CONSTRUCTION OF CORNER SINGULARITIES FOR AGMON-DOUGLIS-NIRENBERG ELLIPTIC SYSTEMS

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Abstract. *Corner singularities in plane domains are characterized by certain singular exponents and angular functions. Our construction of singularities is based on two Cauchy integrals at two different levels of symbolic calculus. This construction yields new information about the angular functions and also more explicit formulas for the computation of the singular exponents.*

Introduction

We consider the classical question of the singularities of the solutions of an elliptic boundary value problem near a corner of a plane domain. The solution has there, for smooth right hand sides, the form of the sum of a *regular part* and a linear combination of certain *singular functions*: the regular part has the maximal regularity allowed by the right hand side and the ellipticity of the problem; the singular functions have unbounded derivatives near the corner but correspond to smooth right hand sides.

As is well known from many now classical papers (see e. g. Wasow [25], Lehman [17], Kondrat'ev [15], Grisvard [13], Maz'ya and Plamenevskii [19], Dauge [9]), the singular functions admit themselves *expansions* of the form:

$$\sum_{p\in\mathbb{N}} \boldsymbol{u}_p \quad \text{with} \quad \boldsymbol{u}_p = \sum_{q=0}^{Q} r^{\nu+p} \log^q r \cdot \varphi_{p,q}(\theta), \quad \forall p \in \mathbb{N} \tag{0.1}$$

where (r,θ) are the local polar coordinates at the corner. The terms $\boldsymbol{u}_p$ with $p=0$ are the principal terms of the singularities and the others come from the curvature of the boundary and from the non principal terms in the operator.

The exponents ν are either solutions of some eigenvalue problem or nonnegative integers. In many instances of practical importance, one knows analytic functions

$\lambda \mapsto F(\lambda)$ whose zeros coincide with the exponents ν : the equation $F(\lambda) = 0$ is called the *characteristic equation* of the problem.

The problem of finding the roots of the characteristic equation has been extensively studied for the standard boundary and transmission problems for Laplace and bi-Laplace equation and for the Stokes and Lamé systems: see e. g. Kondrat'ev [15], Seif [24], Lozi [18], Blum and Rannacher [3], Costabel and Stephan [8], Grisvard [14], Bernardi and Raugel [2], Dauge [10], Sändig, Richter and Sändig [23], Nicaise and Sändig [21].

Thus the dependence on r in the singular functions is described by the functions $r \mapsto r^{\nu+p} \log^q r$. On the other hand, the angular functions $\varphi(\theta)$ are less well studied: the only general information available about their dependence on θ is their regularity: they are analytic functions.

In some of the standard examples quoted above, these functions of the angular variable are also known explicitly for the *principal* terms $\boldsymbol{u}_0$ in the singular functions. They are eigenfunctions of some Sturm–Liouville problems and it turns out that they are always linear combinations of functions of the form

$$\theta^k \cos(a\theta + b)\,. \tag{0.2}$$

We show in this paper that in fact such a simple description holds for the whole expansion of the singular functions of any elliptic boundary value problem for systems of linear partial differential equations elliptic in the sense of Agmon–Douglis–Nirenberg ("ADN-elliptic systems"). We give explicit formulas for the construction of these singular functions in terms of powers of the complex variable

$$\zeta = r\, e^{i\theta}\,. \tag{0.3}$$

Until now, such a description was only known for the Laplace operator (and second order operators with real coefficients): see Wasow [25], Lehman [17], Maz'ya and Rossmann [20], Costabel and Dauge [6, 7]. These formulas for the singular functions describe in particular also their dependence on the angular variable θ.

Numerical method for computing the exponents ν (and also the angular functions $\varphi_{p,q}(\theta)$ for $p = 0$) have recently been developpped in the literature: see [16] and [22]. These methods apply when no *analytic* expressions of the characteristic equation is known (for instance for anisotropic elasticity) and are based on the numerical solution of the Sturm-Liouville problem. Our formulas can provide a new method for computing the exponents ν, where the numerical solution of the Sturm-Liouville problem is no more needed. This method would be based on our expressions for the homogeneous solutions of an ADN system.

The paper is organized as follows.

In §1, we describe the setting more precisely and give the recurrence relations that govern the asymptotic expansion (0.1) of each singular function — see Conclusion 1.1.

In §2 and 3, we study solutions $\boldsymbol{u}$ homogeneous of degree $\lambda \in \mathbb{C}$ of an equation $\boldsymbol{M}\boldsymbol{u} = 0$ where $\boldsymbol{M}$ is an ADN system which is homogeneous and has constant coefficients. We derive explicit formulas for a basis of the space of these solutions in the form of complex contour integrals — see Theorem 2.1 and Lemmabasis.

In §4, we use this solution basis for the construction of the characteristic equation and the resolvent of the associated Sturm–Liouville problem. That yields the principal terms $\boldsymbol{u}_0$ in the singular functions — see Theorems 4.4 and 4.5.

In §5, we study the higher order terms in the singular functions, using the recurrence relations given in §1. We prove that all terms $\boldsymbol{u}_p$ have a common form — see Theorem 5.2.

The formulas we derive in this paper can also provide *stable* expressions with respect to a parameter: we investigate this question in the forthcoming paper [**?**].

1. Recurrence relations for singular functions

1.a Domains with corners. We study the behavior of the singular solutions of an elliptic boundary value problem in a neighborhood of a corner of a plane piecewise smooth domain Ω. In this neighborhood, Ω has the following description in polar coordinates (r, θ):

$$\Omega = \{(r, \theta) \mid \alpha^0(r) < \theta < \alpha^1(r)\}\,.$$

Here α^0 and α^1 are functions in $C^\infty([0, R])$ for some $R > 0$, and we assume

$$\alpha^0(0) < \alpha^1(0)\,.$$

The latter condition excludes a zero angle at the corner, that is an outward cusp, but it admits a reentrant cusp with corner angle 2π.

By applying a C^∞ diffeomorphism, we can assume that $\alpha^0(r) \equiv 0$, and we write

$$\omega(r) := \alpha^1(r)$$

in this case. Note that except in the case $\omega(0) = \pi$ and $\omega(0) = 2\pi$, we could even assume that $\omega(r)$ is constant.

We introduce the following notation with $\omega_0 := \omega(0)$:

$$\Gamma = \{(r, \theta) \mid 0 < \theta < \omega_0\}$$

is the plane sector tangent to Ω at the origin.

$$\partial^0\Omega,\ \ \partial^1\Omega \quad \text{and} \quad \partial^0\Gamma,\ \ \partial^1\Gamma$$

denote the boundary components of Ω and Γ, respectively, where the superscript 0 corresponds to $\theta = 0$.

1.b ADN elliptic systems. We consider the boundary value problem

$$\begin{cases} \boldsymbol{L}\boldsymbol{u} = \boldsymbol{f} & \text{in} \quad \Omega \\ \boldsymbol{B}^j\boldsymbol{u} = \boldsymbol{g}^j & \text{on} \quad \partial^j\Omega \quad \text{for } j = 0,\ 1\,. \end{cases} \tag{1.1}$$

We assume that $\boldsymbol{L}$ is a $N \times N$ system of linear partial differential operators with $C^\infty(\overline{\Omega})$ coefficients, properly elliptic in the sense of Agmon, Douglis and Nirenberg [1], and that each of the two boundary systems $\boldsymbol{B}^j$ satisfies the Shapiro–Lopatinski covering condition. Then the order of the determinant of $\boldsymbol{L}$ is an even number, which we denote $2m$, and we have

$$\boldsymbol{u} = (u_1, \dots, u_N)\,; \quad \boldsymbol{f} = (f_1, \dots, f_N)\,; \quad \boldsymbol{g}^j = (g^j_1, \dots, g^j_m)\,;$$

and

$$\boldsymbol{L} = (L_{k,l})_{1\le k,l\le N}\,; \qquad \boldsymbol{B}^j = (B^j_{h,l})_{\substack{1\le h\le m\\ 1\le l\le N}}\,.$$

In the standard way, we have nonnegative integers

$$(\sigma_k)_{k=1,\dots,N}\,, \quad (\tau_l)_{l=1,\dots,N}\,, \quad (\sigma^j_h)_{h=1,\dots,m} \quad \text{for } j = 0,\ 1,$$

such that

$$2m = \sum_{k=1}^{N} (\sigma_k - \tau_k)$$

and for the orders of the differential operators there holds

$$\operatorname{ord}(L_{k,l}) \le \sigma_k - \tau_l \qquad \text{and} \qquad \operatorname{ord}(B^j_{h,l}) \le \sigma^j_h - \tau_l\,.$$

The constant coefficient principal parts of $\boldsymbol{L}$ and $\boldsymbol{B}^j$ at the origin are denoted by

$$\boldsymbol{M} = (M_{k,l})_{1\le k,l\le N} \qquad \text{and} \qquad \boldsymbol{C}^j = (C^j_{h,l})_{\substack{1\le h\le m\\ 1\le l\le N}}$$

respectively, where now

$$\operatorname{ord}(M_{k,l}) = \sigma_k - \tau_l \qquad \text{and} \qquad \operatorname{ord}(C^j_{h,l}) = \sigma^j_h - \tau_l\,.$$

We note that

$$D := \det \boldsymbol{M}$$

is a properly elliptic scalar operator of order $2m$.

1.c Singular solutions. Our aim in this paper is to give a simple and *constructive description* of the singularities of the solution $\boldsymbol{u}$ at the origin. For regular right hand sides $\boldsymbol{f}$ and $\boldsymbol{g}^j$, any solution $\boldsymbol{u}$ of the boundary value problem (1.1) can be decomposed into a regular part and a linear combination of singular functions that do not depend on the right hand side but only on the domain and the differential operators. It is not our intention here to prove any new result about the *fact* that any solution of (1.1) can be decomposed into regular and singular parts. Such decomposition theorems are well known from the classical papers quoted in the introduction.

We begin by reproducing here the standard wisdom about the form of the singular functions. The singular functions are solutions of the boundary value problem (1.1) with zero or polynomial right hand sides. In the case of the constant coefficient model problem

$$\begin{cases} \boldsymbol{M}\boldsymbol{u} = \boldsymbol{f} & \text{in} \quad \Gamma \\ \boldsymbol{C^j}\boldsymbol{u} = \boldsymbol{g}^j & \text{on} \quad \partial^j\Gamma \quad \text{for } j = 0,\ 1\,. \end{cases} \tag{1.2}$$

a basis for the singular functions is given by functions of the form

$$\boldsymbol{u} = (u_1, \dots, u_N)\,, \quad u_l = r^{\lambda-\tau_l} \sum_{q=0}^{Q} \log^q r \cdot \varphi_{lq}(\theta)\,, \quad \varphi_{lq} \in C^\infty([0, \omega_0])\,.$$

Actually, the functions φ_{lq} are restrictions to $[0, \omega_0]$ of functions in $C^\infty(\mathbb{R})$. We introduce the following spaces of singular functions.

For scalar functions in a plane sector:

$$S^\lambda(\mathbb{R}_+ \times \mathbb{R}) := \{v \mid v(r, \theta) = r^\lambda \sum_{q=0}^{Q} \log^q r \cdot \varphi_q(\theta)\,; \quad Q \in \mathbb{N}, \quad \varphi_q \in C^\infty(\mathbb{R})\ \}\,. \tag{1.3}$$

For vector functions of multi-degree $\vec{\tau} := (\tau_1, \dots, \tau_N)$ in a plane sector:

$$\boldsymbol{S}^{\lambda-\vec{\tau}} := \{\boldsymbol{u} = (u_1, \dots, u_N) \ \mid\ u_l \in S^{\lambda-\tau_l}(\mathbb{R}_+ \times \mathbb{R}) \quad \text{for } l = 1, \dots, N\ \}\,. \tag{1.4}$$

For scalar functions on the half-line:

$$S^\lambda(\mathbb{R}_+) := \{v \mid v(r) = r^\lambda \sum_{q=0}^{Q} \log^q r \cdot \psi_q\,; \quad Q \in \mathbb{N}, \quad \psi_q \in \mathbb{C}\,\}\,. \tag{1.5}$$

Note that for any fixed θ_0, $S^\lambda(\mathbb{R}_+) = S^\lambda(\mathbb{R}_+ \times \mathbb{R})\Big|_{\theta=\theta_0}$ and that $S^\lambda(\mathbb{R}_+ \times \mathbb{R}) = S^\lambda(\mathbb{R}_+) \otimes C^\infty(\mathbb{R})$.

For right hand sides of multidegree $\vec{\sigma} := (\sigma_1, \dots, \sigma_N; \sigma_1^0, \dots, \sigma_N^0; \sigma_1^1, \dots, \sigma_N^1)$:

$$\begin{aligned} \boldsymbol{T}^{\lambda-\vec{\sigma}} := \{(\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1) \mid \ & \boldsymbol{f} = (f_1, \dots, f_N)\,, \quad \boldsymbol{g}^j = (g_1^j, \dots, g_m^j)\,; \\ & f_k \in S^{\lambda-\sigma_k}(\mathbb{R}_+ \times \mathbb{R}) \quad \text{for } k = 1, \dots, N\,; \\ & g_h^j \in S^{\lambda-\sigma_h^j}(\mathbb{R}_+) \quad \text{for } h = 1, \dots, m\ \}\,. \end{aligned} \tag{1.6}$$

Thus the boundary value problem (1.2) acts from $\boldsymbol{S}^{\lambda-\vec{\tau}}$ into $\boldsymbol{T}^{\lambda-\vec{\sigma}}$ for any $\lambda \in \mathbb{C}$.

In the case of the boundary value problem (1.1) with variable coefficients and curved boundary, the singular functions are perturbations of the previous ones. They have asymptotic expansions of the form:

$$\boldsymbol{U} \sim \sum_{p=0}^{\infty} \boldsymbol{u}_p \qquad \boldsymbol{u}_p \in \boldsymbol{S}^{\lambda+p-\vec{\tau}}\,. \tag{1.7}$$

Here the first term $\boldsymbol{u}_0 \in \boldsymbol{S}^{\lambda-\vec{\tau}}$ is a singular function of the boundary value problem (1.2) and the higher order terms $\boldsymbol{u}_p$ are given by the following recurrence relation whose terms $\mathbb{A}_p$ we are going to describe in the next section:

$$\mathbb{A}_0 \boldsymbol{u}_p = -\sum_{n=0}^{p-1} \mathbb{A}_{p-n} \boldsymbol{u}_n \,. \tag{1.8}$$

1.d Expansion of the boundary value system. Let γ^0, γ^1, $\tilde{\gamma}^1$ denote the restriction operators on $\partial^0\Gamma = \partial^0\Omega$, $\partial^1\Gamma$ and $\partial^1\Omega$, respectively. $\mathbb{A}_0$ is the principal part frozen in 0 :

$$\mathbb{A}_0 = (\boldsymbol{M}, \gamma^0 \boldsymbol{C}^0, \gamma^1 \boldsymbol{C}^1)\,. \tag{1.9}$$

The operators $\mathbb{A}_n$ arise from the Taylor expansion in 0 of the coefficients of $\boldsymbol{L}$, $\boldsymbol{B}^0, \boldsymbol{B}^1$ and of $\omega(r)$ and they have the mapping property for any $\lambda \in \mathbb{C}$:

$$\mathbb{A}_n : \boldsymbol{S}^{\lambda-\vec{\tau}} \longrightarrow \boldsymbol{T}^{\lambda+n-\vec{\sigma}}\,. \tag{1.10}$$

They are such that for any $\boldsymbol{u} \in \boldsymbol{S}^{\lambda-\vec{\tau}}$, the following relation holds in the sense of asymptotic developments when $r \to 0$:

$$(\boldsymbol{L}, \gamma^0 \boldsymbol{B}^0, \tilde{\gamma}^1 \boldsymbol{B}^1)\, \boldsymbol{u} \;\sim\; \sum_{n=0}^{\infty} \mathbb{A}_n \boldsymbol{u}\,. \tag{1.11}$$

A rather precise description of these operators $\mathbb{A}_n$ is possible using ideas from the examples given in [8], [9] and [6] (Prop. 2.7): let $\mathfrak{H}_n^{(d)}$ denote the operator that extracts from a differential operator of order $\le d$ with $C^\infty(\mathbb{R}^2)$ - coefficients its component of degree $n-d$: if

$$H(x, \partial_x) = \sum_{|\alpha| \le d} h_\alpha(x)\, \partial_x^\alpha\,,$$

then

$$(\mathfrak{H}_n^{(d)} H)(x, \partial_x) \quad = \sum_{\substack{|\alpha| \le d \\ |\beta|-|\alpha| = n-d}} \frac{x^\beta}{\beta!}\, \partial_x^\beta h_\alpha(0)\, \partial_x^\alpha\,.$$

We have then

$$\mathbb{A}_n = (\boldsymbol{L}_{(n)}, \gamma^0 \boldsymbol{B}^0_{(n)}, \gamma^1 \widetilde{\boldsymbol{B}}^1_{(n)})\,.$$

Here

$$\boldsymbol{L}_{(n)} = \Big(\mathfrak{H}_n^{(\sigma_k - \tau_l)} L_{kl}\Big)_{1 \le k,l \le N} \qquad \text{and} \qquad \boldsymbol{B}^0_{(n)} = \Big(\mathfrak{H}_n^{(\sigma_k^0 - \tau_l)} B^0_{hl}\Big)_{\substack{1 \le h \le m \\ 1 \le l \le N}}\,.$$

In order to define $\widetilde{\boldsymbol{B}}^1_{(n)}$, we need the asymptotic expansion of the operator $\tilde{\gamma}^1$ of restriction on the curved boundary $\partial^1\Omega$ with respect to the restriction on its tangent $\partial^1\Gamma$.

