MAXWELL EIGENMODES IN TENSOR PRODUCT DOMAINS

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ABSTRACT. We describe eigenpairs of the Maxwell system with normalized constant coefficients in a tensor product three-dimensional domain. As an application, we find eigenpairs in a cube, in a cylinder, and in a cylinder with a coaxial circular hole.

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1. Introduction

Let Ω be a domain in $\mathbb{R}^3$. Let ε and μ are the electric permittivity and the magnetic permeability of the material inside Ω. We assume that the boundary of Ω represents *perfectly conducting* or *perfectly insulating* walls:

$$\partial\Omega = \partial\Omega_{\mathsf{cd}} \cup \partial\Omega_{\mathsf{ins}}, \quad \partial\Omega_{\mathsf{cd}} \cap \partial\Omega_{\mathsf{ins}} = \emptyset, \tag{1.1}$$

where $\partial_{\mathsf{cd}}\Omega$ is the perfectly conducting part and $\partial_{\mathsf{ins}}\Omega$ the perfectly insulating part.

The cavity resonator problem is to find the frequencies $\varpi \in \mathbb{R}_+$ and the non-zero electromagnetic fields $(\hat{\mathbf{E}}, \hat{\mathbf{H}}) \in L^2(\Omega)^6$ such that

$$\begin{cases} \operatorname{curl} \hat{\mathbf{E}} - i\varpi\mu\hat{\mathbf{H}} = 0 & \text{in} \quad \Omega, & \text{(Faraday law)} \\ \operatorname{curl} \hat{\mathbf{H}} + i\varpi\varepsilon\hat{\mathbf{E}} = 0 & \text{in} \quad \Omega, & \text{(Ampère law)} \\ \hat{\mathbf{E}} \times \mathbf{n} = 0 \quad \text{and} \quad \hat{\mathbf{H}} \cdot \mathbf{n} = 0, & \text{on} \quad \partial\Omega_{\mathsf{cd}}, & \text{(perfect conductor b. c.)} \\ \hat{\mathbf{E}} \cdot \mathbf{n} = 0 \quad \text{and} \quad \hat{\mathbf{H}} \times \mathbf{n} = 0, & \text{on} \quad \partial\Omega_{\mathsf{ins}}, & \text{(perfect insulator b. c.)} \\ \operatorname{div} \varepsilon\hat{\mathbf{E}} = 0 \quad \text{and} \quad \operatorname{div} \mu\hat{\mathbf{H}} = 0 & \text{in} \quad \Omega, & \text{(gauge conditions).} \end{cases} \tag{1.2}$$

Here, as usual, $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$. The gauge conditions on the divergence are a consequence of the first two equations if $\varpi \neq 0$. Nevertheless we look for solutions of (1.2) including $\varpi = 0$. In the constant coefficient case and perfectly conducting boundary, the occurence of $\varpi = 0$ happens if and only if the domain Ω is topologically non-trivial, i.e. if Ω is not simply connected, or if $\partial\Omega$ is not connected, see Propositions 3.14 & 3.18 in the reference [1].

Remark 1.1. *(i)* We consider here the situation with zero conductivity (case of the air or of a dielectric material). Then ε and μ are real. Therefore, without restriction, the fields $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ can be supposed real valued.
(ii) In presence of a non-zero conductivity, ϖ should be searched in $\mathbb{C}$, and the fields would be complex valued.

Definition 1.2. *The triples* $(\varpi^2, \hat{\mathbf{E}}, \hat{\mathbf{H}})$ *solution of* (1.2) *with* $(\hat{\mathbf{E}}, \hat{\mathbf{H}}) \neq 0$ *are called* Maxwell eigenmodes*,* ϖ *is called* eigenfrequency*,* ϖ^2 eigenvalue *and* $(\hat{\mathbf{E}}, \hat{\mathbf{H}})$ eigenfield.

In sections 2 to 6 of this paper, we consider the case when $\varepsilon \equiv \varepsilon_0$ and $\mu \equiv \mu_0$ in Ω. We also in general assume that the perfectly conducting conditions are applied on the whole boundary of Ω, except when we explicitly mention it. Then (1.2) reduces to

$$\begin{cases} \operatorname{curl} \hat{\mathbf{E}} - i\varpi\mu_0\hat{\mathbf{H}} = 0 & \text{in} \quad \Omega, & \text{(Faraday law)} \\ \operatorname{curl} \hat{\mathbf{H}} + i\varpi\varepsilon_0\hat{\mathbf{E}} = 0 & \text{in} \quad \Omega, & \text{(Ampère law)} \\ \hat{\mathbf{E}} \times \mathbf{n} = 0 \quad \text{and} \quad \hat{\mathbf{H}} \cdot \mathbf{n} = 0, & \text{on} \quad \partial\Omega, & \text{(perfect conductor b. c.)} \\ \operatorname{div} \varepsilon\hat{\mathbf{E}} = 0 \quad \text{and} \quad \operatorname{div} \mu\hat{\mathbf{H}} = 0 & \text{in} \quad \Omega, & \text{(gauge conditions).} \end{cases} \tag{1.3}$$

We introduce the following normalization

$$\kappa = \varpi\sqrt{\varepsilon_0\mu_0} \quad \text{(wave number)}, \qquad \mathbf{E} = \sqrt{\varepsilon_0}\,\hat{\mathbf{E}} \quad \text{and} \quad \mathbf{H} = \sqrt{\mu_0}\,\hat{\mathbf{H}}. \tag{1.4}$$

Then (1.3) is transformed into

$$\begin{cases} \operatorname{curl}\mathbf{E} - i\kappa\mathbf{H} = 0 & \text{in} \quad \Omega, \\ \operatorname{curl}\mathbf{H} + i\kappa\mathbf{E} = 0 & \text{in} \quad \Omega, \\ \mathbf{E}\times\mathbf{n} = 0 \quad \text{and} \quad \mathbf{H}\cdot\mathbf{n} = 0, & \text{on} \quad \partial\Omega, \\ \operatorname{div}\mathbf{E} = 0 \quad \text{and} \quad \operatorname{div}\mathbf{H} = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.5}$$

Remark 1.3. *(i)* Stricto sensu, ϖ is not the frequency but the *"pulsation"*[1]: It corresponds to the time dependency $t \mapsto \exp(i\varpi t)$. The associated period is $T = \frac{2\pi}{\varpi}$. The *frequency* f is then $f = \frac{1}{T}$, and is measured in Hz. Therefore

$$\boxed{\varpi = 2\pi f}$$

(ii) The constants ε_0 and μ_0 satisfy

$$\varepsilon_0\mu_0 = \frac{1}{c^2} \quad (c \text{ speed of light}).$$

We recall that $\mu_0 = 4\pi\,10^{-7}$ Wb A^{-1} m^{-1} and $c \simeq 2.99792458 \times 10^8$ m/s. Hence the relation between the wave number and the pulsation:

$$\boxed{\varpi = c\kappa \simeq 3\times 10^8\,\kappa}$$

This paper is organized as follows. In sections 2 and 3 we give formulas for the normalized Maxwell eigenmodes $(\kappa^2, \mathbf{E}, \mathbf{H})$ solution of the normalized equation (1.5) in the case when Ω has the tensor form $\omega \times I$ with $\omega \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$, separating the modes in TE and TM types. A sort of common type TEM appears when ω is not simply connected. We mention generalizations to special combinations of conducting and insulating boundary conditions.

As an application of our formulas, we consider in section 4 the case when Ω is a cube (or a parallelepiped), and in section 5 and 6 the case when Ω is a cylinder. We bring special attention to the case when the cylinder has a coaxial cylindrical hole. This serves as a limit model for the situation of a cylindrical conductor body inside a cavity. Then the TEM modes appear and are of special importance.

Finally, in sections 7 and 8, still in teh tensor product case, we investigate the variable coefficient case, namely when ε is varying independently of the axial variable. Then the TE and TM structures are no longer a valid Ansatz, in general. In replacement, we obtain wave guide formulations.

[1] "Pulsation" is the French word for *"angular frequency"*. We prefer "pulsation" because of possible mixing up with angular Fourier transformation for axisymmetric domains!

2. Preliminary notions and notation

We recall that all functions are real valued.

2.1. **Electric and magnetic formulations for the Maxwell spectrum.** We first recall the definition of the standard continuous spaces associated with Maxwell equations on a domain $\Omega \subset \mathbb{R}^3$: $\mathbf{H}(\mathrm{curl}, \Omega)$ is the space of $L^2(\Omega)$ fields with curl in $L^2(\Omega)$, while $\mathbf{H}_0(\mathrm{curl}, \Omega)$ is the subspace of $\mathbf{H}(\mathrm{curl}, \Omega)$ with perfectly conducting electric boundary conditions; $\mathbf{H}(\mathrm{div}, \Omega)$ is the space of $L^2(\Omega)$ fields with divergence in $L^2(\Omega)$ and $\mathbf{H}_0(\mathrm{div}, \Omega)$ the subspace of $\mathbf{H}(\mathrm{div}, \Omega)$ with perfectly conducting magnetic boundary conditions. We recall the formula for the curl in 3D:

$$\mathrm{curl}\,\mathbf{u} = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix} \quad \text{for} \quad \mathbf{u} = (u_1, u_2, u_3).$$

Spaces associated with electric and magnetic variational formulations of problem (1.5) are

$$\mathbf{X}_\mathsf{N}(\Omega) := \mathbf{H}_0(\mathrm{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega) \quad \text{and} \quad \mathbf{X}_\mathsf{T}(\Omega) := \mathbf{H}(\mathrm{curl}, \Omega) \cap \mathbf{H}_0(\mathrm{div}, \Omega)\,.$$

The electric variational formulation of (1.5) is:

Find the eigenpairs $(\Lambda = \kappa^2, \mathbf{u})$ *with* $\mathbf{u} \neq 0$ *and* $\mathrm{div}\,\mathbf{u} = 0$ *such that*

$$\mathbf{u} \in \mathbf{X}_\mathsf{N}(\Omega) : \quad \int_\Omega \mathrm{curl}\,\mathbf{u}\ \mathrm{curl}\,\mathbf{v}\ \mathrm{d}\mathbf{x} = \Lambda \int_\Omega \mathbf{u} \cdot \mathbf{v}\ \mathrm{d}\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{X}_\mathsf{N}(\Omega), \tag{2.1}$$

while the magnetic formulation is:

Find the eigenpairs $(\Lambda = \kappa^2, \mathbf{u})$ *with* $\mathbf{u} \neq 0$ *and* $\mathrm{div}\,\mathbf{u} = 0$ *such that*

$$\mathbf{u} \in \mathbf{X}_\mathsf{T}(\Omega) : \quad \int_\Omega \mathrm{curl}\,\mathbf{u}\ \mathrm{curl}\,\mathbf{v}\ \mathrm{d}\mathbf{x} = \Lambda \int_\Omega \mathbf{u} \cdot \mathbf{v}\ \mathrm{d}\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{X}_\mathsf{T}(\Omega), \tag{2.2}$$

We gather the equivalence results in the next lemma:

Lemma 2.1. (i) *If* $(\kappa, \mathbf{E}, \mathbf{H})$ *is a Maxwell eigenmode solution of* (1.5) *with* $\kappa \neq 0$*, then, with* $\Lambda = \kappa^2$*,* $\mathbf{E}$ *is solution of* (2.1) *and* $\mathbf{H}$ *is solution of* (2.2).

