LMS Durham Symposium: Computational methods for wave propagation in direct scattering. 15-25 July 2002, Durham, UK

Computation of resonance frequencies for Maxwell equations in non smooth domains

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Outline

- Maxwell frequencies of the cavity problem.
- Description of singularities of the Maxwell eigenmodes.
- Difficulties for the approximation due to
  - ★ The infinite dimensional kernel,
  - ★ Non-H^1 singularities in reentrant corners and edges.
- Edge elements.
- Regularization :
  - ★ Plain regularization does not work.
  - ★★ Regularization with augmented FEM spaces by non-H^1 singularities works for 2D problems.
  - ★★★ Weighted regularization works in 2D and 3D.

Maxwell eigenvalue problem

Permittivity ε and permeability μ. Find **non-zero frequency ω** such that $\exists\,(E,H)\neq 0$

(Maxwell)
$$\begin{cases} \operatorname{curl} E - i\omega\,\mu H = 0 \quad \& \quad \operatorname{curl} H + i\omega\,\varepsilon E = 0 \quad \text{in} \quad \Omega, \\ E\times n = 0 \quad \& \quad H\cdot n = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

Perfect conductor boundary conditions [a]

To simplify the exposition, consider homogeneous and isotropic medium: ε, μ constant > 0. May assume $\varepsilon = \mu = 1$.

$$\Longrightarrow \qquad \operatorname{div} E = 0 \quad \& \quad \operatorname{div} H = 0$$

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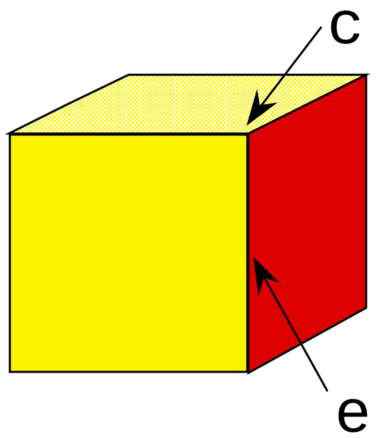
[a] One could also consider impedance b. c. The regularity of the eigenmodes would be the same as with perfect conductor b. c. The approximation would be, in principle, better. But, from a practical point of view, the methods suitable for perfect conductor b. c. give improved results for impedance b. c.

3D Edge Singular Functions of the Laplace Operator



Cavity or domain Ω is a 3D polyhedron.

Look at singular functions of Δ with Dirichlet because

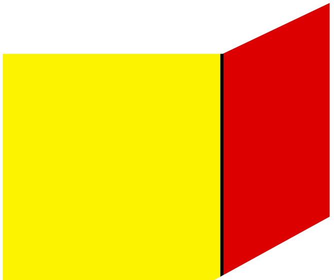
1. **This is similar to Maxwell but simpler,**
2. **Maxwell sing. f. almost all derive from Laplace sing. f.**

Edge e. Locally $\Omega \simeq \Gamma_e \times \mathbb{R}$ with coordinates

$$(r, \theta) \in \Gamma_e = \{r > 0,\ 0 < \theta < \omega_e\} \quad \text{in sector,} \quad \text{and} \quad z \in \mathbb{R}.$$

Local expression of the singular functions: integer $\ell \geq 1$

$$\boxed{\Phi_e = \gamma_e(z)\, r^{\ell\pi/\omega_e} \sin \frac{\ell\pi\theta}{\omega_e}}$$



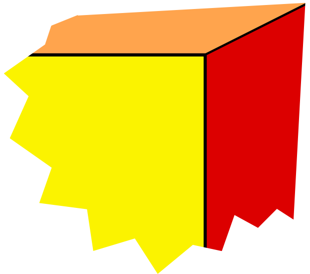
3D Corner Singular Functions of the Laplace Operator

Corner c. Locally $\Omega \simeq \Gamma_c$ with

$$\Gamma_c = \{(\rho, \vartheta) \mid \rho > 0,\ \vartheta \in G_c \subset \mathbb{S}^2\} \quad \text{cone.}$$

Singular functions at the corner c:

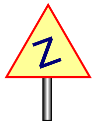
$$\boxed{\Phi_c = \gamma_c\, \rho^\lambda\, \phi_c(\vartheta)}$$



defining the space Z^λ_{Dir} and where the exponent $\lambda \in \Lambda^{\mathrm{Dir}}_c$ satisfies

$$\lambda = -\frac{1}{2} + \sqrt{\mu + \frac{1}{4}}$$

with μ eigenvalue of the Laplace-Beltrami Dirichlet problem on G_c.



