NUMERICAL INVESTIGATION OF A BOUNDARY PENALIZATION METHOD FOR MAXWELL EQUATIONS

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It is well known that, in the presence of non-convex corners or edges on the boundary, nodal finite elements associated with a conformal curl-div formulation do not converge to the correct limit when the electric or magnetic boundary conditions are also imposed in the discrete space. We formulate and investigate in a simple two-dimensional situation a method where the boundary conditions are not imposed in the discrete space but obtained by a penalization method, which amounts to a sort of impedance condition.

1 Regularization by a divergence term and penalization of the boundary condition

We investigate the spectral problem for Maxwell equations with perfect conductor boundary conditions in a bounded domain Ω which we assume for the moment to be 3-dimensional. This problem consists in finding non-zero L^2 electric and magnetic eigenfields $\boldsymbol{E}$ and $\boldsymbol{H}$, and non-zero eigenfrequency ω such that

$$\begin{array}{lll} \mathbf{curl}\,\boldsymbol{E} - i\omega\,\boldsymbol{H} = 0, & \mathbf{curl}\,\boldsymbol{H} + i\omega\,\boldsymbol{E} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{E}\times\boldsymbol{n} = 0, & H_n = 0 & \text{on} \quad \partial\Omega. \end{array} \tag{1}$$

Here $\boldsymbol{n}$ denotes the unit outer normal on $\partial\Omega$ and H_n is the normal component of $\boldsymbol{H}$ on the boundary.

One of the two fields can be eliminated from equations (1), let us say $\boldsymbol{E}$, and we obtain for the magnetic field the problem $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{H} = \omega^2\boldsymbol{H}$ with the divergence constraint $\operatorname{div}\boldsymbol{H} = 0$ and the boundary condition $H_n = 0$. This latter problem admits a variational formulation in the space $\mathrm{X_T}(\Omega)$ of $\mathrm{L}^2(\Omega)$ fields $\boldsymbol{u}$ with $\mathrm{L}^2(\Omega)$ divergence and curl, and zero normal trace u_n:

Find non-zero $\boldsymbol{H} \in \mathrm{X_T}(\Omega)$ and non-zero ω such that:

$$\forall \boldsymbol{H}' \in \mathrm{X_T}(\Omega), \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{H}' = \omega^2 \int_\Omega \boldsymbol{H}\cdot\boldsymbol{H}'. \tag{2}$$

The above bilinear form $(\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)$ is not coercive on $\mathrm{X_T}(\Omega)$. To cure this, a standard procedure is the penalization by the $(\operatorname{div}\cdot, \operatorname{div}\cdot)$ form: for any $s > 0$, we introduce the new problem:

Find non-zero $\boldsymbol{u}$ and ω such that:

$$\forall \boldsymbol{v} \in \mathrm{X_T}(\Omega), \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} + s \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} = \omega^2 \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}. \tag{3}$$

Any solution $(\boldsymbol{u}, \omega)$ of problem (2) has zero divergence, thus is solution of (3) for all $s > 0$. But if Ω has non-convex edges (which is a rather standard situation if Ω is a region outside a conductor) then solutions $\boldsymbol{u}$ do not belong to H^1, in general. If one wants[a] to use curl-div conforming elements (thus continuous) for the FEM Galerkin approximation of problem (3), the discrete solution converges to the spectrum of a Lamé problem posed in the subspace $\mathrm{H_T}(\Omega)$ of $\mathrm{H}^1(\Omega)$ fields $\boldsymbol{u}$ satisfying the boundary condition $u_n = 0$, see [4] where the case of electric boundary conditions is investigated.

The reason for this phenomenon is the following: the space $\mathrm{H_T}(\Omega)$ is closed in $\mathrm{X_T}(\Omega)$ for the natural norm of this latter space. Therefore any Galerkin method using a discrete space of continuous piecewise polynomial continuous fields, thus included in $\mathrm{H_T}(\Omega)$, yields a discrete solution in $\mathrm{H_T}(\Omega)$, and is consequently unable to approach a solution of problem (3) which does not belong to $\mathrm{H_T}(\Omega)$.