Let $v \in S^\lambda(\mathbb{R}_+ \times \mathbb{R})$; we have $\tilde\gamma^1 v(r) = v(r, \omega(r))$. Then the composition of the two Taylor expansions

$$v(r,\omega(r)) \;\sim\; v(r,\omega_0) + \sum_{k=1}^{\infty} \frac{(\omega(r)-\omega_0)^k}{k!}\; \partial_\theta^k v(r,\omega_0)$$

and

$$\omega(r) \;\sim\; \omega_0 + \sum_{l=1}^{\infty} \frac{r^l}{l!}\, \partial_r^l \omega(0)$$

gives an asymptotic expansion

$$\tilde\gamma^1 v(r) \sim \sum_{p=0}^{\infty} v_p(r) \quad \text{with} \quad v_p \in S^{\lambda+p}(\mathbb{R}_+)\,.$$

Thus for example,

$$\begin{aligned} v_0(r) &= v(r,\omega_0)\,, \\ v_1(r) &= r\cdot\omega'(0)\cdot\partial_\theta v(r,\omega_0)\,, \\ v_2(r) &= \frac{r^2}{2}\Big(\omega''(0)\partial_\theta + \omega'(0)^2\partial_\theta^2\Big) v(r,\omega_0)\,. \end{aligned}$$

Now we introduce this expansion into the expansion of the operator $\boldsymbol{B}^1$

$$\boldsymbol{B}^1 \sim \sum \boldsymbol{B}^1_{(n)} \quad \text{with} \quad \boldsymbol{B}^1_{(n)} = \Big(\mathfrak{H}_n^{(\sigma_k^1-\tau_l)} B^1_{hl}\Big)_{\substack{1\le h\le m\\ 1\le l\le N}}$$

and obtain the asymptotic expansion of the operator $\tilde\gamma^1\boldsymbol{B}^1 = \boldsymbol{B}^1(x,\partial_x)\Big|_{\theta=\omega(r)}$:

$$\Big(\tilde\gamma^1\boldsymbol{B}^1 v\Big)(r) \;\sim \Big(\gamma^1\widetilde{\boldsymbol{B}}^1 v\Big)(r) \;\sim \sum_{n=0}^{\infty} \widetilde{\boldsymbol{B}}^1_{(n)}(x,\partial_x)\; v(r,\omega_0)\,.$$

It is easy to check that if $\boldsymbol{u}_0$ is a solution of (1.2) with a certain right hand side $(\boldsymbol{f},\boldsymbol{g}^0,\boldsymbol{g}^1)$ and if $\boldsymbol{u}_p$ for $p \ge 1$ are solutions of the recurrence relations (1.8), then, in the sense of asymptotic developments when $r \to 0$:

$$(\boldsymbol{L},\gamma^0\boldsymbol{B}^0,\tilde\gamma^1\boldsymbol{B}^1)\;(\sum_{p=0}^{\infty}\boldsymbol{u}_p) \;\;\sim\;\; (\boldsymbol{f},\boldsymbol{g}^0,\boldsymbol{g}^1)\,.$$

Conclusion 1.1 A generating set for all singular functions of the boundary value problem (1.1) is given by the formal series (1.7)

$$\boldsymbol{U} \sim \sum_{p=0}^{\infty} \boldsymbol{u}_p$$

where:

- there is $\lambda \in \mathbb{C}$ such that $\forall p \in \mathbb{N}$, $\boldsymbol{u}_p$ belongs to $\boldsymbol{S}^{\lambda+p-\vec{\tau}}$;
- $\boldsymbol{u}_0$ spans the set of *all* solutions in $\boldsymbol{S}^{\lambda-\vec{\tau}}$ of the boundary value problem (1.2) — the principal part of problem (1.1) —

$$\mathbb{A}_0 \boldsymbol{u}_0 = (\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1) \tag{1.12}$$

  with zero or polynomial right hand sides $(\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1)$;
- for $p \geq 1$, $\boldsymbol{u}_p$ is *any* solution of the boundary value problem (1.2) with right hand side $-\sum_{n=0}^{p-1} \mathbb{A}_{p-n} \boldsymbol{u}_n$ — cf (1.8).

Our aim is to give a description of the above functions $\boldsymbol{u}_p$. For this we do not need the above explicit form of the operators $\mathbb{A}_n$. What has to be retained is

(i) the mapping property (1.10),

(ii) the fact that the interior operators $\boldsymbol{L}_{(n)}$ have all the form $\sum_\alpha h_\alpha(x)\, \partial_x^\alpha$ with polynomial coefficients,

(iii) the fact that the boundary operators act on the sides of the sector Γ and have the form $\sum_\alpha h_\alpha(r)\, (r\partial_r)^{\alpha_1}\, \partial_\theta^{\alpha_2}$ with polynomial coefficients.

The first step of our work is to study solutions homogeneous of multi-degree $\lambda - \vec{\tau}$ of the interior equation

$$\boldsymbol{M}\boldsymbol{u} = 0$$

which will then be used to solve the model problem (1.12) with zero right hand side and finally the recurrence relations (1.8).

2. Solutions of the interior system with zero right hand side

2.a The main result. Let us recall that $\boldsymbol{M} = (M_{kl})_{1 \leq k,l \leq N}$ is a properly elliptic ADN system, with constant coefficients, homogeneous of "multi-degree":

$$(\sigma_k - \tau_l)_{1 \leq k,l \leq N} \qquad \text{with} \qquad \sum_k (\sigma_k - \tau_k) = 2m\,.$$

For any complex number λ we are going to construct a basis of the space

$$\mathfrak{W}(\lambda) := \{\, \boldsymbol{u} \in \boldsymbol{S}_0^{\lambda-\vec{\tau}} \mid \boldsymbol{M}\boldsymbol{u} = 0 \quad \text{in} \quad \Gamma\} \tag{2.1}$$

where the space of "multi-homogeneous" functions $\boldsymbol{S}_0^{\lambda-\vec{\tau}}$ is defined as:

$$\boldsymbol{S}_0^{\lambda-\vec{\tau}} := \{\boldsymbol{u} = (u_1, \ldots, u_N)\,, \quad u_l = r^{\lambda-\tau_l} \varphi_{l0}(\theta)\,, \quad \varphi_{l0} \in C^\infty(\mathbb{R}) \quad \text{for } l = 1, \ldots, N\}\,. \tag{2.2}$$

This basis will be used to obtain formulas for both the exponents λ of the singularities and their angular components $\varphi_{lq}(\theta)$.

For the elementary example when $\boldsymbol{M}$ is the scalar Laplace operator, such a basis is given simply by ζ^λ and $\overline{\zeta}^\lambda$ for any $\lambda \neq 0$, where ζ is the complex writing of the cartesian variables

$$\zeta = x_1 + ix_2 = r\,e^{i\theta}\,.$$

This fact is well known and was first used in [25], [17], and recently in [20] and [6, 7].

We are going to prove that such a result can be extended in a certain sense to *any* properly elliptic ADN system as $\boldsymbol{M}$.

Noting that

$$\partial_{x_1} = \partial_{\zeta} + \partial_{\overline{\zeta}} \qquad \text{and} \qquad \partial_{x_2} = i\,(\partial_{\zeta} - \partial_{\overline{\zeta}})$$

where ∂_{ζ} , $\partial_{\overline{\zeta}}$ satisfy

$$\partial_{\zeta}\zeta\,,\ \partial_{\overline{\zeta}}\overline{\zeta} = 1\,, \qquad \partial_{\zeta}\overline{\zeta}\,,\ \partial_{\overline{\zeta}}\zeta = 0\,,$$

we obtain for a scalar homogeneous operator M of order $2m$ and for any complex numbers α and α^* :

$$M(\partial_{x_1},\partial_{x_2})\,(\alpha\zeta+\alpha^*\overline{\zeta})^\lambda = \lambda(\lambda-1)\cdots(\lambda-2m+1)\;(\alpha\zeta+\alpha^*\overline{\zeta})^{\lambda-2m}M(\alpha+\alpha^*,i(\alpha-\alpha^*)) \tag{2.3}$$

Setting:

$$M_+(\alpha) := M(\alpha+1, i(\alpha-1)) \qquad \text{and} \qquad M_-(\alpha) := M(1+\alpha, i(1-\alpha)) \tag{2.4}$$

we see that

$$\begin{array}{llll} \text{if} \ \ \alpha^+ & \text{is a root of} \ \ M_+(\alpha)=0 & \text{then} & (\alpha^+\zeta+\overline{\zeta})^\lambda \in \mathfrak{W}(\lambda) \\ \text{if} \ \ \alpha^- & \text{is a root of} \ \ M_-(\alpha)=0 & \text{then} & (\zeta+\alpha^-\overline{\zeta})^\lambda \in \mathfrak{W}(\lambda)\,. \end{array} \tag{2.5}$$

In the case of $M = \Delta$, we have $M_+(\alpha) = M_-(\alpha) = \alpha$ and α^+, $\alpha^- = 0$.

We extend the ansatz (2.5) so that:

(i) we cover the case when there are multiple roots,

(ii) we can admit systems instead of scalar operators.

We introduce the following diagonal $N \times N$ matrices:

$$\begin{aligned} \boldsymbol{Z}^+_{\vec\tau}(\lambda;\zeta,\zeta^*;\alpha) &= \Big(\lambda(\lambda-1)\cdots(\lambda-\tau_l+1)\;(\alpha\zeta+\zeta^*)^{\lambda-\tau_l}\,\delta_{kl}\Big)_{1\le k,l\le N} \\ \boldsymbol{Z}^-_{\vec\tau}(\lambda;\zeta,\zeta^*;\alpha) &= \Big(\lambda(\lambda-1)\cdots(\lambda-\tau_l+1)\;(\zeta+\alpha\zeta^*)^{\lambda-\tau_l}\,\delta_{kl}\Big)_{1\le k,l\le N} \end{aligned} \tag{2.6}$$

The choice of a branch of the complex power will be discussed below (see (2.9)). Defining $\boldsymbol{M}_+(\alpha)$ and $\boldsymbol{M}_-(\alpha)$ as in formula (2.4), we see that for any ζ and λ in $\mathbb{C}$:

$$\boldsymbol{M}(\partial_{x_1},\partial_{x_2})\,\boldsymbol{Z}^\pm_{\vec\tau}(\lambda;\zeta,\overline{\zeta};\alpha) = \boldsymbol{Z}^\pm_{\vec\sigma}(\lambda;\zeta,\overline{\zeta};\alpha)\,\boldsymbol{M}_\pm(\alpha)\,. \tag{2.7}$$

The elements of $\mathfrak{W}(\lambda)$ will be obtained as Cauchy integrals in the variable α around some roots of

$$D_\pm(\alpha) := \det \boldsymbol{M}_\pm(\alpha)\,.$$

A very general ansatz could be:

$$\int_\gamma \boldsymbol{Z}^\pm_\mp(\lambda;\zeta,\overline{\zeta};\alpha)\,\boldsymbol{M}^{-1}_\pm(\alpha)\,\boldsymbol{f}(\alpha)\,d\alpha \tag{2.8}$$

with γ any contour avoiding the roots of $D_\pm(\alpha)$ and $\boldsymbol{f}(\alpha) \in \mathfrak{A}[\alpha]\otimes\mathbb{C}^N$ (that is, a vector-valued entire analytic function). We will show that less general objects are sufficient.

(i) Due to the proper ellipticity of $\boldsymbol{M}$ and the properties of the Cayley transform $\alpha \mapsto \frac{i(\alpha-1)}{\alpha+1}$, both determinants D_+ and D_- have no roots on the unit circle

$$\tilde{\gamma} = \{\alpha\in\mathbb{C} \mid \ \ |\alpha| = 1\}$$

and each one has m roots (counted with multiplicity) inside $\tilde{\gamma}$. We can therefore use $\tilde{\gamma}$ as contour of integration in the following way: Define $(\alpha\zeta+\overline{\zeta})^\lambda$ as the product $\overline{\zeta}^\lambda(1+\alpha\frac{\zeta}{\overline{\zeta}})^\lambda$. This is well defined for $\alpha\le 1$ and for ζ on the Riemann surface of log. In particular, if ζ is in a sector $0\le\arg\zeta<\omega_0\le 2\pi$, we can choose the branch of $\overline{\zeta}^\lambda$ that coincides with $|\zeta|^\lambda$ on the positive real axis. Then for h meromorphic in $\mathbb{C}$ without poles on $\tilde{\gamma}$ and therefore without poles in $\{\alpha\in\mathbb{C}\mid\rho\le\alpha\le 1\}$ for some $\rho<1$, we define

$$\int_{\tilde{\gamma}} (\alpha\zeta+\overline{\zeta})^\lambda\,h(\alpha)\,d\alpha := \int_{|\alpha|=\rho} \overline{\zeta}^\lambda\left(1+\alpha\frac{\zeta}{\overline{\zeta}}\right)^\lambda h(\alpha)\,d\alpha\,. \tag{2.9}$$

The function $(\zeta+\alpha\overline{\zeta})^\lambda$ is treated by a similar argument.

(ii) Instead of $\mathfrak{A}[\alpha]\otimes\mathbb{C}^N$ we can use special polynomials in $\mathbb{P}_{d-1}[\alpha]\otimes\mathbb{C}^N$ (d is the maximal degree of the matrix elements M_{kl} of $\boldsymbol{M}$) constructed from the shifted polynomials $\boldsymbol{M}^\sharp_{\pm,\delta}$, $\delta=1,\ldots,d$ where, for

$$A(\alpha) = \sum_{n=0}^{d} A_n\,\alpha^n$$

we denote

$$A^\sharp_\delta(\alpha) := \sum_{n=\delta}^{d} A_n\,\alpha^{n-\delta}\,.$$

The main result of this section is the following theorem whose proof will be given in §3. Here we exhibit a finite dimensional generator for the space (2.1). We will construct a basis more explicitly in Lemma 4.2.

