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On the Cosserat spectrum in polygons and polyhedra

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Outline

- **Different formulations of Cosserat problem. Comparison.**
- **Essential spectrum.**
- **Discrete spectrum.**
- **End.**

Two Cosserat problems (Dirichlet)

Domain $\Omega \subset \mathbb{R}^n$ with $n = 2, 3$.

Elastic form: COSSERATS (18)98. Find $\sigma \in \mathbb{R}$ and $u \in H^1_0(\Omega)^n$, $u \neq 0$,

(Coss) $$\sigma \Delta u - \operatorname{grad} \operatorname{div} u = 0 \quad \text{in } \Omega.$$

Variational, with strain tensor e: Find $\sigma \in \mathbb{R}$ and $u \in H^1_0(\Omega)^n$, $u \neq 0$,

(Ela) $$\forall v \in H^1_0(\Omega), \quad \int_\Omega -2\sigma\, e(u) : e(v) + (1+\sigma) \operatorname{div} u \operatorname{div} v \,\mathrm{d}x = 0.$$

(Usual Lamé with $\lambda = \sigma + 1$ and $\mu = -\sigma$).

Pressure form: CROUZEIX '97. Find $\sigma \in \mathbb{R}$ and $p \in L^2(\Omega)$, $p \neq 0$

(D) $$Dp = \sigma p \quad \text{with} \quad D = \operatorname{div} \Delta^{-1} \operatorname{grad},$$

where Δ^{-1} is the inverse $H^{-1}(\Omega) \to H^1_0(\Omega)$ of the Laplace-Dirichlet problem.

Same σ for (Coss) and (D)

One more Cosserat problem

Holomorphic form: **FRIEDRICHS '37.**

Let $\mathfrak{F}(\Omega)$ be the space of $L^2(\Omega)$ holomorphic functions in $\Omega \subset \mathbb{R}^2 \sim \mathbb{C}$.

Find $\mu \in \mathbb{R}$, $w \in \mathfrak{F}(\Omega)$, $w \neq 0$,

(F) $$\forall w' \in \mathfrak{F}(\Omega), \quad \mathrm{Re} \int_\Omega w\, w' \,\mathrm{d}x = \mu \;\mathrm{Re} \int_\Omega w\, \overline{w}' \,\mathrm{d}x.$$

Result: STOYAN '96

$\boxed{\sigma \text{ (Coss) and } \mu \text{ (F) satisfy: } \quad \mu = 1 - 2\sigma}$

Theorem. CODA '99

The underlying operator of (F) is $(1 - 2D) \circ \mathfrak{C}$ acting from $\mathfrak{F}(\Omega)$ into $\mathfrak{F}(\Omega)$, where $\mathfrak{C}$ is the conjugacy operator $u + iv \;\mapsto\; u - iv$.

Regular domain Ω

Theorem. MIKHLIN '73

The system

$$\sigma\Delta u - \operatorname{grad}\operatorname{div} u$$

is a (properly) elliptic $n \times n$ Agmon – Douglis – Nirenberg system iff

$$\boxed{\sigma \neq 0 \quad \text{and} \quad \sigma \neq 1}$$

It is covered by Dirichlet conditions iff

$$\boxed{\sigma \neq 0 \quad \text{and} \quad \sigma \neq \tfrac{1}{2} \quad \text{and} \quad \sigma \neq 1}$$

Theorem. CROUZEIX '97. $L^2(\Omega) = M \oplus N$ with

$M = \{p \in L^2(\Omega),\ \Delta p = 0\}$, $N = \{p \in L^2_0(\Omega),\ \exists\, \theta \in H^2_0(\Omega),\ \Delta\theta = p\}$.

$$\boxed{D\big|_N = \mathrm{Id} \quad \text{and} \quad (D - \tfrac{1}{2}\mathrm{Id})\big|_M \ \text{ is regularizing of order } \infty}$$

Polygonal domain Ω

Let $\mathfrak{S}_{\text{ess}}(\text{Coss})$ be the essential spectrum, i.e. the set of σ for which

$$\sigma\Delta u - \operatorname{grad}\operatorname{div} u \quad \text{is not Fredholm} \quad H^1_0(\Omega)^2 \to H^{-1}(\Omega)^2.$$

Openings of angles ω_k, $k = 1, \ldots, K$.