But smooth fields are dense [2,3] in the larger space W defined as

$$\mathrm{W} = \big\{\boldsymbol{u} \in L^2(\Omega);\ \operatorname{div} \boldsymbol{u} \in L^2(\Omega),\ \mathbf{curl}\, \boldsymbol{u} \in L^2(\Omega),\ u_n \in L^2(\partial\Omega)\big\}.$$

Therefore, there is no theoretical obstruction to the discretization by continuous elements in the space W. But we have to retrieve the boundary conditions. This can be done by the introduction of the new bilinear form $a[s, \lambda]$ defined on $\mathrm{W} \times \mathrm{W}$ for $s > 0$ and $\lambda > 0$ as:

$$a[s, \lambda](\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} + s \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} + \lambda \int_{\partial\Omega} u_n\, v_n. \tag{4}$$

Then the boundary conditions satisfied by solutions of the problem

$$\boldsymbol{u} \in \mathrm{W}, \quad \forall \boldsymbol{v} \in \mathrm{W}, \quad a[s, \lambda](\boldsymbol{u}, \boldsymbol{v}) = \omega^2 \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}, \tag{5}$$

are all “natural” and given by

$$\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{and} \quad s \operatorname{div} \boldsymbol{u} + \lambda u_n = 0 \quad \text{on } \partial\Omega, \tag{6}$$

whereas the tangential boundary conditions associated with problem (3) are still $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = 0$ but the normal one is simply $u_n = 0$.

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[a] Possible reasons for trying to use nodal elements instead edge elements [5,1] can be
1) The wish to adapt pre-existing nodal codes,
2) The need to couple eletromagnetic data with hydrodynamics,
3) The development of simple p or hp versions,
4) Mere curiosity.

2 Spectrum of the penalized problem

Taking as test functions in problem (5) the fields gradients of a potential $\boldsymbol{v} = \mathbf{grad}\,\varphi$ where φ is any function in the domain $\mathrm{D}(\Delta^{\mathrm{Neu}})$ consisting of the functions $\psi \in \mathrm{H}^1(\Omega)$ satisfying $\Delta\psi \in \mathrm{L}^2(\Omega)$ and $\partial_n\psi = 0$ on $\partial\Omega$, we find that the L^2 function $p := \operatorname{div}\boldsymbol{u}$ is solution of

$$\forall\varphi \in \mathrm{D}(\Delta^{\mathrm{Neu}}), \quad s\int_\Omega p\,\Delta\varphi = \omega^2\Big(-\int_\Omega p\,\varphi + \int_{\partial\Omega} u_n\,\varphi\Big)\,. \tag{7}$$

Next we note that the solution $q \in \mathrm{H}^1(\Omega)$ of the Neumann problem, $-s\Delta q = \omega^2 p$ in Ω with $s\partial_n q = \omega^2 u_n$ on $\partial\Omega$, satisfies

$$\forall\varphi \in \mathrm{D}(\Delta^{\mathrm{Neu}}), \quad s\int_\Omega q\,\Delta\varphi = \omega^2\Big(-\int_\Omega p\,\varphi + \int_{\partial\Omega} u_n\,\varphi\Big)\,. \tag{8}$$

Comparing (7) and (8) we obtain that $p - q$ is orthogonal to the range of Δ from its domain $\mathrm{D}(\Delta^{\mathrm{Neu}})$, that is $p - q$ is a constant. Combining with the boundary condition $s\operatorname{div}\boldsymbol{u} + \lambda u_n = 0$ in (6), we obtain that p solves the Robin problem $-s\Delta p = \omega^2 p$ in Ω with $s\partial_n p + \omega^2\frac{s}{\lambda}\,p = 0$ on $\partial\Omega$. Going back to the variational formulation we have obtained

Lemma 1 *If (u, ω) solves problem* (5)*, then $p := \operatorname{div}\boldsymbol{u}$ belongs to $\mathrm{H}^1(\Omega)$ and solves*

$$\forall\varphi \in \mathrm{H}^1(\Omega), \quad s\int_\Omega \mathbf{grad}\,p\;\mathbf{grad}\,\varphi = \omega^2\Big(\int_\Omega p\,\varphi + \frac{s}{\lambda}\int_{\partial\Omega} p\,\varphi\Big)\,. \tag{9}$$

Theorem 2 *Let $s > 0$ and $\lambda > 0$ be fixed.*
If $(\boldsymbol{u}, \omega)$ solves problem (5)*, then* (i) *or* (ii) *holds:*
(i) $\operatorname{div}\boldsymbol{u} = 0$ *and $(\boldsymbol{u}, \omega)$ solves problem* (2)*.*
(ii) $p := \operatorname{div}\boldsymbol{u}$ *is an eigenvector of the Robin problem* (9) *and* $\mathbf{curl}\,\boldsymbol{u} = 0$*.*

