Asymptotic expansion of the solution of an interface problem in a polygonal domain with thin layer

Gabriel Caloz[*], Martin Costabel[*], Monique Dauge[*], and Grégory Vial[†]

Abstract

We consider the solution of an interface problem posed in a bounded domain coated with a layer of thickness ε and with external boundary conditions of Dirichlet or Neumann type. Our aim is to build a multi-scale expansion as ε goes to 0 for that solution.

After presenting a complete multi-scale expansion in a smooth coated domain, we focus on the case of a corner domain. Singularities appear, obstructing the construction of the expansion terms in the same way as in the smooth case. In order to take these singularities into account, we construct profiles in an infinite coated sectorial domain.

Combining expansions in the smooth case with splittings in regular and singular parts involving the profiles, we construct two families of multi-scale expansions for the solution in the coated domain with corner. We prove optimal estimates for the remainders of the multi-scale expansions.

1 Introduction

The interface problem investigated in this paper originates from an electromagnetic model for bodies coated with a dielectric layer. In many practical situations, the layer thickness ε is small compared to the characteristic lengths of the body and the domain has corner points.

The problem is of practical importance and has been widely studied in the mathematical literature, in particular with respect to the question of approximately replacing the effect of the thin layer by effective boundary conditions (cf. e.g. [4], [9], [12], [13], [5], [3]). The usual technique is to build the first terms of an asymptotic expansion of the solution of the problem in powers of the thickness ε. In the previous works, the body is required to have a smooth boundary, which is often not true for the situations encountered in the applications.

The purpose of our paper is to provide an ε-expansion for corner domains in the two-dimensional case. We point out the arising mathematical difficulties and the difference from the smooth case in the structure of the asymptotics. Our method has similarities with [7], [6], and [20] in which asymptotic problems involving singularities are discussed. A detailed comparison of the effect of the thin layer with impedance boundary conditions, together with numerical simulations can be found in [26]. Similar problems can arise in other applications, for instance in elasticity for bonded joints, see [10].

Although we have restricted ourselves to the case of the Laplace operator with Dirichlet and Neumann boundary conditions, our study keeps the fundamental features useful for the applications. The basic tools introduced in this paper have a wider range of applications.

Our paper is organized as follows:

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[*] Université de Rennes 1, IRMAR, Campus de Beaulieu – 35042 Rennes cedex, France.
[†] ENS Cachan Bretagne, Campus de Ker-Lann – 35170 Bruz, France.

After formulating the problems that we are going to investigate, we present an outline of our results, in both situations of a smooth domain and a corner domain. Each time, we consider Dirichlet or Neumann external boundary conditions.

Section 2 is devoted to the smooth case: We improve results of [9] by the proof of an optimal remainder estimate. Moreover, the treatment of external Neumann boundary conditions requires in our case to deal with compatibility conditions on the data, which is not the case in [9], since the domains considered there are unbounded. The description of the structure of the ε-expansion in the interior domain and its coating, together with uniform estimates is one of the fundamental tools for the study of the coated corner domains.

After recalling some well-known results about the splitting in regular and singular parts of the solution of Dirichlet or Neumann problems in a corner domain, we build in Section 3 new objects called *profiles* and denoted by $\mathfrak{K}^\lambda$. These objects enter the ε-expansion as contributions in the rapid variable $\frac{x}{\varepsilon}$. They interpolate between the singularities of the transmission problem and the singularities $\mathfrak{s}^\lambda$ of the limit problem.

In Section 4, relying on the results of the two previous sections, we achieve our goal, which consists in the construction of two families of multi-scale ε-expansions of the solution of our problem in a coated domain with corner. This result will be outlined in formulas (1.6)-(1.8) and presented with full details in Theorems 4.7 and 4.14.

We draw a few conclusions in Section 5 before developing in the appendix the proof of a uniform (in ε) a priori estimate for the transmission problem with a smooth thin layer.

For any positive integer N, $\mathrm{H}^N(\Omega)$ is the standard Sobolev space of $\mathrm{L}^2(\Omega)$ functions with derivatives of order less than N in $\mathrm{L}^2(\Omega)$, and its norm is denoted by $\|\cdot\|_{N,\Omega}$. For positive real N, $\mathrm{H}^N(\Omega)$ is the standard Sobolev space defined by interpolation.

