Oberwolfach meeting on Computational Electromagnetism

Feb 22 - Feb 28

Singularities of electromagnetic fields in the Maxwell and eddy current formulations

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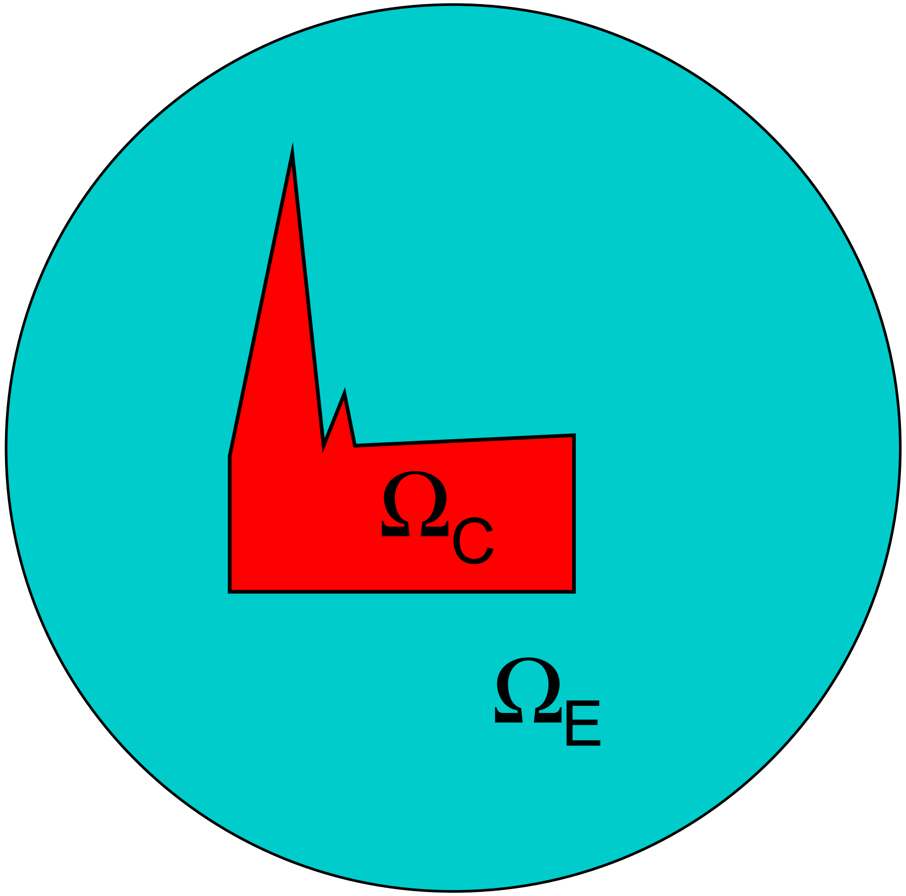
Situation



Conductor body Ω_C :

polyhedron with boundary B .

Conductivity σ_C ,

permittivity ε_C and permeability μ_C .

Exterior region $\Omega_E := \mathcal{B}(0, R) \setminus \Omega_C$.

Conductivity $\sigma_E = 0$,

permittivity ε_E and permeability μ_E .

Region of interest

$$\Omega = \mathcal{B}(0, R) = \overline{\Omega}_C \cup \overline{\Omega}_E$$

with σ , ε and μ defined on Ω .

Perfect conductor boundary conditions on

$$\partial\Omega = \partial\Omega_E \setminus B$$

Outline

★ **Maxwell problem for an imperfect conductor.**

- **Gauge conditions – Regularized variational form.**
- **Singularities of Type 1 and 2.**
- **Sobolev and weighted Sobolev regularity.**

★★ **Eddy current problem.**

- **Gauge conditions – Regularized variational form.**
- **Eddy current pb as a limit of Maxwell (as $\delta \simeq \omega\varepsilon\sigma^{-1} \longrightarrow 0$).**
- **Singularities of Type 1 and 2 (are also limits as $\delta \longrightarrow 0$).**
- **Regularity: The field inside a convex conductor is more regular than outside.**

Maxwell problem for a conductor

Conductor body Ω_{C} : polyhedron with boundary B (connected components B_i).
Conductivity σ_{C} , permittivity ε_{C} and permeability μ_{C} .

Exterior region $\Omega_{\mathsf{E}} := \mathcal{B}(0,R) \setminus \Omega_{\mathsf{C}}$ for R large enough.
Conductivity $\sigma_{\mathsf{E}} = 0$, permittivity ε_{E} and permeability μ_{E} .

