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Polynomial extension operators for spaces $\mathbb{H}^1$, $\mathrm{H}(\mathrm{curl})$ and $\mathrm{H}(\mathrm{div})$ on a cube

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Elementary reference domains and spaces ($p \geq 0$ integer):

- Interval $I := (-1, 1)$
  $\mathbb{P}^p(I)$ polynomials of degree $\leq p$
  Fractional order Sobolev spaces $\mathbb{H}^{\frac{1}{2}}(I)$, $\widetilde{\mathbb{H}}^{\frac{1}{2}}(I)$, $\mathbb{H}^{-\frac{1}{2}}(I)$, $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(I)$
- Square $\Sigma := I \times I$
  $\mathbb{W}_p(\Sigma) = \mathbb{P}^p(I) \otimes \mathbb{P}^p(I)$ polynomials of partial degree $\leq p$
  $\mathbb{W}_p(\partial\Sigma)$ polynomial traces (deg. p on each side & continuity at corners)
- Cube $\Omega := I \times I \times I$
  Discrete and continuous spaces for the De Rahm complex
  - $\mathbb{W}_p(\Omega) = \mathbb{P}^p(I) \otimes \mathbb{P}^p(I) \otimes \mathbb{P}^p(I)$ for $\mathbb{H}^1(\Omega)$
  - $\mathbf{Q}_p(\Omega)$ (Nédélec first kind) for $\mathbf{H}(\mathrm{curl}, \Omega)$
  - $\mathbf{V}_p(\Omega)$ (Raviart-Thomas) for $\mathbf{H}(\mathrm{div}, \Omega)$
  - $\mathbb{Y}_p(\Omega) = \mathbb{W}_{p-1}(\Omega)$ for $\mathbb{L}^2(\Omega)$

  Trace operators and spaces on the boundary $\partial\Omega$
  - Standard γ_0, spaces $\mathbb{W}_p(\partial\Omega)$ & $\mathbb{H}^{\frac{1}{2}}(\partial\Omega)$
  - Tangential γ_{T}, spaces $\mathbf{Q}_p(\partial\Omega)$ & $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}, \partial\Omega)$
  - Normal γ_{N}, spaces $\mathbb{V}_p(\partial\Omega)$ & $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$
  - Average γ_{avg}, spaces $\mathbb{R}$ & $\mathbb{R}$ ...

Q1 Is it possible to find a polynomial characterization of

- Norm $\mathbb{H}^{\frac{1}{2}}(I)$ in $\mathbb{P}^p(I)$
- Norm $\widetilde{\mathbb{H}}^{\frac{1}{2}}(I)$ in $\mathbb{P}^p_0(I) := \mathbb{P}^p(I) \cap \mathbb{H}^1_0(I)$

i.e. find polynomial bases playing a similar role as $e^{ik\theta}$ on the torus.

Q2 Is it possible to find polynomial extension operators

- From $\mathbb{W}_p(\partial\Omega)$ into $\mathbb{W}_p(\Omega)$ lifting trace op. γ_0
- From $\mathbb{Q}_p(\partial\Omega)$ into $\mathbf{Q}_p(\Omega)$ lifting trace op. γ_{T}
- From $\mathbb{V}_p(\partial\Omega)$ into $\mathbf{V}_p(\Omega)$ lifting trace op. γ_{N}

with operator norms bounded independently of degree p?

- Answer to Q1 is important
  - for hp adaptivity
  - for answering Q2
- Answer to Q2 is important
  - for analysing p and hp versions of Finite Element (Maxwell case)
  - for hp adaptivity

Elementary answers ??

Dirichlet

The norms in $\mathbb{H}^1_0(I)$ and $\mathbb{L}^2(I)$ can be characterized by the normalized Dirichlet eigenfunctions Φ_n and their eigenvalues $\lambda_n = \pi^2 n^2/4$:

For $u \in \mathbb{H}^1_0(I)$ represented as $\sum_{n=1}^{\infty} u_n \Phi_n$:

$$\|u\|^2_{\mathbb{H}^1(I)} = \sum_{n=1}^{\infty} (1+\lambda_n)\,|u_n|^2 \quad \text{and} \quad \|u\|^2_{\mathbb{L}^2(I)} = \sum_{n=1}^{\infty} |u_n|^2$$

By interpolation

$$\boxed{\|u\|^2_{\widetilde{\mathbb{H}}^{\frac12}(I)} \simeq \sum_{n=1}^{\infty} (1+\sqrt{\lambda_n})\,|u_n|^2}$$