1.1 Formulation of the problem

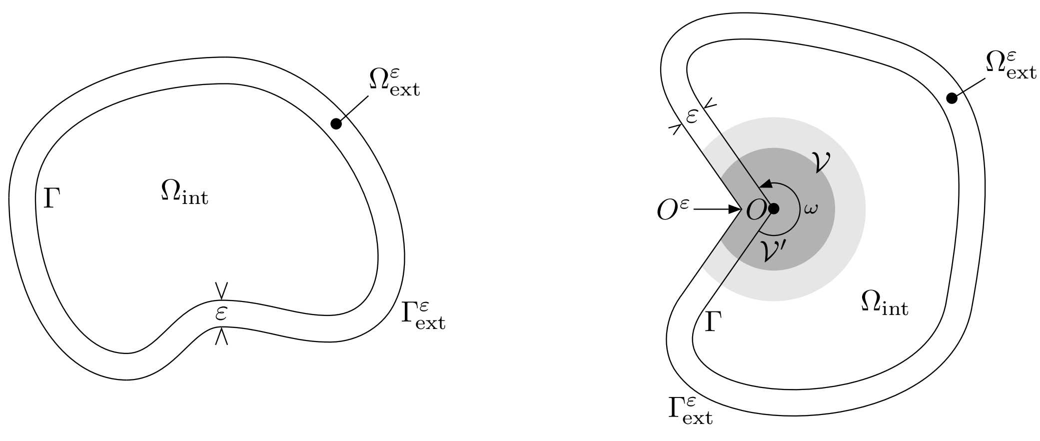

Figure 1: Smooth and corner domains with thin layer Ω^ε.

As already mentioned we consider both smooth and corner situations. The "smooth case" corresponds to the following situation: Let $\Omega_{\text{int}} \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ. For any $t \in \Gamma$ let $\mathbf{n}(t)$ denote the unit outward normal at t. For $\varepsilon > 0$ small enough, let $\Omega^\varepsilon_{\text{ext}}$ be the layer of uniform thickness ε around Ω_{int} given by

$$\Omega^\varepsilon_{\text{ext}} = \{x \in \mathbb{R}^2; \quad x = t + s\,\mathbf{n}(t),\ t \in \Gamma,\ s \in (0,\varepsilon)\}. \tag{1.1}$$

The "corner case" involves the situation where $\Omega_{\rm int}$ is a polygonal domain of $\mathbb{R}^2$. By a standard argument of localization, it is enough to consider one corner at a time: In order to simplify the presentation, we deal with a single corner point in the domain. Now $\Omega_{\rm int} \subset \mathbb{R}^2$ is a bounded domain whose boundary Γ is smooth except at the origin O: We assume that inside a neighborhood of O, Γ coincides with a plane sector of opening ω ($\neq 0, \pi, 2\pi$). Let us fix some notations:

Definition 1.1 *Let $\mathcal{V}' \subset \mathcal{V}$ be the two balls centered in O with radii $0 < \rho' < \rho$ such that $\Omega_{\rm int} \cap \mathcal{V}$ is a sector. Let $\chi \in \mathcal{C}_0^\infty(\mathcal{V})$ be a cut-off function, $\chi \equiv 1$ in $\mathcal{V}'$.*

We assume that, for $0 < \varepsilon \le \varepsilon_0$ small enough, inside $\mathcal{V}$ the external boundary of $\Omega^\varepsilon_{\rm ext}$ is a sector of opening ω too, at a distance ε from Γ, with vertex $O^\varepsilon \in \mathcal{V}'$, see Figure 1. Outside $\mathcal{V}$, the external layer $\Omega^\varepsilon_{\rm ext}$ is defined as (1.1) above in the smooth case.

In both regular and corner cases, the whole domain $\Omega_{\rm int} \cup \Gamma \cup \Omega^\varepsilon_{\rm ext}$ is denoted by Ω^ε and its boundary (the "external" boundary) by $\Gamma^\varepsilon_{\rm ext}$.

Let α be a fixed positive real number. We are interested in the following transmission problem: Find u_ε, defined by $u_{\varepsilon,\rm int}$ in $\Omega_{\rm int}$ and $u_{\varepsilon,\rm ext}$ in $\Omega^\varepsilon_{\rm ext}$ satisfying the equations

$$\left\{\begin{array}{rcll} \alpha\Delta u_{\varepsilon,\rm int} &=& f_{\rm int} & \text{in } \Omega_{\rm int}, \\ \Delta u_{\varepsilon,\rm ext} &=& f_{\rm ext} & \text{in } \Omega^\varepsilon_{\rm ext}, \\ u_{\varepsilon,\rm int} - u_{\varepsilon,\rm ext} &=& 0 & \text{on } \Gamma, \\ \alpha\partial_n u_{\varepsilon,\rm int} - \partial_n u_{\varepsilon,\rm ext} &=& g & \text{on } \Gamma, \\ \text{external b.c.} && & \text{on } \Gamma^\varepsilon_{\rm ext}, \end{array}\right. \tag{P$_\varepsilon$}$$