<u>Theorem</u>. FRIEDRICHS '37 (lub), CROUZEIX '97 (lub), CODA '99

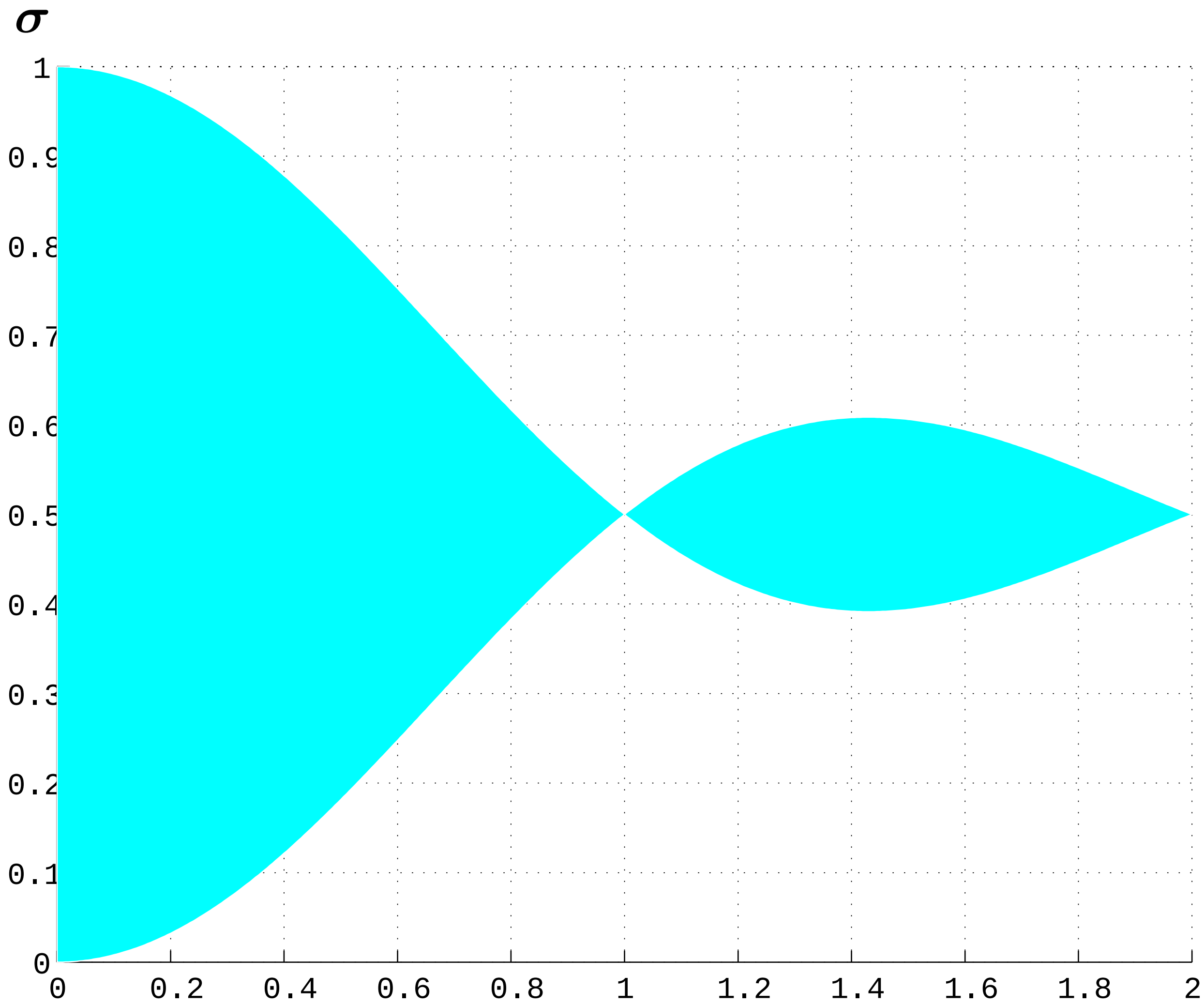
$$\boxed{\mathfrak{S}_{\text{ess}}(\text{Coss}) = \{0\} \cup \bigcup_{k=1}^{K} \left[\frac{1}{2} - \frac{|\sin\omega_k|}{2\omega_k}, \frac{1}{2} + \frac{|\sin\omega_k|}{2\omega_k}\right] \cup \{1\}}$$

<u>Our Proof</u>: Look for σ s.t. the Mellin transform at a corner of opening ω has pole(s) ν with $\operatorname{Re}\nu = 0$ (pure imaginary). The poles ν are solution of

$$\sin^2 \nu\omega = (1 - 2\sigma)^{-2}\nu^2 \sin^2\omega.$$

With $z = \nu\omega$ and $\kappa = \sin\omega/(1-2\sigma)\omega$, the equation is $\sin^2 z = \kappa^2 z^2$.
It has roots $z = iy$ iff $|\kappa| \geq 1$.