(ii) *If* $\Lambda \neq 0$ *and* $\mathbf{u}$ *is solution of* (2.1)*, then with* $\kappa = \sqrt{\Lambda}$*,* $\mathbf{E} = \mathbf{u}$ *and* $\mathbf{H} = \frac{1}{i\kappa}\,\mathrm{curl}\,\mathbf{E}$*, we obtain an eigenmode of* (1.5).

(ii) *If* $\Lambda \neq 0$ *and* $\mathbf{u}$ *is solution of* (2.2)*, then with* $\kappa = \sqrt{\Lambda}$*,* $\mathbf{H} = \mathbf{u}$ *and* $\mathbf{E} = -\frac{1}{i\kappa}\,\mathrm{curl}\,\mathbf{H}$*, we obtain an eigenmode of* (1.5).

The situation $\kappa = 0$ (still with the constraint that the fields are divergence free) occurs when the domain is not simply connected, or if its boundary is not connected, see [1].

We investigate the electric boundary condition first. The case of the magnetic field is considered later.

2.2. **Tensor product domain.** Let $\Omega \subset \mathbb{R}^3$ be of tensor product form

$$\Omega = \omega \times I, \quad \omega \subset \mathbb{R}^2, \quad I \text{ interval in } \mathbb{R}. \tag{2.3}$$

We assume that ω is a bounded Lipschitz domain. We note that the boundary of Ω is connected. But, if ω is not simply connected, the same holds for Ω.

We denote Cartesian coordinates by

$$x = (x_1, x_2, x_3) = (x_\perp, x_3).$$

and, correspondingly, components by

$$\mathbf{u} = (u_1, u_2, u_3) = (\mathbf{u}_\perp, u_3).$$

Likewise, the exterior unit normal $\mathbf{n}$ to $\partial\Omega$ is written $(\mathbf{n}_\perp, n_3)$. On $\omega \times \partial I$, $\mathbf{n}_\perp = 0$ and $n_3 = \pm 1$. On $\partial\omega \times I$, $\mathbf{n}_\perp$ is the exterior unit normal to $\partial\omega$, $n_3 = 0$, and the tangential component of $\mathbf{u}_\perp$ is $\mathbf{u}_\perp \times \mathbf{n}_\perp = u_1 n_2 - u_2 n_1$.

The gradient and the Laplacian in the transverse plane are denoted by $\mathbf{grad}_\perp$ and $\Delta_\perp$:

$$\mathbf{grad}_\perp v = \begin{pmatrix} \partial_1 v \\ \partial_2 v \end{pmatrix} \quad \text{and} \quad \Delta_\perp v = \partial_1^2 v + \partial_2^2 v.$$

The vector and scalar curls in 2D are given by:

$$\mathbf{curl}_\perp v = \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix} \quad \text{and} \quad \operatorname{curl}_\perp \mathbf{v} = \partial_1 v_2 - \partial_2 v_1.$$

We have the formula

$$\operatorname{curl} \mathbf{u} = \begin{pmatrix} \mathbf{curl}_\perp u_3 \\ \operatorname{curl}_\perp \mathbf{u}_\perp \end{pmatrix} + \partial_3 \begin{pmatrix} -u_2 \\ u_1 \\ 0 \end{pmatrix}. \tag{2.4}$$

The *electric boundary conditions* $\mathbf{u} \times \mathbf{n} = 0$ *on* $\partial\Omega$ are equivalent to

$$\begin{aligned} &\mathbf{u}_\perp \times \mathbf{n}_\perp = 0 \quad \text{and} \quad u_3 = 0 \quad \text{on} \quad \partial\omega \times I, \\ &\mathbf{u}_\perp = 0 \quad \text{on} \quad \omega \times \partial I, \end{aligned} \tag{2.5}$$

The interior partial differential equation satisfied by eigenpairs is the system:

$$\operatorname{curl}\operatorname{curl} \mathbf{u} = \Lambda\, \mathbf{u} \quad \text{in} \quad \Omega. \tag{2.6}$$

2.3. **TE and TM modes.** We start the investigation of the solutions of (1.5) in a tensor product domain by introducing special Ansätze for the *electric part*:

Definition 2.2. *For the* electric part *of an eigenmode let:*
(i) *a TE (*Transverse Electric*) mode be a solution* $\mathbf{u}$ *of* (2.1) *of the form*

$$\mathbf{u}(x_\perp, x_3) = \begin{pmatrix} \mathbf{curl}_\perp v(x_\perp) \\ 0 \end{pmatrix} w(x_3), \tag{2.7}$$

with scalar functions $v \in H^1(\omega)$ *and* $w \in L^2(I)$.
(ii) *a TM (*Transverse Magnetic*) mode be a solution* $\mathbf{u}$ *of* (2.1) *of the form*

$$\mathbf{u}(x_\perp, x_3) = \begin{pmatrix} \mathbf{grad}_\perp v(x_\perp) \\ 0 \end{pmatrix} \partial_3 w(x_3) - \begin{pmatrix} 0 \\ \Delta_\perp v(x_\perp) \end{pmatrix} w(x_3), \tag{2.8}$$

with scalar functions $v \in H^1(\omega; \Delta_\perp)$ *and* $w \in H^1(I)$.

As a straightforward consequence of the definitions we obtain:

Lemma 2.3. *If* $\mathbf{u}$ *is a TE or a TM mode, it is divergence free:* $\operatorname{div} \mathbf{u} = 0$ *in* Ω.

Remark 2.4. If ω is not simply connected, there exist extended TE modes of the form

$$\mathbf{u}(x_\perp, x_3) = \begin{pmatrix} \widetilde{\mathbf{curl}}_\perp v(x_\perp) \\ 0 \end{pmatrix} w(x_3), \tag{2.9}$$

with v in the space $\Theta(\omega)$ defined as follows, cf [1]: Let ω° be $\omega \setminus \Sigma$, where $\Sigma = \cup_{l=1}^{L} \Sigma_l$ is a minimal set of cuts so that ω° is simply connected. Then

$$\Theta(\omega) = \{\varphi \in H^1(\omega^\circ) | \quad [\varphi]_{\Sigma_l} = \mathrm{const}(l), \;\; l = 1, \ldots, L\}.$$

For $\varphi \in \Theta(\omega)$, its $\widetilde{\mathbf{curl}}_\perp \varphi$ is its $\mathbf{curl}_\perp$ in ω°, considered as an element of $L^2(\omega)$.

3. The TE and TM modes in a tensor product domain

3.1. **TE modes.** Let $\mathbf{u}$ be a TE mode. We find that $\operatorname{div} \mathbf{u} = 0$ and, using (2.4)

$$\operatorname{curl} \mathbf{u} = \begin{pmatrix} 0 \\ \operatorname{curl}_\perp \mathbf{curl}_\perp v(x_\perp) \end{pmatrix} w(x_3) + \begin{pmatrix} \mathbf{grad}_\perp v(x_\perp) \\ 0 \end{pmatrix} \partial_3 w(x_3),$$

and next:

$$\operatorname{curl} \operatorname{curl} \mathbf{u} = \begin{pmatrix} \mathbf{curl}_\perp \operatorname{curl}_\perp \mathbf{curl}_\perp v(x_\perp) \\ 0 \end{pmatrix} w(x_3) - \begin{pmatrix} \mathbf{curl}_\perp v(x_\perp) \\ 0 \end{pmatrix} \partial_3^2 w(x_3).$$

Since $\operatorname{curl}_\perp \mathbf{curl}_\perp = -\Delta_\perp$, we find that equation $\operatorname{curl}\operatorname{curl}\mathbf{u} = \Lambda\mathbf{u}$ becomes

$$-\begin{pmatrix}\mathbf{curl}_\perp \Delta_\perp v(x_\perp)\\ 0\end{pmatrix} w(x_3) - \begin{pmatrix}\mathbf{curl}_\perp v(x_\perp)\\ 0\end{pmatrix} \partial_3^2 w(x_3) = \Lambda \begin{pmatrix}\mathbf{curl}_\perp v(x_\perp)\\ 0\end{pmatrix} w(x_3). \tag{3.1}$$

Then we find that (3.1) holds if v and w satisfy

$$-\Delta_\perp v = \lambda v \ \text{ in } \ \omega \quad \text{and} \quad -\partial_3^2 w = \mu w \ \text{ in } \ I \quad \text{with} \quad \lambda + \mu = \Lambda. \tag{3.2}$$

Boundary conditions on the TE mode $\mathbf{u}$ are satisfied if, cf (2.5),

$$\partial_n v = 0 \ \text{ on } \ \partial\omega \quad \text{and} \quad w = 0 \ \text{ on } \ \partial I. \tag{3.3}$$

Thus we have found the following families of TE modes:

Lemma 3.1. *Let $\left(\lambda_j^{\mathsf{neu}}, v_j^{\mathsf{neu}}\right)_{j\geq 0}$ be the sequence of eigenpairs of the Neumann problem in ω for the operator $-\Delta_\perp$, with $\lambda_0^{\mathsf{neu}} = 0$ and $v_0^{\mathsf{neu}} = 1$. Let $\left(\mu_k^{\mathsf{dir}}, w_k^{\mathsf{dir}}\right)_{k\geq 1}$ be the sequence of eigenpairs of the Dirichlet problem in I for the operator $-\partial_3^2$. Then, for all $j \geq 1$, $k \geq 1$, the field*

$$\mathbf{E}_{jk}^{\mathsf{TE}}(x_\perp, x_3) = \begin{pmatrix}\mathbf{curl}_\perp v_j^{\mathsf{neu}}(x_\perp)\\ 0\end{pmatrix} w_k^{\mathsf{dir}}(x_3), \tag{3.4}$$

is a TE mode for problem (2.1) *associated with the eigenvalue $\Lambda_{jk}^{\mathsf{TE}} = \lambda_j^{\mathsf{neu}} + \mu_k^{\mathsf{dir}}$.*

3.2. **TM modes.** Let $\mathbf{u}$ be a TM mode. Using (2.4) we find

$$\operatorname{curl}\mathbf{u} = -\begin{pmatrix}\mathbf{curl}_\perp v(x_\perp)\\ 0\end{pmatrix} \partial_3^2 w(x_3) - \begin{pmatrix}\mathbf{curl}_\perp \Delta_\perp v(x_\perp)\\ 0\end{pmatrix} w(x_3)$$

and next

$$\operatorname{curl}\operatorname{curl}\mathbf{u} = -\begin{pmatrix}0\\ \operatorname{curl}_\perp \mathbf{curl}_\perp v\end{pmatrix} \partial_3^2 w - \begin{pmatrix}0\\ \operatorname{curl}_\perp \mathbf{curl}_\perp \Delta_\perp v\end{pmatrix} w - \begin{pmatrix}\mathbf{grad}_\perp v\\ 0\end{pmatrix} \partial_3^3 w - \begin{pmatrix}\mathbf{grad}_\perp \Delta_\perp v\\ 0\end{pmatrix} \partial_3 w.$$