Neumann

Normalized Neumann eigenfunctions Ψ_n and eigenvalues $\mu_n = \pi^2 n^2/4$:

For $u \in \mathbb{H}^1(I)$ represented as $\sum_{n=0}^{\infty} u_n \Psi_n$:

$$\|u\|^2_{\mathbb{H}^1(I)} = \sum_{n=0}^{\infty} (1+\mu_n)\,|u_n|^2 \quad \text{and} \quad \|u\|^2_{\mathbb{L}^2(I)} = \sum_{n=0}^{\infty} |u_n|^2$$

By interpolation

$$\boxed{\|u\|^2_{\mathbb{H}^{\frac12}(I)} \simeq \sum_{n=0}^{\infty} (1+\sqrt{\mu_n})\,|u_n|^2}$$

Answer to Q1 for $\mathbb{P}^p_0(I)$

(i) The norms $\mathbb{H}^1(I)$ and $\mathbb{L}^2(I)$ in $\mathbb{P}^p_0(I)$ can be characterized by the discrete normalized Dirichlet eigenfunctions $\Phi_n^{(p)}$ and their eigenvalues $\lambda_n^{(p)}$:
For polynomial $u \in \mathbb{P}^p_0(I)$ represented as

$$\sum_{n=2}^{p} u_n^{(p)} \Phi_n^{(p)}$$

there holds (stable basis, cf also [Beuchler,Schöberl'05]) :

$$\|u\|^2_{\mathbb{H}^1(I)} = \sum_{n=2}^{p} (1+\lambda_n^{(p)})\,|u_n^{(p)}|^2 \quad \text{and} \quad \|u\|^2_{\mathbb{L}^2(I)} = \sum_{n=2}^{p} |u_n^{(p)}|^2$$

By interpolation ?? Needs :

$$\left[\mathbb{P}^p_0(I)_{\,\mathbb{H}^1_0(I)}\,,\mathbb{P}^p_0(I)_{\,\mathbb{L}^2(I)}\right] = \mathbb{P}^p_0(I)_{\,\widetilde{\mathbb{H}}^{\frac12}(I)}$$

Proved by [Bernardi,Dauge,Maday'07] using

- trace method
- weighted estimates for [Babuška,Suri'87] extension operator.

Conclusion

$$\boxed{\|u\|^2_{\widetilde{\mathbb{H}}^{\frac12}(I)} \simeq \sum_{n=2}^{p} \left(1+\sqrt{\lambda_n^{(p)}}\right) |u_n^{(p)}|^2} \quad \text{uniformly}$$

(ii) Denote the discrete normalized Neumann eigenfunctions on $\mathbb{P}^p(I)$ by $\Psi_n^{(p)}$ and their eigenvalues by $\mu_n^{(p)}$.

For $u \in \mathbb{P}^p(I)$:

$$u = \sum_{n=0}^{p} u_n^{(p)} \Psi_n^{(p)} \quad \Longrightarrow \quad \boxed{\|u\|^2_{\mathbb{H}^{\frac{1}{2}}(I)} \simeq \sum_{n=0}^{p} \left(1 + \sqrt{\mu_n^{(p)}}\right) |u_n^{(p)}|^2}$$

(iii) Using the isomorphism

$$\partial \ : \ \mathbb{H}^{\frac{1}{2}}_{\text{avg}}(I) \longrightarrow \mathbb{H}^{-\frac{1}{2}}(I)$$

implies a characterization of norm $\mathbb{H}^{-\frac{1}{2}}(I)$ on $\mathbb{P}^{p-1}(I)$:

$$u = \sum_{n=1}^{p} u_n^{(p)} \frac{\Psi_n^{(p)\,\prime}}{\sqrt{\mu_n^{(p)}}} \quad \Longrightarrow \quad \boxed{\|u\|^2_{\mathbb{H}^{-\frac{1}{2}}(I)} \simeq \sum_{n=1}^{p} \frac{1}{\sqrt{\mu_n^{(p)}}} |u_n^{(p)}|^2}$$

(iv) Similarly, the isomorphism

$$\partial \ : \ \widetilde{\mathbb{H}}^{\frac{1}{2}}(I) \longrightarrow \widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\text{avg}}(I)$$