PROOF. We consider $p := \operatorname{div}\boldsymbol{u}$. If $p = 0$, then $(\boldsymbol{u}, \omega)$ obviously solves problem (2). If $p \neq 0$, by Lemma 1, p is an eigenvector of the Robin problem (9). Let us introduce $\boldsymbol{w}$ defined as $-s\,\mathbf{grad}\,p/\omega^2$. We check that $\boldsymbol{w}$ belongs to W and that $(\boldsymbol{w}, \omega)$ solves problem (5). Thus the field $\boldsymbol{w}$ is in situation *(ii)*. Finally, the field $\boldsymbol{u} - \boldsymbol{w}$, if non-zero, is in situation *(i)*. ■

3 Two-dimensional case

We now assume that the domain Ω is two-dimensional. We consider the magnetic eigenproblem corresponding to (2)

$$\forall\boldsymbol{H}' \in \mathrm{X_T}(\Omega), \qquad \int_\Omega \operatorname{curl}\boldsymbol{H}\cdot\operatorname{curl}\boldsymbol{H}' = \omega^2\int_\Omega \boldsymbol{H}\cdot\boldsymbol{H}', \tag{10}$$

where $\operatorname{curl} \boldsymbol{u}$ is the scalar curl $\partial_1 u_2 - \partial_2 u_1$ and the space $\mathrm{X_T}(\Omega)$ is defined similarly with **curl** replaced by curl. Note that such solutions correspond to solutions of (1) in the cylinder domain $\Omega \times \mathbb{R}$ with an electric field oriented along the axis of the cylinder and a transverse magnetic field, both being invariant by translation. We associate to (10) its regularized-penalized version (6) with $a[s,\lambda]$ defined as

$$a[s,\lambda](\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v} + s \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} + \lambda \int_{\partial\Omega} u_n \, v_n. \qquad (11)$$

Then $\psi := \operatorname{curl} \boldsymbol{u}$ plays a similar role as the divergence and we can study ψ separately by considering test functions of the form $\mathbf{curl}\, \varphi$ with φ in the domain $\mathrm{D}(\Delta^{\mathrm{Dir}})$ of the Dirichlet problem, i.e. $\varphi \in \mathrm{H}^1_0(\Omega)$ satisfying $\Delta\varphi \in \mathrm{L}^2(\Omega)$. Theorem 2 has now a more precise version.

Theorem 3 *Let $s > 0$ and $\lambda > 0$ be fixed.*
If $(\boldsymbol{u}, \omega)$ solves problem (5)*, then* (i) *or* (ii) *holds:*
(i) $\operatorname{div} \boldsymbol{u} = 0$ *and $(\boldsymbol{u}, \omega)$ solves problem* (10)*. Moreover $\psi := \operatorname{curl} \boldsymbol{u}$ is an eigenvector of Δ^{Dir} with eigenvalue ω^2 and $\boldsymbol{u}$ is proportional to $\mathbf{curl}\, \psi$.*
(ii) $p := \operatorname{div} \boldsymbol{u}$ *is an eigenvector of the Robin problem* (9) *with eigenvalue ω^2 and* $\operatorname{curl} \boldsymbol{u} = 0$*. Moreover $\boldsymbol{u}$ is proportional to $\mathbf{grad}\, p$.*

As a consequence, in two-dimensional domains there exists an alternative way to determine the solutions of problem (5) because they all derive from potentials (**grad** or **curl**). We will take advantage of this to estimate the errors of the computations.

4 Numerical tests

The domain Ω is the *symmetric* L-shape domain $\Omega = \Sigma_0 \setminus \Sigma_1$ where Σ_0 is the square $[0,1] \times [0,1]$ and Σ_1 the square $[\frac{3}{4}, 1] \times [\frac{3}{4}, 1]$.

