathbb{P}) = -\frac{1}{2} + \sqrt{\mu^{\mathsf{Dir}}_{\boldsymbol{c},1} + \frac{1}{4}}, \tag{5}$$

$$b^*_{\boldsymbol{c}}(\Omega,\mathbb{P}) = \min\{2, b_{\boldsymbol{c}}(\Omega,\mathbb{P})\}, \tag{6}$$

where $\mu^{\mathsf{Dir}}_{\boldsymbol{c},1}$ is the first eigenvalue of the Laplace-Beltrami operator with Dirichlet boundary conditions on $\Theta_{\boldsymbol{c}}$,
(ii) *for the Laplace equation with zero Neumann boundary conditions,*

$$b_{\boldsymbol{e}}(\Omega,\mathbb{P}) = \frac{\pi}{\omega_{\boldsymbol{e}}}, \quad b_{\boldsymbol{c}}(\Omega,\mathbb{P}) = 0, \tag{7}$$

$$b^*_{\boldsymbol{c}}(\Omega,\mathbb{P}) \geq \min\{2, b^{(2)}_{\boldsymbol{c}}(\Omega,\mathbb{P})\} \quad \textit{with} \quad b^{(2)}_{\boldsymbol{c}}(\Omega,\mathbb{P}) = -\frac{1}{2} + \sqrt{\mu^{\mathsf{Neu}}_{\boldsymbol{c},2} + \frac{1}{4}}, \tag{8}$$

where $\mu^{\mathsf{Neu}}_{\boldsymbol{c},2}$ is the second eigenvalue of the Laplace-Beltrami operator with Neumann boundary conditions on $\Theta_{\boldsymbol{c}}$. Note that $b_{\boldsymbol{c}}(\Omega,\mathbb{P})$ corresponds to $b^{(1)}_{\boldsymbol{c}}(\Omega,\mathbb{P})$ defined with the first Neumann eigenvalue $\mu^{\mathsf{Neu}}_{\boldsymbol{c},1}$ which is always equal to 0.

5.2.1. Edge domains

For domains with edges only (i.e. without the corners), Theorems of type S (Cauchy-type) in isotropic spaces and hence of type A, can be proved without difficulty, see [14, section 5.1]. But in connection with the *h-p* version of finite elements, this would not help. Indeed the 3D *h-p* FEM works necessarily with *anisotropic* meshes taking anisotropic regularity of the solution into account, and exponential convergence with respect to the number of degrees of freedom can be proved only if improved regularity along the edges holds [17, 7, 22, 27] (see also [4, 10] for the h FEM).

So weights w_ℓ providing isotropic semi-norms $\sum_{|\alpha|=\ell} \|w_\ell \partial_{\boldsymbol{x}}^\alpha \boldsymbol{u}\|_{L^2(\Omega)}$ have to be replaced by anisotropic weights $w_{\boldsymbol{e},\alpha}$ defined in a neighborhood $\mathcal{V}_{\boldsymbol{e}}$ of the edges $\boldsymbol{e}$, leading to

$$|\boldsymbol{u}|_{\mathbb{X}^\ell} = \sum_{\boldsymbol{e}\in\mathscr{E}} \sum_{|\alpha|=\ell} \|w_{\boldsymbol{e},\alpha} \partial_{\boldsymbol{x}}^\alpha \boldsymbol{u}\|_{L^2(\mathcal{V}_{\boldsymbol{e}})} .$$

By choosing tubular coordinates $\boldsymbol{x}_{\boldsymbol{e}} = (x_{\boldsymbol{e}}^\perp, x_{\boldsymbol{e}}^\parallel)$ and corresponding multi-indices $\alpha_{\boldsymbol{e}} = (\alpha_{\boldsymbol{e}}^\perp, \alpha_{\boldsymbol{e}}^\parallel)$, — perpendicular and parallel to $\boldsymbol{e}$, we typically take

$$\boxed{w_{\boldsymbol{e},\alpha} = r_{\boldsymbol{e}}^{\beta_{\boldsymbol{e}}+|\alpha_{\boldsymbol{e}}^\perp|}}$$

which is independent of the order of the derivatives $\partial_{\boldsymbol{x}}^{\alpha_{\boldsymbol{e}}^\parallel}$ parallel to $\boldsymbol{e}$.