Theorem 2.1 *Let λ be any complex number. We set:*

$$\mathfrak{W}^{\pm}(\lambda) := \Big\{ \int_{\tilde\gamma} \boldsymbol{Z}^{\pm}_{\mathcal{T}}(\lambda;\zeta,\overline{\zeta};\alpha)\, \boldsymbol{M}_{\pm}^{-1}(\alpha) \left(\boldsymbol{M}^{\sharp}_{\pm,d}\, \boldsymbol{q}^1 + \cdots + \boldsymbol{M}^{\sharp}_{\pm,1}\, \boldsymbol{q}^d \right) d\alpha \ \mid \ \boldsymbol{q}^1,\ldots,\boldsymbol{q}^d \in \mathbb{C}^N \Big\}.$$

There holds:

(i) for all $\lambda \in \mathbb{C}$, the dimension of the solution space $\mathfrak{W}(\lambda)$ — see (2.1) *— is equal to $2m$;*

(ii) for all $\lambda \in \mathbb{C} \setminus \mathbb{N}$, the dimension of $\mathfrak{W}^{\pm}(\lambda)$ is equal to m and

$$\mathfrak{W}^{+}(\lambda) \oplus \mathfrak{W}^{-}(\lambda) = \mathfrak{W}(\lambda)\,.$$

Remark 2.2 The definition of the spaces $\mathfrak{W}^{\pm}(\lambda)$ is independent of the choice of a branch of the complex power functions as dicussed above. Note, however, that the elements of $\mathfrak{W}^{\pm}(\lambda)$ are well defined analytic functions on the Riemann surface of the logarithm and therefore on a neighborhood of the corner of our domain Ω.

Remark 2.3 In the expression of $\mathfrak{W}^{\pm}(\lambda)$, we can replace d, the maximal degree of $\boldsymbol{M}$, by $d_{\pm}$ which we define as the maximal degree of $\boldsymbol{M}_{\pm}$. We saw in the example of the Laplace operator and we shall see from the example of the Stokes operator that $d_{\pm}$ can be strictly less than d.

2.b Particular cases and examples.

The generic elements of the spaces $\mathfrak{W}^{\pm}(\lambda)$ have a simpler expression when the operator is scalar or when it is a first order system (see Corollary 3.13 where $\mathfrak{W}^{+}(\lambda)$ is given when $\boldsymbol{M}^{+}(\alpha) = \alpha\boldsymbol{A} + \boldsymbol{B}$). We also get some simplifications when the system is rotationally invariant.

2.b (i) Scalar operators. When $N = 1$, the order $d_{\pm}$ of $M_{\pm}(\alpha)$ is equal to its degree as a scalar polynomial and one can assume that its leading coefficient is 1. Then all the shifted polynomials $M^{\sharp}_{\pm,\delta}$ for $\delta = 1,\ldots,d_{\pm}$ have their leading coefficients equal to 1, so they form a basis of $\mathbb{P}_{d_{\pm}-1}[\alpha]$, and we have

$$\mathfrak{W}^{+}(\lambda) = \Big\{ \int_{\tilde\gamma} (\alpha\zeta + \overline{\zeta})^{\lambda}\, M_{+}^{-1}(\alpha)\, f(\alpha)\, d\alpha \ \mid \ f(\alpha) \in \mathbb{P}_{d_{+}-1}[\alpha] \Big\};$$

$$\mathfrak{W}^{-}(\lambda) = \Big\{ \int_{\tilde\gamma} (\zeta + \alpha\overline{\zeta})^{\lambda}\, M_{-}^{-1}(\alpha)\, f(\alpha)\, d\alpha \ \mid \ f(\alpha) \in \mathbb{P}_{d_{-}-1}[\alpha] \Big\}.$$

If $f(\alpha)$ is chosen as the product of all the roots of $M_{\pm}(\alpha)$ except one inside the unit disc, say $\alpha^{\pm}$, then the above Cauchy integrals give back the ansatz (2.5). But such expressions are not necessarily stable with respect to parameters on which the coefficients of M may depend. On the other hand, if $M_{\pm}(\alpha)$ and $f(\alpha)$ depends smoothly on a parameter, so does the Cauchy integral.

2.b (ii) Polyharmonic operators. If $M = \Delta^m$, then $M_\pm = 4^m\alpha^m$. It is straightforward that in such a situation:

$$\mathfrak{W}^+(\lambda) = \Big\{\overline{\zeta}^{\lambda-j}\,\zeta^j \ \mid\ j = 0,\dots,m-1\Big\};$$

$$\mathfrak{W}^-(\lambda) = \Big\{\zeta^{\lambda-j}\,\overline{\zeta}^j \ \mid\ j = 0,\dots,m-1\Big\}.$$

2.b (iii) Rotationally invariant systems. We denote by R_η the rotation in $\mathbb{R}^2$ with angle $\eta \in [0, 2\pi)$. We say that the system $\boldsymbol{M}(\partial_x)$ is *rotationally invariant* if for each $\eta \in [0, 2\pi)$ there exists a linear isomorphism $\boldsymbol{J}_\eta : \mathbb{C}^N \to \mathbb{C}^N$ such that for any $\boldsymbol{u} \in C^\infty \otimes \mathbb{C}^N$:

$$\boldsymbol{M}(\partial_x)\Big(\boldsymbol{J}_\eta \boldsymbol{u} \circ R_\eta\Big) = \Big(\boldsymbol{J}_\eta\, \boldsymbol{M}(\partial_x)\ \boldsymbol{u}\Big) \circ R_\eta\,. \tag{2.10}$$

As there holds

$$\boldsymbol{M}(\partial_x)\Big(\boldsymbol{u} \circ R_\eta\Big) = \Big(\boldsymbol{M}(R_{-\eta}\partial_x)\ \boldsymbol{u}\Big) \circ R_\eta\,,$$

relation (2.10) yields that

$$\boldsymbol{M}(R_{-\eta}\xi) \circ \boldsymbol{J}_\eta = \boldsymbol{J}_\eta \circ \boldsymbol{M}(\xi) \qquad \forall \xi \in \mathbb{R}^2\,. \tag{2.11}$$

Hence the determinant of $\boldsymbol{M}(\xi)$ is a radial function. As its degree is $2m$, we deduce that there is a nonzero constant c such that

$$\det \boldsymbol{M}(\xi) = c\,(\xi_1^2 + \xi_2^2)^m.$$

From this we deduce that $D_\pm(\alpha) = 4^m c\,\alpha^m$ and that the components of the elements of $\mathfrak{W}(\lambda)$ have a form similar to what we found for the m-harmonic operator. Examples of rotationally invariant systems are given by the Stokes and Lamé systems.

2.b (iv) Stokes system. The two-dimensional Stokes operator associates to (u_1, u_2, p) the triple (f_1, f_2, g) by

$$\begin{cases} \Delta u_1 & & + & \partial_{x_1} p & = & f_1 \\ & \Delta u_2 & + & \partial_{x_2} p & = & f_2 \\ \partial_{x_1} u_1 & + \ \partial_{x_2} u_2 & & & = & g\,. \end{cases}$$

Using the differential operators ∂_ζ and $\partial_{\overline{\zeta}}$ and the change of functions

$$u_\zeta := u_1 + i\,u_2 \qquad u_{\overline{\zeta}} := u_1 - i\,u_2$$

$$f_\zeta := f_1 + i\,f_2 \qquad f_{\overline{\zeta}} := f_1 - i\,f_2$$

the Stokes system can be written:

$$\begin{cases} 2\partial_\zeta \partial_{\overline{\zeta}} u_\zeta & & + & \partial_{\overline{\zeta}} p & = & \frac{1}{2}\, f_\zeta \\ & 2\partial_\zeta \partial_{\overline{\zeta}} u_{\overline{\zeta}} & + & \partial_\zeta p & = & \frac{1}{2}\, f_{\overline{\zeta}} \\ \partial_\zeta u_\zeta & + \ \partial_{\overline{\zeta}} u_{\overline{\zeta}} & & & = & \frac{1}{2}\, g\,. \end{cases}$$

Hence

$$\boldsymbol{M}_+(\alpha) = \begin{pmatrix} 2\alpha & 0 & 1 \\ 0 & 2\alpha & \alpha \\ \alpha & 1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{M}_-(\alpha) = \begin{pmatrix} 2\alpha & 0 & \alpha \\ 0 & 2\alpha & 1 \\ 1 & \alpha & 0 \end{pmatrix}.$$

The degree of $\boldsymbol{M}_\pm$ is 1, and according to Remark 2.3 we have only to compute

$$\int_{\tilde\gamma} \boldsymbol{Z}^{\pm}_{(0,0,1)}(\lambda;\zeta,\overline{\zeta};\alpha)\, \boldsymbol{M}_\pm^{-1}(\alpha) \circ \Big(\frac{d}{d\alpha}\, \boldsymbol{M}_\pm(\alpha)\Big)\, \boldsymbol{q}\; d\alpha$$

for $\boldsymbol{q} \in \mathbb{C}^3$. We find that the following triples $\boldsymbol{w}^\pm_j(\lambda,\cdot)$ for $j = 1, 2$ are bases of $\mathfrak{W}^\pm(\lambda)$:

$$\boldsymbol{w}_1^+(\lambda;\zeta) = (\overline{\zeta}^\lambda, 0, 0)\,; \qquad \boldsymbol{w}_2^+(\lambda;\zeta) = (-\lambda\overline{\zeta}^{\lambda-1}\zeta,\ \overline{\zeta}^\lambda,\ 2\lambda\overline{\zeta}^{\lambda-1})\,;$$

$$\boldsymbol{w}_1^-(\lambda;\zeta) = (0, \zeta^\lambda, 0)\,; \qquad \boldsymbol{w}_2^-(\lambda;\zeta) = (\zeta^\lambda,\ -\lambda\zeta^{\lambda-1}\overline{\zeta},\ 2\lambda\zeta^{\lambda-1})\,.$$

3. Proof of Theorem 2.1

3.a The dimension of the space $\mathfrak{W}(\lambda)$. In this section, we are going to prove the point *(i)* of Theorem 2.1, i. e. that for any $\lambda \in \mathbb{C}$, $\dim \mathfrak{W}(\lambda) = 2m$.

We write the system $\boldsymbol{M}$ in polar coordinates: for each k, l there exists an operator $\mathscr{M}_{kl}$ with smooth coefficients such that:

$$r^{\sigma_k} M_{kl}(\partial_x)\, r^{-\tau_l} = \mathscr{M}_{kl}(\theta; \partial_\theta, r\partial_r)\,. \tag{3.1}$$

We introduce the system $\mathscr{M}(\lambda) = \mathscr{M}(\theta;\partial_\theta,\lambda)$ whose coefficients are the $\mathscr{M}_{kl}(\theta;\partial_\theta,\lambda)$. We denote by $\mathscr{W}(\lambda)$ the space

$$\mathscr{W}(\lambda) = \{\boldsymbol{\varphi} \in C^\infty(\mathbb{R}) \otimes \mathbb{C}^N \mid \mathscr{M}(\lambda)\, \boldsymbol{\varphi} = 0\}\,. \tag{3.2}$$

The following equivalence holds for any $\boldsymbol{u}$ in $\boldsymbol{S}_0^{\lambda-\vec\tau}$:

$$\boldsymbol{u} = \Big(r^{\lambda-\tau_1}\varphi_1(\theta), \ldots, r^{\lambda-\tau_N}\varphi_N(\theta)\Big) \in \mathfrak{W}(\lambda) \Longleftrightarrow \boldsymbol{\varphi} := (\varphi_1,\ldots,\varphi_N) \in \mathscr{W}(\lambda)\,. \tag{3.3}$$

Thus we are going to study $\mathscr{W}(\lambda)$.

We use the ADN linearization procedure (see [1]), to reduce the problem to the case when the order d of $\boldsymbol{M}$ is equal to 1: the new operator $\widetilde{\boldsymbol{M}}$ is an ADN-elliptic system of size $\tilde N \geq N$ and multi-degree $(\tilde\sigma_k - \tilde\tau_l)_{kl}$ with the sum $\sum_k(\tilde\sigma_k - \tilde\tau_k)$ still equal to $2m$. Note that $\tilde\sigma_k - \tilde\tau_l$ can be different from 0 or 1, but that in this case $\widetilde{M}_{kl} = 0$. The linearization operator $\boldsymbol{u} \mapsto \tilde{\boldsymbol{u}}$ induces a bijective mapping from $\mathfrak{W}(\lambda)$ onto $\widetilde{\mathfrak{W}}(\lambda)$.

From now on, we assume in this section 3.a that $\boldsymbol{M}$ is of order 1 (and ADN-elliptic, homogeneous with constant coefficients). Then there are $N \times N$ matrices $\boldsymbol{M}^0$, $\boldsymbol{M}^1$ and $\boldsymbol{M}^2$ such that:

$$\boldsymbol{M}(\partial_{x_1},\partial_{x_2}) = \boldsymbol{M}^1\partial_{x_1} + \boldsymbol{M}^2\partial_{x_2} + \boldsymbol{M}^0\,.$$

Hence $\mathscr{M}$ is of order 1 and $\mathscr{M}(\lambda)$ can be written as

$$\mathscr{M}(\lambda) = \mathscr{A}(\theta)\partial_\theta + \mathscr{B}(\theta,\lambda)\,.$$

We check that:

$$\begin{aligned}\mathscr{A}(\theta) &= -\sin\theta\,\boldsymbol{M}^1 + \cos\theta\,\boldsymbol{M}^2 \\ \mathscr{B}(\theta,\lambda) &= \Big(\cos\theta\,\boldsymbol{M}^1 + \sin\theta\,\boldsymbol{M}^2\Big)\Big(\operatorname{diag}(\lambda-\tau_l)\Big) + \boldsymbol{M}^0\end{aligned}$$

where $\operatorname{diag}(\lambda-\tau_l)$ denotes the diagonal matrix $\Big(\delta_{kl}(\lambda-\tau_l)\Big)_{k,l}$.

As a direct consequence of the fact that the system $\boldsymbol{M}(\partial_{x_1},\partial_{x_2})$ is ADN-elliptic of multi-degree $(\sigma_k-\tau_l)_{kl}$ with the sum $\sum_k(\sigma_k-\tau_k)$ equal to $2m$, we obtain the following lemma.

Lemma 3.1 *Let $p_0(\theta;\eta)$ denote the determinant of $\mathscr{A}(\theta)\eta+\boldsymbol{M}^0$. The degree $d^\circ_\eta p_0$ of $p_0(\theta;\cdot)$ as a polynomial in η is equal to $2m$ for all $\theta\in\mathbb{R}$ and $\lambda\in\mathbb{C}$.*

Hence for all $\theta_0\in\mathbb{R}$, the equation

$$\mathscr{A}(\theta_0)\,\partial_\theta\boldsymbol{\varphi} + \boldsymbol{M}^0\boldsymbol{\varphi} = \boldsymbol{\psi} \tag{3.4}$$

is a (singular) system of algebraic-differential equations (singular as soon as $2m < N$), which can be solved with the help of the reduction to the *Smith form* :

$$\mathscr{A}(\theta_0)\,\eta + \boldsymbol{M}^0 = \boldsymbol{E}(\eta)\,\boldsymbol{D}(\eta)\,\boldsymbol{F}(\eta) \tag{3.5}$$

where $\boldsymbol{D}$, $\boldsymbol{E}$ and $\boldsymbol{F}$ have polynomial coefficients, $\boldsymbol{E}$ and $\boldsymbol{F}$ have their determinants equal to 1 and $\boldsymbol{D}$ is diagonal, see [11]. Hence $\det\boldsymbol{D}(\eta) = p_0(\theta_0,\eta)$. From this, we deduce that the equation (3.4) has exactly $2m$ independent solutions $\boldsymbol{\varphi}\in C^\infty(\mathbb{R})\otimes\mathbb{C}^N$ for any $\boldsymbol{\psi}\in C^\infty(\mathbb{R})\otimes\mathbb{C}^N$ (for $\boldsymbol{\psi}=0$, this result is referred as Chrystal's theorem). Moreover, there exists $2m$ Cauchy conditions which determine completely the solution of (3.4).

On the other hand, the ADN-ellipticity of $\mathscr{A}(\theta_0)\,\partial_\theta+\boldsymbol{M}^0$ yields a priori estimates between appropriate Sobolev spaces. As a consequence of the previous facts we obtain:

Lemma 3.2 *Let $\theta_0\in\mathbb{R}$. Let I be a bounded interval and let $\theta_1\in\overline{I}$. There exists $2m$ independent Cauchy conditions in θ_1 : $\chi^1_{\theta_1},\ldots,\chi^{2m}_{\theta_1}$ such that for any $s\geq s_0$ with s_0 large enough, the operator*

$$\begin{array}{ccccc}\boldsymbol{H}^{s-\vec\tau}(I) & \to & \boldsymbol{H}^{s-\vec\sigma}(I) & \oplus & \mathbb{C}^{2m} \\ \boldsymbol{\varphi} & \mapsto & \mathscr{A}(\theta_0)\,\partial_\theta\boldsymbol{\varphi}+\boldsymbol{M}^0\boldsymbol{\varphi} & , & \Big(\chi^j_{\theta_1}(\boldsymbol{\varphi})\Big)_{j=1,\ldots,2m}\end{array}$$

is an isomorphism. (Here $\boldsymbol{H}^{s-\vec\tau}$ denotes the product $\prod_{l=1}^N H^{s-\tau_l}$).