Region of interest $\Omega = \mathcal{B}(0,R) = \overline{\Omega}_{\mathsf{C}} \cup \overline{\Omega}_{\mathsf{E}}$.
Conductivity σ , permittivity ε and permeability μ defined on Ω . Frequency ω , fixed.
Perfect conductor boundary conditions on $\partial\Omega = \partial\Omega_{\mathsf{E}} \setminus B$.

Source current density j_0 in $\mathsf{L}^2(\mathbb{R}^3)^3$, with support in Ω_{C} and divergence free, i.e.
$\operatorname{div} j_0 = 0$ in Ω_{C} and $j_0 \cdot n = 0$ on B .

Find the electromagnetic field (E, H)

(Maxwell)
$$\begin{cases} (i) & \operatorname{curl} E = -i\omega\,\mu H & \text{in} \quad \Omega, \\ (ii) & \operatorname{curl} H = (\sigma + i\omega\,\varepsilon)E + j_0 & \text{in} \quad \Omega, \\ (iii) & E \times n = 0 \quad \& \quad H \cdot n = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Conditions on the divergence of the electric field

Taking the divergence of equation (ii)

$$\operatorname{div} \alpha E = 0, \qquad \text{with} \quad \alpha_{\mathsf{C}} = \sigma_{\mathsf{C}} + i\omega\,\varepsilon_{\mathsf{C}} \quad \text{and} \quad \alpha_{\mathsf{E}} = i\omega\,\varepsilon_{\mathsf{E}}.$$

Therefore

(1) $$\operatorname{div} E_{\mathsf{C}} = 0 \quad \text{and} \quad \boxed{\operatorname{div} E_{\mathsf{E}} = 0}$$

(2) $$\alpha_{\mathsf{C}} E_{\mathsf{C}} \cdot n = \alpha_{\mathsf{E}} E_{\mathsf{E}} \cdot n \qquad \text{on} \quad B.$$

Equation (ii) also yields that $E_{\mathsf{C}} = \operatorname{curl} \psi$ with $\psi = (i\omega\varepsilon_{\mathsf{C}} + \sigma_{\mathsf{C}})^{-1}(H - J_0)$ where J_0 is a vector potential for j_0. By localization around B_i:

$$\int_{B_i} E_{\mathsf{C}} \cdot n \,\mathrm{d}S = \int_{\Omega_C} \operatorname{div}\{\operatorname{curl}(\mu_i \psi)\}\,\mathrm{d}x = 0.$$

Combining with (2)

$$\boxed{\int_{B_i} E_{\mathsf{E}} \cdot n \,\mathrm{d}S = 0}$$

Regularized variational formulation

We propose a variational space independent of σ, ε and ω:

$$\mathbf{Y}(\Omega) = \Big\{ u \in \mathsf{L}^2(\Omega)^3 : \quad \operatorname{curl} u \in \mathsf{L}^2(\Omega)^3, \quad \operatorname{div} u_{\mathsf{C}} \in \mathsf{L}^2(\Omega_{\mathsf{C}}), \quad \operatorname{div} u_{\mathsf{E}} \in \mathsf{L}^2(\Omega_{\mathsf{E}}) \quad u \times n = 0 \ \text{ on } \ \partial\Omega, \quad \int_{B_i} u_{\mathsf{E}} \cdot n \, \mathrm{d}S = 0 \Big\}.$$

The associate variational forms are for u, $v \in \mathbf{Y}(\Omega)$

$$a(u,v) = \int_{\Omega} \left(\mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} \overline{v} - \omega^2 \varepsilon u \cdot \overline{v} \right) \mathrm{d}x + i\omega \int_{\Omega_{\mathsf{C}}} \sigma_{\mathsf{C}} u_{\mathsf{C}} \cdot \overline{v}_{\mathsf{C}} \, \mathrm{d}x$$

$$a_{\mathrm{reg}}(u,v) = a(u,v) + \int_{\Omega_{\mathsf{C}}} \operatorname{div} u_{\mathsf{C}} \ \operatorname{div} \overline{v}_{\mathsf{C}} \, \mathrm{d}x + \int_{\Omega_{\mathsf{E}}} \operatorname{div} u_{\mathsf{E}} \ \operatorname{div} \overline{v}_{\mathsf{E}} \, \mathrm{d}x.$$

The electric field E solution of (Maxwell) is the only solution of

$$E \in \mathbf{Y}(\Omega), \quad \forall v \in \mathbf{Y}(\Omega), \quad a_{\mathrm{reg}}(E, v) = -i\omega (j_0, v)_{\Omega_{\mathsf{C}}}.$$

Corner singularities, general recipe

Fix a corner O (here O $\in B$). Let (ρ, ϑ) be polar coordinates centered in O.