5.2.2. Polyhedral domains

In the general situation, i.e., for a domain with edges and corners, anisotropic weights have also to be used. To simplify the presentation, we assume here that all edges are parallel to the coordinate axes (we refer to [14] for the general situation). The non-homogeneous version of anisotropic weights is

$$w_\alpha = \prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\max\{0,\beta_{\boldsymbol{c}}+|\alpha|\}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \Big(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\Big)^{\max\{0,\beta_{\boldsymbol{e}}+|\alpha_{\boldsymbol{e}}^\perp|\}}, \tag{9}$$

defining the anisotropic seminorms and norms for solutions $\boldsymbol{u}$

$$|\boldsymbol{u}|_{\mathbb{X}^\ell} = \sum_{|\alpha|=\ell} \|w_\alpha \partial_{\boldsymbol{x}}^\alpha \boldsymbol{u}\|_{L^2(\Omega)} \quad \text{and} \quad \|\boldsymbol{u}\|_{\mathbb{X}^k} = \max_{\ell=0,\dots,k} |\boldsymbol{u}|_{\mathbb{X}^\ell} \tag{10}$$

and right hand sides $\boldsymbol{f}$

$$|\boldsymbol{f}|_{\mathbb{Y}^\ell} = \sum_{|\alpha|=\ell-2} \|w_\alpha \partial_{\boldsymbol{x}}^\alpha \boldsymbol{f}\|_{L^2(\Omega)} \quad \text{and} \quad \|\boldsymbol{f}\|_{\mathbb{Y}^k} = \max_{\ell=2,\dots,k} |\boldsymbol{f}|_{\mathbb{Y}^\ell} . \tag{11}$$

Now we can state our result of type S for anisotropic norms.

Theorem 5.3. *(Type S with Cauchy estimates and anisotropic norms)* [14] *Assume that the operator $\mathbb{P}$ is homogeneous with constant coefficients. Let $\underline{\beta} = (\beta_{\boldsymbol{c}}, \beta_{\boldsymbol{e}})$ be such that*

$$\forall \boldsymbol{e} \in \mathscr{E}, \quad 0 < -\beta_{\boldsymbol{e}} - 1 \text{ and } -\beta_{\boldsymbol{e}} - 1 \notin \operatorname{Re}\sigma(\mathfrak{P}_{\boldsymbol{e}}),$$

where $\sigma(\mathfrak{P}_{\boldsymbol{e}})$ is the spectrum of the Mellin edge symbol $\mathfrak{P}_{\boldsymbol{e}}$ of $\mathbb{P}$. Let $m \geq 1$ and $m \geq \max\{-\beta_{\boldsymbol{e}}, -\beta_{\boldsymbol{c}}\}$, and take the weights (9) *defining the anisotropic semi-norms* (10)–(11)*. Then there exists $A > 0$ such that for any $k \geq m$ and $\boldsymbol{u} \in \mathbb{X}^m$*

$$\frac{1}{k!}\,|\boldsymbol{u}|_{\mathbb{X}^k} \leq A^{k+1}\Big(\sum_{\ell=0}^{k+2} \frac{1}{\ell!}\,|\mathbb{P}\boldsymbol{u}|_{\mathbb{Y}^\ell} + |\boldsymbol{u}|_{\mathbb{X}^m}\Big).$$

Example 5.4. *For the Laplace equation with Dirichlet boundary conditions, the spectrum of the edge Mellin symbol is explicit: $\sigma(\mathfrak{P}_{\boldsymbol{e}}) = \{\frac{k\pi}{\omega_{\boldsymbol{e}}} : k \in \mathbb{Z}^*\}$.*

As an immediate consequence of results of type B (Theorem 5.1) and of type AS (Theorem 5.3), we directly deduce results of type AA (anisotropic analytic) in polyhedral domains.