Contribution to the essential spectrum of an angle ω



ω in π radians

Discrete spectrum

Let $\mathfrak{S}_{\mathrm{d}}(\text{Coss})$ be the discrete spectrum.

<u>Theorem</u>. FRIEDRICHS '37, CROUZEIX '97

$$\boxed{\mathfrak{S}_{\mathrm{d}}(\text{Coss}) \subset [a, 1-a], \quad a > 0}$$

Disc $(n=2)$: $\mathfrak{S}_{\mathrm{d}}(\text{Coss}) = \emptyset$. **Ball** $(n=3)$: $\mathfrak{S}_{\mathrm{d}}(\text{Coss}) = \left\{\frac{\ell}{2\ell+1},\ \ell \geq 1\right\}$.

Ellipse $(n=2)$ of equation $\frac{x^2}{\cosh^2\alpha} + \frac{y^2}{\sinh^2\alpha} \leq 1$ (aspect ratio $\tanh\alpha$):

$$\mathfrak{S}_{\mathrm{d}}(\text{Coss}) = \left\{\sigma_\ell(\alpha),\ 1-\sigma_\ell(\alpha),\ \ell \geq 2\right\}$$

with

$$\sigma_\ell(\alpha) = \frac{1}{2}\left(1 - \frac{\ell\sinh 2\alpha}{\sinh 2\ell\alpha}\right).$$

As $\alpha \to \infty$, the ellipse tends to the disc and $\sigma_\ell(\alpha) \longrightarrow \frac{1}{2}$.

As $\alpha \to 0$, the aspect ratio $\simeq \alpha$ and $\boxed{\sigma_\ell(\alpha) \longrightarrow \alpha^2\,\frac{\ell^2-1}{3}}$

Quasi-modes in rectangles

Rectangle $(-1,1)\times(-\varepsilon,\varepsilon)$ with aspect ratio $\varepsilon\in(0,1]$.

Scale the problem to the square $(-1,1)^2$ and look for expansions

$$\Sigma_\ell[\varepsilon] = \varepsilon^2\Sigma_{\ell,0} + \varepsilon^3\Sigma_{\ell,1} + \dots \text{ for eigenvalues}$$

$$U_\ell[\varepsilon] = U_\ell^0 + \varepsilon U_\ell^1 + \dots \text{ for eigenvectors.}$$

The problem in $(-1,1)^2$ is

$$\begin{cases} \partial_x^2 U_x[\varepsilon] + \varepsilon^{-2}\partial_{xy}^2 U_y[\varepsilon] & = & \Sigma[\varepsilon](\partial_x^2 + \varepsilon^{-2}\partial_y^2)U_x[\varepsilon] \\ \partial_{xy}^2 U_x[\varepsilon] + \varepsilon^{-2}\partial_y^2 U_y[\varepsilon] & = & \Sigma[\varepsilon](\partial_x^2 + \varepsilon^{-2}\partial_y^2)U_y[\varepsilon] \end{cases}$$

Find $U_y^0 = 0$. First “non-trivial” equation

$$\partial_x^2 U_x^0 + \partial_{xy}^2 U_y^2 = \Sigma_0 \partial_y^2 U_x^0,$$

reducing to the eigenvalue problem

$$\frac{1}{2}\int_{-1}^{1} \partial_x^2 U_x^0 \,\mathrm{d}y = \Sigma_0 \partial_y^2 U_x^0.$$