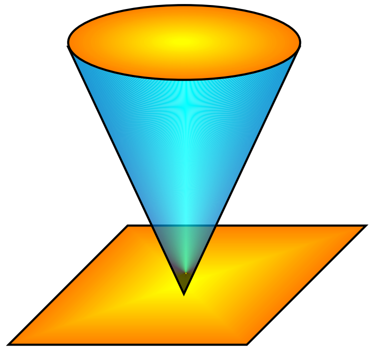
ϕ_c has singularities at the vertices v_e of G_c – with obvious notation $v_e \in e$.

More on the web *M. Dauge. "Simple" Corner-Edge Asymptotics.*

3D Corner Singular Functions of the Maxwell eigenproblem

Most of sing. f. derive from Laplace sing. f.

$Z^{\lambda}_{\mathrm{Dir,\ Neu}}$ **spaces of sing. f. homogeneous with deg.** λ **.**



Topological sing. f. appear when ∂G **is multiply connected.**

$P_{\mathrm{Dir}} = \{\Phi \in H^1(G) \mid \ \Delta_G \Phi = 0 \text{ in } G, \Phi = c_j \text{ on } \partial_j G\}$

Type	Generator	λ	E	H
Δ (elect.)	$\Phi \in Z^{\lambda}_{\mathrm{Dir}}$	$\lambda \in \Lambda^{\mathrm{Dir}}$	$\mathrm{grad}\,\Phi$	$-ik\ \mathrm{grad}\,\Phi \times x$
Δ (magn.)	$\Psi \in Z^{\lambda}_{\mathrm{Neu}}$	$\lambda \in \Lambda^{\mathrm{Neu}}$	$ik\ \mathrm{grad}\,\Psi \times x$	$\mathrm{grad}\,\Psi$
Top (elect.)	$\Phi \in P_{\mathrm{Dir}}$	0	$\mathrm{grad}\,\Phi$	$-ik\ \mathrm{grad}\,\Phi \times x$
Top (magn.)	$\Psi \in P_{\mathrm{Neu}}$	0	$ik\,\widetilde{\mathrm{grad}}\Psi \times x$	$\widetilde{\mathrm{grad}}\Psi$

$$k = \omega(\lambda + 1)^{-1}$$

(Poor) regularity of Maxwell eigenmodes

Define, with $\mathfrak{C}$ the set of corners c and $\mathfrak{E}$ the set of edges e

$$\tau_{\mathfrak{C}}^{\mathrm{Dir}} = \min_{c\in\mathfrak{C}} (\Lambda_c^{\mathrm{Dir}} \cap \mathbb{R}_+)\,, \quad \tau_{\mathfrak{C}}^{\mathrm{Neu}} = \min_{c\in\mathfrak{C}} (\Lambda_c^{\mathrm{Neu}} \cap \mathbb{R}_+)\,, \quad \tau_{\mathfrak{E}} = \min_{e\in\mathfrak{E}} \frac{\pi}{\omega_e}$$

Regularity $E \in H^\tau(\Omega)$ for all τ such that (δ arbitrarily small)

Ω	$< \tau$	$\tau <$
Screens, or not locally simply connected	$\frac{1}{2} - \delta$	$\frac{1}{2}$
No screen, locally simply connected, but not convex	$\frac{1}{2}$	$\min\{\tau_{\mathfrak{E}},\ \tau_{\mathfrak{C}}^{\mathrm{Dir}} + \frac{1}{2}\} < 1$
Convex	1	$\min\{\tau_{\mathfrak{E}},\ \tau_{\mathfrak{C}}^{\mathrm{Neu}} + \frac{3}{2}\}$
Parallelepiped	$3 - \delta$	3

Thorough analysis of singular functions in *CoDA, 1997, 1998, 2000.*

Poorer regularity of Maxwell eigenmodes for inhomogeneous materials

For piecewise constant ε and μ on polyhedral subdomains of Ω, it is still possible to describe all sing. f. (*CoDa-Nicaise*, *1999*).