Since $\operatorname{curl}_\perp \mathbf{curl}_\perp = -\Delta_\perp$, we find that equation $\operatorname{curl}\operatorname{curl}\mathbf{u} = \Lambda\mathbf{u}$ becomes

$$\begin{aligned}(3.5)\quad \begin{pmatrix}0\\ \Delta_\perp v\end{pmatrix}\partial_3^2 w + \begin{pmatrix}0\\ \Delta_\perp^2 v\end{pmatrix} w - \begin{pmatrix}\mathbf{grad}_\perp v\\ 0\end{pmatrix}\partial_3^3 w - \begin{pmatrix}\mathbf{grad}_\perp \Delta_\perp v\\ 0\end{pmatrix}\partial_3 w =\\ -\Lambda\begin{pmatrix}0\\ \Delta_\perp v\end{pmatrix} w + \Lambda\begin{pmatrix}\mathbf{grad}_\perp v\\ 0\end{pmatrix}\partial_3 w.\end{aligned}$$

Then, like in the TE case, we find that (3.5) holds if v and w satisfy

$$(3.6)\qquad -\Delta_\perp v = \lambda v \ \text{ in } \omega \quad \text{and} \quad -\partial_3^2 w = \mu w \ \text{ in } I \quad \text{with} \quad \lambda + \mu = \Lambda.$$

Concerning the boundary conditions, (2.5) yields

$$(3.7)\qquad \begin{cases} v = \text{const. on each } \partial_l\omega & \text{or} \quad \partial_3 w \equiv 0 \ \text{ in } I,\\ \mathbf{grad}_\perp v \equiv 0 \ \text{ in } \omega & \text{or} \quad \partial_3 w = 0 \ \text{ on } \partial I,\\ \Delta_\perp v = 0 \ \text{ on } \partial\omega & \text{or} \quad w \equiv 0 \ \text{ in } I.\end{cases}$$

Here, $\partial_l\omega$, $l = 1, \ldots, L$, are the connected components of $\partial\omega$.

The conditions $\mathbf{grad}_\perp v \equiv 0$ and $w \equiv 0$ have to be discarded since they imply $\mathbf{u} \equiv 0$. Therefore we should have $\partial_3 w = 0$ on ∂I and $\Delta_\perp v = 0$ on $\partial\omega$. The latter condition implies that $v = 0$ on $\partial\omega$ in the case when $\lambda \neq 0$. When $\lambda = 0$, the condition $v = \text{const.}$ on each $\partial_l\omega$ is sufficient. Thus we can show that (3.6)-(3.7) can be summarized as follows: Either

$$(3.8)\qquad \begin{cases} -\Delta_\perp v = \lambda v \ \text{ in } \omega & \text{and} \quad v = 0 \ \text{ on } \partial\omega\\ -\partial_3^2 w = \mu w \ \text{ in } I & \text{and} \quad \partial_3 w = 0 \ \text{ on } \partial I\end{cases} \quad \text{with} \quad \lambda \neq 0,\ \ \lambda + \mu = \Lambda,$$

or

$$(3.9)\qquad \begin{cases} -\Delta_\perp v = 0 \ \text{ in } \omega & \text{and} \quad v = \text{const} \ \text{ on each } \partial_l\omega\\ -\partial_3^2 w = \mu w \ \text{ in } I & \text{and} \quad \partial_3 w = 0 \ \text{ on } \partial I\end{cases} \quad \text{with} \quad \mu = \Lambda.$$

Thus we have found the following families of TM modes:

Lemma 3.2. *Let $\big(\lambda_j^{\mathsf{dir}}, v_j^{\mathsf{dir}}\big)_{j\geq 1}$ be the sequence of eigenpairs of the Dirichlet problem in ω for the operator $-\Delta_\perp$. Let $\big(\mu_k^{\mathsf{neu}}, w_k^{\mathsf{neu}}\big)_{k\geq 0}$ be the sequence of eigenpairs of the Neumann problem in I for the operator $-\partial_3^2$, with $\mu_0^{\mathsf{neu}} = 0$ and $w_0^{\mathsf{neu}} = 1$. Then, for all $j \geq 1$, $k \geq 0$, the field*

$$(3.10)\qquad \mathbf{E}_{jk}^{\mathsf{TM}}(x_\perp, x_3) = \begin{pmatrix}\mathbf{grad}_\perp v_j^{\mathsf{dir}}(x_\perp)\\ 0\end{pmatrix}\partial_3 w_k^{\mathsf{neu}}(x_3) - \begin{pmatrix}0\\ \Delta_\perp v_j^{\mathsf{dir}}(x_\perp)\end{pmatrix} w_k^{\mathsf{neu}}(x_3),$$

is a TM mode for problem (2.1) *associated with the eigenvalue* $\Lambda_{jk}^{\mathsf{TM}} = \lambda_j^{\mathsf{dir}} + \mu_k^{\mathsf{neu}}$.

- *If, moreover,* $\partial\omega$ *has more than one connected components* ($L \geq 2$)*, there exist* $L-1$ *independent harmonic potentials* v_l^{top}*,* $l = 1, \ldots, L-1$ *with constant traces on each connected components of* $\partial\omega$*. They generate the* $L-1$ *families of TEM modes defined for all* $l = 1, \ldots, L-1$ *and* $k \geq 1$ *by*

$$\mathbf{E}_{lk}^{\mathsf{TEM}}(x_\perp, x_3) = \begin{pmatrix} \mathbf{grad}_\perp\, v_l^{\mathsf{top}}(x_\perp) \\ 0 \end{pmatrix} w_k^{\mathsf{dir}}(x_3). \tag{3.11}$$

Remark 3.3. *(i)* In (3.11) we have used that the derivatives $\partial_3 w_k^{\mathsf{neu}}$ for $k \geq 1$ are an eigenvector basis for the Dirichlet problem on the interval I.
(ii) There exists potentials $\tilde{v}_l^{\mathsf{top}} \in \Theta(\omega)$, cf Remark 2.4, such that for any $l \leq L-1$, there holds

$$\widetilde{\mathbf{curl}}_\perp\, \tilde{v}_l^{\mathsf{top}} = \mathbf{grad}_\perp\, v_l^{\mathsf{top}}. \tag{3.12}$$

Therefore for all $k \geq 1$, the mode $\mathbf{E}_{lk}^{\mathsf{TEM}}$ is an extended TE mode. This is why it is called a TEM mode.

3.3. **Completeness.** The aim of this section is to prove

Lemma 3.4. *Let* $\mathbf{u} \in \mathbf{X}_{\mathsf{N}}(\Omega)$ *such that* $\operatorname{div} \mathbf{u} = 0$*. We assume that for all integers* $j \geq 1$ *and* $l \in [1, L-1]$

$$\langle \mathbf{u}, \mathbf{E}_{jk}^{\mathsf{TE}} \rangle = 0 \ \ (\forall k \geq 1), \quad \langle \mathbf{u}, \mathbf{E}_{jk}^{\mathsf{TM}} \rangle = 0 \ \ (\forall k \geq 0) \quad \textit{and} \quad \langle \mathbf{u}, \mathbf{E}_{lk}^{\mathsf{TEM}} \rangle = 0 \ \ (\forall k \geq 1).$$

Here $\langle \cdot, \cdot \rangle$ *is the* L^2 *scalar product on* Ω*. Then* $\mathbf{u} = 0$*.*

Proof. We first draw consequences from the orthogonality properties against the TM modes: We fix j and k and set $v = v_j^{\mathsf{dir}}$, $w = w_k^{\mathsf{neu}}$ and integrate by parts:

$$\begin{aligned}
0 &= \int_I \int_\omega \mathbf{u}_\perp(x_\perp, x_3)\, \mathbf{grad}_\perp\, v(x_\perp) \partial_3 w(x_3) - u_3(x_\perp, x_3)\, \Delta_\perp v(x_\perp) w(x_3)\, \mathrm{d}x_\perp \mathrm{d}x_3 \\
&= \int_I \int_\omega -\operatorname{div}_\perp \mathbf{u}_\perp(x_\perp, x_3)\, v(x_\perp) \partial_3 w(x_3) - u_3(x_\perp, x_3)\, \Delta_\perp v(x_\perp) w(x_3)\, \mathrm{d}x_\perp \mathrm{d}x_3 \\
&= \int_I \int_\omega \partial_3 u_3(x_\perp, x_3)\, v(x_\perp) \partial_3 w(x_3) - u_3(x_\perp, x_3)\, \Delta_\perp v(x_\perp) w(x_3)\, \mathrm{d}x_\perp \mathrm{d}x_3 \\
&= \int_I \int_\omega -u_3(x_\perp, x_3)\, v(x_\perp) \partial_3^2 w(x_3) - u_3(x_\perp, x_3)\, \Delta_\perp v(x_\perp) w(x_3)\, \mathrm{d}x_\perp \mathrm{d}x_3.
\end{aligned}$$

Here we have used that $\operatorname{div}\mathbf{u} = 0$, replacing $\operatorname{div}_\perp \mathbf{u}_\perp$ by $-\partial_3 u_3$. Coming back to the properties of $v = v_j^{\mathsf{dir}}$ and $w = w_k^{\mathsf{neu}}$ we find for all $j \geq 1$ and $k \geq 0$

$$\int_I \int_\omega u_3(x_\perp, x_3)\,(\lambda_j^{\mathsf{dir}} + \mu_k^{\mathsf{neu}}) v_j^{\mathsf{dir}}(x_\perp) w_k^{\mathsf{neu}}(x_3)\;\mathrm{d}x_\perp \mathrm{d}x_3 = 0.$$

Since $\lambda_j^{\mathsf{dir}} + \mu_k^{\mathsf{neu}}$ is never 0, we deduce that for all $j \geq 1$ and $k \geq 0$

$$\int_I \int_\omega u_3(x_\perp, x_3)\, v_j^{\mathsf{dir}}(x_\perp) w_k^{\mathsf{neu}}(x_3)\;\mathrm{d}x_\perp \mathrm{d}x_3 = 0.$$

The set $v_j^{\mathsf{dir}}(x_\perp) w_k^{\mathsf{neu}}(x_3)$ being a complete basis in $L^2(\Omega)$, we deduce that $u_3 = 0$.