Now we can prove:

Theorem 3.3 *Let $I = (0,\omega)$ be an interval. Then for any $s \geq s_0$ with s_0 as in Lemma 3.2, the operator*

$$\begin{array}{ccccc} \boldsymbol{H}^{s-\vec{\tau}}(I) & \to & \boldsymbol{H}^{s-\vec{\sigma}}(I) & \oplus & \mathbb{C}^{2m} \\ \boldsymbol{\varphi} & \mapsto & \mathscr{M}(\lambda)\boldsymbol{\varphi} & , & \Big(\chi_0^j(\boldsymbol{\varphi})\Big)_{j=1,\ldots,2m} \end{array}$$

is an isomorphism.

Remark 3.4 We see that the space of Cauchy data in $\theta = 0$ for the system

$$\Big(\mathscr{A}(\theta)\partial_\theta + \mathscr{B}(\theta,\lambda)\Big)\boldsymbol{\varphi} = \boldsymbol{\psi} \tag{3.6}$$

is the same as the space of Cauchy data for the constant coefficient system (3.4) with $\theta_0 = 0$. We shall see in Remark 3.15 how to write the space of these Cauchy data explicitly.

Proof. We use Lemma 3.2 with localization arguments. For any $\theta_0 \in \mathbb{R}$ and $\rho \geq 0$, let us introduce:

$$\begin{array}{lcll} \mathscr{N}_{\theta_0}(\rho;\theta,\partial_\theta) & := & \Big(\mathrm{diag}(\rho^{\sigma_k})\Big) \times \mathscr{M}(\theta_0 + \rho(\theta-\theta_0), \dfrac{\partial_\theta}{\rho}, \lambda) \times \Big(\mathrm{diag}(\rho^{-\tau_l})\Big) & \text{if } \rho > 0 \\ \mathscr{N}_{\theta_0}(0;\theta,\partial_\theta) & := & \mathscr{A}(\theta_0)\,\partial_\theta + \boldsymbol{M}^0 & \text{if } \rho = 0 \end{array}$$

Note that, as $\mathscr{A}(\theta_0)\,\partial_\theta + \boldsymbol{M}^0$ is *homogeneous* of muti-degree $(\sigma_k - \tau_l)_{kl}$, we have for any $\rho > 0$:

$$\mathscr{N}_{\theta_0}(0;\theta,\partial_\theta) = \Big(\mathrm{diag}(\rho^{\sigma_k})\Big) \times \mathscr{N}_{\theta_0}(0;\theta_0,\frac{\partial_\theta}{\rho}) \times \Big(\mathrm{diag}(\rho^{-\tau_l})\Big).$$

Let us fix an interval I containing θ_0. By standard arguments, we obtain the estimate:

$$\| \mathscr{N}_{\theta_0}(\rho;\theta,\partial_\theta)\boldsymbol{\varphi} - \mathscr{N}_{\theta_0}(0;\theta,\partial_\theta)\boldsymbol{\varphi} \|_{\boldsymbol{H}^{s-\vec{\sigma}}(I)} \leq C\rho\, \|\boldsymbol{\varphi}\|_{\boldsymbol{H}^{s-\vec{\tau}}(I)}$$

with C independent from ρ (see for instance [9, Lemma 10.13]). This estimate and Lemma 3.2 yield that for $\rho > 0$ small enough, $\mathscr{N}_{\theta_0}(\rho;\theta,\partial_\theta)$ joined with the Cauchy conditions in θ_0 induces an isomorphism. Defining

$$\boldsymbol{\varphi}_\rho(\theta) := \Big(\mathrm{diag}(\rho^{-\tau_l})\Big)\boldsymbol{\varphi}(\theta_0 + \frac{\theta-\theta_0}{\rho}) \quad \text{and} \quad \boldsymbol{\psi}_\rho(\theta) := \Big(\mathrm{diag}(\rho^{-\sigma_k})\Big)\boldsymbol{\psi}(\theta_0 + \frac{\theta-\theta_0}{\rho})$$

we can see that

$$\mathscr{N}_{\theta_0}(\rho;\theta,\partial_\theta)\boldsymbol{\varphi} = \boldsymbol{\psi} \text{ on } I \quad \Longleftrightarrow \quad \mathscr{M}(\lambda)\boldsymbol{\varphi}_\rho = \boldsymbol{\psi}_\rho \text{ on } I_{\theta_0}$$

where $I_{\theta_0} = \{\theta_0 + \rho(\theta-\theta_0) \mid \theta \in I\}$. Hence, $\boldsymbol{M}(\lambda)$ joined with $2m$ Cauchy conditions in θ_0 induces an isomorphism on I_{θ_0}.

Thus we find locally $2m$ linearly independent solutions of the homogeneous equation $\mathscr{M}(\lambda)\boldsymbol{\varphi} = 0$. By the unique solvability of the Cauchy problem just shown, these solutions can be extended over the whole interval I as a basis for the kernel of the operator $\mathscr{M}(\lambda)$. By the same matching arguments, any solution of the inhomogeneous equation $\mathscr{M}(\lambda)\boldsymbol{\varphi} = \boldsymbol{\psi}$ can be extended to I. ∎

Remark 3.5 The degree of the determinant of $\mathscr{A}(\theta)\,\eta + \mathscr{B}(\theta,\lambda)$ as a polynomial in η is $2m$ for any θ and any λ. This condition on the determinant is sufficient in the constant coefficient case to have exactly $2m$ independent solutions to the homogeneous equation. But, in the variable coefficient case, such a condition *is not sufficient* to have the same result, as shown by counterexamples in [4, ch.6]. Here, we need supplementary information to obtain the result. This is supplied by the ellipticity in our case.

Remark 3.6 If all the multi-degrees $\sigma_k - \tau_l$ are ≥ 0, we can show that for all θ, $\operatorname{rank}\mathscr{A}(\theta) = 2m$. This provides an alternative proof to Theorem 3.3 by a direct solution by block elimination of the equation $\mathscr{M}(\lambda)\boldsymbol{\varphi} = \boldsymbol{\psi}$. But in the general case we consider here, it may happen that $\operatorname{rank}\mathscr{A}(\theta) > 2m$ and this solution method does no longer work.

3.b The spaces $\mathfrak{W}^+(\lambda)$ and $\mathfrak{W}^-(\lambda)$. As an obvious consequence of relations (2.7), we obtain that $\mathfrak{W}^{\pm}(\lambda) \subset \mathfrak{W}(\lambda)$. Now we only have to prove that $\dim \mathfrak{W}^{\pm}(\lambda) = m$ and that $\mathfrak{W}^+(\lambda) \cap \mathfrak{W}^-(\lambda) = \{0\}$ to make the proof of Theorem 2.1 complete.

We introduce the following notations: $\mathfrak{A}_1$ is the space of functions holomorphic in a neighborhood of the unit disk in $\mathbb{C}$ and, in relation with the ansatz (2.8), $\mathfrak{W}_1^{\pm}(\lambda)$ is defined as

$$\mathfrak{W}_1^{\pm}(\lambda) = \Big\{ \int_{\tilde{\gamma}} \boldsymbol{Z}^{\pm}_{\boldsymbol{\tau}}(\lambda;\zeta,\overline{\zeta};\alpha)\, \boldsymbol{M}_{\pm}^{-1}(\alpha)\, \boldsymbol{f}(\alpha)\, d\alpha \;\mid\; \boldsymbol{f}(\alpha) \in \mathfrak{A}_1^N \Big\}.$$

We have:

$$\mathfrak{W}^{\pm}(\lambda) \subset \mathfrak{W}_1^{\pm}(\lambda) \subset \mathfrak{W}(\lambda)\,.$$

Lemma 3.7 *If $\lambda \notin \mathbb{N}$, then the intersection $\mathfrak{W}_1^+(\lambda) \cap \mathfrak{W}_1^-(\lambda)$ is reduced to $\{0\}$.*

Proof. Let $\boldsymbol{u}$ belong to the intersection. Then, using (2.9), we have for any component u_l of $\boldsymbol{u}$:

$$u_l = \int_{|\alpha|=\rho} \overline{\zeta}^{\lambda-\tau_l} \left(1 + \alpha\frac{\zeta}{\overline{\zeta}}\right)^{\lambda-\tau_l} h_l^+(\alpha)\, d\alpha$$

and

$$u_l = \int_{|\alpha|=\rho} \zeta^{\lambda-\tau_l} \left(1 + \alpha\frac{\overline{\zeta}}{\zeta}\right)^{\lambda-\tau_l} h_l^-(\alpha)\, d\alpha\,,$$

where $h_l^{\pm}(\alpha)$ are meromorphic in the unit disk and ρ is < 1. Expanding $\left(1 + \alpha\frac{\zeta}{\overline{\zeta}}\right)^{\lambda-\tau_l}$ and $\left(1 + \alpha\frac{\overline{\zeta}}{\zeta}\right)^{\lambda-\tau_l}$ we obtain two series

$$u_l = \sum_{n=0}^{\infty} c_n^+\, \overline{\zeta}^{\lambda-\tau_l-n}\zeta^n$$

and

$$u_l = \sum_{n=0}^{\infty} c_n^-\, \zeta^{\lambda-\tau_l-n}\overline{\zeta}^n$$

with $c_n^{\pm} = \binom{\lambda-\tau_l}{n} \int_{|\alpha|=\rho} \alpha^n h_l^{\pm}(\alpha)\, d\alpha\,.$

The coefficients $c_n^\pm$ satisfying the estimate

$$|c_n^\pm| \leq C\,\rho^n$$

the two above series are converging for ζ such that $\rho < \left|\frac{\zeta}{\overline{\zeta}}\right| < \frac{1}{\rho}$. Thus, setting $\eta := \frac{\zeta}{\overline{\zeta}}$, we have:

$$\sum_{n=0}^{\infty} c_n^+\,\eta^n = \sum_{n=0}^{\infty} c_n^-\,\eta^{\lambda-\tau_l-n} \quad \text{for} \quad \rho < |\eta| < \frac{1}{\rho}\,.$$

If $\lambda \notin \mathbb{N}$, all exponents are different from each other. Hence all coefficients are 0.
∎

Lemma 3.8 *If $\lambda \notin \mathbb{N}$, then the mapping*

$$\begin{array}{ccc} \mathfrak{A}_1^N/_{\boldsymbol{M}_+\,\mathfrak{A}_1^N} & \longrightarrow & \mathfrak{W}_1^+(\lambda) \\ \boldsymbol{f} & \longmapsto & \displaystyle\int_{\tilde{\gamma}} \boldsymbol{Z}_{\vec{\tau}}^+(\lambda;\zeta,\overline{\zeta};\alpha)\,\boldsymbol{M}_+^{-1}(\alpha)\,\boldsymbol{f}(\alpha)\,d\alpha \end{array} \tag{3.7}$$

is an isomorphism and the corresponding result holds for + replaced by −.

Proof. This mapping is obviously well defined and onto. It remains to prove that it is injective. If $\boldsymbol{f}$ is such that $\int_{\tilde{\gamma}} \boldsymbol{Z}_{\vec{\tau}}^+(\alpha)\,\boldsymbol{M}_+^{-1}(\alpha)\,\boldsymbol{f}(\alpha)\,d\alpha \equiv 0$, then, setting $(h_1(\alpha),\ldots,h_N(\alpha)) := \boldsymbol{M}_+^{-1}(\alpha)\,\boldsymbol{f}(\alpha)$ computations as in the previous proof yield that for each l :

$$\int_{|\alpha|=\rho} \alpha^n h_l(\alpha)\,d\alpha = 0 \qquad \forall n \in \mathbb{N}\,,$$

Here we have used that $\binom{\lambda-\tau_l}{n} \neq 0$ for all $n \in \mathbb{N}$ if $\lambda \notin \mathbb{N}$. Hence each h_l is holomorphic in the unit disk. Thus $\boldsymbol{f} = \boldsymbol{M}_+\boldsymbol{h}$ with $\boldsymbol{h} \in \mathfrak{A}_1^N$. ∎

From now on, we only consider the case of the + sign for $\boldsymbol{M}_\pm$. The case of the − sign is exactly similar.

Let us denote by $\mathfrak{M}^+$ the subspace of $\mathfrak{A}_1^N$ defined by

$$\mathfrak{M}^+ := \left\{\boldsymbol{M}_{+,d}^\sharp\,\boldsymbol{q}^1 + \cdots + \boldsymbol{M}_{+,1}^\sharp\,\boldsymbol{q}^d \;\;|\;\; \boldsymbol{q}^1,\ldots,\boldsymbol{q}^d \in \mathbb{C}^N\right\}$$

where d is the order of $\boldsymbol{M}_+$. As a straightforward consequence of Lemma 3.8, we obtain:

Corollary 3.9 *If $\lambda \notin \mathbb{N}$, then the following mapping is an isomorphism:*

$$\begin{array}{ccc} \mathfrak{M}^+/_{\boldsymbol{M}_+\,\mathfrak{A}_1^N} & \longrightarrow & \mathfrak{W}^+(\lambda) \\ \boldsymbol{f} & \longmapsto & \displaystyle\int_{\tilde{\gamma}} \boldsymbol{Z}_{\vec{\tau}}^+(\lambda;\zeta,\overline{\zeta};\alpha)\,\boldsymbol{M}_+^{-1}(\alpha)\,\boldsymbol{f}(\alpha)\,d\alpha\,. \end{array} \tag{3.8}$$

Our proof is organized in two parts now:

(i) The study of the case when the order d is equal to 1.

(ii) The linearization to pass from the order 1 to any order.