But the regularity may be much lower

$$E \in H^{\tau}(\Omega), \quad \forall \tau < \tau_0(\varepsilon, \mu)$$

and for all $\delta > 0$, $\exists \varepsilon$ such that $\boxed{\tau_0(\varepsilon, \mu) < \delta}$

(Non-Hodge) Decomposition of eigenmodes

An outcome of the sing. f. analysis is the decomposition of electric eigenmodes E :

$$E = E_0 + \operatorname{grad} \Phi : \quad \begin{cases} E_0 \in H_N := \{u \in H^1(\Omega)^3,\ u \times n\big|_{\partial\Omega} = 0\}, \\ \Phi \in H^1(\Omega),\ \Delta\Phi \in L^2(\Omega),\ \Phi\big|_{\partial^*\Omega} \in H^{3/2}(\partial^*\Omega) \end{cases}$$

where $\partial^*\Omega$ is the unfolded version of the boundary $\partial\Omega$ (contact points – topological singularities or screen surfaces – are doubled). Extension of *Birman-Solomyak*, *1993*.

Note that the singular part satisfies

$$\operatorname{grad} \Phi \in X_N := H_0(\operatorname{curl}) \cap H(\operatorname{div}).$$

A cause of trouble is the fact – *Co*, *1991* :

$\boxed{H_N \text{ is closed in } X_N}$

for the topology of X_N : $\|\cdot\|_{L^2} + \|\operatorname{curl}\cdot\|_{L^2} + \|\operatorname{div}\cdot\|_{L^2}$.

?

Equations of the electric eigenmode $E = u$

$$\operatorname{curl}\operatorname{curl} u = \omega^2 u, \quad \operatorname{div} u = 0, \quad u \times n\big|_{\partial\Omega} = 0.$$

Look for space $\mathfrak{X}$ and form a so that u is solution of the $\boxed{\text{Galerkin problem}}$

$$(\mathfrak{P}) \qquad u \in \mathfrak{X}, \quad \forall v \in \mathfrak{X}, \qquad a(u,v) = \omega^2 \int_\Omega u \cdot v\,,$$

with $\boxed{\text{possible discretizations}}$ in FEM subspaces $\mathfrak{X}_h$ of $\mathfrak{X}$

$$(\mathfrak{P}_h) \qquad u_h \in \mathfrak{X}_h, \quad \forall v_h \in \mathfrak{X}_h, \qquad a(u_h,v_h) = \omega_h^2 \int_\Omega u_h \cdot v_h\,,$$

so that $\boxed{\omega_h \longrightarrow \omega}$ in the sense that the m-th non-zero eigenfreq. $\omega_{h,m}$ of $(\mathfrak{P}_h)$ converges to the m-th non-zero eigenfreq. ω_m of $(\mathfrak{P})$.

Minimal space is not a good choice

Minimal space $\mathfrak{X} = H_0(\mathrm{curl}) \cap H(\mathrm{div}; 0)$ i.e. divergence free fields in $H_0(\mathrm{curl})$.

The form $a = a_0$ with a_0 the curl bilinear form

$$a_0(u, v) = \int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\, v \,.$$

Continuous Problem $(\mathfrak{P})$ has exactly the Maxwell electric eigenmodes as solutions.

But . . .

any FEM space $\mathfrak{X}_h$ contained in such $\mathfrak{X}$ would be curl and div conforming, therefore grad conforming, therefore contained in H_N.

For non-convex polyhedra, $H_N \neq X_N$ and H_N is closed in X_N

$\Longrightarrow$ ***obstruction to*** $\omega_{h,m} \to \omega_m$

Maximal space is not a good choice

Maximal space $\mathfrak{X} = H_0(\mathrm{curl})$. The form a is still the curl bilinear form a_0

$$a_0(u,v) = \int_\Omega \mathrm{curl}\, u \cdot \mathrm{curl}\, v\,.$$

The non-zero eigenfreq. of Continuous Pb. $(\mathfrak{P})$ are exactly the Maxwell eigenfreq.
But $(\mathfrak{P})$ has the kernel

$$K = \{u \in H_0(\mathrm{curl})\ , \quad \mathrm{curl}\, u = 0\}.$$

Suppose for simplicity Ω simply connected. Then

$$K = \{u = \mathrm{grad}\, p\ , \quad p \in H^1_0(\Omega)\}.$$

Thus K is infinite dimensional.

The interesting spectrum lies between the two “points” 0 and $+\infty$ of the essential spectrum of $(\mathfrak{P})$. *Descloux*, *1981*, tells you that you are in bad shape from the approximation point of view.