Around O, the 3D space is shared into two complementary cones Γ_C and Γ_E reproducing the sharing by Ω_C and Ω_E:

$$\mathbb{R}^3 = \overline{\Gamma}_\mathsf{C} \cup \overline{\Gamma}_\mathsf{E}.$$

The general recipe consists in looking for solutions u of the form $\rho^\lambda U(\vartheta)$ of the principal part of the associated elliptic bvp, with zero RHS:

$$\left\{\begin{array}{ll}
\operatorname{curl}(\mu_\mathsf{C}^{-1} \operatorname{curl} u_\mathsf{C}) - \nabla \operatorname{div} u_\mathsf{C} = 0 & \text{in } \Gamma_\mathsf{C}, \\
\operatorname{curl}(\mu_\mathsf{E}^{-1} \operatorname{curl} u_\mathsf{E}) - \nabla \operatorname{div} u_\mathsf{E} = 0 & \text{in } \Gamma_\mathsf{E}, \\
[u \times n] = 0, \quad [\alpha u \cdot n] = 0 & \text{on } \mathrm{I} := \partial\Gamma_\mathsf{C} = \partial\Gamma_\mathsf{E}, \\
[\mu^{-1} \operatorname{curl} u \times n] = 0, \quad [\operatorname{curl} u \cdot n] = 0 & \text{on } \mathrm{I}, \\
[\operatorname{div} \alpha u] = 0, \quad [\alpha^{-1} \partial_n \operatorname{div} \alpha u] = 0 & \text{on } \mathrm{I}.
\end{array}\right.$$

Corner singularities, Maxwell recipe

Set $q_{\mathsf{C}} = \operatorname{div} \alpha_{\mathsf{C}} u_{\mathsf{C}}$, $q_{\mathsf{E}} = \operatorname{div} \alpha_{\mathsf{E}} u_{\mathsf{E}}$ **and** $\psi_{\mathsf{C}} = \mu_{\mathsf{C}}^{-1} \operatorname{curl} u_{\mathsf{C}}$, $\psi_{\mathsf{E}} = \mu_{\mathsf{E}}^{-1} \operatorname{curl} u_{\mathsf{E}}$.

$$\left\{ \begin{array}{lll} \Delta q_{\mathsf{C}} = 0 & \text{in} & \Gamma_{\mathsf{C}}, \\ \Delta q_{\mathsf{E}} = 0 & \text{in} & \Gamma_{\mathsf{E}}, \\ [q] = 0, \quad [\alpha^{-1} \partial_n q] = 0 & \text{on} & \mathrm{I}. \end{array} \right. \qquad \text{Type 3:} \left[\begin{array}{l} \text{general } q \\ \text{particular } \psi, \\ \text{particular } u \end{array} \right.$$

$$\left\{ \begin{array}{lll} \operatorname{curl} \psi_{\mathsf{C}} = \nabla q_{\mathsf{C}}, \quad \operatorname{div}(\mu_{\mathsf{C}} \psi_{\mathsf{C}}) = 0 & \text{in} & \Gamma_{\mathsf{C}}, \\ \operatorname{curl} \psi_{\mathsf{E}} = \nabla q_{\mathsf{E}}, \quad \operatorname{div}(\mu_{\mathsf{E}} \psi_{\mathsf{E}}) = 0 & \text{in} & \Gamma_{\mathsf{E}}, \\ [\psi \times n] = 0, \quad [\mu \psi \cdot n] = 0 & \text{on} & \mathrm{I}. \end{array} \right. \qquad \text{Type 2:} \left[\begin{array}{l} q = 0, \\ \text{general } \psi, \\ \text{particular } u \end{array} \right.$$

$$\left\{ \begin{array}{lll} \operatorname{curl} u_{\mathsf{C}} = \mu_{\mathsf{C}} \psi_{\mathsf{C}}, \quad \operatorname{div} \alpha_{\mathsf{C}} u_{\mathsf{C}} = q_{\mathsf{C}} & \text{in} & \Gamma_{\mathsf{C}}, \\ \operatorname{curl} u_{\mathsf{E}} = \mu_{\mathsf{E}} \psi_{\mathsf{E}}, \quad \operatorname{div} \alpha_{\mathsf{E}} u_{\mathsf{E}} = q_{\mathsf{E}} & \text{in} & \Gamma_{\mathsf{E}}, \\ [u \times n] = 0, \quad [\alpha u \cdot n] = 0 & \text{on} & \mathrm{I}. \end{array} \right. \qquad \text{Type 1:} \left[\begin{array}{l} q = 0, \\ \psi = 0, \\ \text{general } u \end{array} \right.$$