3.b (i) The case of the order one. We assume that the order of $\boldsymbol{M}_+$ is 1; hence there are two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{M}_+(\alpha) = \alpha\boldsymbol{A} + \boldsymbol{B}$. Only one shifted polynomial is defined from $\boldsymbol{M}_+$: this is $\boldsymbol{M}^\sharp_{+,1}$ which coincides with $\boldsymbol{A}$. Thus the space $\mathfrak{M}^+$ coincides with $\boldsymbol{A}(\mathbb{C}^N)$. Define the operator:

$$\boldsymbol{P} := \frac{1}{2i\pi}\int_{\tilde\gamma} (\alpha\boldsymbol{A} + \boldsymbol{B})^{-1}\boldsymbol{A}\, d\alpha\,.$$

Lemma 3.10 *The operator $\boldsymbol{P}$ is a projector in $\mathbb{C}^N$ and its rank is equal to m.*

Proof. The fact that $\boldsymbol{P}$ is a projector can be deduced in a classical way, see for example [12], from the *resolvent identity*:

$$(\alpha\boldsymbol{A} + \boldsymbol{B})^{-1} - (\alpha'\boldsymbol{A} + \boldsymbol{B})^{-1} = -(\alpha - \alpha')(\alpha\boldsymbol{A} + \boldsymbol{B})^{-1}\boldsymbol{A}\,(\alpha'\boldsymbol{A} + \boldsymbol{B})^{-1}. \tag{3.9}$$

Hence we have:

$$\begin{aligned}\operatorname{rank}\boldsymbol{P} &= \operatorname{Tr}\tfrac{1}{2i\pi}\int_{\tilde\gamma} (\alpha\boldsymbol{A} + \boldsymbol{B})^{-1}\boldsymbol{A}\, d\alpha\\ &= \tfrac{1}{2i\pi}\int_{\tilde\gamma} \operatorname{Tr}(\alpha\boldsymbol{A} + \boldsymbol{B})^{-1}\tfrac{d}{d\alpha}(\alpha\boldsymbol{A} + \boldsymbol{B})\, d\alpha\,.\end{aligned}$$

Using the reduction of $(\alpha\boldsymbol{A} + \boldsymbol{B})$ to the Smith form $\boldsymbol{E}(\alpha)\,\boldsymbol{D}(\alpha)\,\boldsymbol{F}(\alpha)$ as in (3.5), and with the help of the commutativity relation $\operatorname{Tr}AB = \operatorname{Tr}BA$ we obtain:

$$\begin{aligned}\operatorname{Tr}(\alpha\boldsymbol{A} + \boldsymbol{B})^{-1}\tfrac{d}{d\alpha}(\alpha\boldsymbol{A} + \boldsymbol{B}) &= \operatorname{Tr}\boldsymbol{F}^{-1}\boldsymbol{D}^{-1}\boldsymbol{E}^{-1}(\boldsymbol{E}'\boldsymbol{D}\boldsymbol{F} + \boldsymbol{E}\boldsymbol{D}'\boldsymbol{F} + \boldsymbol{E}\boldsymbol{D}\boldsymbol{F}')\\ &= \operatorname{Tr}\boldsymbol{D}^{-1}\boldsymbol{D}' + \operatorname{Tr}\boldsymbol{E}^{-1}\boldsymbol{E}' + \operatorname{Tr}\boldsymbol{F}^{-1}\boldsymbol{F}'\,.\end{aligned}$$

Since the determinants of $\boldsymbol{E}$ and $\boldsymbol{F}$ are equal to 1, the functions $\operatorname{Tr}\boldsymbol{E}^{-1}\boldsymbol{E}'$ and $\operatorname{Tr}\boldsymbol{F}^{-1}\boldsymbol{F}'$ are holomorphic and:

$$\operatorname{rank}\boldsymbol{P} = \frac{1}{2i\pi}\int_{\tilde\gamma} \operatorname{Tr}\boldsymbol{D}^{-1}(\alpha)\boldsymbol{D}'(\alpha)\, d\alpha\,.$$

As $\boldsymbol{D}$ is diagonal, the above integral is the number of the roots of the determinant of $\boldsymbol{D}$ which are inside $\tilde\gamma$. But $\det\boldsymbol{D} = \det\boldsymbol{M}_+$. We know that the proper ellipticity of $\boldsymbol{M}$ implies that $\det\boldsymbol{M}_+$ has exactly m roots inside $\tilde\gamma$. Hence $\operatorname{rank}\boldsymbol{P} = m$. ∎

Lemma 3.11 *Let $\boldsymbol{f}$ belong to $\mathfrak{A}^N_1$. We define an element $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{f}) \in \mathbb{C}^N$ by*

$$\boldsymbol{q} := \frac{1}{2i\pi}\int_{\tilde\gamma} (\alpha\boldsymbol{A} + \boldsymbol{B})^{-1}\boldsymbol{f}(\alpha)\, d\alpha\,.$$

Then we have for all ζ and λ :

$$\int_{\tilde\gamma} \boldsymbol{Z}^\pm_{\mathcal{T}}(\lambda;\zeta,\overline{\zeta};\alpha)\,(\alpha\boldsymbol{A} + \boldsymbol{B})^{-1}\boldsymbol{f}(\alpha)\, d\alpha = \int_{\tilde\gamma} \boldsymbol{Z}^\pm_{\mathcal{T}}(\lambda;\zeta,\overline{\zeta};\alpha)\,(\alpha\boldsymbol{A} + \boldsymbol{B})^{-1}\boldsymbol{A}\boldsymbol{q}\, d\alpha\,.$$

Proof. We omit the variables ζ and λ in the notations. We introduce ρ and ρ', $0 < \rho < \rho' < 1$ such that $(\alpha\boldsymbol{A} + \boldsymbol{B})^{-1}$ is holomorphic in the ring $\rho \le |\alpha| \le 1$. Using the resolvent identity (3.9), we obtain:

$$\int_{\tilde\gamma} \boldsymbol{Z}^+_{\mathcal{T}}(\alpha)\,(\alpha\boldsymbol{A}+\boldsymbol{B})^{-1}\boldsymbol{A}\boldsymbol{q}\;d\alpha =$$

$$= \tfrac{1}{2i\pi}\int_{|\alpha|=\rho} \boldsymbol{Z}^+_{\mathcal{T}}(\alpha)\,(\alpha\boldsymbol{A}+\boldsymbol{B})^{-1}\boldsymbol{A}\int_{|\alpha|=\rho'} (\alpha'\boldsymbol{A}+\boldsymbol{B})^{-1}\boldsymbol{f}(\alpha')\;d\alpha'\;d\alpha$$

$$= \tfrac{1}{2i\pi}\int_{|\alpha|=\rho}\int_{|\alpha|=\rho'} \boldsymbol{Z}^+_{\mathcal{T}}(\alpha)\;\frac{(\alpha\boldsymbol{A}+\boldsymbol{B})^{-1}-(\alpha'\boldsymbol{A}+\boldsymbol{B})^{-1}}{\alpha'-\alpha}\;\boldsymbol{f}(\alpha')\;d\alpha'\;d\alpha\,.$$

The contribution of the term with $(\alpha'\boldsymbol{A}+\boldsymbol{B})^{-1}$ is 0 because for any α', $|\alpha'|=\rho'$, the function

$$\alpha \;\longmapsto\; \boldsymbol{Z}^+_{\mathcal{T}}(\alpha)\,(\alpha'-\alpha)^{-1}\,(\alpha'\boldsymbol{A}+\boldsymbol{B})^{-1}\boldsymbol{f}(\alpha')$$

is holomorphic for $|\alpha|\le\rho$. Then the Cauchy residue formula applied on $|\alpha'|=\rho'$ for $\boldsymbol{f}$ yields the wanted equality. ∎

Corollary 3.12 *For all $\lambda\in\mathbb{C}$, the spaces $\mathfrak{W}^+(\lambda)$ and $\mathfrak{W}^+_1(\lambda)$ coincide.*

Corollary 3.13 *If $\lambda\notin\mathbb{N}$, then the following mapping is an isomorphism:*

$$\begin{array}{lcl} \boldsymbol{P}(\mathbb{C}^N) & \longrightarrow & \mathfrak{W}^+(\lambda) \\ \boldsymbol{q} & \longmapsto & \displaystyle\int_{\tilde\gamma} \boldsymbol{Z}^+_{\mathcal{T}}(\lambda;\zeta,\overline{\zeta};\alpha)\,(\alpha\boldsymbol{A}+\boldsymbol{B})^{-1}\boldsymbol{A}\boldsymbol{q}\;d\alpha\,. \end{array} \tag{3.10}$$

Corollary 3.14 *In the case when $d=1$, if $\lambda\notin\mathbb{N}$ the dimension of the space $\mathfrak{W}^+(\lambda)$ is equal to m.*

Proof of the corollaries. The first corollary is a straightforward consequence of Lemma 3.11. The third corollary is a direct consequence of the second one and of Lemma 3.10. Let us prove Corollary 3.13.

Due to Lemma 3.11, the mapping (3.10) is onto. As in the proof of Lemma 3.8 we obtain that:

$$\int_{\tilde\gamma} \boldsymbol{Z}^+_{\mathcal{T}}(\zeta;\alpha)\,(\alpha\boldsymbol{A}+\boldsymbol{B})^{-1}\boldsymbol{A}\boldsymbol{q}\;d\alpha\equiv 0 \Longrightarrow \int_{\tilde\gamma}(\alpha\boldsymbol{A}+\boldsymbol{B})^{-1}\boldsymbol{A}\boldsymbol{q}\;d\alpha=0\,,\quad\text{i. e. }\;\boldsymbol{P}(\boldsymbol{q})=0\,.$$

Conversely, Lemma 3.11 yields that:

$$\int_{\tilde\gamma} \boldsymbol{Z}^+_{\mathcal{T}}(\zeta;\alpha)\,(\alpha\boldsymbol{A}+\boldsymbol{B})^{-1}\boldsymbol{A}\boldsymbol{q}\;d\alpha = \int_{\tilde\gamma} \boldsymbol{Z}^+_{\mathcal{T}}(\zeta;\alpha)\,(\alpha\boldsymbol{A}+\boldsymbol{B})^{-1}\boldsymbol{A}\,\boldsymbol{P}(\boldsymbol{q})\;d\alpha\,.$$

Hence the kernel of the mapping (3.10) is $\ker\boldsymbol{P}$. As $\boldsymbol{P}$ is a projector, the corollary is proved. ∎

Remark 3.15 Let $\mathscr{A}$ and $\mathscr{B}$ be two $N\times N$ matrices such that the degree of the determimant of $\mathscr{A}\eta+\mathscr{B}$ as a polynomial in η is equal to $2m$. Let γ be a contour surrounding all the roots of the determinant. Then defining

$$\mathscr{P} := \frac{1}{2i\pi}\int_{\gamma}(\eta\mathscr{A}+\mathscr{B})^{-1}\mathscr{A}\;d\eta\,,$$

by the same arguments as above for Corollary 3.13, we obtain that the following mapping is an isomorphism

$$\begin{array}{ccc} \mathscr{P}(\mathbb{C}^N) & \longrightarrow & \{\boldsymbol{\varphi} \mid (\mathscr{A}\partial_\theta + \mathscr{B})\boldsymbol{\varphi}(\theta) \equiv 0\} \\ \boldsymbol{q} & \longmapsto & \boldsymbol{\varphi}(\theta) := \int_{\tilde{\gamma}} \Big(\mathrm{diag}(e^{\eta\theta})\Big) (\eta\mathscr{A} + \mathscr{B})^{-1}\mathscr{A}\boldsymbol{q}\, d\eta\,. \end{array}$$

We see that the natural Cauchy conditions in $\theta = 0$ are, for any fixed $\boldsymbol{q} \in \mathscr{P}(\mathbb{C}^N)$:

$$\mathscr{P}\Big(\boldsymbol{\varphi}\Big|_{\theta=0}\Big) = \boldsymbol{q}\,.$$

As an example, we can consider $\mathscr{A} = \boldsymbol{M}^2$ and $\mathscr{B} = \boldsymbol{M}^0$ and according to Theorem 3.3, the space of Cauchy data for the system $\mathscr{M}(\lambda)$ (3.6) is given as above for any λ with the projector

$$\mathscr{P} := \frac{1}{2i\pi}\int_\gamma (\eta\boldsymbol{M}^2 + \boldsymbol{M}^0)^{-1}\boldsymbol{M}^2\, d\eta\,.$$

We obtain as consequence of the Corollaries 3.9 and 3.12:

Corollary 3.16 *In the case when $d = 1$, the following natural injection is an isomorphism:*

$$\mathfrak{M}^+ \big/ \boldsymbol{M}_+\mathfrak{A}_1^N \quad\hookrightarrow\quad \mathfrak{A}_1^N \big/ \boldsymbol{M}_+\mathfrak{A}_1^N$$

3.b (ii) The linearization. We treat now the general case when

$$\boldsymbol{M}_+(\alpha) = \sum_{\delta=0}^{d} \boldsymbol{M}_\delta\, \alpha^\delta \quad \text{with} \quad d \geq 1\,.$$

For any $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ in $\mathfrak{A}_1^N$, we check that the equation $\boldsymbol{M}_+\boldsymbol{\varphi} = \boldsymbol{\psi}$ is equivalent to the following "linearized" equation in $\mathbb{C}^N \otimes \mathbb{C}^d$:

$$\left[\alpha\begin{pmatrix} \boldsymbol{M}_d & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{pmatrix} + \begin{pmatrix} \boldsymbol{M}_{d-1} & \cdots & \boldsymbol{M}_1 & \boldsymbol{M}_0 \\ -I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -I & 0 \end{pmatrix}\right]\begin{pmatrix} \alpha^{d-1}\boldsymbol{\varphi} \\ \alpha^{d-2}\boldsymbol{\varphi} \\ \vdots \\ \boldsymbol{\varphi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\psi} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

which we write in a condensed form

$$(\alpha\boldsymbol{A} + \boldsymbol{B})\widetilde{\boldsymbol{\varphi}} = \widetilde{\boldsymbol{\psi}}\,.$$

The system $\alpha\boldsymbol{A} + \boldsymbol{B}$ defines $\widetilde{\boldsymbol{M}_+}$, the first order system associated with $\boldsymbol{M}_+$. The space associated with $\widetilde{\boldsymbol{M}_+}$ is $\widetilde{\mathfrak{M}}^+$:

$$\widetilde{\mathfrak{M}}^+ = \Big\{\boldsymbol{A}\tilde{\boldsymbol{q}} \mid \tilde{\boldsymbol{q}} = (\boldsymbol{q}^1, \ldots, \boldsymbol{q}^d) \in \mathbb{C}^N \otimes \mathbb{C}^d\Big\}\,.$$

The link between the case when $d = 1$ and the general case is made by:

Proposition 3.17 *The following mapping is an isomorphism:*

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}^+ / \widetilde{\boldsymbol{M}}_+ \mathfrak{A}_1^{dN} & \longrightarrow & \mathfrak{M}^+ / \boldsymbol{M}_+ \mathfrak{A}_1^N \\ \boldsymbol{A}\tilde{\boldsymbol{q}} & \longmapsto & \boldsymbol{M}^\sharp_{+,d}\,\boldsymbol{q}^1 + \cdots + \boldsymbol{M}^\sharp_{+,1}\,\boldsymbol{q}^d\,. \end{array} \tag{3.11}$$

Proof. The mapping (3.11) being onto by construction, it suffices to prove that it is well defined and injective.

Lemma 3.18 *Let* $\tilde{\boldsymbol{q}} = (\boldsymbol{q}^1, \ldots, \boldsymbol{q}^d) \in \mathbb{C}^N \otimes \mathbb{C}^d$ *and* $\tilde{\boldsymbol{\varphi}} = (\boldsymbol{\varphi}^1, \ldots, \boldsymbol{\varphi}^d) \in \mathfrak{A}_1^N \otimes \mathbb{C}^d$. *We suppose that*

$$\boldsymbol{A}\tilde{\boldsymbol{q}} = (\alpha \boldsymbol{A} + \boldsymbol{B})\tilde{\boldsymbol{\varphi}}(\alpha)\,.$$

Then:

$$\boldsymbol{M}^\sharp_{+,d}\,\boldsymbol{q}^1 + \cdots + \boldsymbol{M}^\sharp_{+,1}\,\boldsymbol{q}^d = \boldsymbol{M}_+ \boldsymbol{\varphi}^d\,.$$

Lemma 3.19 *Let* $(\boldsymbol{q}^1, \ldots, \boldsymbol{q}^d) = \tilde{\boldsymbol{q}} \in \mathbb{C}^N \otimes \mathbb{C}^d$ *and* $\boldsymbol{\varphi}^d \in \mathfrak{A}_1^N$. *We suppose that*

$$\boldsymbol{M}^\sharp_{+,d}\,\boldsymbol{q}^1 + \cdots + \boldsymbol{M}^\sharp_{+,1}\,\boldsymbol{q}^d = \boldsymbol{M}_+ \boldsymbol{\varphi}^d\,.$$

We define $\boldsymbol{\varphi}^1, \ldots, \boldsymbol{\varphi}^{d-1} \in \mathfrak{A}_1^N$ *by*

$$\left\{\begin{array}{lcccccccc} \boldsymbol{\varphi}^1 & = & \alpha^{d-1}\boldsymbol{\varphi}^d & + & \alpha^{d-2}\boldsymbol{q}^d & + \cdots + & \alpha \boldsymbol{q}^3 & + & \boldsymbol{q}^2 \\ \boldsymbol{\varphi}^2 & = & \alpha^{d-2}\boldsymbol{\varphi}^d & + & \alpha^{d-3}\boldsymbol{q}^d & + \cdots + & \boldsymbol{q}^3 & & \\ & \vdots & & & & & & & \\ \boldsymbol{\varphi}^{d-1} & = & \alpha \boldsymbol{\varphi}^d & + & \boldsymbol{q}^d & & & & \end{array}\right.$$

Then, setting $\tilde{\boldsymbol{\varphi}} := (\boldsymbol{\varphi}^1, \ldots, \boldsymbol{\varphi}^d)$, *we have:*

$$\boldsymbol{A}\tilde{\boldsymbol{q}} = (\alpha \boldsymbol{A} + \boldsymbol{B})\tilde{\boldsymbol{\varphi}}(\alpha)\,.$$

Lemma 3.18 and Lemma 3.19 prove that the mapping (3.11) is well defined and injective, respectively. ∎

The proof of the two lemmas being similar, we only prove the first one.