where ∂_n denotes the normal derivative (outer for $\Omega_{\rm int}$, inner for $\Omega^\varepsilon_{\rm ext}$). The right-hand sides $f_{\rm int}$ and g do not depend on ε and $f_{\rm ext}$ is supposed to be the restriction to $\Omega^\varepsilon_{\rm ext}$ of an ε-independent function. All data are real and supposed to be smooth, i.e., $f_{\rm int} \in \mathcal{C}^\infty(\overline{\Omega}_{\rm int})$, $f_{\rm ext} \in \mathcal{C}^\infty(\overline{\Omega}^\varepsilon_{\rm ext})$ and $g \in \mathcal{C}^\infty(\Gamma)$. In the corner case we assume moreover that $f_{\rm ext}$ is zero near O.

The external boundary conditions (b.c.) which we consider are either Dirichlet or Neumann conditions.

1.2 Dirichlet external b.c.

Here the external b.c. in (P$_\varepsilon$) is $u_{\varepsilon,\rm ext} = 0$ on $\Gamma^\varepsilon_{\rm ext}$. Problem (P$_\varepsilon$) is a well-posed elliptic problem in $\mathrm{H}^1_0(\Omega^\varepsilon)$ whose variational formulation is

$$\alpha \int_{\Omega_{\rm int}} \nabla u_{\varepsilon,\rm int} \cdot \nabla v_{\rm int} \,\mathrm{d}x + \int_{\Omega^\varepsilon_{\rm ext}} \nabla u_{\varepsilon,\rm ext} \cdot \nabla v_{\rm ext} \,\mathrm{d}x = \\ - \int_{\Omega_{\rm int}} f_{\rm int}\, v_{\rm int} \,\mathrm{d}x - \int_{\Omega^\varepsilon_{\rm ext}} f_{\rm ext}\, v_{\rm ext} \,\mathrm{d}x + \int_\Gamma g v \,\mathrm{d}\sigma, \quad \forall v \in \mathrm{H}^1_0(\Omega^\varepsilon). \tag{1.2}$$

Existence and uniqueness of a weak solution directly follow from the Lax-Milgram lemma. We also have an a priori estimate with a constant C independent of ε:

$$\|u_\varepsilon\|_{1,\Omega^\varepsilon} \le C \left[\|f_{\rm int}\|_{0,\Omega_{\rm int}} + \|f_{\rm ext}\|_{0,\Omega^\varepsilon_{\rm ext}} + \|g\|_{0,\Gamma} \right]. \tag{1.3}$$

The limit problem as $\varepsilon \to 0$ is the following Dirichlet problem without the thin layer:

$$\left\{ \begin{array}{rcll} \alpha \Delta u^0_{\rm int} & = & f & \text{in } \Omega_{\rm int}, \\ u^0_{\rm int} & = & h & \text{on } \Gamma, \end{array} \right. \tag{P$_0$}$$

with $f = f_{\rm int}$ and $h = 0$.

1.2.1 The smooth case

In the smooth case the interior part expansion of the solution of problem (P$_\varepsilon$) has the simple form, cf. [9] or [5],

$$u_{\varepsilon,{\rm int}}(x) = u^0_{\rm int}(x) + \varepsilon u^1_{\rm int}(x) + \cdots + \varepsilon^N u^N_{\rm int}(x) + \mathcal{O}(\varepsilon^N), \tag{1.4}$$

each $u^k_{\rm int}$ being independent of ε. We further investigate expansion (1.4) in two directions: *(i)* We prove optimal estimates for the remainder, and *(ii)* exhibit a construction algorithm for all terms $u^k_{\rm int}$. We will see that for any $k \geq 1$, $u^k_{\rm int}$ is solution of the Dirichlet problem on $\Omega_{\rm int}$ with $f = 0$ and $h = h^k$, with h^k the trace of differential operators acting on the previous terms $u^\ell_{\rm int}$ for $\ell < k$.

1.2.2 The corner case: Expansion into regular parts and profiles

In the case of a corner domain, the expansion (1.4) is not valid anymore, because the generic presence of singularities prevents the traces h^k produced by the construction algorithm of the smooth case to belong to the right trace space $\mathrm{H}^{\frac{1}{2}}(\Gamma)$.

Let us present the form of our results. To avoid unnecessary complications, we assume here that the data $f_{\rm int}$ and g are 0 at the corner point O, and the same for all their derivatives, and that the ratio $\frac{\pi}{\omega}$ is *irrational*. The general case is addressed in Section 4.