Maxwell Corner Singularities of Type 1

$u_C = \nabla\Phi_C$ and $u_E = \nabla\Phi_E$ with

$$\begin{cases} \operatorname{div} \nabla\Phi_C = 0 & \text{in } \Gamma_C, \\ \operatorname{div} \nabla\Phi_E = 0 & \text{in } \Gamma_E, \\ [\Phi] = 0, \quad [\alpha\partial_n\Phi] = 0 & \text{on } \mathrm{I}. \end{cases}$$

The singular density Φ is in $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ and is a singularity of the Laplace transmission problem with α: $\alpha = \alpha_C = \sigma_C + i\omega\,\varepsilon_C$ in Γ_C and $\alpha = \alpha_E = i\omega\,\varepsilon_E$ in Γ_E.

The singularities $\Phi = \rho^\lambda\varphi(\vartheta)$ with $\operatorname{Re}\lambda > 0$ and $\lambda(\lambda+1) = \nu$ eigenvalue and φ eigenvector of the problem (with $G_C = \Gamma_C \cap \mathbb{S}^2$ and $G_E = \Gamma_E \cap \mathbb{S}^2$)

$\varphi \in \mathrm{H}^1(\mathbb{S}^2), \quad \forall\psi \in \mathrm{H}^1(\mathbb{S}^2)$

$$\int_{G_C} \alpha_C \nabla_\top\varphi \cdot \nabla_\top\psi + \int_{G_E} \alpha_E \nabla_\top\varphi \cdot \nabla_\top\psi = \nu\left\{\int_{G_C} \alpha_C\varphi\psi + \int_{G_E} \alpha_E\varphi\psi\right\}$$

For wedges, this reduces to a 1D angular version: $\nabla_\top \longrightarrow \partial_\theta$ and $\lambda^2 = \nu$.

Maxwell Corner Singularities of Type 2

$\psi_{\mathsf{C}} = \nabla\Psi_{\mathsf{C}}$ and $\psi_{\mathsf{E}} = \nabla\Psi_{\mathsf{E}}$ with

$$\begin{cases} \operatorname{div}\nabla\Psi_{\mathsf{C}} = 0 & \text{in } \Gamma_{\mathsf{C}}, \\ \operatorname{div}\nabla\Psi_{\mathsf{E}} = 0 & \text{in } \Gamma_{\mathsf{E}}, \\ [\Psi] = 0, \quad [\mu\partial_n\Psi] = 0 & \text{on } \mathrm{I}. \end{cases}$$

The density Ψ is a singularity $\rho^{\lambda}\psi(\vartheta)$ of the Laplace transmission problem with μ :
$\mu = \mu_{\mathsf{C}}$ in Γ_{C} and $\mu = \mu_{\mathsf{E}}$ in Γ_{E}.

Then $u = (\lambda + 1)^{-1}(\mu\nabla\Psi \times x - \nabla r)$ with r solution of

$$\begin{cases} \Delta r_{\mathsf{C}} = 0 & \text{in } \Gamma_{\mathsf{C}}, \\ \Delta r_{\mathsf{E}} = 0 & \text{in } \Gamma_{\mathsf{E}}, \\ [r] = 0, \quad [\alpha\partial_n r] = [\alpha\mu](\nabla\Psi \times x)\cdot n & \text{on } \mathrm{I}. \end{cases}$$

Sobolev regularity

Let β_α and β_μ be the limiting regularity Sobolev exponents for the transmission Laplace operators $\operatorname{div} \alpha\nabla$ and $\operatorname{div} \mu\nabla$ respectively. Note that

$$\frac{3}{2} < \beta_\mu < 2 \quad \text{and} \quad 1 < \beta_\alpha.$$

Then

$$E_\mathsf{C} \in \mathsf{H}^s(\Omega_\mathsf{C}) \quad \text{and} \quad E_\mathsf{E} \in \mathsf{H}^s(\Omega_\mathsf{E}), \quad \forall s < \min\{\beta_\alpha - 1, \beta_\mu\}.$$

Moreover

$$E = \nabla\Phi + E^{\mathsf{reg}}$$

with

$$E_\mathsf{C}^{\mathsf{reg}} \in \mathsf{H}^s(\Omega_\mathsf{C}) \quad \text{and} \quad E_\mathsf{E}^{\mathsf{reg}} \in \mathsf{H}^s(\Omega_\mathsf{E}), \quad \forall s < \min\{\beta_\alpha\, , \beta_\mu\}.$$