Proof of Lemma 3.18. The equality $\boldsymbol{A}\tilde{\boldsymbol{q}} = (\alpha \boldsymbol{A} + \boldsymbol{B})\tilde{\boldsymbol{\varphi}}(\alpha)$ yields the relations:

(1) $$\alpha \boldsymbol{M}_d \boldsymbol{\varphi}^1 + \boldsymbol{M}_{d-1}\boldsymbol{\varphi}^1 + \cdots + \boldsymbol{M}_0 \boldsymbol{\varphi}^d = \boldsymbol{M}_d \boldsymbol{q}^1$$

and

(2) $$-\boldsymbol{\varphi}^{j-1} + \alpha \boldsymbol{\varphi}^j = \boldsymbol{q}^j \qquad \text{for} \quad j = 2, \ldots, d\,.$$

Let us compute $\boldsymbol{\psi} := \boldsymbol{M}^{\sharp}_{+,d}\,\boldsymbol{q}^1 + \cdots + \boldsymbol{M}^{\sharp}_{+,1}\,\boldsymbol{q}^d$. We first use the definition of $\boldsymbol{M}^{\sharp}_{+,\delta}$, next the relations (1) and (2).

$$\begin{aligned}
\boldsymbol{\psi} &= \boldsymbol{M}_d\boldsymbol{q}^1 + \sum_{\delta=1}^{d-1}\sum_{j=\delta}^{d}\alpha^{j-\delta}\boldsymbol{M}_j\boldsymbol{q}^{d+1-j} \\
&= \alpha\boldsymbol{M}_d\boldsymbol{\varphi}^1 + \sum_{j=0}^{d-1}\boldsymbol{M}_j\boldsymbol{\varphi}^{d-j} + \sum_{\delta=1}^{d-1}\sum_{j=\delta}^{d}\alpha^{j-\delta}\boldsymbol{M}_j(\alpha\boldsymbol{\varphi}^{d+1-\delta} - \boldsymbol{\varphi}^{d-\delta}) \\
&= \sum_{j=0}^{d-1}\boldsymbol{M}_j\Big(\boldsymbol{\varphi}^{d-j} + \sum_{\delta=1}^{j}(\alpha^{j+1-\delta}\boldsymbol{\varphi}^{d+1-\delta} - \alpha^{j-\delta}\boldsymbol{\varphi}^{d-\delta})\Big) \\
&\quad + \alpha\boldsymbol{M}_d\boldsymbol{\varphi}^1 + \boldsymbol{M}_d\sum_{\delta=1}^{d-1}(\alpha^{d+1-\delta}\boldsymbol{\varphi}^{d+1-\delta} - \alpha^{d-\delta}\boldsymbol{\varphi}^{d-\delta}) \\
&= \sum_{j=0}^{d-1}\boldsymbol{M}_j\alpha^j\boldsymbol{\varphi}^d + \boldsymbol{M}_d\alpha^d\boldsymbol{\varphi}^d\,.
\end{aligned}$$

The last equality yields that $\boldsymbol{\psi} = \boldsymbol{M}_+\boldsymbol{\varphi}^d$. ∎

Conclusion 3.20 Lemma 3.9 and Proposition 3.17 give that $\forall\lambda \notin \mathbb{N}$ the spaces $\mathfrak{W}^+(\lambda)$ and $\widetilde{\mathfrak{W}}^+(\lambda)$ (associated with the linearized operator $\widetilde{\mathfrak{M}}_+$) are isomorphic, via the commutative diagram below, where all arrows are isomorphisms.

$$\begin{array}{ccc}
\widetilde{\mathfrak{M}}^+\big/_{\widetilde{\boldsymbol{M}}_+\mathfrak{A}_1^{dN}} & \longrightarrow & \widetilde{\mathfrak{W}}^+(\lambda) \\
\downarrow & & \downarrow \\
\mathfrak{M}^+\big/_{\boldsymbol{M}_+\mathfrak{A}_1^{N}} & \longrightarrow & \mathfrak{W}^+(\lambda)
\end{array}$$

Combining this with Corollary 3.14, we obtain that $\dim\mathfrak{W}^+(\lambda) = m$.

Remark 3.21 We have just proved a little more than necessary: the consideration of the spaces $\mathfrak{W}_1^+(\lambda)$ and $\mathfrak{A}_1^N\big/_{\boldsymbol{M}_+\mathfrak{A}_1^N}$ could have been omitted to reach the aim of proving Theorem 2.1. Indeed we have the following diagram of commutative isomorphism which complete the above one:

$$\begin{array}{ccccc}
\widetilde{\mathfrak{M}}^+\big/_{\widetilde{\boldsymbol{M}}_+\mathfrak{A}_1^{dN}} & \longrightarrow & \mathfrak{A}_1^{dN}\big/_{\widetilde{\boldsymbol{M}}_+\mathfrak{A}_1^{dN}} & \longrightarrow & \widetilde{\mathfrak{W}}_1^+(\lambda) = \widetilde{\mathfrak{W}}^+(\lambda) \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{M}^+\big/_{\boldsymbol{M}_+\mathfrak{A}_1^{N}} & \longrightarrow & \mathfrak{A}_1^{N}\big/_{\boldsymbol{M}_+\mathfrak{A}_1^{N}} & \longrightarrow & \mathfrak{W}_1^+(\lambda) = \mathfrak{W}^+(\lambda)
\end{array}$$

To check that, it suffices to see that the mapping (3.11) is onto on the space $\mathfrak{A}_1^N\big/_{\boldsymbol{M}_+\mathfrak{A}_1^N}$ which can be shown like Lemma 3.18.

4. Solutions of the boundary value system with zero right hand side

The next step of the description of the singular functions $\sum_p \boldsymbol{u}_p$ — see Conclusion 1.1 — is the construction of the first terms $\boldsymbol{u}_0$ which correspond to a right hand side equal to 0 : thus we search for all solutions $\boldsymbol{u}$ of

$$\mathbb{A}_0\boldsymbol{u} = 0 \quad \text{and} \quad \boldsymbol{u} \in \boldsymbol{S}^{\lambda_0-\vec{\tau}} \text{ for a certain } \lambda_0 \in \mathbb{C}\,. \tag{4.1}$$

4.a Mellin transform. If $\boldsymbol{u}$ satisfies (4.1), multiplying $\boldsymbol{u}$ by a cut-off function $\chi \in \mathscr{D}(\mathbb{R})$ which is $\equiv 1$ in the neighborhood of 0 and computing the Mellin transform:

$$\mathscr{U}(\lambda) = (\mathscr{U}_1, \ldots, \mathscr{U}_N) \quad \text{with} \quad \mathscr{U}_l(\lambda,\theta) := \int_0^\infty r^{-\lambda+\tau_l}\,\chi(r)\,u_l(r,\theta)\,\frac{dr}{r} \quad l = 1,\ldots,N$$

we see that $\mathscr{M}(\lambda)\,\mathscr{U}(\lambda)$ is an entire analytic function of λ, where we recall that $\mathscr{M}(\lambda)$ is the system whose coefficients are the $\mathscr{M}_{kl}(\theta;\partial_\theta,\lambda)$ defined in (3.1). In a similar way, for $j = 0,1$ we define $\mathscr{C}^j(\lambda)$ as the system whose coefficients are the $\mathscr{C}^j_{hl}(\theta;\partial_\theta,\lambda)$ where $\mathscr{C}^j_{hl}$ satisfies

$$r^{\sigma^j_k} C^j_{hl}(\partial_x)\, r^{-\tau_l} = \mathscr{C}^j_{hl}(\theta;\partial_\theta, r\partial_r)\,. \tag{4.2}$$

Thus for each $\lambda \in \mathbb{C}$, $\mathscr{U}(\lambda)$ is a solution of the following boundary value problem in $(0,\omega_0)$:

$$\begin{cases} \mathscr{M}(\lambda)\,\mathscr{U}(\lambda) & = \quad \boldsymbol{\psi}(\lambda) \quad \text{in} \quad (0,\omega_0) \\ \mathscr{C}^0(\lambda)\,\mathscr{U}(\lambda)\Big|_{\theta=0} & = \quad \boldsymbol{\psi}^0(\lambda) \\ \mathscr{C}^1(\lambda)\,\mathscr{U}(\lambda)\Big|_{\theta=\omega_0} & = \quad \boldsymbol{\psi}^1(\lambda) \end{cases} \tag{4.3}$$

with $\boldsymbol{\psi}^0$, $\boldsymbol{\psi}^1$ analytic with values in $\mathbb{C}^m$. We define $\mathscr{A}(\lambda)$ as the following analytic family of boundary value problems on $(0,\omega_0)$:

$$\mathscr{A}(\lambda) = \Big(\mathscr{M}(\lambda), \mathscr{C}^0(\lambda), \mathscr{C}^1(\lambda)\Big). \tag{4.4}$$

We write (4.3) in the condensed form:

$$\mathscr{A}(\lambda)\,\mathscr{U}(\lambda) = \Psi(\lambda)\,.$$

As it is well known — see also the next section, it follows from the ADN ellipticity that $\mathscr{A}(\lambda)^{-1}$ is meromorphic in $\mathbb{C}$ and since $\boldsymbol{u} \in \boldsymbol{S}^{\lambda_0-\vec{\tau}}$ we reconstruct $\boldsymbol{u}$ by the residue formula

$$\boldsymbol{u} = \frac{1}{2i\pi}\,\Big(\mathrm{diag}(r^{-\tau_l})\Big)\int_\gamma r^\lambda \mathscr{A}(\lambda)^{-1}\Psi(\lambda)\,d\lambda$$

where γ is a simple curve surrounding λ_0. It is easy to see that, conversely, any function $\boldsymbol{u}$ of the above form with Ψ holomorphic, is a solution of $\mathbb{A}_0\boldsymbol{u} = 0$. Let us summarize:

Lemma 4.1 *Let $\mathscr{X}(\lambda_0)$ be the space of all solutions of* (4.1). *Then $\mathscr{X}(\lambda_0)$ is not reduced to* $\{0\}$ *if and only if λ_0 is a pole of $\mathscr{A}(\lambda)^{-1}$ where $\mathscr{A}(\lambda)$ is the operator of problem* (4.3) *and*

$$\mathscr{X}(\lambda_0) = \Big\{\frac{1}{2i\pi}\,\Big(\mathrm{diag}(r^{-\tau_l})\Big)\int_{\gamma(\lambda_0)} r^\lambda \mathscr{A}(\lambda)^{-1}\Psi(\lambda)\,d\lambda \quad | \quad \Psi(\lambda) \textit{ holomorphic }\Big\}$$

where $\gamma(\lambda_0)$ is a simple curve surrounding only one pole λ_0 of $\mathscr{A}(\lambda)^{-1}$.

4.b The meromorphic inverse of $\mathscr{A}(\lambda)$ We are going to describe $\mathscr{A}(\lambda)^{-1}$ by solving the equation

$$\begin{cases} \mathscr{M}(\lambda)\mathscr{U} & = \boldsymbol{\psi} \quad \text{in} \quad (0,\omega_0) \\ \mathscr{C}^0(\lambda)\mathscr{U}\Big|_{\theta=0} & = \boldsymbol{\psi}^0 \\ \mathscr{C}^1(\lambda)\mathscr{U}\Big|_{\theta=\omega_0} & = \boldsymbol{\psi}^1 \end{cases} \tag{4.5}$$

for $\boldsymbol{\psi}$ given in $\boldsymbol{H}^{s-\vec{\sigma}}(0,\omega_0)$ — for $s \geq s_0$ large enough — and $\boldsymbol{\psi}_0$, $\boldsymbol{\psi}_1$ in $\mathbb{C}^m$. The strategy for solving (4.5) is classical: we solve in a first step the interior equation $\mathscr{M}(\lambda)\,\mathscr{U} = \boldsymbol{\psi}$ with the help of a right inverse $\mathscr{R}(\lambda)$ to $\mathscr{M}(\lambda)$ acting between the spaces $\boldsymbol{H}^{s-\vec{\sigma}}(0,\omega_0)$ and $\boldsymbol{H}^{s-\vec{\tau}}(0,\omega_0)$. The second step is to solve the boundary conditions in the space $\mathscr{W}(\lambda) = \ker \mathscr{M}(\lambda)$ — cf (3.2). Let us give details for these constructions.

Let us define the space $\mathscr{H}(\lambda)$ as the orthogonal complement of the space $\mathscr{W}(\lambda)$ in $\boldsymbol{H}^{s-\vec{\tau}}(0,\omega_0)$. As a direct consequence of Theorem 3.3, the operator $\mathscr{M}(\lambda)$ induces an isomorphism from $\mathscr{H}(\lambda)$ onto $\boldsymbol{H}^{s-\vec{\sigma}}(0,\omega_0)$ with a locally bounded inverse $\mathscr{M}(\lambda)^{-1}$. We define $\mathscr{R}(\lambda)$ as the composed operator $\mathscr{I}_{\mathscr{H}(\lambda)} \circ \mathscr{M}(\lambda)^{-1}$ where $\mathscr{I}_{\mathscr{H}(\lambda)}$ is the canonical injection of $\mathscr{H}(\lambda)$ into $\boldsymbol{H}^{s-\vec{\tau}}(0,\omega_0)$.

Let us denote by $\mathscr{C}(\lambda)$ the system of the boundary operators

$$\mathscr{C}(\lambda) : \mathscr{U} \longmapsto \Big(\mathscr{C}^0_1(\lambda)\,\mathscr{U}(0),\ldots,\mathscr{C}^0_m(\lambda)\,\mathscr{U}(0),\ \mathscr{C}^1_1(\lambda)\,\mathscr{U}(\omega_0),\ldots,\mathscr{C}^1_m(\lambda)\,\mathscr{U}(\omega_0)\Big)$$

and by π_M and π_C the canonical projections

$$\begin{aligned} \pi_M &: \boldsymbol{H}^{s-\vec{\sigma}}(0,\omega_0) \oplus \mathbb{C}^{2m} \longrightarrow \boldsymbol{H}^{s-\vec{\sigma}}(0,\omega_0) \\ \pi_C &: \boldsymbol{H}^{s-\vec{\sigma}}(0,\omega_0) \oplus \mathbb{C}^{2m} \longrightarrow \mathbb{C}^{2m} \end{aligned}$$

We see that the solution of (4.5) will be given, if it exists, by the sum $\mathscr{U} = \mathscr{U}_1 + \mathscr{U}_2$ where $\mathscr{U}_1 = \mathscr{R}(\lambda)\,\boldsymbol{\psi}$ and $\mathscr{U}_2$ is the solution (if it exists) of

$$\mathscr{U}_2 \in \mathscr{W}(\lambda) \quad \text{and} \quad \mathscr{C}(\lambda)\mathscr{U}_2 = (\boldsymbol{\psi}^0, \boldsymbol{\psi}^1) - \mathscr{C}(\lambda)\mathscr{U}_1\,.$$

To choose a basis of $\mathscr{W}(\lambda)$, we use the equivalence (3.3), Theorem 2.1 and Corollary 3.9. We have

Lemma 4.2 *Let $\mu \in \mathbb{C}\setminus\mathbb{N}$. There exist $\boldsymbol{q}^1_{\pm,h},\ldots,\boldsymbol{q}^d_{\pm,h}$ in $\mathbb{C}^N$ for $h = 1,\ldots,m$ such that the*

$$\boldsymbol{w}^{\pm}_h(\mu,\zeta) := \int_{\tilde{\gamma}} \boldsymbol{Z}^{\pm}_{\vec{\tau}}(\mu;\zeta,\overline{\zeta};\alpha)\,\boldsymbol{M}^{-1}_{\pm}(\alpha)\left(\boldsymbol{M}^{\sharp}_{\pm,d}\,q^1_{\pm,h} + \cdots + \boldsymbol{M}^{\sharp}_{\pm,1}\,\boldsymbol{q}^d_{\pm,h}\right)d\alpha$$

for $h = 1,\ldots,m$ are a basis of $\mathfrak{W}^{\pm}(\mu)$.