Let (r, θ) be polar coordinates centered at O and such that $-\frac{\omega}{2} \leq \theta \leq \frac{\omega}{2}$ in $\Omega_{\rm int} \cap \mathcal{V}$. The singularities of the Dirichlet problem (P$_0$) take the form

$$\mathfrak{s}^\lambda = \left\{ \begin{array}{ll} r^\lambda \cos(\lambda\theta) & \text{if } \lambda = \frac{q\pi}{\omega} \text{ with } q \text{ odd}, \\ r^\lambda \sin(\lambda\theta) & \text{if } \lambda = \frac{q\pi}{\omega} \text{ with } q \text{ even}, \end{array} \right. \qquad (q \in \mathbb{N}).$$

For the limit term $u^0_{\rm int}(x)$, we have the well-known expansion into *regular and singular* parts, cf. [17, 11]: Since the right-hand side is $\mathcal{C}^\infty(\overline{\Omega}_{\rm int})$, it takes the form for each fixed integer $N > 0$

$$u^0_{\rm int}(x) = u^{0,N}_{\rm int}(x) + \sum_{q;\, 0 < \frac{q\pi}{\omega} < N} c_q\, \chi(x)\, \mathfrak{s}^{\frac{q\pi}{\omega}}(x), \quad c_q \in \mathbb{R} \text{ and } u^{0,N}_{\rm int} \in \mathrm{H}^{N+1}(\Omega_{\rm int}). \tag{1.5}$$

Here χ is the cut-off function of Definition 1.1. In fact, with our simplifying hypotheses, the remainder is also flat: $u^{0,N}_{\rm int} = \mathcal{O}(r^N)$ as $r \to 0$. Thus (1.5) can also be seen as an expansion in powers of r as r tends to 0.

The main result of our paper is a complete ε-expansion for $u_{\varepsilon,{\rm int}}$. We have found different ways to assemble terms together, resulting into two distinct formulas. The first one is an expansion into regular and singular terms. But, in contrast with (1.5), the singular terms cannot be simply a linear combination of the $\mathfrak{s}^\lambda$. They now involve singular profiles $\mathfrak{K}^\lambda$ depending on the rapid variable $\frac{x}{\varepsilon}$: For each fixed integer $N > 0$

$$\begin{aligned} u_{\varepsilon,\mathrm{int}}(x) &= u_{\mathrm{int}}^{0,N}(x) + \varepsilon u_{\mathrm{int}}^{1,N-1}(x) + \varepsilon^2 u_{\mathrm{int}}^{2,N-2}(x) + \cdots \\ &+ \varepsilon^{\frac{2\pi}{\omega}} u_{\mathrm{int}}^{\frac{2\pi}{\omega},N-\frac{2\pi}{\omega}}(x) + \varepsilon^{1+\frac{2\pi}{\omega}} u_{\mathrm{int}}^{1+\frac{2\pi}{\omega},N-1-\frac{2\pi}{\omega}}(x) + \varepsilon^{\frac{3\pi}{\omega}} u_{\mathrm{int}}^{\frac{3\pi}{\omega},N-\frac{3\pi}{\omega}}(x) + \cdots \\ &+ \varepsilon^{\frac{\pi}{\omega}}(c_1 + c_1'\varepsilon + \cdots)\,\chi(x)\,\mathfrak{K}^{\frac{\pi}{\omega}}(\tfrac{x}{\varepsilon}) \\ &+ \varepsilon^{\frac{2\pi}{\omega}}(c_2 + c_2'\varepsilon + \cdots)\,\chi(x)\,\mathfrak{K}^{\frac{2\pi}{\omega}}(\tfrac{x}{\varepsilon}) + \cdots + \mathcal{O}(\varepsilon^N), \end{aligned} \tag{1.6}$$