Next, we use the orthogonality against the TE modes: for all $j \geq 1$ and $k \geq 1$ there holds:

$$\int_I w_k^{\mathsf{dir}}(x_3) \int_\omega \mathbf{u}_\perp(x_\perp, x_3) \cdot \mathbf{curl}_\perp\, v_j^{\mathsf{neu}}(x_\perp)\;\mathrm{d}x_\perp \mathrm{d}x_3 = 0.$$

Therefore, for all $j \geq 1$:

$$\int_\omega \mathbf{u}_\perp(x_\perp, x_3) \cdot \mathbf{curl}_\perp\, v_j^{\mathsf{neu}}(x_\perp)\;\mathrm{d}x_\perp = 0, \quad \forall x_3 \in I.$$

We deduce that $\operatorname{curl}_\perp \mathbf{u}_\perp(\cdot, x_3)$ is orthogonal to all v_j^{neu} for $j \geq 1$, which means that $\operatorname{curl}_\perp \mathbf{u}_\perp(\cdot, x_3)$ is constant with respect to $x_\perp$. There exists a function $z = z(x_3)$ such that

$$(*) \qquad \operatorname{curl}_\perp \mathbf{u}_\perp(x_\perp, x_3) = z(x_3).$$

Since $\operatorname{div}\mathbf{u} = 0$ and $u_3 = 0$, we have $\operatorname{div}_\perp \mathbf{u}_\perp = 0$. Besides, the orthogonality relations against the TEM modes yields for all $k \geq 1$ and $l \leq L - 1$

$$\int_I w_k^{\mathsf{dir}}(x_3) \int_\omega \mathbf{u}_\perp(x_\perp, x_3) \cdot \mathbf{grad}_\perp\, v_l^{\mathsf{top}}(x_\perp)\;\mathrm{d}x_\perp \mathrm{d}x_3 = 0.$$

We deduce that

$$\int_\omega \mathbf{u}_\perp(x_\perp, x_3) \cdot \mathbf{grad}_\perp\, v_l^{\mathsf{top}}(x_\perp)\;\mathrm{d}x_\perp = 0, \quad \forall x_3 \in I,$$

from which we find that

$$\int_{\partial\omega_l} \mathbf{u}_\perp \cdot \mathbf{n}_\perp\;\mathrm{d}\sigma = 0, \quad l = 1, \dots, L.$$

Combined with $\operatorname{div}_\perp \mathbf{u}_\perp = 0$, this provides the existence of a potential $y \in L^2(I, H^1(\omega))$ satisfying the Neumann boundary condition on $\partial\omega$ such that

$$\mathbf{u}_\perp(x_\perp, x_3) = \mathbf{curl}_\perp\, y(x_\perp, x_3).$$

With $(*)$ we find

$$-\Delta_\perp y(x_\perp, x_3) = z(x_3).$$

Since y satisfies the homogeneous Neumann condition with respect to $x_\perp$, this implies that $z(x_3) = 0$ for all x_3. Finally we have obtained that $\mathbf{u}_\perp = 0$. □

3.4. **Eigenmodes.** Summarizing, we have proved:

Theorem 3.5. *Let $\Omega = \omega \times I$. The eigenpairs* (2.1) *of the Maxwell operator with electric boundary conditions are the three families:*

$$\mathbf{E}^{\mathsf{TE}}_{jk} = \begin{pmatrix} \mathbf{curl}_\perp\, v^{\mathsf{neu}}_j(x_\perp) \\ 0 \end{pmatrix} w^{\mathsf{dir}}_k(x_3) \quad \textit{with} \quad \Lambda^{\mathsf{TE}}_{jk} = \lambda^{\mathsf{neu}}_j + \mu^{\mathsf{dir}}_k,\ \ j \geq 1,\ k \geq 1,$$

$$\mathbf{E}^{\mathsf{TM}}_{jk} = \begin{pmatrix} \mathbf{grad}_\perp\, v^{\mathsf{dir}}_j(x_\perp) \\ 0 \end{pmatrix} \partial_3 w^{\mathsf{neu}}_k(x_3) - \begin{pmatrix} 0 \\ \Delta_\perp v^{\mathsf{dir}}_j(x_\perp) \end{pmatrix} w^{\mathsf{neu}}_k(x_3)$$
$$\textit{with} \quad \Lambda^{\mathsf{TM}}_{jk} = \lambda^{\mathsf{dir}}_j + \mu^{\mathsf{neu}}_k,\ \ j \geq 1,\ k \geq 0,$$

$$\mathbf{E}^{\mathsf{TEM}}_{lk} = \begin{pmatrix} \mathbf{grad}_\perp\, v^{\mathsf{top}}_l(x_\perp) \\ 0 \end{pmatrix} w^{\mathsf{dir}}_k(x_3) \quad \textit{with} \quad \Lambda^{\mathsf{TEM}}_{lk} = \mu^{\mathsf{dir}}_k,\ \ 1 \leq l \leq L-1,\ k \geq 1.$$

See Lemma 3.1 and 3.2 for the definitions of λ^{neu}_j, λ^{dir}_j, μ^{dir}_k, μ^{neu}_k, etc... All the associated eigenvalues $\Lambda^{\mathsf{TE}}_{jk}$, $\Lambda^{\mathsf{TM}}_{jk}$ and $\Lambda^{\mathsf{TEM}}_{jk}$ are non-zero.

Since the magnetic field $\mathbf{H}$ associated with the electric field $\mathbf{E}$ is given by

$$\mathbf{H} = \frac{1}{i\sqrt{\Lambda}}\ \operatorname{curl} \mathbf{E},$$

for any non-zero eigenvalue Λ, we deduce:

Corollary 3.6. *Under the conditions of Theorem 3.5, we set $\kappa = \sqrt{\Lambda}$. The associated magnetic fields are given by*

$$\mathbf{H}^{\mathsf{TE}}_{jk} = \frac{1}{i\kappa^{\mathsf{TE}}_{jk}} \left\{ \begin{pmatrix} \mathbf{grad}_\perp\, v^{\mathsf{neu}}_j(x_\perp) \\ 0 \end{pmatrix} \partial_3 w^{\mathsf{dir}}_k(x_3) - \begin{pmatrix} 0 \\ \Delta_\perp v^{\mathsf{neu}}_j(x_\perp) \end{pmatrix} w^{\mathsf{dir}}_k(x_3) \right\} \quad j,k \geq 1,$$

$$\mathbf{H}^{\mathsf{TM}}_{jk} = -i\kappa^{\mathsf{TM}}_{jk} \begin{pmatrix} \mathbf{curl}_\perp\, v^{\mathsf{dir}}_j(x_\perp) \\ 0 \end{pmatrix} w^{\mathsf{neu}}_k(x_3) \quad j \geq 1,\ k \geq 0,$$

$$\mathbf{H}^{\mathsf{TEM}}_{lk} = \frac{i}{\kappa^{\mathsf{TEM}}_{lk}} \begin{pmatrix} \mathbf{curl}_\perp\, v^{\mathsf{top}}_l(x_\perp) \\ 0 \end{pmatrix} \partial_3 w^{\mathsf{dir}}_k(x_3) \quad 1 \leq l \leq L-1,\ k \geq 1.$$

Remark 3.7. *(i)* The *electric* fields in the pairs $(\mathsf{E}^{\mathsf{TE}}, \mathsf{H}^{\mathsf{TE}})$ are transverse to the axis x_3, whilst in the pairs $(\mathsf{E}^{\mathsf{TM}}, \mathsf{H}^{\mathsf{TM}})$ the *magnetic* fields are transverse to the axis x_3.

(ii) We notice that for all $k \geq 1$, $\mathsf{H}^{\mathsf{TEM}}_{lk}$ can also be written as

$$\mathsf{H}^{\mathsf{TEM}}_{lk} = i \begin{pmatrix} \mathbf{curl}_\perp\, v^{\mathsf{top}}_l(x_\perp) \\ 0 \end{pmatrix} w^{\mathsf{neu}}_k(x_3)$$

The expression above also makes sense for $k = 0$. The associated eigenvalue is 0 and the corresponding electric field is 0. These eigenmodes are those produced by the 3D topological non-triviality of Ω. Note that for all $k \geq 1$ we can write

$$\mathsf{E}^{\mathsf{TEM}}_{lk} = -\frac{1}{\kappa} \begin{pmatrix} \mathbf{curl}_\perp\, v^{\mathsf{top}}_l(x_\perp) \\ 0 \end{pmatrix} \partial_3 w^{\mathsf{neu}}_k(x_3).$$

Remark 3.8. If ω contains holes, i.e. if TEM modes are present, they often contribute the smallest positive eigenvalues. Let us make formulas for eigenvalues more explicit: Let ℓ be the length of the inerval I and let us assume that ω has *one hole*. Besides the magnetostatic zero eigenvalue, we find

$$\Lambda^{\mathsf{TE}}_{jk} = \lambda^{\mathsf{neu}}_j + \left(\frac{k\pi}{\ell}\right)^2 \ (\forall j, k \geq 1), \quad \Lambda^{\mathsf{TM}}_{jk} = \lambda^{\mathsf{dir}}_j + \left(\frac{k\pi}{\ell}\right)^2 \ (\forall j \geq 1, k \geq 0),$$

and

$$\Lambda^{\mathsf{TEM}}_{k} = \left(\frac{k\pi}{\ell}\right)^2 \ (\forall k \geq 1).$$

Then the smallest positive eigenvalue is either $\Lambda^{\mathsf{TM}}_{1,0}$ or $\Lambda^{\mathsf{TEM}}_{1}$. If ω is fixed and ℓ large enough, $\Lambda^{\mathsf{TEM}}_{1}$ is smaller than $\Lambda^{\mathsf{TM}}_{1,0}$, see also Remark 6.3.

Remark 3.9. Similar results hold for *mixed boundary conditions*, i.e. when the perfectly conducting or insulating parts $\partial\Omega_{\mathsf{cd}}$ and $\partial\Omega_{\mathsf{ins}}$ are chosen to be either $\partial\omega \times I$ or $\omega \times \partial I$:

(i) Let us consider the case when

$$\partial\Omega_{\mathsf{cd}} = \partial\omega \times I \quad \text{and} \quad \partial\Omega_{\mathsf{ins}} = \omega \times \partial I.$$

Then, the essential boundary condition for the electric field E on $\omega \times \partial I$ is $\mathsf{E}_3 = 0$ and the natural boundary condition is $\operatorname{curl} \mathsf{E} \times \mathsf{n} = 0$, reducing to $\partial_3 \mathsf{E}_\perp = 0$. Thus we find the three families of eigenfields:

$$\mathsf{E}^{\mathsf{TE}}_{jk} = \begin{pmatrix} \mathbf{curl}_\perp\, v^{\mathsf{neu}}_j(x_\perp) \\ 0 \end{pmatrix} w^{\mathsf{neu}}_k(x_3) \ \text{ with } \ j \ge 1,\ k \ge 0,$$

$$\mathsf{E}^{\mathsf{TM}}_{jk} = \begin{pmatrix} \mathbf{grad}_\perp\, v^{\mathsf{dir}}_j(x_\perp) \\ 0 \end{pmatrix} \partial_3 w^{\mathsf{dir}}_k(x_3) - \begin{pmatrix} 0 \\ \Delta_\perp v^{\mathsf{dir}}_j(x_\perp) \end{pmatrix} w^{\mathsf{dir}}_k(x_3), \ \text{ with } \ j \ge 1,\ k \ge 1,$$

$$\mathsf{E}^{\mathsf{TEM}}_{lk} = \begin{pmatrix} \mathbf{grad}_\perp\, v^{\mathsf{top}}_l(x_\perp) \\ 0 \end{pmatrix} w^{\mathsf{neu}}_k(x_3) \ \text{ with } \ 1 \le l \le L-1,\ k \ge 0.$$

associated with the eigenvalues $\Lambda^{\mathsf{TE}}_{jk} = \lambda^{\mathsf{neu}}_j + \mu^{\mathsf{neu}}_k$, $\Lambda^{\mathsf{TM}}_{jk} = \lambda^{\mathsf{dir}}_j + \mu^{\mathsf{dir}}_k$, and $\Lambda^{\mathsf{TEM}}_{lk} = \mu^{\mathsf{neu}}_k$.
(ii) We set $I = (0, \ell)$. Let us consider the case when

$$\partial\Omega_{\mathsf{cd}} = (\partial\omega \times I) \cup (\omega \times \{0\}) \quad \text{and} \quad \partial\Omega_{\mathsf{ins}} = \omega \times \{\ell\}.$$