Moreover, for any $\lambda \in \mathbb{C}\setminus\mathbb{N}$, the $\boldsymbol{w}_h^\pm(\lambda,\zeta)$ for $h = 1,\dots,m$ are a basis of $\mathfrak{W}^\pm(\lambda)$ and the functions $\boldsymbol{\varphi}_h(\lambda,\cdot)$ defined for $h = 1,\dots,2m$ by

$$\boldsymbol{\varphi}_h(\lambda,\theta) := \begin{cases} \boldsymbol{w}_h^+(\lambda,e^{i\theta}) & \text{for} \quad h = 1,\dots,m \\ \boldsymbol{w}_{h-m}^-(\lambda,e^{i\theta}) & \text{for} \quad h = m+1,\dots,2m \end{cases}$$

are a basis of $\mathscr{W}(\lambda)$.

We introduce the following three matrices.

Definition 4.3 *We define:*

- *$\mathscr{N}(\lambda)$ as the $2m \times 2m$ matrix whose $2m$ columns are the $\mathscr{C}(\lambda)\boldsymbol{\varphi}_h(\lambda,\cdot)$ for $h = 1,\dots,2m$;*
- *$\mathscr{F}(\lambda)$ as the $N \times 2m$ matrix whose $2m$ columns are the $\boldsymbol{\varphi}_h(\lambda,\cdot)$ for $h = 1,\dots,2m$.*
- *$\boldsymbol{W}(\lambda)(r,\theta)$ as the $N \times 2m$ matrix $\Big(\mathrm{diag}(r^{-\tau_l})\Big)r^\lambda\,\mathscr{F}(\lambda)(\theta)$. Thus the $2m$ columns $\boldsymbol{W}_h(\lambda)$ of $\boldsymbol{W}(\lambda)$ are the basis vectors $\boldsymbol{w}_{h'}^\pm(\lambda,\cdot)$ defined in Lemma 4.2 according to*

$$\boldsymbol{W}_h(\lambda) := \begin{cases} \boldsymbol{w}_h^+(\lambda,\cdot) & \text{for} \quad h = 1,\dots,m \\ \boldsymbol{w}_{h-m}^-(\lambda,\cdot) & \text{for} \quad h = m+1,\dots,2m\,. \end{cases}$$

Note that the elements of the matrix $\mathscr{N}(\lambda)$ are complex numbers, whereas the elements of $\mathscr{F}(\lambda)$ are functions of θ and the elements of $\boldsymbol{W}(\lambda)$ are functions of $\zeta = re^{i\theta}$.

Theorem 4.4 *We have for any $\lambda \in \mathbb{C}$ (in the sense of the equality between two meromorphic functions):*

$$\mathscr{A}(\lambda)^{-1} = \mathscr{R}(\lambda)\circ\pi_M + \mathscr{F}(\lambda)\,\mathscr{N}(\lambda)^{-1}\Big(\pi_C - \mathscr{C}(\lambda)\,\mathscr{R}(\lambda)\circ\pi_M\Big)$$

Proof. The equality being obvious if one knows that $\mathscr{N}(\lambda)$ has a meromorphic inverse, let us prove that $\det\mathscr{N}(\lambda)$ is not identically zero. This fact is a consequence of the ellipticity which implies that for any $\xi \ge 0$, there exists $\eta \ge 0$ such that for any λ satisfying $\mathrm{Re}\,\lambda \le \xi$ and $\mathrm{Im}\,\lambda \ge \eta$, the operator $\mathscr{A}(\lambda)$ is injective. Hence, for such λ, $\mathscr{N}(\lambda)$ is injective, thus invertible. ∎

4.c The principal part of the singularities. We are now able to exhibit finite dimensional generators for the space $\mathscr{X}(\lambda_0)$ of all solutions of (4.1). First we join together Lemmas 4.1, 4.2 and Theorem 4.4. We find that

$$\mathscr{X}(\lambda_0) = \Big\{\frac{1}{2i\pi}\Big(\mathrm{diag}(r^{-\tau_l})\Big)\int_{\gamma(\lambda_0)} r^\lambda\,\mathscr{F}(\lambda)\,\mathscr{N}(\lambda)^{-1}\mathscr{G}(\lambda)\,d\lambda \mid \mathscr{G} : \mathbb{C} \mapsto \mathbb{C}^{2m} \text{ holomorphic}\Big\}.$$

Theorem 4.5 *Let λ_0 in $\mathbb{C}$. There holds:*

(i) If λ_0 is a pole of $\mathscr{A}(\lambda)^{-1}$ then λ_0 is a pole of $\mathscr{N}(\lambda)^{-1}$; let d be the order of this pole of $\mathscr{N}(\lambda)^{-1}$. Then

$$\mathscr{X}(\lambda_0) = \Big\{ \frac{1}{2i\pi} \int_{\gamma(\lambda_0)} \boldsymbol{W}(\lambda)\, \mathscr{N}(\lambda)^{-1} \mathscr{G}(\lambda)\, d\lambda \quad | \quad \mathscr{G} \in \mathbb{P}_{d-1}[\lambda] \otimes \mathbb{C}^{2m} \Big\}.$$

(ii) If a non-integer λ_0 is a pole of $\mathscr{N}(\lambda)^{-1}$ then λ_0 is a pole of $\mathscr{A}(\lambda)^{-1}$.

Proof. The point *(i)* is a consequence of the previous considerations and of the fact that any holomorphic function can be written as the sum of a polynomial of degree $d-1$ and of a term of the form $(\lambda - \lambda_0)^d h(\lambda)$ where h is holomorphic.

Let us prove the point *(ii)*. If $\lambda \notin \mathbb{N}$, the basis vectors $\boldsymbol{\varphi}_h(\lambda, \cdot)$ for $h = 1 \ldots, 2m$ are independent — according to the notation of Lemma 4.2. Theorem 3.3 and the linearization procedure of section 3.a yields the existence of $2m$ independent Cauchy conditions $\chi_h(\lambda)$ in $\theta = 0$ with analytic dependence in λ. Let $\chi(\lambda)$ be the matrix of the $2m$ Cauchy conditions $\chi_h(\lambda)$. As $\chi(\lambda)\, \mathscr{F}(\lambda)$ is non singular, composing the representation formula of Theorem 4.4 with $\chi(\lambda)$ yields that $\mathscr{A}(\lambda)^{-1}$ has a pole in λ_0 if $\mathscr{N}(\lambda)^{-1}$ has a pole in λ_0. ∎

Corollary 4.6 *If λ_0 is a pole of $\mathscr{N}(\lambda)^{-1}$ of order 1, then*

$$\mathscr{X}(\lambda_0) = \Big\{ \boldsymbol{W}(\lambda_0)\, \boldsymbol{p} \quad | \quad \boldsymbol{p} \in \ker \mathscr{N}(\lambda_0) \Big\}.$$

Corollary 4.7 *If λ_0 is a pole of $\mathscr{N}(\lambda)^{-1}$ of order d, then*

$$\mathscr{X}(\lambda_0) \subset \Big\{ \frac{1}{2i\pi} \int_{\gamma(\lambda_0)} \frac{\boldsymbol{W}(\lambda)\, \mathscr{G}(\lambda)}{(\lambda - \lambda_0)^d}\, d\lambda \quad | \quad \mathscr{G} \in \mathbb{P}_{d-1}[\lambda] \otimes \mathbb{C}^{2m} \Big\}.$$

5. Solutions of the boundary value system with polynomial or singular right hand side

The last step of the description of the singular functions $\sum_p \boldsymbol{u}_p$ — see Conclusion 1.1 — is the construction of the terms $\boldsymbol{u}_p$ which correspond to a polynomial or a singular right hand side:

- In the first case $p = 0$ and $\lambda_0 \in \mathbb{N}$ and we search for solutions $\boldsymbol{u}$ of

$$\mathbb{A}_0 \boldsymbol{u} = (\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1) \ \text{ and } \ \boldsymbol{u} \in \boldsymbol{S}^{\lambda_0 - \vec{\tau}} \ \text{ with } (\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1) \in \boldsymbol{T}^{\lambda_0 - \vec{\sigma}} \text{ polynomial.} \tag{5.1}$$

- In the second case, $p \geq 1$ and we search for solutions $\boldsymbol{u}$ of

$$\mathbb{A}_0 \boldsymbol{u} = (\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1) \ \text{ and } \ \boldsymbol{u} \in \boldsymbol{S}^{\lambda_0 + p - \vec{\tau}} \ \text{ with } (\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1) \in \boldsymbol{T}^{\lambda_0 + p - \vec{\sigma}}. \tag{5.2}$$

  In this case, λ_0 is an integer or a pole of $\mathscr{A}(\lambda)^{-1}$ where $\mathscr{A}(\lambda)$ is defined in (4.4) and the right hand side $(\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1)$ is determined by the recurrence relation — cf (1.8):

$$(\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1) = - \sum_{n=0}^{p-1} \mathbb{A}_{p-n} \boldsymbol{u}_n \, .$$

5.a Solvability.

For $(\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1) \in \boldsymbol{S}^{\lambda_0 - \vec{\sigma}}$, we introduce the Mellin transforms:

$$\boldsymbol{\psi}(\lambda) = (\psi_1, \dots, \psi_N) \quad \text{with} \quad \psi_k(\lambda, \theta) := \int_0^\infty r^{-\lambda+\sigma_k}\, \chi(r)\, f_k(r, \theta)\, \frac{dr}{r} \quad k = 1, \dots, N$$

and for $j = 0, 1$:

$$\boldsymbol{\psi}^j(\lambda) = (\psi_1^j, \dots, \psi_m^j) \quad \text{with} \quad \psi_h^j(\lambda) := \int_0^\infty r^{-\lambda+\sigma_h^j}\, \chi(r)\, g_h^j(r)\, \frac{dr}{r} \quad h = 1, \dots, m$$

with $\chi \in \mathscr{D}(\mathbb{R})$ a cut-off function which is $\equiv 1$ in the neighborhood of 0. We set

$$\Psi(\lambda) = \Big(\boldsymbol{\psi}(\lambda), \boldsymbol{\psi}^0(\lambda), \boldsymbol{\psi}^1(\lambda)\Big).$$

The vector function Ψ is meromorphic in $\mathbb{C}$ and admits only one pole in $\lambda = \lambda_0$.

Lemma 5.1 *Let $(\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1) \in \boldsymbol{S}^{\lambda_0 - \vec{\sigma}}$. $\Psi(\lambda)$ being defined as above, a solution $\boldsymbol{u}$ of*

$$\mathbb{A}_0 \boldsymbol{u} = (\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1)$$

is given by the Cauchy integral

$$\boldsymbol{u} = \frac{1}{2i\pi} \Big(\operatorname{diag}(r^{-\tau_l})\Big) \int_{\gamma(\lambda_0)} r^\lambda \mathscr{A}(\lambda)^{-1} \Psi(\lambda)\, d\lambda$$

where $\gamma(\lambda_0)$ is a simple curve surrounding λ_0 and avoiding the poles of $\mathscr{A}(\lambda)^{-1}$. We can assume that the only possible pole of $\mathscr{A}(\lambda)^{-1}$ inside $\gamma(\lambda_0)$ is λ_0.

The proof of the Lemma is a straightforward consequence of the definitions and of the inverse Mellin formula:

$$(\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1) = \frac{1}{2i\pi} \Big(\operatorname{diag}(r^{-\vec{\sigma}})\Big) \int_{\gamma(\lambda_0)} r^\lambda \Psi(\lambda)\, d\lambda\,. \tag{5.3}$$

Note that one can take as $\Psi(\lambda)$ an explicit rational fraction determined from the expansion of $(\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1)$ in polar coordinates. All possible definitions of $\Psi(\lambda)$ have the same polar part in $\lambda = \lambda_0$ and the inverse formula (5.3) is still valid.

5.b Conclusion.

We summarize the results we have obtained.

The asymptotic expansion near the corner of any solution of the system (1.1) is a sum of terms $\sum_{\lambda_0} \boldsymbol{U}_{\lambda_0}$ where $\lambda_0 \in \mathbb{N}$ or λ_0 is a pole of the inverse of $\mathscr{A}(\lambda)$ defined in (4.4). The relevant λ_0 are determined by the regularity of the solution and the regularity of the right hand side of (1.1) — cf [15].

Each $\boldsymbol{U}_{\lambda_0}$ is a formal sum

$$\boldsymbol{U}_{\lambda_0} = \sum_p \boldsymbol{u}_{\lambda_0, p} \quad \text{with} \quad \boldsymbol{u}_{\lambda_0, p} \in \boldsymbol{S}^{\lambda_0 + p - \vec{\tau}}.$$

Note that only a finite number of terms has to be taken into account, depending on the regularity of the right hand side.

If $\lambda_0 \notin \mathbb{N}$, $\boldsymbol{u}_{\lambda_0,0}$ is a solution of (4.1), i. e. is an element of the space $\mathscr{X}(\lambda_0)$, and has the form determined in Theorem 4.5.

If $\lambda_0 \in \mathbb{N}$, $\boldsymbol{u}_{\lambda_0,0}$ is a solution of (5.1); if λ_0 is not a pole of $\mathscr{A}(\lambda)^{-1}$, $\boldsymbol{u}_{\lambda_0,0}$ is completely determined by Lemma 5.1; if λ_0 is a pole of $\mathscr{A}(\lambda)^{-1}$, $\boldsymbol{u}_{\lambda_0,0}$ is the sum of an element of $\mathscr{X}(\lambda_0)$ and the term defined by Lemma 5.1.

The term $\boldsymbol{u}_{\lambda_0,p}$ for $p \geq 1$ is a solution of (1.8), determined with the help of Lemma 5.1.

Any $\boldsymbol{u}_{\lambda_0,p}$ for any λ_0 in $\mathbb{N}$ or pole of $\mathscr{A}(\lambda)^{-1}$ and for any $p \in \mathbb{N}$ has the general form described in the following Theorem 5.2.

5.c General form of the singular functions.

Theorem 5.2 *Any function $\boldsymbol{u}_{\lambda_0,p}$ defined above can be written as the sum of a finite number of terms of the form $\boldsymbol{u}^+_{\lambda_0+p}$ and $\boldsymbol{u}^-_{\lambda_0+p}$ as follows. $\gamma(\lambda_0+p)$ denotes a simple curve surrounding λ_0+p and avoiding the poles of $\mathscr{A}(\lambda)^{-1}$. We can assume that the only possible pole of $\mathscr{A}(\lambda)^{-1}$ inside $\gamma(\lambda_0+p)$ is λ_0+p. There exists three positive integers d_0, d_1 and d_2 such that*

$$\boldsymbol{u}^{\pm}_{\lambda_0+p}(\zeta) = \frac{1}{2i\pi} \int_{\gamma(\lambda_0+p)} \frac{\boldsymbol{V}^{\pm}_{\vec{\tau}}(\lambda,\zeta)\, \mathscr{H}^{\pm}(\lambda)}{(\lambda-\lambda_0-p)^{d_0}}\, d\lambda \quad \textit{with} \quad \mathscr{H}^{\pm} \in \mathbb{P}_{d_0-1}[\lambda]\otimes\mathbb{C}^N \tag{5.4}$$

where $\boldsymbol{V}^{\pm}_{\vec{\tau}}(\lambda,\zeta)$ is the diagonal matrix $\operatorname{diag}(v^{\pm}_l(\lambda,\zeta))$ *with $(v^{\pm}_1,\dots,v^{\pm}_N) =: \boldsymbol{v}^{\pm}_{\vec{\tau}}$ defined by*

$$\boldsymbol{v}^{\pm}_{\vec{\tau}}(\lambda,\zeta) = \frac{1}{2i\pi} \int_{\tilde{\gamma}} \frac{\boldsymbol{Z}^{\pm}_{\vec{\tau}}(\lambda;\zeta,\overline{\zeta};\alpha)\, \boldsymbol{h}^{\pm}(\alpha)}{\alpha^{d_1}\, \det \boldsymbol{M}_{\pm}(\alpha)^{d_2}}\, d\alpha \quad \textit{with} \quad \boldsymbol{h}^{\pm} \in \mathbb{P}_{d_1+md_2-1}[\lambda]\otimes\mathbb{C}^N. \tag{5.5}$$

The matrices $\boldsymbol{Z}^{\pm}_{\vec{\tau}}$ are defined in (2.6) *and $\boldsymbol{M}_{\pm}(\alpha)$ in* (2.4) *— compare with Theorem* 2.1.