Weighted Sobolev spaces

Isotropic spaces (Kondrat'ev) and $m \in \mathbb{N}$, $\beta \in \mathbb{R}$

$$\mathsf{W}^m_\beta(\Omega) = \Big\{ u \in \mathsf{L}^2_{\mathsf{loc}}(\Omega) \;:\; \forall\,\alpha,\ |\alpha| \le m,\quad \partial^\alpha u \in \mathsf{L}^2(\mathscr{V}^0),$$
$$\forall\, c \text{ corner},\quad r_c^{\beta+|\alpha|}\partial^\alpha u \in \mathsf{L}^2(\mathscr{V}_c) \Big\}.$$

Isotropic spaces (Nazarov-Plamenevskii) $\mathsf{D} = \mathsf{C}$ or E, and $m \in \mathbb{N}$, $\beta \in \mathbb{R}$

$$\mathsf{K}^m_\beta(\Omega_{\mathsf{D}}) = \Big\{ u \in \mathsf{L}^2_{\mathsf{loc}}(\Omega_{\mathsf{D}}) \;:\; \forall\,\alpha,\ |\alpha| \le m,\quad \partial^\alpha u \in \mathsf{L}^2(\mathscr{V}^0),$$
$$\forall\, c \text{ corner},\quad r_c^{\beta+|\alpha|}\partial^\alpha u \in \mathsf{L}^2(\mathscr{V}_c^0)$$
$$\forall\, e \text{ edge},\quad r_e^{\beta+|\alpha|}\partial^\alpha u \in \mathsf{L}^2(\mathscr{V}_e^0) \quad\text{and}\quad r_e^{\beta+|\alpha|}\partial^\alpha u \in \mathsf{L}^2(\mathscr{V}_e^c) \Big\}.$$

Anisotropic spaces (with $\alpha = (\alpha_\perp, \alpha_3)$ transverse - longitudinal to the edge)

$$\mathsf{M}^m_\beta(\Omega_{\mathsf{D}}) = \Big\{ u \in \mathsf{L}^2_{\mathsf{loc}}(\Omega_{\mathsf{D}}) \;:\; \forall\,\alpha,\ |\alpha| \le m,\quad \partial^\alpha u \in \mathsf{L}^2(\mathscr{V}^0),$$
$$\forall\, c \text{ corner},\quad r_c^{\beta+|\alpha|}\partial^\alpha u \in \mathsf{L}^2(\mathscr{V}_c^0)$$
$$\forall\, e \text{ edge},\quad r_e^{\beta+|\alpha_\perp|}\partial^\alpha u \in \mathsf{L}^2(\mathscr{V}_e^0) \quad\text{and}\quad r_c^{\alpha_3} r_e^{\beta+|\alpha_\perp|}\partial^\alpha u \in \mathsf{L}^2(\mathscr{V}_e^c) \Big\}$$

Weighted (anisotropic) Sobolev regularity

Suppose j_0 smooth.

Let β_α and β_μ be the limiting regularity Sobolev exponents for the transmission Laplace operators $\operatorname{div}\alpha\nabla$ and $\operatorname{div}\mu\nabla$ respectively.

Splitting of the electric field

$$\boldsymbol{E} = \nabla\Phi + \boldsymbol{E}^{\mathsf{reg}}$$

with

$$\boldsymbol{E}_{\mathsf{C}}^{\mathsf{reg}} \in \mathsf{M}^{\infty}_{-\beta}(\Omega_{\mathsf{C}}) \quad \text{and} \quad \boldsymbol{E}_{\mathsf{E}}^{\mathsf{reg}} \in \mathsf{M}^{\infty}_{-\beta}(\Omega_{\mathsf{E}}), \quad \forall\beta < \min\{\beta_\alpha, \beta_\mu\}$$

and the potential

$$\Phi = \Phi^0 + \Phi^1 + \Phi^{\mathsf{reg}}, \qquad \Phi^{\mathsf{reg}} \in \mathsf{H}^{\infty}(\Omega)$$

and

$$\Phi^0_{\mathsf{C}} \in \mathsf{M}^{\infty}_{-\beta}(\Omega_{\mathsf{C}}), \quad \Phi^0_{\mathsf{E}} \in \mathsf{M}^{\infty}_{-\beta}(\Omega_{\mathsf{E}}), \quad \Phi^1 \in \mathsf{W}^{\infty}_{-\beta}(\Omega).$$

Eddy current equations

Recall Maxwell equations.