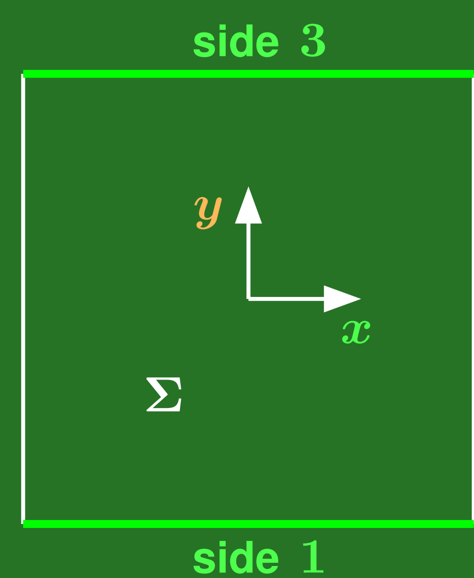
allows a characterization of norm $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(I)$ on $\mathbb{P}^{p-1}(I)$ using the expansion along a constant function and derivatives of $\Phi_n^{(p)}$.

Let $u \in \mathbb{W}_p(\partial\Sigma)$. Construction of a lifting operator, cf [Pavarino,Widlund'96]

- Lift independently the restrictions $\underline{u}_{(1,3)}$ of u on two opposite sides by discrete Neumann expansions:

For $\underline{u}(x) = \sum_{j=0}^{p} u_n \Psi_n^{(p)}(x)$, set $U(x,y) = \sum_{j=0}^{p} u_n \Psi_n^{(p)}(x)\, \beta_n^{(p)}(y)$

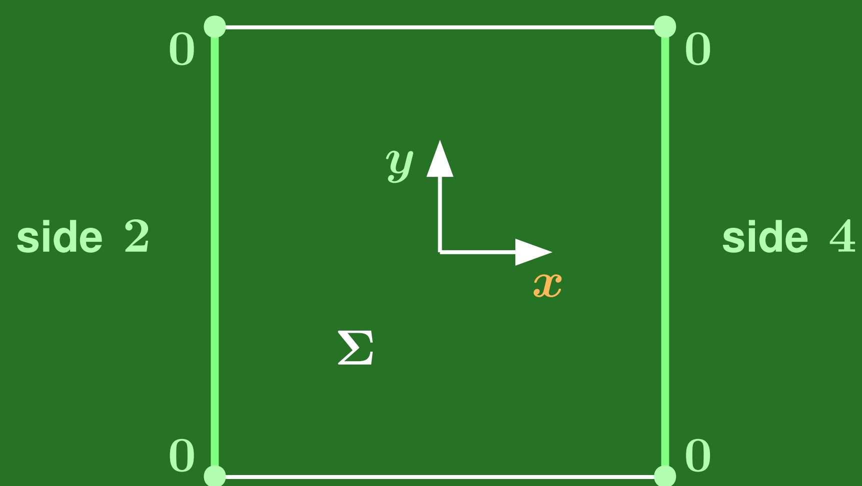
where $\beta_n^{(p)}$ is a polynomial analogue on I of the decreasing exponential $t \longmapsto \exp(-\mu_n^{(p)} t)$ on $\mathbb{R}_+$.



- Set $v = u - U_1|_{\partial\Sigma} - U_3|_{\partial\Sigma}$.
  Lift independently the restrictions $\underline{v}_{(2,4)}$ of v on the two remaining sides by discrete Dirichlet expansions:

  For $\underline{v}(y) = \sum_{j=2}^{p} v_n \Phi_n^{(p)}(y)$, set $V(x,y) = \sum_{j=2}^{p} v_n \, \alpha_n^{(p)}(x) \, \Phi_n^{(p)}(y)$

  where $\alpha_n^{(p)}$ is a polynomial analogue of $t \longmapsto \exp(-\lambda_n^{(p)} t)$.



- Set finally $U = U_1 + U_3 + V_2 + V_4$ and obtain uniform estimates

$$\|U\|_{\mathbb{H}^1(\Sigma)} \leq C \|u\|_{\mathbb{H}^{\frac{1}{2}}(\partial\Sigma)}$$

thanks to answer to Q1.

Let $h \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\underline{h} \in H^{-\frac{1}{2}}(F_1)$ its restriction to face 1. For

$$\underline{h}(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} h_{nm}\, \Phi_n(x)\, \Phi_m(y)$$

set

$$U(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} h_{nm}\, \Phi_n(x)\, \Phi_m(y)\, \gamma_{nm}(z) \tag{1}$$

so that $\Delta U = 0$, $\partial_n U = \underline{h}$ on F_1, and 0 on opposite F_4,
and $U = 0$ on the 4 remaining faces.