with the following features

- The terms $u_{\mathrm{int}}^{\lambda,\mu}$ are independent of ε and flat in O at the order μ, i.e., $u_{\mathrm{int}}^{\lambda,\mu} = \mathcal{O}(r^\mu)$ as $r \to 0$. The exponent λ indicates the power of ε in front of $u_{\mathrm{int}}^{\lambda,\mu}$. It is an integer or a number of the form $\frac{q\pi}{\omega} + p$ with $q \geq 2$, $p \geq 0$ integers. In the above expansion $\mu = N - \lambda$, which means in particular that these terms depend on the given precision N of the expansion.
- The numbers c_q, $c_q', \ldots$ are real coefficients independent of N.
- The profiles $X \mapsto \mathfrak{K}^\lambda(X)$ are defined for $\lambda = \frac{q\pi}{\omega}$ in a model infinite sector with layer of thickness 1, see Figure 2, p.15. They solve a transmission problem with zero data and behave like $\mathfrak{s}^\lambda$ as $R \to \infty$. In expansion (1.6), only those with $\lambda \leq N$ are involved. They play a similar role as the singularities $\mathfrak{s}^\lambda$ arising in (1.5), which solve a Dirichlet problem with zero data in the infinite sector without layer. Note also that, owing to its homogeneity, $\mathfrak{s}^\lambda$ can be written as $\varepsilon^{\frac{q\pi}{\omega}} \mathfrak{s}^{\frac{q\pi}{\omega}}(\frac{x}{\varepsilon})$ in rapid variables.

The different terms in (1.6) satisfy the following energy estimates:

$$\big\| u_{\mathrm{int}}^{\lambda,\mu} \big\|_{\mathrm{H}^1(\Omega_{\mathrm{int}})} = \mathcal{O}(1) \quad \text{and} \quad \big\| \chi(\cdot)\mathfrak{K}^\lambda(\tfrac{\cdot}{\varepsilon}) \big\|_{\mathrm{H}^1(\Omega_{\mathrm{int}})} = \mathcal{O}(\varepsilon^{-\lambda}). \tag{1.7}$$

There are fundamental differences between the expansions (1.4) and (1.6): Non-integer powers of ε appear and a new scale is introduced in the functions $\mathfrak{K}^\lambda$.

1.2.3 The corner case: Corner layer expansion

The expansion (1.6) has two features which can be considered as inconvenient: *(i)* The limit term u_{int}^0 is not clearly visible, and *(ii)* the "flat" terms are of the form $u^{\lambda,N-\lambda}$, thus depend on the given precision N. To go from N to $N+1$, these terms have to be split themselves into flatter terms and singularities, to produce the $u^{\lambda,N+1-\lambda}$ and contribute to coefficients on profiles $\mathfrak{K}^\nu$.

It is possible to construct a different type of ε-expansion, by a mere rearrangement of terms inside the former expansion (1.6). This rearrangement relies on the asymptotic structure at infinity of the "canonical" profiles $\mathfrak{K}^\lambda$, which consists of a finite number of homogeneous functions $\mathfrak{K}^{\lambda,\lambda-\ell}$ of positive degree $\lambda - \ell$ with integer ℓ. Setting

$$\mathfrak{Y}^\lambda := \mathfrak{K}^\lambda - \sum\nolimits_\ell \mathfrak{K}^{\lambda,\lambda-\ell},$$

we find the new asymptotics for u_ε :

$$\begin{aligned} u_{\varepsilon,\text{int}}(x) \;=\;& u^0_{\text{int}}(x) + \varepsilon u^1_{\text{int}}(x) + \varepsilon^2 u^2_{\text{int}}(x) + \cdots \\ &+ \varepsilon^{\frac{2\pi}{\omega}} u^{\frac{2\pi}{\omega}}_{\text{int}}(x) + \varepsilon^{1+\frac{2\pi}{\omega}} u^{1+\frac{2\pi}{\omega}}_{\text{int}}(x) + \varepsilon^{\frac{3\pi}{\omega}} u^{\frac{3\pi}{\omega}}_{\text{int}}(x) + \cdots \\ &+ \varepsilon^{\frac{\pi}{\omega}} (c_1 + c_1'\varepsilon + \cdots)\, \chi(x)\, \mathfrak{Y}^{\frac{\pi}{\omega}}(\tfrac{x}{\varepsilon}) \\ &+ \varepsilon^{\frac{2\pi}{\omega}} (c_2 + c_2'\varepsilon + \cdots)\, \chi(x)\, \mathfrak{Y}^{\frac{2\pi}{\omega}}(\tfrac{x}{\varepsilon}) + \cdots + \mathcal{O}(\varepsilon^N), \end{aligned} \tag{1.8}$$

where, now, the terms u^ν, for $\nu = 0, 1, \ldots$ are no more “flat” nor regular, but they are independent of the target precision $\mathcal{O}(\varepsilon^N)$. Moreover u^0_{int} is the solution of problem (P_0). As opposed to the profiles $\mathfrak{K}^\lambda$, the $\mathfrak{Y}^\lambda$ tend to zero at infinity and, if λ is not integer, have a bounded H^1 energy on Ω_{int} :

$$\big\|\chi(\cdot)\mathfrak{Y}^\lambda(\tfrac{\cdot}{\varepsilon})\big\|_{\mathrm{H}^1(\Omega_{\text{int}})} = \mathcal{O}(1). \tag{1.9}$$

They deserve the appellation of *corner layer* although they do not decrease exponentially, but as a negative power of the distance to the origin. The expansion (1.8) fits better the standard idea of asymptotic expansion, where one only adds terms in $\mathcal{O}(\varepsilon^\nu)$ with $\nu \in (N, N+1]$ to get from a remainder in $\mathcal{O}(\varepsilon^N)$ to a remainder in $\mathcal{O}(\varepsilon^{N+1})$.