We use four different meshes which are regular and uniform, with triangular $\mathbb{P}_1$ or $\mathbb{P}_2$ elements. We fix $s = 30$ and vary λ by geometrical increments

Table 1. Combinations of meshes and elements

Name	Elements	h	# of triangles
Mesh 1	$\mathbb{P}_1$ or $\mathbb{P}_2$	$\frac{1}{4}$	40
Mesh 2	$\mathbb{P}_1$ or $\mathbb{P}_2$	$\frac{1}{8}$	160
Mesh 3	$\mathbb{P}_1$ or $\mathbb{P}_2$	$\frac{1}{16}$	640
Mesh 4	$\mathbb{P}_1$	$\frac{1}{32}$	2560

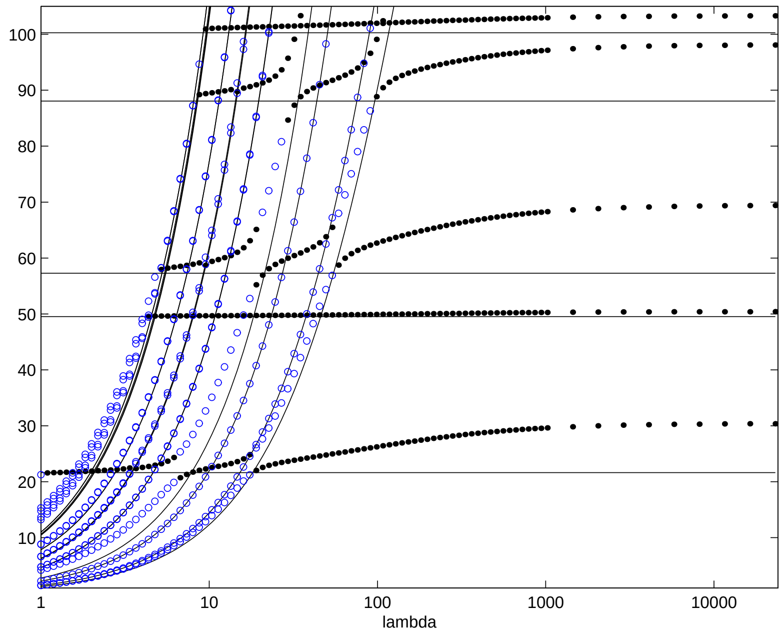

Figure 1. Lowest eigenvalues with Mesh 2 and $\mathbb{P}_2$ elements

from 1 to 24000. We compute once for all the (scalar) Dirichlet and Robin eigenvalues, then compute the Galerkin approximation of problem (5). For each computed eigenpair $(\boldsymbol{u}_h, \omega_h)$, we also compute the L^2 norms of $\operatorname{curl} \boldsymbol{u}_h$, $\operatorname{div} \boldsymbol{u}_h$ and of the normal trace on the boundary u_{hn}, each of them normalized by the $\mathrm{L}^2(\Omega)$-norm of $\boldsymbol{u}_h$. Thus we can sort the eigenpairs according to the value of the ratio

$$\frac{\| \operatorname{curl} \boldsymbol{u}_h \|^2_{\mathrm{L}^2(\Omega)}}{s \| \operatorname{div} \boldsymbol{u}_h \|^2_{\mathrm{L}^2(\Omega)} + \lambda \| u_{hn} \|^2_{\mathrm{L}^2(\partial\Omega)}} \,.$$

In Figure 1, we plot ω^2 versus λ and we represent by bullets and circles the computed eigenvalues for which this ratio is larger (curl type) and smaller (gradient type) than 1 respectively. The solid horizontal lines are the eigenvalues of Δ^{Dir} (case *(i)* in Theorem 3) and the curved solid lines are the Robin eigenvalues (case *(ii)*).

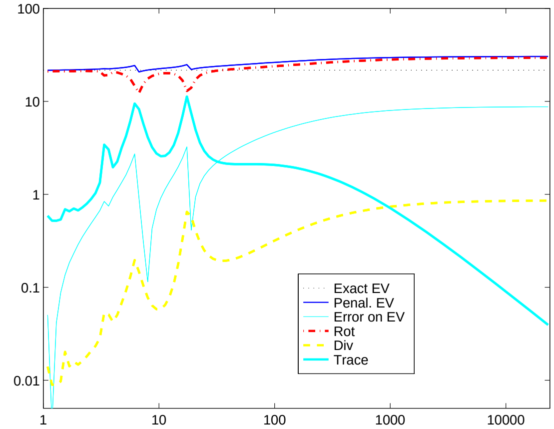

Figure 2. First eigenvalue of curl type (Mesh 2 and $\mathbb{P}_2$)