The axial generators w_k can be described thanks to the eigenvectors w^{mix}_k, $k \ge 1$, of the *mixed* problem in ω:

$$-\partial_3^2 w = \mu w, \quad w(0) = 0, \quad \partial_3 w(\ell) = 0.$$

We find

$$\mathsf{E}^{\mathsf{TE}}_{jk} = \begin{pmatrix} \mathbf{curl}_\perp\, v^{\mathsf{neu}}_j(x_\perp) \\ 0 \end{pmatrix} w^{\mathsf{mix}}_k(x_3) \ \text{ with } \ j \ge 1,\ k \ge 1,$$

$$\mathsf{E}^{\mathsf{TM}}_{jk} = \begin{pmatrix} \mathbf{grad}_\perp\, v^{\mathsf{dir}}_j(x_\perp) \\ 0 \end{pmatrix} \partial_3^2 w^{\mathsf{mix}}_k(x_3) - \begin{pmatrix} 0 \\ \Delta_\perp v^{\mathsf{dir}}_j(x_\perp) \end{pmatrix} \partial_3 w^{\mathsf{mix}}_k(x_3), \ \text{ with } \ j \ge 1,\ k \ge 1,$$

$$\mathsf{E}^{\mathsf{TEM}}_{lk} = \begin{pmatrix} \mathbf{grad}_\perp\, v^{\mathsf{top}}_l(x_\perp) \\ 0 \end{pmatrix} w^{\mathsf{mix}}_k(x_3) \ \text{ with } \ 1 \le l \le L-1,\ k \ge 1.$$

If ω contains holes, TEM modes are present and contribute the smallest positive eigenvalue $\left(\frac{\pi}{2\ell}\right)^2$.

4. Application 1: Maxwell eigenvalues of the cube

Let Ω be the cube $(0, \pi)^3$. We can apply Theorem 3.5 with $\omega = (0, \pi)^2$ and $I = (0, \pi)$. Since ω is simply connected we have TE and TM modes only. Therefore the normalized Maxwell eigenvalues are

$$\lambda^{\mathsf{neu}}_j + \mu^{\mathsf{dir}}_k, \ \ j \ge 1,\ k \ge 1 \quad \text{and} \quad \lambda^{\mathsf{dir}}_j + \mu^{\mathsf{neu}}_k, \ \ j \ge 1,\ k \ge 0.$$

We have
$$\mu_k^{\mathsf{dir}} = k^2,\ \ k \geq 1 \quad \text{and} \quad \mu_k^{\mathsf{neu}} = k^2,\ \ k \geq 0.$$
The Dirichlet eigenvalues on ω are
$$k_1^2 + k_2^2, \quad k_1, k_2 \geq 1.$$
The non-zero Neumann eigenvalues are
$$k_1^2 + k_2^2, \quad k_1, k_2 \geq 0,\ \ k_1 \text{ or } k_2 \neq 0.$$
Therefore the TE eigenvalues are
$$k_1^2 + k_2^2 + k_3^2, \quad k_1, k_2 \geq 0,\ \ k_1 \text{ or } k_2 \neq 0,\ \ k_3 \geq 1.$$
The TM eigenvalues are
$$k_1^2 + k_2^2 + k_3^2, \quad k_1, k_2 \geq 1,\ \ k_3 \geq 0.$$
Therefore we have once
$$k_1^2 + k_2^2 + k_3^2,\ \ k_1, k_2, k_3 \geq 0 \ \text{ with only one index } \nu \in \{1,2,3\} \text{ such that } \ k_\nu = 0,$$
and twice
$$k_1^2 + k_2^2 + k_3^2, \quad k_1, k_2, k_3 \geq 1.$$
The first eigenvalues are
$$2\ \text{(mult. 3)}, \quad 3\ \text{(mult. 2)}, \quad 5\ \text{(mult. 6)}, \quad 6\ \text{(mult. 6)}, \quad 8\ \text{(mult. 3)},\ldots$$
A larger multiplicity of 12 is attained for example for $14 = 1 + 4 + 9$. But 12 is not the maximal multiplicity (e.g. the multiplicity of $26 = 25 + 1 + 0 = 16 + 9 + 1$ is 18).

The Dirichlet eigenvectors on $(0, \pi)$ are $\zeta \mapsto \sin k\zeta$, $k \geq 1$, and the Neumann eigenvectors are $\cos k\zeta$, $k \geq 0$. The components of the electric eigenvectors in the cube are (sums of) products of two $\sin$ terms by one $\cos$ term.

For a *rectangular parallelepiped*
$$\Omega = (0, \ell_1) \times (0, \ell_2) \times (0, \ell_3),$$
we find the eigenvalues: Once
$$\left(\frac{k_1\pi}{\ell_1}\right)^2 + \left(\frac{k_2\pi}{\ell_2}\right)^2 + \left(\frac{k_3\pi}{\ell_3}\right)^2,$$
$$\forall k_1, k_2, k_3 \geq 0 \quad \text{with only one index } \nu \in \{1,2,3\} \text{ such that } \ k_\nu = 0,$$

and twice

$$\left(\frac{k_1\pi}{\ell_1}\right)^2 + \left(\frac{k_2\pi}{\ell_2}\right)^2 + \left(\frac{k_3\pi}{\ell_3}\right)^2, \quad \forall k_1, k_2, k_3 \geq 1.$$

Compare with the (slightly wrong) formulas in
`http://scienceworld.wolfram.com/physics/ResonantCavity.html`.

5. Application 2: Maxwell eigenvalues in a cylinder

We assume that, besides the assumption that $\Omega = \omega \times I$, the domain Ω is axisymmetric. This implies that ω is a disc, or a disc with a concentric hole. We investigate both situations. Let R be the external radius of ω and r_0 be its internal radius, with the convention that $r_0 = 0$ corresponds to the case when ω is a disc.

We use *cylindrical coordinates* $(r, \theta, x_3) \in (r_0, R) \times (0, 2\pi) \times I$. Setting $\breve{u}(r, \theta, x_3) = u(x)$, we introduce *cylindrical components* (u_r, u_θ, u_3) of the field $\mathbf{u} = (u_1, u_2, u_3)$,

$$u_r = \breve{u}_1 \cos\theta + \breve{u}_2 \sin\theta \quad \text{and} \quad u_\theta = -\breve{u}_1 \sin\theta + \breve{u}_2 \cos\theta.$$

Therefore, for a scalar function v, the radial and angular components of $\mathbf{grad}_\perp\, v$ are $\partial_r v$ and $\frac{1}{r}\partial_\theta v$, and those of $\mathbf{curl}_\perp\, v$ are $\frac{1}{r}\partial_\theta v$ and $-\partial_r v$. Thus the TE electromagnetic modes given by Theorem 3.5 and Corollary 3.6 have the form $(\mathbf{E}, \frac{1}{i\kappa}\mathbf{H})$ with $\mathbf{E}$ and $\mathbf{H}$ given by

$$\left\{\begin{aligned} E_r &= \tfrac{1}{r}\partial_\theta v(r,\theta)\, w(x_3), \\ E_\theta &= -\partial_r v(r,\theta)\, w(x_3), \\ E_3 &= 0 \end{aligned}\right. \quad \text{and} \quad \left\{\begin{aligned} H_r &= \partial_r v(r,\theta)\, \partial_3 w(x_3), \\ H_\theta &= \tfrac{1}{r}\partial_\theta v(r,\theta)\, \partial_3 w(x_3), \\ H_3 &= -\tfrac{1}{r^2}((r\partial_r)^2 + \partial_\theta^2) v(r,\theta)\, w(x_3), \end{aligned}\right. \tag{5.1}$$

while TM electromagnetic modes have the form $(\mathbf{E}, -i\kappa\mathbf{H})$ with $\mathbf{E}$ and $\mathbf{H}$ given by

$$\left\{\begin{aligned} E_r &= \partial_r v(r,\theta)\, \partial_3 w(x_3), \\ E_\theta &= \tfrac{1}{r}\partial_\theta v(r,\theta)\, \partial_3 w(x_3), \\ E_3 &= -\tfrac{1}{r^2}((r\partial_r)^2 + \partial_\theta^2) v(r,\theta)\, w(x_3), \end{aligned}\right. \quad \text{and} \quad \left\{\begin{aligned} H_r &= \tfrac{1}{r}\partial_\theta v(r,\theta)\, w(x_3), \\ H_\theta &= -\partial_r v(r,\theta)\, w(x_3), \\ H_3 &= 0 \end{aligned}\right. \tag{5.2}$$

Definition 5.1. *Let u be a scalar function, $u \in L^2(\Omega)$ and let $\breve{u}$ the function defined on $(r_0, R) \times (0, 2\pi) \times I$ by $\breve{u}(r, \theta, x_3) = u(x)$. For any $n \in \mathbb{Z}$, the angular Fourier coefficient of order n of u is denoted by u^n and is defined as:*

$$u^n(r, x_3) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \breve{u}(r, \theta, x_3)\, e^{-in\theta}\, \mathrm{d}\theta, \quad r_0 < r < R, \;\; x_3 \in I. \tag{5.3}$$

Let $\mathbf{u} = (u_1, u_2, u_3)$ be a vector field, $\mathbf{u} \in L^2(\Omega)^3$. For any $n \in \mathbb{Z}$, the angular Fourier coefficient of order n of $\mathbf{u}$ are those of the scalar functions u_r, u_θ and u_3 and denoted by u_r^n, u_θ^n and u_3^n. See [2] *for more details.*

The Fourier coefficients of a TE electromagnetic modes of the form $(\mathsf{E}, \frac{1}{i\kappa}\mathsf{H})$ are

$$\left\{\begin{aligned} E^n_r &= \tfrac{in}{r} v^n(r)\, w(x_3),\\ E^n_\theta &= -\partial_r v^n(r)\, w(x_3),\\ E^n_3 &= 0\end{aligned}\right. \quad\text{and}\quad \left\{\begin{aligned} H^n_r &= \partial_r v^n(r)\, \partial_3 w(x_3),\\ H^n_\theta &= \tfrac{in}{r} v^n(r)\, \partial_3 w(x_3),\\ H^n_3 &= -\tfrac{1}{r^2}((r\partial_r)^2 - n^2)v^n(r)\, w(x_3),\end{aligned}\right. \tag{5.4}$$

and likewise for the TM modes of the form $(\mathsf{E}, -i\kappa\mathsf{H})$:

$$\left\{\begin{aligned} E^n_r &= \partial_r v^n(r)\, \partial_3 w(x_3),\\ E^n_\theta &= \tfrac{in}{r} v^n(r)\, \partial_3 w(x_3),\\ E^n_3 &= -\tfrac{1}{r^2}((r\partial_r)^2 - n^2)v^n(r)\, w(x_3),\end{aligned}\right. \quad\text{and}\quad \left\{\begin{aligned} H^n_r &= \tfrac{in}{r} v^n(r)\, w(x_3),\\ H^n_\theta &= -\partial_r v^n(r)\, w(x_3),\\ H^n_3 &= 0.\end{aligned}\right. \tag{5.5}$$