Mimicking the kernel: edge elements

Edge elements initiated by *NÉDÉLEC*, *1980*, contain several families of compatible FEM spaces and projectors for 0-forms (potential p) and 1-forms (electric field u):

For p	space P_h of Lagrange elements	grad conforming	projector π_h
For u	space V_h of edge elements	curl conforming	projector r_h

and the compatibility is the commuting diagram property

$$\boxed{r_h(\mathrm{grad}\, p) = \mathrm{grad}(\pi_h p)}$$

Take – with $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(\Omega)$

$$\mathfrak{X} = \{u \in H_0(\mathrm{curl}) \ , \quad \forall v \in K, \ \langle u, v\rangle = 0\}$$

$$\mathfrak{X}_h = \{u_h \in V_h \ , \quad \forall v_h \in K \cap V_h, \ \langle u_h, v_h\rangle = 0\}$$

Relations $K = \mathrm{grad}(H^1_0)$ & $\boxed{K \cap V_h = \mathrm{grad}(P_h)}$ allow convergence proofs . . .

Note: non-conforming approximation: $\mathfrak{X} \subset H(\mathrm{div}; 0)$ but $\mathfrak{X}_h \not\subset H(\mathrm{div}; 0)$.

Edge elements: Questions to the Audience

Implementation: which one among

- **Direct use of $\mathfrak{X}_h$ (preconstrained finite element space?)**
- **Saddle point problem in $V_h \times P_h$**
  (possibly combined with discrete regularization)
- **Triple field formulation ?**

State of the art for proofs:

- **h-version: optimal if $u \in H^{1/2+\delta}(\Omega)$ and $p \in H^{3/2+\delta}(\Omega)$.**
  ***cf survey* HIPTMAIR, *2002*.**
- **p- and hp-version: On the way *BOFFI-DEMKOWICZ-CO, 2002*.**

A third bad idea: the Plain Regularization

Take $\mathfrak{X} = H_0(\mathrm{curl}) \cap H(\mathrm{div})$ and $a = a_s$ for a $s > 0$ where

$$a_s(u, v) = a_0(u, v) + s \int_\Omega \mathrm{div}\, u \; \mathrm{div}\, v \, .$$

Yields problem $\mathfrak{P}^{[s]}$.

Denote by $\sigma(\mathfrak{P})$ the spectrum of a problem $\mathfrak{P}$.

$\mathfrak{M}$ the Maxwell problem and Δ^{Dir} the Dirichlet problem for Δ.

$$\sigma(\mathfrak{P}^{[s]}) = \sigma(\mathfrak{M}) \cup s\, \sigma(\Delta^{\mathrm{Dir}}).$$

But any FEM space $\mathfrak{X}_h$ contained in such $\mathfrak{X}$ would be curl and div conforming, therefore grad conforming, therefore contained in H_N, etc...

Plain Regularization with a Singular Function Method

An idea to repair the Plain Regularization is to replace the FEM spaces with augmented spaces by those singular functions which cannot be approximated by the (nodal) FEM spaces. Two different realizations of this idea.

(i) Bonnet-Hazard-Lohrengel, 1999. In 2D polygonal domains.

Add to nodal FEM spaces X_h the field $S_c^{\mathrm{Dir}} := \mathrm{grad}\left(r^{\pi/\omega_c}\sin\frac{\pi\theta}{\omega_c}\right)$ for each reentrant corner c. No cut-off. Correction on the boundary instead: to $X_{h,0}$ (i.e. with boundary condition) add $U_c^{\mathrm{Dir}} = S_c^{\mathrm{Dir}} - T_c$ with $T_c \in X_h$ solution of

$$\forall v_h \in X_{h,0}, \quad a_s(T_c, v_h) = 0, \quad T_c \times n = \gamma_h(S_c^{\mathrm{Dir}} \times n).$$

(ii) Assous-Ciarlet Jr-Sonnendrücker, 1998. In 2D polygonal domains.

Continuous analogue: $H_0(\mathrm{curl}) \cap H(\mathrm{div};0) = H_N \cap H(\mathrm{div};0) \overset{\perp}{\oplus} \mathrm{curl}(\Phi^{\mathrm{Neu}})$ where Φ^{Neu} is a special space of singular solutions of the Neumann problem for Δ.

The Weighted Regularization

Costabel–Dauge, *2002*.