Find the electromagnetic field (E, H)

(Maxwell)

$$\begin{cases} \operatorname{curl} E = -i\omega\,\mu H & \text{in } \Omega, \\ \operatorname{curl} H = (\sigma + i\omega\,\varepsilon)E + j_0 & \text{in } \Omega, \\ E \times n = 0 \quad \& \quad H \cdot n = 0 & \text{on } \partial\Omega. \end{cases}$$

The Eddy Current equations are defined for $\sigma >> \varepsilon$ by setting ε to 0 in (Maxwell).

Find the electromagnetic field (E, H)

(Eddy c.)

$$\begin{cases} \operatorname{curl} E = -i\omega\,\mu H & \text{in } \Omega, \\ \operatorname{curl} H = \sigma E + j_0 & \text{in } \Omega, \\ E \times n = 0 \quad \& \quad H \cdot n = 0 & \text{on } \partial\Omega. \end{cases}$$

Conditions on the divergence of the electric field

Taking the divergence of the 2d equation

$$\operatorname{div} \sigma E = 0, \qquad \text{with} \quad \sigma = \sigma_{\mathsf{C}} \ \text{in} \ \Omega_{\mathsf{C}} \quad \text{and} \quad \sigma_{\mathsf{E}} = 0.$$

Therefore

$$\operatorname{div} E_{\mathsf{C}} = 0 \quad \text{and} \quad E_{\mathsf{C}} \cdot n = 0 \qquad \text{on} \quad B$$

We have nothing on $\operatorname{div} E_{\mathsf{E}}$ **.**

We impose as gauge conditions, the conditions we obtained from (Maxwell) before passing to the limit:

$$\boxed{\operatorname{div} E_{\mathsf{E}} = 0} \quad \text{and} \quad \boxed{\int_{B_i} E_{\mathsf{E}} \cdot n \, \mathrm{d}S = 0}$$

The eddy current equations as limiting equations

Set $\omega\varepsilon = \delta\breve{\varepsilon}$ where

- $\delta > 0$ is a small parameter
- $\breve{\varepsilon}$ has the same order of magnitude as σ_{C} .

The assumption that δ is small reflects the fact that

the product $\omega\varepsilon$ is small with respect to σ_{C} .

We write (Maxwell) and (Eddy c.) in a unified way, for $\delta > 0$ and $\delta = 0$, resp.:

$$(\mathfrak{P}^\delta) \qquad \begin{cases} \operatorname{curl} E^\delta = -i\omega\,\mu H^\delta & \text{in} \quad \Omega, \\ \operatorname{curl} H^\delta = (\sigma + i\delta\,\breve{\varepsilon})E^\delta + j_0 & \text{in} \quad \Omega, \\ E^\delta \times n = 0 \quad \& \quad H^\delta \cdot n = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Regularized variational formulations

We propose a variational space independent of σ, ε and ω:

$$\mathsf{Y}(\Omega) = \Big\{ u \in \mathsf{L}^2(\Omega)^3 : \quad \operatorname{curl} u \in \mathsf{L}^2(\Omega)^3, \\ \operatorname{div} u_{\mathsf{C}} \in \mathsf{L}^2(\Omega_{\mathsf{C}}), \quad \operatorname{div} u_{\mathsf{E}} \in \mathsf{L}^2(\Omega_{\mathsf{E}}) \\ u \times n = 0 \ \text{ on } \ \partial\Omega, \quad \int_{B_i} u_{\mathsf{E}} \cdot n \, \mathrm{d}S = 0 \Big\}.$$

The associate variational forms are for u, $v \in \mathsf{Y}(\Omega)$

$$a^\delta(u,v) = \int_\Omega \big(\mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} \overline{v} - \omega\delta\breve{\varepsilon} u \cdot \overline{v}\big) \, \mathrm{d}x + i\omega \int_{\Omega_{\mathsf{C}}} \sigma_{\mathsf{C}} u_{\mathsf{C}} \cdot \overline{v}_{\mathsf{C}} \, \mathrm{d}x$$

$$a^\delta_{\mathsf{reg}}(u,v) = a^\delta(u,v) + \int_{\Omega_{\mathsf{C}}} \square \ \operatorname{div} u_{\mathsf{C}} \ \operatorname{div} \overline{v}_{\mathsf{C}} \, \mathrm{d}x + \int_{\Omega_{\mathsf{E}}} \square \ \operatorname{div} u_{\mathsf{E}} \ \operatorname{div} \overline{v}_{\mathsf{E}} \, \mathrm{d}x.$$