Theorem 5.5. *(Type A with anisotropic norms)* [14] *Assume that the operator $\mathbb{P}$ is homogeneous with constant coefficients. With the same numbers $b^*_{\boldsymbol{c}}(\Omega, \mathbb{P})$ and $b_{\boldsymbol{e}}(\Omega, \mathbb{P})$ as in Theorem 5.1, let $\underline{\beta} < -1$ be such that*

$$-\beta_{\boldsymbol{c}} - \frac{3}{2} < b^*_{\boldsymbol{c}}(\Omega, \mathbb{P}) \quad \text{and} \quad -\beta_{\boldsymbol{e}} - 1 < b_{\boldsymbol{e}}(\Omega, \mathbb{P})$$

and take the weights (9) *defining the anisotropic semi-norms* (10)–(11)*. Then*

$$\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^\varpi \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^\varpi.$$

Remark 5.6. Homogeneous weights $w_\alpha = \prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\ell+\beta_{\boldsymbol{c}}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \big(\frac{r_{\boldsymbol{e}}}{r_{\mathscr{C}}}\big)^{|\alpha^\perp_{\boldsymbol{e}}|+\beta_{\boldsymbol{e}}}$ can be used if

$$\boldsymbol{u} \in \mathbb{V} \Longrightarrow \frac{\boldsymbol{u}}{r} \in L^2(\Omega)$$

holds. Then the relevant bound for $-\beta_{\boldsymbol{c}} - \frac{3}{2}$ is $b_{\boldsymbol{c}}(\Omega, \mathbb{P})$.

To illustrate these results, we provide in Table 1 values of $\min_{\boldsymbol{e}} b_{\boldsymbol{e}}(\Omega, \mathbb{P})$, $\min_{\boldsymbol{c}} b_{\boldsymbol{c}}(\Omega, \mathbb{P})$, and $\min_{\boldsymbol{c}} b^*_{\boldsymbol{c}}(\Omega, \mathbb{P})$ for the following domains Ω:

1. the cube $(-1, 1)^3$,
2. the thick L-shaped domain $\{(-1, 1)^2 \setminus (0, 1)^2\} \times (-1, 1)$,
3. the Fichera corner $(-1, 1)^3 \setminus (0, 1)^3$,

and in each case, we consider as boundary value problem $\mathbb{P}$ the Laplace operator with either zero Dirichlet or zero Neumann boundary conditions.

Domain Ω	$\min_{\boldsymbol{e}} b_{\boldsymbol{e}}(\Omega)$	$\min_{\boldsymbol{c}} b_{\boldsymbol{c}}(\Omega)$	$\min_{\boldsymbol{c}} b^*_{\boldsymbol{c}}(\Omega)$
Cube, Dirichlet	2	3	2
Cube, Neumann	2	0	4
Thick L, Dirichlet	2/3	5/3	5/3
Thick L, Neumann	2/3	0	2/3
Fichera corner, Dirichlet	2/3	0.45418	0.45418
Fichera corner, Neumann	2/3	0	0.84001

Table 1: Limiting numbers for Δ on example domains

All the values of Table 1 are obtained as an application of Example 5.2:

- The Dirichlet and Neumann Laplace-Beltrami eigenvalues attached to the vertex $(0, 0, 0)$ of the Fichera corner have been computed for us by Thomas Apel with the method of [3].

- The Dirichlet Laplace-Beltrami eigenvalues for a half-dihedron of opening ω are deduced from [16, § 18.C], which yields that $b_{\boldsymbol{c}}(\Omega) = \frac{\pi}{\omega} + 1$. By the same method one shows that for the Neumann problem the quantity $b^{(2)}_{\boldsymbol{c}}(\Omega)$ defined in (8) equals $\frac{\pi}{\omega}$.

- The value $b^*_{\boldsymbol{c}}(\Omega)$ for the cube is not completely determined by formula (8). One has to evaluate dimensions of homogeneous polynomial spaces to check the condition of "injectivity modulo polynomials", cf [16, Ch. 4] and [12].

Finally, to make the connection easier with our previous works, we present in Table 2 a lexicon with the names of the different spaces used in our papers [9, 10, 11, 12, 14].

Type	Homogeneous norms	Non-homogeneous norms
Isotropic	$K^k_\beta(\Omega)$	$J^k_\beta(\Omega)$
Anisotropic	$M^k_\beta(\Omega)$	$N^k_\beta(\Omega)$
Anisotropic Analytic	$A^k_\beta(\Omega)$	$B^k_\beta(\Omega)$

Table 2: 3D Lexicon of weighted spaces

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