Quasi-modes in rectangles, continued

Representation of U^0_x on the Laplace-Dirichlet eigenmode basis $\big(e_n(x)\big)_{n\ge1}$ on $(-1,1)$

$$U^0_x = \sum_{n\ge1} e_n(x)\, v_n(y).$$

The reduced eigenvalue problem becomes

$$\forall n\ge1,\quad -\frac{n^2\pi^2}{8}\int_{-1}^{1} v_n(y)\,\mathrm{d}y = \Sigma_0 v_n''(y),$$

with boundary conditions $v_n(\pm1)=0$. Solutions : $\exists\ell\ge1$

$$v_n(y) = \delta_{n\ell}(y^2-1)\quad\text{and}\quad \Sigma_0 = \frac{\ell^2\pi^2}{12}.$$

The construction can be continued... Thus, as $\varepsilon\to0$, $\exists\,\ell(\varepsilon)$ such that

$$\boxed{\sigma_{\ell(\varepsilon)}(\varepsilon) \;=\; \varepsilon^2\frac{\ell^2\pi^2}{12} + \mathcal{O}(\varepsilon^4)\ \text{belongs to}\ \mathfrak{S}_{\mathrm{d}}(\text{Coss})}$$

Eigenmodes in rectangles: a majorant and a minorant

With $\sigma_\ell(\varepsilon)$ the $\ell-$ th Cosserat eigenvalue on the rectangle $\Omega^\varepsilon := (-1,1) \times (-\varepsilon,\varepsilon)$, by Min-Max principle we prove that

$$(*) \qquad \forall 0 < \alpha \leq \varepsilon, \quad \forall \ell \geq 1, \qquad \alpha^{-2}\,\sigma_\ell(\alpha) \geq \varepsilon^{-2}\,\sigma_\ell(\varepsilon).$$

The quasi-mode result yields for α small enough, $\sigma_\ell(\alpha) \leq \alpha^2\,\frac{\ell^2\pi^2}{12} + c_\ell\,\alpha^4$.
Combining with $(*)$ we obtain

$$\boxed{\sigma_\ell(\varepsilon) \leq \varepsilon^2\,\frac{\ell^2\pi^2}{12}.}$$

Minorant, HORGAN – PAYNE'83 : with $\rho = \rho(z) = 1/n_r(z)$, $n_r(z)$ the radial component of the unit normal in $z \in \partial\Omega$:
$\sigma_1(\Omega) \geq \min_{z\in\partial\Omega} 1/\{1 + \left[\rho + (\rho^2-1)^{1/2}\right]^2\}$, whence for the rectangle Ω^ε

$$\boxed{\sigma_1(\varepsilon) \geq \frac{\sin(\arctan\varepsilon)^2}{\sin(\arctan\varepsilon)^2 + \left[1 + \left(1-\sin(\arctan\varepsilon)^2\right)^{1/2}\right]^2}}$$

Eigenmodes in rectangles: a conjecture

Fix $\ell \geq 1$. The terms of the (formal) series $\Sigma_\ell[\varepsilon]$ are given by recurrence formulas linking for $j \leq \ell$ the Σ_j and polynomials v_x^j and v_y^j such that

$$U_x^j(x,y) = e_\ell(x)\, v_x^j(y) \quad \text{and} \quad U_y^j(x,y) = e'_\ell(x)\, v_y^j(y).$$

The series Σ_ℓ has the form $\sum_{k\geq 1} \varepsilon^{2k}\ell^{2k}\, b_k$ with universal coefficients b_k.