1.3 Neumann external b.c.

The external b.c. in (P_ε) is now $\partial_n u_{\varepsilon,\text{ext}} = 0$. Since the problem has now the constant functions in its kernel, a compatibility condition is needed on the right-hand side:

$$-\int_{\Omega_{\text{int}}} f_{\text{int}}\,\mathrm{d}x + \int_\Gamma g\,\mathrm{d}\sigma - \int_{\Omega^\varepsilon_{\text{ext}}} f_{\text{ext}}\,\mathrm{d}x = 0. \tag{1.10}$$

Since we want (1.10) to be satisfied for every $\varepsilon > 0$, it requires

$$-\int_{\Omega_{\text{int}}} f_{\text{int}}\,\mathrm{d}x + \int_\Gamma g\,\mathrm{d}\sigma = 0 \quad \text{and} \quad \forall\varepsilon > 0, \;\int_{\Omega^\varepsilon_{\text{ext}}} f_{\text{ext}}\,\mathrm{d}x = 0. \tag{1.11}$$

Under the assumptions (1.11), we can ensure uniqueness of a solution to the Neumann interface problem by imposing the following mean-value property:

$$\int_{\Omega_{\text{int}}} u_{\varepsilon,\text{int}}\,\mathrm{d}x = 0. \tag{1.12}$$

A expansion similar to (1.4) holds in this situation, u^0_{int} solving the interior Laplace problem in Ω_{int} with homogeneous Neumann boundary conditions on Γ. In the corner case, we have expansions analogous to (1.6) and (1.8). The main new difficulty is the construction of a suitable variational space for the profiles.

2 Asymptotics for a smooth coated domain

This section is devoted to the smooth case whose understanding is necessary for the treatment of a corner domain. In other words, we first focus on the situation “layer without corner” before treating in the next sections the situation "corner without layer" and, next, "corner with layer" we are interested in.

In the smooth case the curve Γ is supposed infinitely differentiable. Let ℓ_Γ be its length. The layer can be represented as the product $[0,\ell_\Gamma)\times(0,\varepsilon)$ thanks to the decomposition

$$\Omega^\varepsilon_{\rm ext} = \{x(t) + s\,\mathbf{n}\,(x(t))\ ;\ x(t)\in\Gamma \text{ and } s\in(0,\varepsilon)\},$$

where t denotes the arclength on Γ. The introduction of the stretched variable

$$S = \varepsilon^{-1}s$$

maps $[0,\ell_\Gamma)\times(0,\varepsilon)$ onto $[0,\ell_\Gamma)\times(0,1)$. The parameter does not appear anymore in the geometry, but in the equations through the expression of the Laplace operator in the layer (in the following formula, $c(t)$ is the curvature at the point of Γ of arclength t):

$$\Delta_{\rm ext} = \varepsilon^{-2}\partial_S^2 + \frac{\varepsilon^{-1}c(t)}{1+\varepsilon Sc(t)}\partial_S + \frac{1}{1+\varepsilon Sc(t)}\partial_t\left(\frac{1}{1+\varepsilon Sc(t)}\partial_t\right). \tag{2.1}$$

Expanding (2.1) into powers of ε, we obtain the formal expansion $\Delta_{\rm ext} = \varepsilon^{-2}\left[\partial_S^2 + \sum_\ell \varepsilon^\ell \mathbf{A}_\ell\right]$. More precisely we can write

$$\Delta_{\rm ext} = \varepsilon^{-2}\Big[\partial_S^2 + \sum_{\ell=1}^{L-1}\varepsilon^\ell\mathbf{A}_\ell + \varepsilon^L\mathbf{R}^L_\varepsilon\Big] \quad \text{for all} \quad L\geq 1. \tag{2.2}$$