The electric field E solution of $(\mathfrak{P}^\delta)$ solves

$$E^\delta \in \mathsf{Y}(\Omega), \quad \forall v \in \mathsf{Y}(\Omega), \quad a^\delta_{\mathsf{reg}}(E,v) = -i\omega(j_0, v)_{\Omega_{\mathsf{C}}}.$$

The eddy current limit

Theorem

(i) $\exists \delta_0 > 0$ **s. t. the sesquilinear forms** $a^{\delta}_{\mathrm{reg}}$ **are uniformly coercive for** $\delta \in [0, \delta_0]$ **.**

(ii) **We have the uniform bound:**

$$\exists C > 0, \quad \forall \delta \in [0, \delta_0], \quad \|E^{\delta}\|_{\mathbf{Y}(\Omega)} \leq C.$$

(iii) **We have the convergence as** $\delta \longrightarrow 0$ **:**

$$\exists C > 0, \quad \forall \delta \in [0, \delta_0], \quad \|E^{\delta} - E^{0}\|_{\mathbf{Y}(\Omega)} \leq C\delta.$$

The singularities are of Type 1 and 2, also in the limit $\delta = 0$ **,**
and they depend continuously on δ

.../...

Eddy current Corner Singularities of Type 1

$u_{\mathsf{C}} = \nabla\Phi_{\mathsf{C}}$ and $u_{\mathsf{E}} = \nabla\Phi_{\mathsf{E}}$ with

$$\begin{cases} \operatorname{div}\nabla\Phi_{\mathsf{C}} = 0 & \text{in } \Gamma_{\mathsf{C}}, \\ \operatorname{div}\nabla\Phi_{\mathsf{E}} = 0 & \text{in } \Gamma_{\mathsf{E}}, \\ [\Phi] = 0, \quad \partial_n\Phi_{\mathsf{C}} = 0 & \text{on } \mathrm{I}. \end{cases}$$

We have either (i) or (ii)

(i) Φ_{C} is a singularity of the Laplace Neumann problem, Φ_{E} has the same degree λ.

(ii) $\Phi_{\mathsf{C}} = 0$ and Φ_{E} is a singularity of the Laplace Dirichlet problem.

Skip to Type 2

Using the harmonic extension P_{E} from $G_{\mathsf{C}} := \Gamma_{\mathsf{C}} \cap \mathbb{S}^2$ into $G_{\mathsf{E}} := \Gamma_{\mathsf{E}} \cap \mathbb{S}^2$, we can write the “singularities eigenvalue pb” in the unified way for $\delta = 0$ and $\delta > 0$:

Find $(\varphi_{\mathsf{C}}, \varphi_0) \in H^1(G_{\mathsf{C}}) \times H^1_0(G_{\mathsf{E}})$, $\forall(\psi_{\mathsf{C}}, \psi_0) \in H^1(G_{\mathsf{C}}) \times H^1_0(G_{\mathsf{E}})$:

$$a_\delta(\varphi_{\mathsf{C}}, \varphi_0\,;\, \psi_{\mathsf{C}}, \psi_0) = \nu b_\delta(\varphi_{\mathsf{C}}, \varphi_0\,;\, \psi_{\mathsf{C}}, \psi_0)$$

The singularity of Type 1 is $u = \nabla\Phi$ with $\Phi_{\mathsf{C}} = r^\lambda \varphi_{\mathsf{C}}$ and $\Phi_{\mathsf{E}} = r^\lambda (P_{\mathsf{E}}\varphi_{\mathsf{C}} + \varphi_0)$.

Let $\eta = \dfrac{i\delta\breve{\varepsilon}_{\mathsf{E}}}{\sigma_{\mathsf{C}} + i\delta\breve{\varepsilon}_{\mathsf{C}}}$.

The forms a_δ and b_δ depend continuously on $\delta \in [0, \delta_0]$:

$$a_\delta = \int_{G_{\mathsf{C}}} \nabla\varphi_{\mathsf{C}} \cdot \nabla\psi_{\mathsf{C}} + \eta \int_{G_{\mathsf{E}}} \nabla P_{\mathsf{E}}\varphi_{\mathsf{C}} \cdot \nabla P_{\mathsf{E}}\psi_{\mathsf{C}} + \int_{G_{\mathsf{E}}} \nabla\varphi_0 \cdot \nabla\psi_0$$

and

$$b_\delta = \int_{G_{\mathsf{C}}} \varphi_{\mathsf{C}}\, \psi_{\mathsf{C}} + \eta \int_{G_{\mathsf{E}}} \big(P_{\mathsf{E}}\varphi_{\mathsf{C}}\, P_{\mathsf{E}}\psi_{\mathsf{C}} + \varphi_0\, P_{\mathsf{E}}\psi_{\mathsf{C}}\big) + \int_{G_{\mathsf{E}}} \big(P_{\mathsf{E}}\varphi_{\mathsf{C}}\, \psi_0 + \varphi_0\, \psi_0\big).$$