The Dirichlet and Neumann problems for $\Delta_\perp$ in ω are axisymmetric problems (the domain and the operators are invariant by rotation). Therefore, they commute with $i\partial_\theta$ and share with $i\partial_\theta$ a common eigenvector basis. Therefore the eigenvectors of the Dirichlet and Neumann problems in ω can be classified according to their angular Fourier coefficient, and we obtain a similar classification for the TE and the TM modes: As a corollary of Theorem 3.5, we have

Corollary 5.2. *Let ω be a disc of radius R. For any $n \in \mathbb{Z}$, the TE modes of order n have only their n-th Fourier coefficient non-zero: It has the form* (5.4) *with w Dirichlet eigenvector on I and v^n (non-constant) eigenvector of the problem*

$$\left\{\begin{aligned} &-\tfrac{1}{r^2}((r\partial_r)^2 - n^2)v^n(r) = \lambda v^n \quad \textit{in} \quad (0,R),\\ &v^n(0) = 0 \quad \textit{if} \quad n\neq 0, \quad \partial_r v^n(0) = 0 \quad \textit{if} \quad n = 0,\\ &\partial_r v^n(R) = 0.\end{aligned}\right. \tag{5.6}$$

Similarly the n-th Fourier coefficients of the TM modes are given by (5.5) *with w Neumann eigenvector on I with v^n eigenvector of the problem*

$$\left\{\begin{aligned} &-\tfrac{1}{r^2}((r\partial_r)^2 - n^2)v^n(r) = \lambda v^n \quad \textit{in} \quad (0,R),\\ &v^n(0) = 0 \quad \textit{if} \quad n\neq 0, \quad \partial_r v^n(0) = 0 \quad \textit{if} \quad n = 0,\\ &v^n(R) = 0.\end{aligned}\right. \tag{5.7}$$

When ω has a hole, the new feature is the appearance of the TEM modes. Indeed, the generator v^{top} can be defined as the function $x \mapsto \log r$. It is axisymmetric, therefore the TEM modes are axisymmetric too. In connection with Remark 3.3, we note that the "conjugate" potential $\tilde v^{\mathsf{top}}$ is the function $x \mapsto \theta$. There holds, *cf* (3.12):

$$\widetilde{\mathbf{curl}}_\perp\, \tilde v^{\mathsf{top}} = \mathbf{grad}_\perp\, v^{\mathsf{top}} = \begin{pmatrix} \frac{1}{r}\\ 0\\ 0\end{pmatrix} \quad\text{and}\quad \mathbf{curl}_\perp\, v^{\mathsf{top}} = -\begin{pmatrix} 0\\ \frac{1}{r}\\ 0\end{pmatrix}. \tag{5.8}$$

We summarize our results for an annulus ω:

Corollary 5.3. *Let ω be an annulus of interior radius r_0 and exterior radius R. For any $n \in \mathbb{Z}$, the TE modes of order n have only their n-th Fourier coefficient non-zero: It has the form* (5.4) *with w Dirichlet eigenvector on I and v^n (non-constant) eigenvector of the problem*

$$\left\{ \begin{array}{l} -\frac{1}{r^2}((r\partial_r)^2 - n^2)v^n(r) = \lambda v^n \quad \textit{in} \quad (r_0, R), \\ \partial_r v^n(r_0) = 0, \\ \partial_r v^n(R) = 0. \end{array} \right. \tag{5.9}$$

Similarly the n-th Fourier coefficients of the TM modes are given by (5.5) *with w Neumann eigenvector on I with v^n eigenvector of the problem*

$$\left\{ \begin{array}{l} -\frac{1}{r^2}((r\partial_r)^2 - n^2)v^n(r) = \lambda v^n \quad \textit{in} \quad (r_0, R), \\ v^n(r_0) = 0, \\ v^n(R) = 0. \end{array} \right. \tag{5.10}$$

Besides, the family of TEM modes is axisymmetric and has the form $(\mathbf{E}, -i\kappa\mathbf{H})$ with

$$\left\{ \begin{array}{l} E_r^0 = \frac{1}{r}\partial_3 w(x_3), \\ E_\theta^0 = 0, \\ E_3^0 = 0, \end{array} \right. \quad \textit{and} \quad \left\{ \begin{array}{l} H_r^0 = 0, \\ H_\theta^0 = -\frac{1}{r} w(x_3), \\ H_3^0 = 0 \end{array} \right. \tag{5.11}$$

with w Neumann eigenvector on I associated with the eigenvalue κ^2. For $\kappa = 0$, the TEM mode is $(\mathbf{E}, \mathbf{H}) = (0, \mathbf{H})$ with $\mathbf{H} = (0\ 1\ 0)^\top$.

Remark 5.4. As r_0 tends to 0, the Dirichlet and Neumann eigenmodes of the annulus tend to the Dirichlet and Neumann eigenvalues of the disc of same radius. Hence the TE and TM modes of the cylinder with hole tend to the TE and TM modes of the cylinder without hole. In contrast, the TEM modes do not depend on r_0 as long as $r_0 \neq 0$, but disappear at the limit when $r_0 = 0$. This fact has a practical importance when thin conductor wires are present.

6. APPENDIX: DIRICHLET AND NEUMANN EIGENVALUES IN A DISC

Let ω be the disc of radius R. The Dirichlet and Neumann eigenvalues for $-\Delta$ in ω can be determined by the solution of problems (5.6) and (5.7). This is based on Bessel functions of the first kind $J_n(z)$, with the same n as in (5.6) and (5.7). The function J_n is the solution of the differential equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 0,$$

which is bounded in $x = 0$. Moreover, $J_0(0) = 1$ and $J_0'(0) = 0$, and $J_n(0) = \mathcal{O}(x^n)$.

Lemma 6.1 ([3]). (i) *Let $z^{\mathsf{dir}}_{n,j}$ be the positive zeros of of J_n. The eigenvalues of* (5.7) *are*

$$\lambda^{\mathsf{dir}}_{n,j} = \left(\frac{z^{\mathsf{dir}}_{n,j}}{R}\right)^2, \quad n \geq 0, \quad j \geq 1. \tag{6.1}$$

and the corresponding eigenvector is $r \mapsto J_n(z^{\mathsf{dir}}_{n,j}\frac{r}{R})$.
(ii) *Let $z^{\mathsf{neu}}_{n,j}$ be the positive zeros of of J'_n. The non-zero eigenvalues of* (5.6) *are*

$$\lambda^{\mathsf{neu}}_{n,j} = \left(\frac{z^{\mathsf{neu}}_{n,j}}{R}\right)^2, \quad n \geq 0, \quad j \geq 1. \tag{6.2}$$

and the corresponding eigenvector is $r \mapsto J'_n(z^{\mathsf{neu}}_{n,j}\frac{r}{R})$.

We give in the next table values for the first three zeros $z^{\mathsf{dir}}_{n,j}$ and $z^{\mathsf{neu}}_{n,j}$ for $n = 0, 1, 2$. We use the relation $J_{\nu-1} - J_{\nu+1} = 2J'_\nu$ to compute $z^{\mathsf{neu}}_{n,j}$. Since $J_{-1} = -J_1$, there holds

$$z^{\mathsf{neu}}_{0,j} = z^{\mathsf{dir}}_{1,j}, \quad \forall j \geq 1.$$

$z^{\mathsf{dir}}_{0,j}$	$z^{\mathsf{dir}}_{1,j}$	$z^{\mathsf{dir}}_{2,j}$	$z^{\mathsf{neu}}_{0,j}$	$z^{\mathsf{neu}}_{1,j}$	$z^{\mathsf{neu}}_{2,j}$
2.4048	3.8317	5.1356	3.8317	1.8412	3.0542
5.5201	7.0156	8.4172	7.0156	5.3314	6.7061
8.6537	10.173	11.620	10.173	8.5363	9.9695

TABLE 1. The first three zeros of J_0, J_1, J_2, J'_0, J'_1, J'_2.

Corollary 6.2. (i) *Let Ω be a cylinder of radius R and length ℓ. Let $n \in \mathbb{Z}$. The TE modes with angular order n are associated with the eigenvalues*

$$\left(\frac{z^{\mathsf{neu}}_{n,j}}{R}\right)^2 + \left(\frac{k\pi}{\ell}\right)^2, \quad j \geq 1, \ k \geq 1. \tag{6.3}$$

The TM modes with angular order n are associated with the eigenvalues

$$\left(\frac{z^{\mathsf{dir}}_{n,j}}{R}\right)^2 + \left(\frac{k\pi}{\ell}\right)^2, \quad j \geq 1, \ k \geq 0. \tag{6.4}$$

(ii) *Let Ω be a cylinder of radius R and length ℓ, with a coaxial circular hole of diameter $r_0 < R$. The TE and TM eigenvalues tend to those of the cylinder without hole as $r_0 \to 0$.*

Moreover, the TEM modes have their angular order equal to 0 *and are associated with the eigenvalues (which are independent of* R *and* r_0*):*

$$\left(\frac{k\pi}{\ell}\right)^2, \quad k \geq 0. \tag{6.5}$$

Remark 6.3. Let Ω be a cylinder of radius R and length ℓ, with a coaxial circular hole of diameter $r_0 < R$. *(i)* If r_0 is small enough and

$$\ell > R\,\frac{\pi}{z^{\mathsf{dir}}_{0,1}} \quad \text{i.e.} \quad \ell > 1.3064\,R, \tag{6.6}$$

the smallest positive Maxwell eigenvalue in Ω corresponds to a TEM mode. The relation between the frequency f (see Introduction, §1) and the first non-zero TEM mode is then

$$\boxed{2\ell f = c}$$

which means that ℓ is the half-wave length.

(ii) In case *(ii)* of Remark 3.9, i.e. when

$$\partial\Omega_{\mathsf{cd}} = (\partial\omega \times I) \cup (\omega \times \{0\}) \quad \text{and} \quad \partial\Omega_{\mathsf{ins}} = \omega \times \{\ell\}.$$

the smallest positive Maxwell eigenvalue in Ω always corresponds to a TEM mode, associated with the frequency f such that

$$\boxed{4\ell f = c}$$

which means that ℓ is the quarter-wave length.

7. Extension to nonconstant electric permittivity

Let us consider the original Maxwell system (1.3) again. We still assume that the magnetic permeability μ is equal to μ_0 in the whole domain Ω. But we allow now that the electric permittivity ε may vary in Ω. We set

$$\varepsilon = \varepsilon_{\mathsf{rel}}\varepsilon_0, \quad \varepsilon_{\mathsf{rel}} \geq 1.$$

We consider domains Ω in the tensor product form $\omega \times I$. We assume that

$$\varepsilon_{\mathsf{rel}}(x) = \varepsilon_{\mathsf{rel}}(x_\perp), \quad \varepsilon_{\mathsf{rel}} \in L^\infty(\omega), \tag{7.1}$$

like in wave guides or optic fibers. Then the splitting of eigenvectors between TE, TM and TEM does not hold any more (at least not in the form given by Theorem 3.5 and Corollary 3.6).