The first piece of the lifting of h is

$$H_1 = \nabla U. \tag{2}$$

Based on relations $\Phi_n = -\mu_n^{-\frac{1}{2}}\, \Psi'_n$ and $\Phi'_n = \sqrt{\mu_n}\, \Psi_n$:

$$\begin{aligned} H_1 \;=\; \Bigg(& -\sum_{n=1}^{p}\sum_{m=1}^{p} h_{nm}\, \sqrt{\mu_n}\Psi_n(x)\, \frac{\Psi'_m(y)}{\sqrt{\mu_m}}\, \gamma_{nm}(z)\,, \\ & -\sum_{n=1}^{p}\sum_{m=1}^{p} h_{nm}\, \frac{\Psi'_n(x)}{\sqrt{\mu_n}}\, \sqrt{\mu_m}\Psi_m(y)\, \gamma_{nm}(z)\,, \\ & \sum_{n=1}^{p}\sum_{m=1}^{p} h_{nm}\, \frac{\Psi'_n(x)}{\sqrt{\mu_n}}\, \frac{\Psi'_m(y)}{\sqrt{\mu_m}}\, \gamma'_{nm}(z) \Bigg) \end{aligned} \tag{3}$$

The trick is to mimic formula (3), and not the two first ones (1) - (2) ...

Let $h \in \mathbb{V}_p(\partial\Omega)$ and $\underline{h} \in \mathbb{P}^{p-1} \otimes \mathbb{P}^{p-1}$ its restriction to face 1.

For

$$\underline{h}(x,y) = \sum_{n=1}^{p} \sum_{m=1}^{p} h_{nm} \frac{\Psi_n^{(p)\prime}(x)}{\sqrt{\mu_n^{(p)}}} \frac{\Psi_m^{(p)\prime}(y)}{\sqrt{\mu_n^{(p)}}}$$

we set

$$H_1 = \Bigg(-\sum_{n=1}^{p} \sum_{m=1}^{p} h_{nm} \sqrt{\mu_n^{(p)}}\,\Psi_n^{(p)}(x) \frac{\Psi_m^{(p)\prime}(y)}{\sqrt{\mu_m^{(p)}}} \frac{\beta_{nm}^{(p)\,\prime}(z)}{\mu_n^{(p)} + \mu_m^{(p)}},$$
$$-\sum_{n=1}^{p} \sum_{m=1}^{p} h_{nm} \frac{\Psi_n^{(p)\prime}(x)}{\sqrt{\mu_n^{(p)}}} \sqrt{\mu_m^{(p)}}\,\Psi_m^{(p)}(y) \frac{\beta_{nm}^{(p)\,\prime}(z)}{\mu_n^{(p)} + \mu_m^{(p)}},$$
$$\sum_{n=1}^{p} \sum_{m=1}^{p} h_{nm} \frac{\Psi_n^{(p)\prime}(x)}{\sqrt{\mu_n^{(p)}}} \frac{\Psi_m^{(p)\prime}(y)}{\sqrt{\mu_m^{(p)}}} \beta_{nm}^{(p)}(z) \Bigg)$$

and continue, and it works...

Construct $H \in \mathrm{V}_p(\Omega)$ such that $\gamma_{\mathrm{N}} H = h$ and such that

$$\|H\|_{\mathrm{H}(\mathrm{div},\Omega)} \leq C \|h\|_{\mathbb{H}^{-\frac{1}{2}}(\partial\Omega)}$$

Conclusion: Commutative diagram in the continous case

Using expansions in Dirichlet and Neumann 1D bases, we have constructed extension operators $\mathcal{L}_0$ and $\mathcal{L}_\mathsf{N}$ for standard γ_0 and normal γ_N traces.

These extension operators are uniquely determined :

- $\mathcal{L}_0$ by condtion $\Delta \circ \mathcal{L}_0 \equiv 0$
- $\mathcal{L}_\mathsf{N}$ by conditions $\mathrm{curl} \circ \mathcal{L}_\mathsf{N} \equiv 0$ and $\mathbf{Rg}(\mathrm{div} \circ \mathcal{L}_\mathsf{N}) = \mathbb{R}$.