Here the differential operators $\mathbf{A}_\ell = \mathbf{A}_\ell(t,S;\partial_t,\partial_S)$ have $\mathcal{C}^\infty$ coefficients in t, polynomial in S of degree $\ell-2$, and contain at most one differentiation with respect to S. Note that, in particular, $\mathbf{A}_1 = c(t)\partial_S$. The operators $\mathbf{R}^L_\varepsilon$ also have $\mathcal{C}^\infty$ coefficients in t and S, bounded in ε. There holds

$$\partial_n = \varepsilon^{-1}\partial_S$$

in the layer. Finally, for a function $v_{\rm ext}$ defined in $\Omega^\varepsilon_{\rm ext}$, we denote by $V_{\rm ext}$ the function such that

$$v_{\rm ext}(x) = V_{\rm ext}(t,S), \quad (t,S)\in[0,\ell_\Gamma)\times(0,1).$$

2.1 Dirichlet external b.c.

After the change of variables $s\mapsto S$ in $\Omega^\varepsilon_{\rm ext}$, problem (P$_\varepsilon$) becomes

$$\left\{\begin{array}{rcll}
\varepsilon^{-2}\left[\partial_S^2 U_{\varepsilon,\rm ext} + \sum_{\ell\geq 1}\varepsilon^\ell\mathbf{A}_\ell U_{\varepsilon,\rm ext}\right] &=& F^\varepsilon_{\rm ext} & \text{in } [0,\ell_\Gamma)\times(0,1),\\
\varepsilon^{-1}\partial_S U_{\varepsilon,\rm ext} &=& \alpha\partial_n u_{\varepsilon,\rm int} - g & \text{on } [0,\ell_\Gamma)\times\{0\},\\
U_{\varepsilon,\rm ext} &=& 0 & \text{on } [0,\ell_\Gamma)\times\{1\},\\
\alpha\Delta u_{\varepsilon,\rm int} &=& f_{\rm int} & \text{in } \Omega_{\rm int},\\
u_{\varepsilon,\rm int} &=& U_{\varepsilon,\rm ext} & \text{on } \Gamma,
\end{array}\right. \tag{2.3}$$

where $F^\varepsilon_{\rm ext}(t,S) = \tilde f_{\rm ext}(t,S\varepsilon)$ with $\tilde f_{\rm ext}(t,s) = f_{\rm ext}(x)$. If the function $f_{\rm ext}$ is sufficiently smooth, the Taylor expansion of $\tilde f_{\rm ext}$ in the variable s at $s=0$ leads to the expansion for all $L\in\mathbb{N}$

$$F^\varepsilon_{\rm ext}(t,S) = \sum_{\ell=0}^{L}\varepsilon^\ell F^\ell_{\rm ext}(t)S^\ell + \varepsilon^{L+1}F^{(L+1)}_{\rm rem} \quad \text{with} \quad F^\ell_{\rm ext}(t) = \frac{1}{\ell!}\partial_s^\ell\tilde f_{\rm ext}(t,0) \tag{2.4}$$

and $F_{\rm rem}^{(L+1)}$ smooth and bounded. Inserting the Ansatz

$$u_{\varepsilon,\rm int} = \sum_{n\in\mathbb{N}} \varepsilon^n u_{\rm int}^n \quad\text{and}\quad U_{\varepsilon,\rm ext} = \sum_{n\in\mathbb{N}} \varepsilon^n U_{\rm ext}^n \tag{2.5}$$

in equations (2.3), we get the following two families of problems, coupled by their boundary conditions on Γ (corresponding to $S=0$):

$$\begin{cases} \partial_S^2 U_{\rm ext}^n &=& F_{\rm ext}^{n-2}(t) S^{n-2} - \sum_{\ell+p=n} \mathsf{A}_\ell U_{\rm ext}^p & \text{for } 0<S<1, \\ \partial_S U_{\rm ext}^n &=& \alpha\partial_n u_{\rm int}^{n-1} - g\,\delta_1^n & \text{for } S=0, \\ U_{\rm ext}^n &=& 0 & \text{for } S=1, \end{cases} \tag{2.6}$$

$$\begin{cases} \alpha\Delta u_{\rm int}^n &=& f_{\rm int}\delta_0^n & \text{in } \Omega_{\rm int}, \\ u_{\rm int}^n &=& U_{\rm ext}^n & \text{on } \Gamma. \end{cases} \tag{2.7}$$

In the cases $n=0$ and $n=1$, the problems (2.6)-(2.7) are simple to solve. From (2.6) with $n=0$ we obtain $U_{\rm ext}^0=0$ and (2.7) yields that $u_{\rm int}^0$ solves the interior Laplace problem (P_0) with $f=f_{\rm int}$ and $h=0$. At step $n=1$, we find successively that $U_{\rm ext}^1=(S-1)[\alpha\partial_n u_{\rm int}^0|_\Gamma - g]$ and that $u_{\rm int}^1$ solves problem (P_0) with $f=0$ and $h=-\alpha\partial_n u_{\rm int}^0+g$.