The splitting of the spectrum according to frequencies with respect to the axial variable x_3 remains possible, as we will see. We are going to investigate the magnetic field $\mathbf{H}$, taking advantage of its local regularity even if $\varepsilon_{\mathsf{rel}}$ is not continuous.

We consider the same normalization as in the introduction. Then, instead of (1.5) we have

$$\begin{cases} \operatorname{curl} \mathbf{E} - i\kappa \mathbf{H} = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} \mathbf{H} + i\kappa\varepsilon_{\mathsf{rel}} \mathbf{E} = 0 & \text{in} \quad \Omega, \\ \mathbf{E} \times \mathbf{n} = 0 \quad \text{and} \quad \mathbf{H} \cdot \mathbf{n} = 0, & \text{on} \quad \partial\Omega, \\ \operatorname{div} \varepsilon_{\mathsf{rel}} \mathbf{E} = 0 \quad \text{and} \quad \operatorname{div} \mathbf{H} = 0 & \text{in} \quad \Omega. \end{cases} \tag{7.2}$$

The magnetic variational formulation becomes, instead of (2.2):

Find the eigenpairs $(\Lambda = \kappa^2, \mathbf{u})$ *with* $\mathbf{u} \neq 0$ *and* $\operatorname{div} \mathbf{u} = 0$ *such that*

$$\mathbf{u} \in \mathbf{X}_{\mathsf{T}}(\Omega) : \quad \int_\Omega \frac{1}{\varepsilon_{\mathsf{rel}}} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{u}' \,\mathrm{d}\mathbf{x} = \Lambda \int_\Omega \mathbf{u} \cdot \mathbf{u}' \,\mathrm{d}\mathbf{x}, \quad \forall \mathbf{u}' \in \mathbf{X}_{\mathsf{T}}(\Omega), \tag{7.3}$$

To simplify notations, let us assume that

$$\boxed{I = (0, \pi)} \tag{7.4}$$

Note that, in the constant material case, considering the Maxwell eigenmodes from the magnetic point of view, we can reformulate the magnetic part of eigenmodes given in Corollary 3.6 in the following way

$$\mathbf{H}^{\mathsf{TE}}_{jk} = \begin{pmatrix} k \,\mathbf{grad}_\perp v_j^{\mathsf{neu}}(x_\perp) \cos(kx_3) \\ -\Delta_\perp v_j^{\mathsf{neu}}(x_\perp) \sin(kx_3) \end{pmatrix} \qquad j \geq 1, \ \ k \geq 1, \tag{7.5}$$

$$\mathbf{H}^{\mathsf{TM}}_{jk} = \begin{pmatrix} \mathbf{curl}_\perp v_j^{\mathsf{dir}}(x_\perp) \cos(kx_3) \\ 0 \end{pmatrix} \qquad j \geq 1, \ \ k \geq 0, \tag{7.6}$$

$$\mathbf{H}^{\mathsf{TEM}}_{lk} = \begin{pmatrix} \mathbf{curl}_\perp v_l^{\mathsf{top}}(x_\perp) \cos(kx_3) \\ 0 \end{pmatrix} \qquad 1 \leq l \leq L-1, \ \ k \geq 0. \tag{7.7}$$

We are going to prove that we still have a similar structure with respect to the axial variable x_3.

Theorem 7.1. *With the assumptions* (7.1) *and* (7.4)*, the magnetic eigenmodes solution of* (7.3) *can be written as*

$$\big(\mathbf{H}^k_j, \Lambda^k_j\big)_{j\geq 1, k\geq 0}$$

with

$$\mathbf{H}^k_j = \begin{pmatrix} \mathbf{v}^k_{\perp,j}(x_\perp) \cos(kx_3) \\ v^k_{3,j}(x_\perp) \sin(kx_3) \end{pmatrix}. \tag{7.8}$$

Here, for all $k \in \mathbb{N}$, $\mathbf{v}_j^k := (\mathbf{v}_{\perp,j}^k, v_{3,j}^k)$ and Λ_j^k are the the eigenvectors and eigenvalues of the problem:

Find $\mathbf{v} = (\mathbf{v}_\perp, v_3) \neq 0$ and $\Lambda \in \mathbb{R}$ with $\operatorname{div}_\perp \mathbf{v}_\perp + kv_3 = 0$ such that

$$
\begin{aligned}
(7.9)\quad & \mathbf{v}_\perp \in \mathbf{X}_\mathsf{T}(\omega),\ v_3 \in H^1(\omega): \\
& \int_\omega \frac{1}{\varepsilon_{\mathsf{rel}}} \Big\{ \operatorname{curl}_\perp \mathbf{v}_\perp\ \operatorname{curl}_\perp \mathbf{v}'_\perp + \big(\mathbf{grad}_\perp v_3 + k\mathbf{v}_\perp\big) \cdot \big(\mathbf{grad}_\perp v'_3 + k\mathbf{v}'_\perp\big) \Big\}\, \mathrm{d}\mathbf{x} \\
& \qquad\qquad = \Lambda \int_\omega \mathbf{v} \cdot \mathbf{v}'\, \mathrm{d}\mathbf{x}, \quad \forall \mathbf{v}' \in \mathbf{X}_\mathsf{T}(\omega) \times H^1(\omega).
\end{aligned}
$$

Proof. Solutions of (7.3) satisfy on $\omega \times \{0\}$ the essential boundary condition $u_3 = 0$, and the natural boundary condition $\frac{1}{\varepsilon_{\mathsf{rel}}} \operatorname{curl} \mathbf{u} \times \mathbf{e}_3 = 0$. Since $u_3 = 0$ on $\omega \times \{0\}$, $\partial_1 u_3$ and $\partial_2 u_3$ are also 0 on $\omega \times \{0\}$, and the natural boundary condition implies that $\partial_3 u_1 = \partial_3 u_2 = 0$ on $\omega \times \{0\}$. Therefore, defining the extension

$$
\widetilde{\mathbf{u}}_\perp(x_\perp, -x_3) = \mathbf{u}_\perp(x_\perp, x_3) \quad \text{and} \quad \tilde{u}_3(x_\perp, -x_3) = -u_3(x_\perp, x_3),\ \ \forall x_3 \in (0,\pi)
$$

we obtain an element $\widetilde{\mathbf{u}} \in \mathbf{X}_\mathsf{T}(\omega \times (-\pi,\pi))$ which satisfies $\operatorname{div} \widetilde{\mathbf{u}} = 0$ and is solution of (7.3) on the extended domain $\omega \times (-\pi,\pi)$. Moreover, $\mathbf{u}(x_\perp, -\pi) = \mathbf{u}(x_\perp, \pi)$ and $\partial_3 \mathbf{u}(x_\perp, -\pi) = \partial_3 \mathbf{u}(x_\perp, \pi)$ for all $x_\perp \in \omega$. We deduce that $\widetilde{\mathbf{u}}$ is solution of (7.3) on the domain $\mathbf{X}_\mathsf{T}(\omega \times \mathbb{T})$ where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Since the coefficient $\varepsilon_{\mathsf{rel}}$ does not depend on x_3, the underlying Maxwell operator commutes with ∂_3. Therefore the spectrum of problem (7.3) can be decomposed according to the eigenvectors of ∂_3 on $\mathbb{T}$, which are the functions $x_3 \mapsto e^{ikx_3}$, $k \in \mathbb{Z}$.

For any positive integer k, we notice that if $\big(\mathbf{v}_\perp(x_\perp), v_3(x_\perp)\big)e^{ikx_3}$ is solution of (7.3) on the domain $\mathbf{X}_\mathsf{T}(\omega \times \mathbb{T})$, then $\big(\mathbf{v}_\perp(x_\perp), -v_3(x_\perp)\big)e^{-ikx_3}$ is also solution of the same problem. Therefore, their sum is also solution of the same problem. Moreover this sum has the form (7.8) and satisfies the boundary conditions (perfectly conducting walls)[2] of the space $\mathbf{X}_\mathsf{T}(\Omega)$. Conversely this sum is, up to a multiplicative constant, the only linear combination of $\big(\mathbf{v}_\perp(x_\perp), v_3(x_\perp)\big)e^{ikx_3}$ and $\big(\mathbf{v}_\perp(x_\perp), -v_3(x_\perp)\big)e^{-ikx_3}$ which satisfies the boundary conditions of the space $\mathbf{X}_\mathsf{T}(\Omega)$.

Calculating

$$
\int_\Omega \frac{1}{\varepsilon_{\mathsf{rel}}} \operatorname{curl} \mathbf{u}\ \operatorname{curl} \mathbf{u}'\, \mathrm{d}\mathbf{x}
$$

for

$$
\mathbf{u} = \begin{pmatrix} \mathbf{v}(x_\perp)\ \cos(kx_3) \\ v_3(x_\perp)\ \sin(kx_3) \end{pmatrix} \quad \text{and} \quad \mathbf{u}' = \begin{pmatrix} \mathbf{v}'(x_\perp)\ \cos(kx_3) \\ v'_3(x_\perp)\ \sin(kx_3) \end{pmatrix},
$$

[2]Considering the *difference* instead the sum, we would find the perfectly insulating boundary conditions on $\omega \times \partial I$.

we find

$$\int_\omega \frac{1}{\varepsilon_{\mathsf{rel}}} \Big\{ \operatorname{curl}_\perp \mathbf{v}_\perp \ \operatorname{curl}_\perp \mathbf{v}'_\perp + \big(\mathbf{curl}_\perp v_3 + k\mathbf{v}_\perp \times \mathbf{e}_3 \big) \cdot \big(\mathbf{curl}_\perp v'_3 + k\mathbf{v}'_\perp \times \mathbf{e}_3 \big) \Big\} \,\mathrm{d}\mathbf{x}$$

which coincides with

$$\int_\omega \frac{1}{\varepsilon_{\mathsf{rel}}} \Big\{ \operatorname{curl}_\perp \mathbf{v}_\perp \ \operatorname{curl}_\perp \mathbf{v}'_\perp + \big(\mathbf{grad}_\perp v_3 + k\mathbf{v}_\perp \big) \cdot \big(\mathbf{grad}_\perp v'_3 + k\mathbf{v}'_\perp \big) \Big\} \,\mathrm{d}\mathbf{x}.$$

□

Remark 7.2. For $k = 0$, problem (7.9) reduces to two uncoupled problems: The magnetic 2D Maxwell eigenvalue problem in ω for $\mathbf{v}_\perp$ and the Neumann eigenvalue problem for $-\Delta_\perp$ in ω for v_3. This last problem does no yield any non-trivial solution of (7.9) since for $k = 0$, the third component in the Ansatz (7.8) is zero. Moreover, we can show that the solutions of the magnetic 2D Maxwell eigenvalue problem in ω are the pairs $(\mathbf{curl}_\perp v_j^{\mathsf{dir}}, \lambda_j^{\mathsf{dir}})$, $j \geq 1$, with the eigenpairs $(v_j^{\mathsf{dir}}, \lambda_j^{\mathsf{dir}})$ of the problem

$$-\Delta_\perp v = \lambda\, \varepsilon v \quad \text{in} \quad \omega, \qquad v \in H^1_0(\omega). \tag{7.10}$$