In Figures 2 and 3, we plot the first and second eigenvalues of curl type *(i)* along with the parts in the energy of their curls, divergence and trace

$$\|\operatorname{curl}\boldsymbol{u}_h\|^2_{\mathrm{L}^2(\Omega)}\,,\quad s\|\operatorname{div}\boldsymbol{u}_h\|^2_{\mathrm{L}^2(\Omega)}\,,\quad \lambda\|u_{hn}\|^2_{\mathrm{L}^2(\partial\Omega)}\,.$$

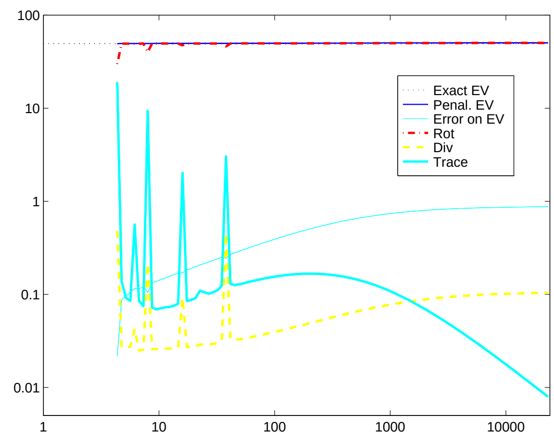

Figure 3. Second eigenvalue of curl type (Mesh 2 and $\mathbb{P}_2$)

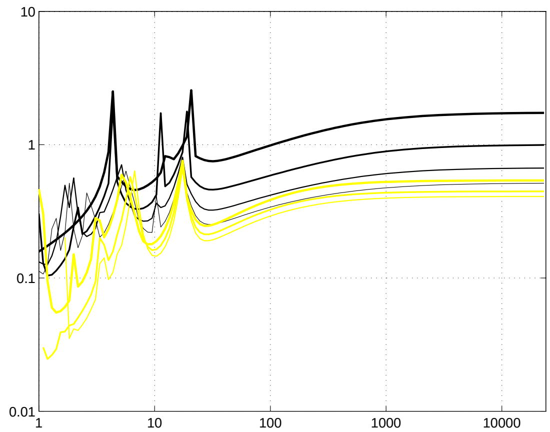

Figure 4. Errors on the first eigenvalue of curl type

In Figures 4 and 5 we plot the relative errors corresponding to the the first and second eigenvalues of curl type, with Mesh 1 to 4 with $\mathbb{P}_1$ elements (dark lines, from thickest to thinnest) and with Mesh 1 to 3 with $\mathbb{P}_2$ elements (lighter lines). We evaluate these errors e_h in the following way:

$$e_h := \Big(|\omega^2 - \omega_h^2| + \|\operatorname{div}\boldsymbol{u}_h\|^2_{\mathrm{L}^2(\Omega)} + \lambda\|u_{hn}\|^2_{\mathrm{L}^2(\partial\Omega)}\Big)/\omega^2.$$

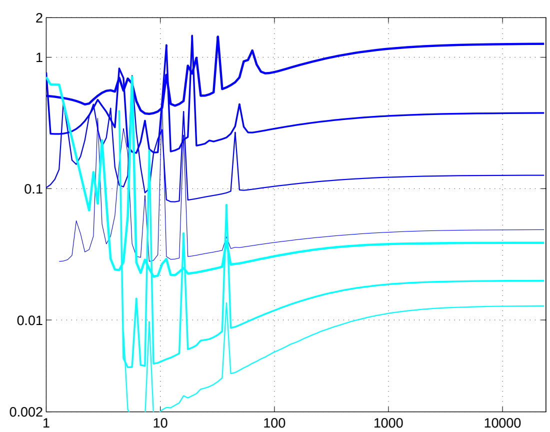

Figure 5. Errors on the second eigenvalue of curl type

The behaviors of the errors in Figures 4 and 5 are very different because the first eigenfunction has the strong non H^1 singularity whereas the coefficient in front of this singularity is zero for the second eigenfunction for symmetry reasons. We see that we have convergence as $h \to 0$ (albeit slow) in the case of the second, regular, eigenfunction, whereas for the first eigenvalue only for low values of λ a sort of convergence is observable. The lack of convergence for large λ cannot be improved even by strong mesh refinements near the reentrant corner. Further studies will be necessary to determine if there is a kind of locking mechanism involved that can be overcome by the choice of higher order elements or *h*-*p* methods.

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