They satisfy the exact sequence and commuting diagram properties :

$$
\begin{array}{ccccccc}
\mathbb{H}^1(\Omega) & \xrightarrow{\nabla} & \mathbf{H}(\mathrm{curl},\Omega) & \underset{K}{\overset{\mathrm{curl}}{\rightleftarrows}} & \mathbf{H}(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & \mathbb{L}^2(\Omega) \\
\gamma_0 \Big\downarrow \Big\uparrow \mathcal{L}_0 & & \gamma_\mathsf{T} \Big\downarrow \Big\uparrow \mathcal{L}_\mathsf{T} & & \gamma_\mathsf{N} \Big\downarrow \Big\uparrow \mathcal{L}_\mathsf{N} & & \gamma_{\mathsf{avg}} \Big\downarrow \Big\uparrow \mathcal{L}_{\mathsf{avg}} \\
\mathbb{H}^{\frac12}(\partial\Omega) & \xrightarrow{\nabla} & \mathbf{H}^{-\frac12}(\mathrm{curl},\partial\Omega) & \xrightarrow{\mathrm{curl}} & \mathbb{H}^{-\frac12}(\partial\Omega) & \xrightarrow{\gamma_{\mathsf{avg}}} & \mathbb{R}
\end{array}
$$

The construction of $\mathcal{L}_\mathsf{T}$ uses the Poincaré map K as defined in [Gopalakrishnan,Demkowicz'04]

and goes through polynomial constructions.

Final conclusion

Construction of extension operators $\mathcal{L}_0^{(p)}$, $\mathcal{L}_{\mathsf{T}}^{(p)}$ and $\mathcal{L}_{\mathsf{N}}^{(p)}$

These extension operators are uniquely determined :

- $U = \mathcal{L}_0^{(p)} u$ by $\displaystyle\int_\Omega \nabla U \cdot \nabla V = 0, \ \forall V \in \mathbb{W}_{p,0}(\Omega)$
- $H = \mathcal{L}_{\mathsf{N}}^{(p)} h$ by $\displaystyle\int_\Omega H \cdot \operatorname{curl} E = 0, \ \forall E \in \mathbf{Q}_{p,0}(\Omega)$ and $\operatorname{div} H \in \mathbb{R}$

Commuting diagram properties

$$
\begin{array}{ccccccc}
\mathbb{W}_p(\Omega) & \xrightarrow{\nabla} & \mathbf{Q}_p(\Omega) & \underset{K}{\overset{\operatorname{curl}}{\rightleftarrows}} & \mathbf{V}_p(\Omega) & \xrightarrow{\operatorname{div}} & \mathbb{L}^2(\Omega) \\
\gamma_0 \Big\downarrow \Big\uparrow \mathcal{L}_0^{(p)} & & \gamma_{\mathsf{T}} \Big\downarrow \Big\uparrow \mathcal{L}_{\mathsf{T}}^{(p)} & & \gamma_{\mathsf{N}} \Big\downarrow \Big\uparrow \mathcal{L}_{\mathsf{N}}^{(p)} & & \gamma_{\mathsf{avg}} \Big\downarrow \Big\uparrow \mathcal{L}_{\mathsf{avg}} \\
\mathbb{W}_p(\partial\Omega) & \xrightarrow{\nabla} & \mathbf{Q}_p(\operatorname{curl},\partial\Omega) & \xrightarrow{\operatorname{curl}} & \mathbb{V}_p(\partial\Omega) & \xrightarrow{\gamma_{\mathsf{avg}}} & \mathbb{R}
\end{array}
$$

Operator norms bounded independently from p

$$
\begin{aligned}
\|\mathcal{L}_0^{(p)} u\|_{\mathbb{H}^1(\Omega)} &\leq C\,\|h\|_{\mathbb{H}^{\frac12}(\partial\Omega)} \\
\|\mathcal{L}_{\mathsf{T}}^{(p)} e_{\mathsf{T}}\|_{\mathbf{H}(\operatorname{curl},\Omega)} &\leq C\,\|e_{\mathsf{T}}\|_{\mathbf{H}^{-\frac12}(\operatorname{curl},\partial\Omega)} \\
\|\mathcal{L}_{\mathsf{N}}^{(p)} h\|_{\mathbf{H}(\operatorname{div},\Omega)} &\leq C\,\|h\|_{\mathbb{H}^{-\frac12}(\partial\Omega)}
\end{aligned}
$$

For $s \geq 0$:

- $\mathbb{H}^s(I)$ space of restrictions to I of functions in $\mathbb{H}^s(\mathbb{R})$.
- $\widetilde{\mathbb{H}}^s(I)$ closure of $\mathcal{C}_0^\infty(I)$ in $\mathbb{H}^s(\mathbb{R})$ (not in $\mathbb{H}^s(I)$!)
- $\mathbb{H}^{-s}(I)$ dual space of $\widetilde{\mathbb{H}}^s(I)$
- $\widetilde{\mathbb{H}}^{-s}(I)$ dual space of $\mathbb{H}^s(I)$

$\big(\mathbb{H}^s(I)\big)_{s\in\mathbb{R}}$ interpolation scale and $\big(\widetilde{\mathbb{H}}^s(I)\big)_{s\in\mathbb{R}}$ interpolation scale

The extension by 0 defines embeddings

- from $\widetilde{\mathbb{H}}^s(I)$ into $\widetilde{\mathbb{H}}^s(\mathbb{R})$
- from $\widetilde{\mathbb{H}}^{-s}(I)$ into $\widetilde{\mathbb{H}}^{-s}(\mathbb{R})$, defined as the action on the restriction to I.

The restriction to I defines surjective operators

- from $\mathbb{H}^s(\mathbb{R})$ onto $\mathbb{H}^s(I)$
- from $\mathbb{H}^{-s}(\mathbb{R})$ onto $\mathbb{H}^{-s}(I)$, defined as the action on the extension by 0.

With tensor product spaces

$$\mathbb{P}^{(p,q,r)}(\Omega) = \mathbb{P}^p(I) \otimes \mathbb{P}^q(I) \otimes \mathbb{P}^r(I)$$

set

$$\begin{aligned}
\mathbb{W}_p(\Omega) &= \mathbb{P}^{(p,p,p)}(\Omega) \\
\mathbb{Q}_p(\Omega) &= \mathbb{P}^{(p-1,p,p)}(\Omega) \times \mathbb{P}^{(p,p-1,p)}(\Omega) \times \mathbb{P}^{(p,p,p-1)}(\Omega) \\
\mathbb{V}_p(\Omega) &= \mathbb{P}^{(p,p-1,p-1)}(\Omega) \times \mathbb{P}^{(p-1,p,p-1)}(\Omega) \times \mathbb{P}^{(p-1,p-1,p)}(\Omega)
\end{aligned}$$

Space of traces: With the faces $F \in \mathcal{F}$ of Ω:

$$\begin{aligned}
\mathbb{W}_p(\partial\Omega) &= \{u \in H^{\frac12}(\partial\Omega) : u|_F \in \mathbb{P}^{(p,p)}(F),\ \forall F \in \mathcal{F}\} \\
\mathbb{Q}_p(\partial\Omega) &= \{e \in \mathrm{H}^{-\frac12}(\mathrm{curl}, \partial\Omega) : e|_F \in \mathbb{Q}_p(F),\ \forall F \in \mathcal{F}\} \\
\mathbb{V}_p(\partial\Omega) &= \{h \in H^{-\frac12}(\partial\Omega) : h|_F \in \mathbb{P}^{(p-1,p-1)}(F),\ \forall F \in \mathcal{F}\}
\end{aligned}$$

Note that

- **In $\mathbb{W}_p(\partial\Omega)$, the face traces share common values along edges,**
- **In $\mathbb{Q}_p(\partial\Omega)$, the face traces share common values for their tangential traces along edges,**
- **In $\mathbb{V}_p(\partial\Omega)$, the face traces are independent of each other.**

- Polynomial trace $e_\mathsf{T} \in \mathbb{Q}_p(\Omega)$
- $h = \operatorname{curl}_{\partial\Omega} e_\mathsf{T}$
- $H = \mathcal{L}_\mathsf{N}^{(p)} h$
- $E = KH$ thus $\operatorname{curl} E = H$
- $\operatorname{curl} \gamma_\mathsf{T} E = \gamma_\mathsf{N} \operatorname{curl} E = \gamma_\mathsf{N} H = h$
- $f_\mathsf{T} := e_\mathsf{T} - \gamma_\mathsf{T} E$ satisfies $\operatorname{curl} f_\mathsf{T} = 0$
- Exists $u \in W_p(\partial\Omega)$: $e_\mathsf{T} - \gamma_\mathsf{T} KH = \nabla_{\partial\Omega} u$
- $U = \mathcal{L}_0^{(p)} u$
- $\mathcal{L}_\mathsf{T}^{(p)} e_\mathsf{T} := \nabla U + KH$