Remark 5.3 For rotationally invariant systems as defined in 2.b (iii), $\det \boldsymbol{M}_{\pm}(\alpha) = c\,\alpha^m$. Thus $\boldsymbol{v}^{+}_{\vec{\tau}}(\lambda,\zeta)$ is a combination of terms of the form

$$\overline{\zeta}^{\lambda} \left(\zeta^{\ell_1}\overline{\zeta}^{-\ell_1-\tau_1},\dots,\zeta^{\ell_N}\overline{\zeta}^{-\ell_N-\tau_N}\right)$$

and $\boldsymbol{v}^{-}_{\vec{\tau}}(\lambda,\zeta)$ is a combination of terms of the form

$$\zeta^{\lambda} \left(\overline{\zeta}^{\ell_1}\zeta^{-\ell_1-\tau_1},\dots,\overline{\zeta}^{\ell_N}\zeta^{-\ell_N-\tau_N}\right)$$

respectively, with $\ell_1,\dots,\ell_N$ in $\mathbb{N}$. Hence, $\boldsymbol{u}^{+}_{\lambda_0+p}(\zeta)$ is a combination of terms of the form

$$\overline{\zeta}^{\lambda_0+p} \log^q \overline{\zeta}\, \left(\zeta^{\ell_1}\overline{\zeta}^{-\ell_1-\tau_1},\dots,\zeta^{\ell_N}\overline{\zeta}^{-\ell_N-\tau_N}\right)$$

and $\boldsymbol{u}^-_{\lambda_0+p}(\zeta)$ is a combination of terms of the form

$$\zeta^{\lambda_0+p}\log^q\zeta\,\left(\overline{\zeta}^{\ell_1}\zeta^{-\ell_1-\tau_1},\ldots,\overline{\zeta}^{\ell_N}\zeta^{-\ell_N-\tau_N}\right)$$

respectively, with $q\in\mathbb{N}$.

Proof of Theorem 5.2. If $p=0$ and if $\boldsymbol{u}_0$ is solution of (4.1), the representation of $\boldsymbol{u}_0$ by the two Cauchy integrals (5.4) and (5.5) is a straightforward consequence of Theorems 4.5 and 2.1.

Otherwise, $\boldsymbol{u}_p$ is a solution of (5.1) or (5.2). We are going to prove in a first stage that the interior right hand side $\boldsymbol{f}$ can also be represented by two Cauchy integrals similar to (5.4) and (5.5).

First case: $\boldsymbol{f}$ is a polynomial in $S^{\lambda_0-\vec{\sigma}}$, with $\lambda_0\in\mathbb{N}$. Then $\boldsymbol{f}$ can be written as the sum of terms of the form

$$\overline{\zeta}^{\lambda_0}\left(q_1\,\zeta^{\ell}\,\overline{\zeta}^{-\ell-\sigma_1},\ldots,q_N\,\zeta^{\ell}\,\overline{\zeta}^{-\ell-\sigma_N}\right)\tag{1}$$

with $\ell\in\mathbb{N}$, $\ell\le\lambda_0$ and $q_1,\ldots,q_N$ some coefficients.

For $k\in\{1\ldots,N\}$, let us define v_k^+ as

$$v_k^+(\lambda,\zeta):=\binom{\lambda-\sigma_k}{\ell}\overline{\zeta}^{\lambda-\sigma_k-\ell}\zeta^{\ell}.$$

We have the integral representation

$$v_k^+(\lambda,\zeta)=\frac{1}{2i\pi}\int_{\tilde{\gamma}}\frac{(\alpha\zeta+\overline{\zeta})^{\lambda-\sigma_k}}{\alpha^{\ell+1}}\,d\alpha\,.$$

Since

$$\overline{\zeta}^{\lambda_0-\sigma_k-\ell}\zeta^{\ell}=\frac{1}{2i\pi}\int_{\gamma(\lambda_0)}\frac{\binom{\lambda-\sigma_k}{\ell}\overline{\zeta}^{\lambda-\sigma_k-\ell}\zeta^{\ell}}{\binom{\lambda-\sigma_k}{\ell}(\lambda-\lambda_0)}\;d\lambda$$

there exists $d_0\in\mathbb{N}$ and a polynomial $g_k\in\mathbb{P}_{d-1}[\lambda]$ such that

$$\overline{\zeta}^{\lambda_0-\sigma_k-\ell}\zeta^{\ell}=\frac{1}{2i\pi}\int_{\gamma(\lambda_0)}\frac{v_k^+(\lambda,\zeta)\,g_k(\lambda)}{(\lambda-\lambda_0)^{d_0}}\;d\lambda\,.$$

Hence the term (1) has a form similar to (5.4) with $\boldsymbol{v}^+_{\vec{\tau}}(\lambda,\zeta)$ replaced by $\boldsymbol{v}^+_{\vec{\sigma}}(\lambda,\zeta)$:

$$\boldsymbol{v}^+_{\vec{\sigma}}(\lambda,\zeta)=\frac{1}{2i\pi}\int_{\tilde{\gamma}}\frac{\boldsymbol{Z}^+_{\vec{\sigma}}(\lambda;\zeta,\overline{\zeta};\alpha)\,\boldsymbol{q}}{\alpha^{\ell+1}}\;d\alpha$$

where $\boldsymbol{q}=(q_1,\ldots,q_N)$.

Second case: Now $p\ge1$ and $\boldsymbol{f}=-\sum_{n=0}^{p-1}\boldsymbol{L}_{(p-n)}\boldsymbol{u}_n$ — see section 1.d for the definition of the operators $\boldsymbol{L}_{(n)}$. We assume that $\boldsymbol{u}_0,\ldots,\boldsymbol{u}_{p-1}$ have the form described by the Theorem 5.2. Let us fix $n\in\{0\ldots,p-1\}$ and k, $l\in\{1\ldots,N\}$. We consider the action of k-th line of the system $\boldsymbol{L}_{(p-n)}$ on the l-th term of $\boldsymbol{u}_n$. The resulting function can be written as a sum of terms of the form

$$\zeta^{\beta_1}\overline{\zeta}^{\beta_2}\partial_\zeta^{i_1}\partial_{\overline{\zeta}}^{i_2}\int_{\gamma(\lambda_0+n)}\frac{v_l^+(\lambda,\zeta)\,g_l(\lambda)}{(\lambda-\lambda_0-n)^{d_0}}\;d\lambda \tag{2}$$

with $\beta_1+\beta_2-i_1-i_2=p-n-\sigma_k+\tau_l$, and

$$v_l^+(\lambda,\zeta)=\int_{\tilde{\gamma}}\frac{(\alpha\zeta+\overline{\zeta})^{\lambda-\tau_l}\;h_l(\alpha)}{\alpha^{d_1}\,D_+(\alpha)^{d_2}}\;d\alpha \tag{3}$$

— where $D_+(\alpha)$ denotes $\det \boldsymbol{M}_+(\alpha)$ — and similar terms with the minus sign. Evaluating the term (2) reduces to calculate $\zeta^{\beta_1}\overline{\zeta}^{\beta_2}\partial_\zeta^{i_1}\partial_{\overline{\zeta}}^{i_2}v_l^+(\lambda,\zeta)$, hence with (3)

$$\zeta^{\beta_1}\overline{\zeta}^{\beta_2}\partial_\zeta^{i_1}\partial_{\overline{\zeta}}^{i_2}(\alpha\zeta+\overline{\zeta})^{\lambda-\tau_l}.$$

We get rid of the term $\overline{\zeta}^{\beta_2}$ by the binomial expansion of $\overline{\zeta}^{\beta_2}=(\alpha\zeta+\overline{\zeta}-\alpha\zeta)^{\beta_2}$. Then we eliminate the powers of ζ by integration by parts in α. The derivatives with respect to α of the fractional part of (3)

$$\frac{d^i}{d\alpha^i}\;\frac{1}{\alpha^{d_1}\,D_+(\alpha)^{d_2}}$$

can be represented as rational fractions with a denominator of the form $\alpha^{d'_1}\,D_+(\alpha)^{d'_2}$. Finally, the power of $(\alpha\zeta+\overline{\zeta})$ is equal to $\lambda-\tau_l+\beta_1+\beta_2-i_1-i_2$, i. e. $\lambda-\sigma_k+p-n$. We change the contour of integration in the integral (2), replacing $\gamma(\lambda_0+n)$ by $\gamma(\lambda_0+n)+p-n=:\gamma(\lambda_0+p)$.

Conclusion: In all situations, the interior right hand side $\boldsymbol{f}$ can be written as a sum of terms

$$\frac{1}{2i\pi}\int_{\gamma(\lambda_0+p)}\frac{\boldsymbol{V}_{\vec{\sigma}}^{\pm}(\lambda,\zeta)\,\mathscr{H}^{\pm}(\lambda)}{(\lambda-\lambda_0-p)^{d_0}}\;d\lambda \tag{4}$$

with $\mathscr{H}^{\pm}\in\mathbb{P}_{d_0-1}[\lambda]\otimes\mathbb{C}^N$ and $\boldsymbol{V}_{\vec{\sigma}}^{\pm}$ the diagonal matrix whose diagonal is

$$\boldsymbol{v}_{\vec{\sigma}}^{\pm}(\lambda,\zeta)=\frac{1}{2i\pi}\int_{\tilde{\gamma}}\frac{\boldsymbol{Z}_{\vec{\sigma}}^{\pm}(\lambda;\zeta,\overline{\zeta};\alpha)\,\boldsymbol{h}^{\pm}(\alpha)}{\alpha^{d_1}\,D_{\pm}(\alpha)^{d_2}}\;d\alpha\,. \tag{5}$$

with $\boldsymbol{h}^{\pm}\in\mathbb{P}_{d_1+md_2-1}[\lambda]\otimes\mathbb{C}^N$.

In the last stage of the proof we construct a solution $\boldsymbol{u}_p$ of (5.1) or (5.2).

Solution of the interior equation $\boldsymbol{M}\boldsymbol{u}=0$: Using the relations (2.7), we obtain for the right hand side (4) the following solution in $\boldsymbol{S}^{\lambda_0+p-\vec{\tau}}$:

$$\frac{1}{2i\pi}\int_{\gamma(\lambda_0+p)}\frac{\boldsymbol{V}_{\vec{\tau}}^{\pm}(\lambda,\zeta)\,\mathscr{H}^{\pm}(\lambda)}{(\lambda-\lambda_0-p)^{d_0}}\;d\lambda \tag{6}$$

with $\boldsymbol{V}_{\vec{\tau}}^{\pm}$ the diagonal matrix whose diagonal is

$$\boldsymbol{v}_{\vec{\tau}}^{\pm}(\lambda,\zeta)=\frac{1}{2i\pi}\int_{\tilde{\gamma}}\frac{\boldsymbol{Z}_{\vec{\tau}}^{\pm}(\lambda;\zeta,\overline{\zeta};\alpha)\,\boldsymbol{M}_{\pm}(\alpha)^{-1}\,\boldsymbol{h}^{\pm}(\alpha)}{\alpha^{d_1}\,D_{\pm}(\alpha)^{d_2}}\;d\alpha\,. \tag{7}$$

The form (6) coincides with (5.4) and we see that (7) has the form (5.5).

Boundary conditions: We solve the equation $\mathbb{A}_0 \boldsymbol{u}_p = (\boldsymbol{f}, \boldsymbol{g}^0, \boldsymbol{g}^1)$ by setting

$$\boldsymbol{u}_p = \boldsymbol{u} + \check{\boldsymbol{u}}$$

where $\boldsymbol{u}$ is the solution of $\boldsymbol{M}\boldsymbol{u} = \boldsymbol{f}$ we have just constructed and $\check{\boldsymbol{u}}$ is a solution of:

$$\begin{cases} \boldsymbol{M}\check{\boldsymbol{u}} &= 0 & \text{in} & \Gamma \\ \boldsymbol{C}^j\check{\boldsymbol{u}} &= \boldsymbol{g}^j - \boldsymbol{C}^j\boldsymbol{u} := \check{\boldsymbol{g}}^j & \text{on} & \partial^j\Gamma \quad \text{for } j = 0,\ 1\,. \end{cases} \tag{5.6}$$

It remains to prove that $\check{\boldsymbol{u}}$ has the desired form. Using the spaces (1.5) we see that

$$\check{\boldsymbol{g}}^j = (\check{g}_1^j, \ldots, \check{g}_m^j) \quad \text{with} \quad \check{g}_h^j \in S^{\lambda_0+p-\sigma_h^j}(\mathbb{R}^+), \quad h = 1, \ldots, m\,.$$

Defining $\boldsymbol{\psi}_C(\lambda)$ as previously for Lemma 5.1 by

$$\boldsymbol{\psi}_C(\lambda) = (\psi_1^0, \ldots, \psi_m^0,\ \psi_1^1, \ldots, \psi_m^1) \quad \text{with}$$

$$\psi_h^j(\lambda) := \int_0^\infty r^{-\lambda+\sigma_h^j}\, \chi(r)\, \check{g}_h^j(r)\, \frac{dr}{r}\,, \quad \text{for } h = 1, \ldots, m$$

and applying Lemma 5.1, we obtain that

$$\check{\boldsymbol{u}} = \frac{1}{2i\pi} \Big(\operatorname{diag}(r^{-\tau_l})\Big) \int_{\gamma(\lambda_0+p)} r^\lambda \mathscr{A}(\lambda)^{-1}(0, \boldsymbol{\psi}_C(\lambda))\, d\lambda$$

is a solution in $S^{\lambda_0+p-\vec{\tau}}$ of (5.6).

Using Theorem 4.4, we deduce that

$$\check{\boldsymbol{u}} = \frac{1}{2i\pi} \Big(\operatorname{diag}(r^{-\tau_l})\Big) \int_{\gamma(\lambda_0+p)} r^\lambda\, \mathscr{F}(\lambda)\, \mathscr{N}(\lambda)^{-1} \boldsymbol{\psi}_C(\lambda)\, d\lambda.$$

With $\boldsymbol{W}(\lambda)$ the matrix introduced in Definition 4.3 we see that

$$\check{\boldsymbol{u}}(\zeta) = \frac{1}{2i\pi} \int_{\gamma(\lambda_0+p)} \boldsymbol{W}(\lambda)(\zeta)\, \mathscr{N}(\lambda)^{-1} \boldsymbol{\psi}_C(\lambda)\, d\lambda.$$

We can split $\check{\boldsymbol{u}}$ into two terms $\check{\boldsymbol{u}}_+$ and $\check{\boldsymbol{u}}_-$ defined as

$$\check{\boldsymbol{u}}_\pm = \sum_{h=1}^m \frac{1}{2i\pi} \int_{\gamma(\lambda_0+p)} \frac{\boldsymbol{w}_h^\pm(\lambda, \zeta)\, \mathscr{G}_h^\pm(\lambda)}{(\lambda - \lambda_0 - p)^{d_0}}\ d\lambda$$

where $\boldsymbol{w}_h^\pm$ are defined in Lemma 4.2, $d_0 \in \mathbb{N}$ is large enough and $\mathscr{G}_h^+(\lambda)$ is the only polynomial in $\mathbb{P}_{d_0-1}[\lambda]$ such that the difference between the h-th component of $\mathscr{N}(\lambda)^{-1}\boldsymbol{\psi}_C(\lambda)$ and $\mathscr{G}_h^+(\lambda) \times (\lambda - \lambda_0 - p)^{-d_0}$ is holomorphic inside $\gamma(\lambda_0 + p)$. Lemma 4.2 yields that $\check{\boldsymbol{u}}_\pm$ has the desired form. ∎

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