Thus we have found for $k = 0$ one family of TM modes:

$$\mathbf{H}_j^{\mathsf{TM}} = \begin{pmatrix} \mathbf{curl}_\perp v_j^{\mathsf{dir}}(x_\perp) \\ 0 \end{pmatrix} \qquad j \geq 1.$$

Remark 7.3. *(i)* The bilinear form a_k of problem (7.9) can be regularized by

$$\int_\omega \frac{1}{\varepsilon_{\mathsf{rel}}} \Big\{ \big(\operatorname{div}_\perp \mathbf{v}_\perp + kv_3 \big) \big(\operatorname{div}_\perp \mathbf{v}'_\perp + kv'_3 \big) \Big\} \,\mathrm{d}\mathbf{x}.$$

Let b_k be the corresponding regularized bilinear form:

$$\begin{aligned} b_k(\mathbf{v}, \mathbf{v}') = \int_\omega \frac{1}{\varepsilon_{\mathsf{rel}}} \Big\{ & \operatorname{curl}_\perp \mathbf{v}_\perp \ \operatorname{curl}_\perp \mathbf{v}'_\perp \\ & + \big(\mathbf{grad}_\perp v_3 + k\mathbf{v}_\perp \big) \cdot \big(\mathbf{grad}_\perp v'_3 + k\mathbf{v}'_\perp \big) \\ & \qquad + \big(\operatorname{div}_\perp \mathbf{v}_\perp + kv_3 \big) \big(\operatorname{div}_\perp \mathbf{v}'_\perp + kv'_3 \big) \Big\} \,\mathrm{d}\mathbf{x}. \end{aligned} \tag{7.11}$$

(ii) If $\varepsilon_{\mathsf{rel}}$ is constant, we can show that

$$\begin{aligned} b_k(\mathbf{v}, \mathbf{v}') = \frac{1}{\varepsilon_{\mathsf{rel}}} \int_\omega & \operatorname{curl}_\perp \mathbf{v}_\perp \ \operatorname{curl}_\perp \mathbf{v}'_\perp + \mathbf{grad}_\perp v_3 \cdot \mathbf{grad}_\perp v'_3 \\ & + \operatorname{div}_\perp \mathbf{v}_\perp \ \operatorname{div}_\perp \mathbf{v}'_\perp + k^2 (\mathbf{v}_\perp \cdot \mathbf{v}'_\perp + v_3 v'_3) \,\mathrm{d}\mathbf{x}. \end{aligned} \tag{7.12}$$

(iii) If $\partial\omega$ is not connected, let v^{top} be a non-zero harmonic potential with constant traces on each connected component of $\partial\omega$. For $\mathbf{v}$ defined by $\mathbf{v}_\perp = \mathbf{curl}_\perp\, v^{\mathsf{top}}$ and $v_3 = 0$, we find $\operatorname{div}_\perp \mathbf{v}_\perp + kv_3 = 0$ for all k and

$$\int_\omega \frac{1}{\varepsilon_{\mathsf{rel}}} \Big\{ \operatorname{curl}_\perp \mathbf{v}_\perp \ \operatorname{curl}_\perp \mathbf{v}'_\perp + \big(\mathbf{grad}_\perp\, v_3 + k\mathbf{v}_\perp\big)\cdot\big(\mathbf{grad}_\perp\, v'_3 + k\mathbf{v}'_\perp\big)\Big\}\,\mathrm{d}\mathbf{x} = k^2 \int_\omega \frac{1}{\varepsilon_{\mathsf{rel}}}\, \mathbf{v}\cdot\mathbf{v}'\,\mathrm{d}\mathbf{x}. \tag{7.13}$$

The corresponding magnetic field is, compare with (7.7)

$$\mathbf{H} = \begin{pmatrix} \mathbf{curl}_\perp\, v^{\mathsf{top}}(x_\perp)\ \cos(kx_3) \\ 0 \end{pmatrix}.$$

It is divergence free and its Rayleigh quotient is $< k^2$. Nevertheless, it is not an eigenvector of problem (7.3), in general: Indeed we have

$$\operatorname{curl}\frac{1}{\varepsilon_{\mathsf{rel}}}\operatorname{curl}\mathbf{H} = -k\operatorname{curl}\frac{1}{\varepsilon_{\mathsf{rel}}}\begin{pmatrix} \mathbf{grad}_\perp\, v^{\mathsf{top}}(x_\perp)\ \sin(kx_3) \\ 0 \end{pmatrix}.$$

8. Axisymmetric nonconstant electric permittivity

We consider now the case when ω is a disc or an annulus, and the situation where $\varepsilon_{\mathsf{rel}}(x_\perp) = \varepsilon_{\mathsf{rel}}(r)$, i.e. $\varepsilon_{\mathsf{rel}}$ depends on the radial variable only. Then we can combine the above decomposition into wave-guide problems indexed by k (Theorem 7.1) with the angular Fourier transformation (Definition 5.1).

We recall first the expression of $\mathbf{grad}$, curl and div operators in cylindrical coordinates and components: For a scalar function v

$$\begin{cases} (\mathbf{grad}\, v)_r = \partial_r v, \\ (\mathbf{grad}\, v)_\theta = \frac{1}{r}\partial_\theta v, \\ (\mathbf{grad}\, v)_3 = \partial_3 v, \end{cases} \tag{8.1}$$

and for a vector function $\mathbf{v}$,

$$\begin{cases} (\operatorname{curl}\mathbf{v})_r = \frac{1}{r}\partial_\theta v_3 - \partial_3 v_\theta, \\ (\operatorname{curl}\mathbf{v})_\theta = \partial_3 v_r - \partial_r v_3, \\ (\operatorname{curl}\mathbf{v})_3 = \partial_r v_\theta + \frac{1}{r}v_\theta - \frac{1}{r}\partial_\theta v_r, \end{cases} \tag{8.2}$$

and

$$\operatorname{div}\mathbf{v} = \partial_r v_r + \tfrac{1}{r}v_r + \tfrac{1}{r}\partial_\theta v_\theta + \partial_3 v_3. \tag{8.3}$$

Let us now write the bilinear form b_k (7.11) with respect to the cylindrical coordintates v_r, v_θ and v_3 of $\mathbf{v}$:

$$
\begin{aligned}
(8.4)\quad b_k(\mathbf{v},\mathbf{v}') = \int_\omega \frac{1}{\varepsilon_{\mathsf{rel}}} \Big\{ &\big(\partial_r v_\theta + \tfrac{1}{r} v_\theta - \tfrac{1}{r}\partial_\theta v_r\big)\big(\partial_r v'_\theta + \tfrac{1}{r} v'_\theta - \tfrac{1}{r}\partial_\theta v'_r\big) \\
&+ \big(\partial_r v_3 + k v_r\big)\big(\partial_r v'_3 + k v'_r\big) + \big(\tfrac{1}{r}\partial_\theta v_3 + k v_\theta\big)\big(\tfrac{1}{r}\partial_\theta v'_3 + k v'_\theta\big) \\
&+ \big(\partial_r v_r + \tfrac{1}{r} v_r + \tfrac{1}{r}\partial_\theta v_\theta + k v_3\big)\big(\partial_r v'_r + \tfrac{1}{r} v'_r + \tfrac{1}{r}\partial_\theta v'_\theta + k v'_3\big) \Big\}\, \mathrm{d}\mathbf{x}.
\end{aligned}
$$

Therefore, the contribution of angular Fourier modes of order n is

$$
\begin{aligned}
(8.5)\quad b_k^n(\mathbf{v},\mathbf{v}') = \int_{r_0}^{R} \frac{1}{\varepsilon_{\mathsf{rel}}} \Big\{ &\big(\partial_r v_\theta^n + \tfrac{1}{r} v_\theta^n - \tfrac{in}{r} v_r^n\big)\big(\partial_r \bar v_\theta'^n + \tfrac{1}{r} \bar v_\theta'^n + \tfrac{in}{r} \bar v_r'^n\big) \\
&+ \big(\partial_r v_3^n + k v_r^n\big)\big(\partial_r \bar v_3'^n + k \bar v_r'^n\big) + \big(\tfrac{in}{r} v_3^n + k v_\theta^n\big)\big(-\tfrac{in}{r} \bar v_3'^n + k \bar v_\theta'^n\big) \\
&+ \big(\partial_r v_r^n + \tfrac{1}{r} v_r^n + \tfrac{in}{r} v_\theta^n + k v_3^n\big)\big(\partial_r \bar v_r'^n + \tfrac{1}{r} \bar v_r'^n - \tfrac{in}{r} \bar v_\theta'^n + k \bar v_3'^n\big) \Big\}\, r\mathrm{d}r.
\end{aligned}
$$

It is a priori not possible to prove any monotonicity property with respect to k or n: Integrating by parts the mixed terms containing k or n allows to eliminate them in the case when $\varepsilon_{\mathsf{rel}}$ is constant, and not otherwise. The simplification which subsists is the uncoupling between v_θ and (v_r, v_3) when $n = 0$:

$$
\begin{aligned}
(8.6)\quad b_k^0(\mathbf{v},\mathbf{v}') = \int_{r_0}^{R} \frac{1}{\varepsilon_{\mathsf{rel}}} \Big\{ &\big(\partial_r v_\theta^0 + \tfrac{1}{r} v_\theta^0\big)\big(\partial_r \bar v_\theta'^0 + \tfrac{1}{r} \bar v_\theta'^0\big) + k^2 v_\theta^0 \bar v_\theta'^0 \\
&+ \big(\partial_r v_3^0 + k v_r^0\big)\big(\partial_r \bar v_3'^0 + k \bar v_r'^0\big) \\
&+ \big(\partial_r v_r^0 + \tfrac{1}{r} v_r^0 + k v_3^0\big)\big(\partial_r \bar v_r'^0 + \tfrac{1}{r} \bar v_r'^0 + k \bar v_3'^0\big) \Big\}\, r\mathrm{d}r.
\end{aligned}
$$

Remark 8.1. *(i)* The Rayleigh quotient of "pseudo-TEM" modes (cf (5.11))

$$
\mathbf{H}^{\mathsf{TEM}} = \begin{pmatrix} 0 \\ \frac{1}{r}\, w(x_3) \\ 0 \end{pmatrix}
$$

is equal to $k^2\big(\int_{r_0}^R \frac{1}{\varepsilon_{\mathsf{rel}}} \frac{\mathrm{d}r}{r}\big)\big(\int_{r_0}^R \frac{\mathrm{d}r}{r}\big)^{-1}$.

(ii) The boundary conditions associated with the axisymmetric magnetic problem (i.e. $n = 0$) are

$$
\begin{array}{ll}
h_\theta = 0 \quad \text{on} \quad z = 0, \pi, & \partial_r h_\theta + \frac{1}{r} h_\theta = 0 \quad \text{on} \quad r = r_0, R \\
h_r, h_3 = 0 \quad \text{on} \quad z = 0, \pi, & h_r, h_3 = 0 \quad \text{on} \quad r = r_0, R.
\end{array}
$$

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