Solving the 30 first relations with SCILAB let the following relation plausible

$$b_k = (-1)^k \sum_{j\geq 1} (-1)^j j^{-2k}$$

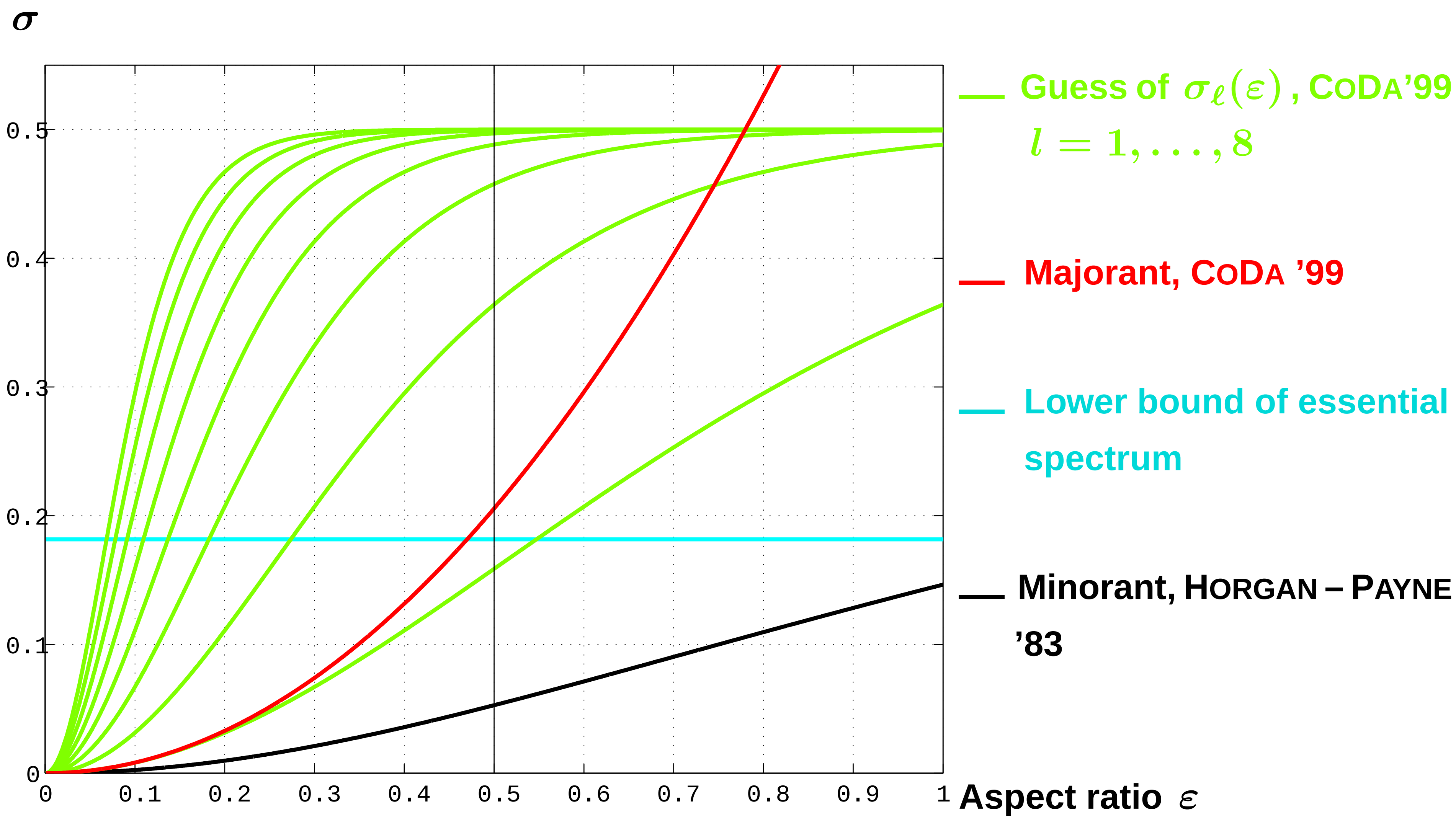
which yields (with the help of a formula in [ABRAMOWITZ–STEGUN]) that Σ_ℓ is a convergent series

$$\Sigma_\ell(\varepsilon) = \frac{1}{2}\left(1 - \frac{\ell\pi\varepsilon}{\sinh \ell\pi\varepsilon}\right).$$