Eddy current Corner Singularities of Type 2

$\psi_{\mathsf{C}} = \nabla \Psi_{\mathsf{C}}$ **and** $\psi_{\mathsf{E}} = \nabla \Psi_{\mathsf{E}}$ **with**

$$\begin{cases} \operatorname{div} \nabla \Psi_{\mathsf{C}} = 0 & \text{in } \Gamma_{\mathsf{C}}, \\ \operatorname{div} \nabla \Psi_{\mathsf{E}} = 0 & \text{in } \Gamma_{\mathsf{E}}, \\ [\Psi] = 0, \quad [\mu \partial_n \Psi] = 0 & \text{on } \mathrm{I}. \end{cases}$$

The density Ψ **is a singularity** $\rho^\lambda \psi(\vartheta)$ **of the Laplace transmission problem with** μ **:**
$\mu = \mu_{\mathsf{C}}$ **in** Γ_{C} **and** $\mu = \mu_{\mathsf{E}}$ **in** Γ_{E} **.**

Then $u = (\lambda + 1)^{-1}(\mu \nabla \Psi \times x - \nabla r)$ **with** r **solution of**

$$\begin{cases} \Delta r_{\mathsf{C}} = 0 & \text{in } \Gamma_{\mathsf{C}}, \\ \Delta r_{\mathsf{E}} = 0 & \text{in } \Gamma_{\mathsf{E}}, \\ [r] = 0, \quad \partial_n r_{\mathsf{C}} = \mu_{\mathsf{C}}(\nabla \Psi_{\mathsf{C}} \times x) \cdot n & \text{on } \mathrm{I}. \end{cases}$$

Sobolev regularity

Let $\beta_{\mathsf{E}}^{\mathsf{Dir}}$, $\beta_{\mathsf{C}}^{\mathsf{Neu}}$ and β_μ be the limiting regularity Sobolev exponents for the Dirichlet pb on Ω_{E}, Neumann pb on Ω_{C} and the transmission pb $\operatorname{div}\mu\nabla$ respectively:

$$\frac{3}{2} < \beta_\mu < 2 \quad \text{and} \quad \frac{3}{2} < \min\{\beta_{\mathsf{C}}^{\mathsf{Neu}}, \beta_{\mathsf{E}}^{\mathsf{Dir}}\} < 2$$

Then

$$E_{\mathsf{C}} \in \mathsf{H}^s(\Omega_{\mathsf{C}}) \quad \forall s < \min\{\beta_{\mathsf{C}}^{\mathsf{Neu}} - 1, \beta_\mu\}. \quad \text{and}$$

$$E_{\mathsf{E}} \in \mathsf{H}^s(\Omega_{\mathsf{E}}), \quad \forall s < \min\Big\{\min\{\beta_{\mathsf{C}}^{\mathsf{Neu}}, \beta_{\mathsf{E}}^{\mathsf{Dir}}\} - 1, \beta_\mu\Big\}.$$

Moreover

$$E = \nabla\Phi + E^{\mathsf{reg}}$$

with $E_{\mathsf{C}}^{\mathsf{reg}} \in \mathsf{H}^s(\Omega_{\mathsf{C}})$ $\forall s < \min\{\beta_{\mathsf{C}}^{\mathsf{Neu}}, \beta_\mu\}$ **and**

$$E_{\mathsf{E}}^{\mathsf{reg}} \in \mathsf{H}^s(\Omega_{\mathsf{E}}), \forall s < \min\{\min\{\beta_{\mathsf{C}}^{\mathsf{Neu}}, \beta_{\mathsf{E}}^{\mathsf{Dir}}\}, \beta_\mu\}.$$

Conclusion: How to approximate these solutions?

Combining with FEM techniques already investigated for Maxwell equations in polyhedral bodies, we may hope that the following will provide "good" approximations (considered the nasty singularities):

- **Curl-Conforming Elements (first *NÉDÉLEC* family of edge elements) – used with a Lagrange multiplier in Ω_E (*KIKUSHI* formulation)**
- **Weighted Regularization Method with nodal FEM in Ω_C and Ω_E. $\longrightarrow$**
- **Singular Complement Method (for axisymmetric domains only).**

The performances of all methods may hopefully be dramatically improved by the use of anisotropic refinement along edges and at corners, and higher degree polynomials.