Take $a = \tilde{a}_s$ for a $s > 0$ where

$$\tilde{a}_s(u,v) = a_0(u,v) + s \int_\Omega w(x)^2 \;\operatorname{div} u \;\operatorname{div} v \,\mathrm{d}x,$$

where $w > 0$ is a bounded weight and

$$\mathfrak{X} = X_N^w := H_0(\operatorname{curl}) \cap \{u \in L^2(\Omega)^3 \;,\; w \operatorname{div} u \in L^2(\Omega)\}.$$

Yields problem $\mathfrak{P}_w^{[s]}$.

Denote by $\sigma(\mathfrak{P})$ the spectrum of a problem $\mathfrak{P}$ and by $\mathfrak{M}$ the Maxwell problem.

Let L_w^{Dir} be the Dirichlet problem for the weighted Laplacian $w \,\Delta\, w$:

$$\sigma(\mathfrak{P}_w^{[s]}) = \sigma(\mathfrak{M}) \cup s\,\sigma(L_w^{\mathrm{Dir}}).$$

The topology of $\mathfrak{X}$ is now : $\|\cdot\|_{L^2} + \|\operatorname{curl}\cdot\|_{L^2} + \|w \;\operatorname{div}\cdot\|_{L^2}$.

The class of weights of the Weighted Regularization

We take the weight w of the form

$$\boxed{w(x) = d(x)^\gamma \quad \text{with} \quad 0 \le \gamma \le 1 \quad \text{and} \quad d(x) = \operatorname{dist}(x, \mathfrak{S})}$$

where $\mathfrak{S}$ is the set of non-convex corners of Ω in 2D and non-convex edges in 3D.

Define the Laplace-Dirichlet operator $\Delta_\gamma^{\mathrm{Dir}}$ as

$$\begin{aligned}
\Delta_\gamma^{\mathrm{Dir}} : \quad \mathcal{D}(\Delta_\gamma^{\mathrm{Dir}}) := \quad \{\varphi \in H_0^1(\Omega) \mid \Delta\varphi \in L_\gamma^2(\Omega)\} \quad &\longrightarrow \quad L_\gamma^2(\Omega) \\
\varphi \quad &\longmapsto \quad \Delta\varphi.
\end{aligned}$$

Theorem : Denote $X_N^\gamma := X_N^{d^\gamma}$.

(i) Any element $u \in X_N^\gamma$ can be decomposed into the sum

$$\boxed{u = w + \operatorname{grad}\varphi, \quad \textit{with} \quad w \in H_N \quad \textit{and} \quad \varphi \in \mathcal{D}(\Delta_\gamma^{\mathrm{Dir}})}$$

(ii) If $H^2 \cap H_0^1(\Omega)$ is dense in $\mathcal{D}(\Delta_\gamma^{\mathrm{Dir}})$, then H_N is dense in X_N^γ.

Choose the weight of the Weighted Regularization

For any polygonal domain (2D) or polyhedral domain (3D), exists $\tau(\Omega) > 0$ such that

$$\forall\gamma,\quad 1-\tau(\Omega) < \gamma \le 1,\quad H_N \text{ is dense in } X_N^\gamma.$$

Precisely, with $\tau_{\mathfrak{E}} = \min\limits_{e\in\mathfrak{E}} \frac{\pi}{\omega_e}$ and $\tau_{\mathfrak{C}}^{\mathrm{Dir}} = \min\limits_{c\in\mathfrak{C}} (\Lambda_c^{\mathrm{Dir}} \cap \mathbb{R}_+)$:

$$\tau(\Omega) = \tau_{\mathfrak{E}} \text{ in 2D} \qquad \text{and} \qquad \tau(\Omega) = \min\{\tau_{\mathfrak{E}},\ \tau_{\mathfrak{C}}^{\mathrm{Dir}} + \frac{1}{2}\} \text{ in 3D.}$$

Denote by L_γ^{Dir} the “spurious” weighted Laplacian $d^\gamma \Delta d^\gamma$.

Moreover, there exists $\sigma_0 > 0$ such that for all $\gamma \le 1$ the spectrum of L_γ^{Dir} satisfies

$$\boxed{\sigma(L_\gamma^{\mathrm{Dir}}) \ge \sigma_0}$$

NB

★ $\forall\gamma < 1$, L_γ^{Dir} has a purely discrete spectrum with only accumulation at $+\infty$.

★★ For $\gamma = 1$, $\sigma(L_\gamma^{\mathrm{Dir}})$ contains a full interval $[\sigma_1, +\infty)$ (essential spectrum).