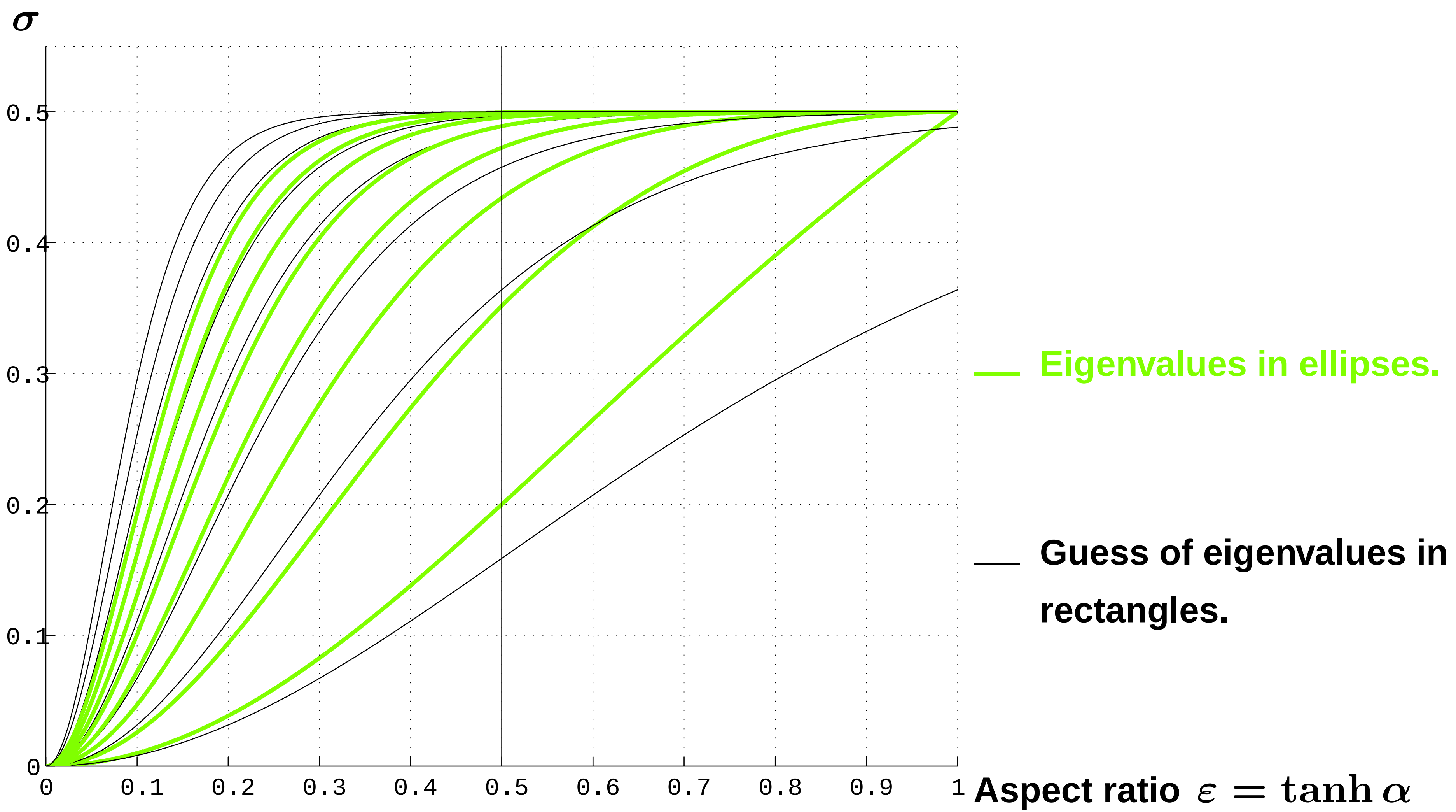
The conjecture is that $\sigma_\ell(\varepsilon) = \Sigma_\ell(\varepsilon)$.

$\boxed{\textbf{Square ??}}$

Rectangles


σ
0.5
0.4
0.3
0.2
0.1
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Aspect ratio ε
Guess of $\sigma_\ell(\varepsilon)$, CODA'99
$l = 1, \ldots, 8$
Majorant, CODA '99
Lower bound of essential spectrum
Minorant, HORGAN – PAYNE '83

Ellipses



σ
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Eigenvalues in ellipses.
Guess of eigenvalues in rectangles.
Aspect ratio $\varepsilon = \tanh \alpha$

Conclusions and Extensions

- Approximation of spectrum by numerical methods difficult due to the presence of **essential spectrum.**
- For Neumann problem in 2D, essential spectrum $\mathfrak{S}_{\mathrm{ess}}(\mathsf{Coss}) = \{0, \frac{1}{2}, 1, +\infty\}$, discrete spectrum $\mathfrak{S}_{\mathrm{d}}(\mathsf{Coss}) \subset (0, +\infty)$.
- For mixed Dirichlet-Neumann problem in 2D (without Dirichlet-Dirichlet corner), essential spectrum $\mathfrak{S}_{\mathrm{ess}}(\mathsf{Coss}) = \{0\} \cup [\frac{1}{2}, +\infty)$,
  discrete spectrum $\mathfrak{S}_{\mathrm{d}}(\mathsf{Coss}) \subset (0, \frac{1}{2})$.
- For polyhedra: Possibility of new contributions to the essential spectrum by corners, and also by the spectrum of the partial Fourier transform operators along edges.