The whole construction follows from a recurrence argument. Suppose the sequences $(u_{\rm int}^n)$ and $(U_{\rm ext}^n)$ known until rank $n=N-1$, then the Sturm-Liouville problem (2.6) uniquely defines $U_{\rm ext}^N$ whose trace is inserted into (2.7) as a Dirichlet data to determine the interior part $u_{\rm int}^N$.

Note that the variable t only appears as a parameter in equations (2.6) which are thus one-dimensional. Therefore there is no elliptic regularization in the tangential direction: $U_{\rm ext}^n$ is not more regular than $\alpha\partial_n u_{\rm int}^{n-1}$, which implies that we loose regularity at each step. However, an assumption of infinite smoothness on the right-hand sides $f_{\rm int}, f_{\rm ext}$, and g ensures that the construction can be performed. This is not true in the case of a corner domain, as we will see later on, and the loss of regularity will be a major difficulty.

Theorem 2.1 *Let $f_{\rm int}$ belong to $\mathcal{C}^\infty(\overline{\Omega}_{\rm int})$, $f_{\rm ext}$ to $\mathcal{C}^\infty(\overline{\Omega}_{\rm ext}^{\varepsilon_0})$ for an $\varepsilon_0>0$, and g to $\mathcal{C}^\infty(\Gamma)$. The solution u_ε of (P_ε) with Dirichlet external b.c. has a two-scale expansion which can be written for each $N\in\mathbb{N}$ in the form*

$$u_\varepsilon = \sum_{n=0}^{N} \varepsilon^n u^n + r_\varepsilon^{N+1}, \quad \textit{with } u^n|_{\Omega_{\rm int}} = u_{\rm int}^n \quad \textit{and} \quad u^n|_{\Omega_{\rm ext}^\varepsilon}(t,s) = U_{\rm ext}^n(t,\varepsilon^{-1}s). \tag{2.8}$$

The remainders satisfy, with a constant C_N independent of $\varepsilon\le\varepsilon_0$:

$$\left\|r_\varepsilon^{N+1}\right\|_{1,\Omega_{\rm int}} + \sqrt{\varepsilon}\left\|r_\varepsilon^{N+1}\right\|_{1,\Omega_{\rm ext}^\varepsilon} \le C_N\,\varepsilon^{N+1}. \tag{2.9}$$

Proof: By construction, the remainder r_ε^{N+1} is solution of problem (P_ε)

$$\begin{cases} \alpha\Delta r_{\varepsilon,\rm int}^{N+1} &=& 0 & \text{in } \Omega_{\rm int}, \\ \Delta r_{\varepsilon,\rm ext}^{N+1} &=& \varepsilon^{N-1}\Big[-\sum_{\ell=0}^{N} \mathsf{R}_\varepsilon^{N+1-\ell} U_{\rm ext}^\ell + F_{\rm rem}^{(N-1)}\Big] & \text{in } \Omega_{\rm ext}^\varepsilon, \\ r_{\varepsilon,\rm int}^{N+1} - r_{\varepsilon,\rm ext}^{N+1} &=& 0 & \text{on } \Gamma, \\ \alpha\partial_n r_{\varepsilon,\rm int}^{N+1} - \partial_n r_{\varepsilon,\rm ext}^{N+1} &=& g\,\delta_0^N - \varepsilon^N\alpha\partial_n u_{\rm int}^N & \text{on } \Gamma, \\ r_{\varepsilon,\rm ext}^{N+1} &=& 0 & \text{on } \Gamma_{\rm ext}^\varepsilon. \end{cases} \tag{2.10}$$

If we denote the data of this system by $f^{N+1}_{\varepsilon,\mathrm{ext}}$ and g^{N+1}_{ε}, we find the estimates

$$\left\|f^{N+1}_{\varepsilon,\mathrm{ext}}\right\|_{0,\Omega^\varepsilon_{\mathrm{ext}}} = \mathcal{O}\big(\varepsilon^{N-\frac{1}{2}}\big) \quad \text{and} \quad \left\|g^{N+1}_{\varepsilon}\right\|_{0,\Gamma} = \mathcal{O}(\varepsilon^{N}).$$

Using the a priori estimate (1.3), we immediately obtain

$$\left\|r^{N+1}_{\varepsilon}\right\|_{1,\Omega^\varepsilon} \leq C\,\varepsilon^{N-\frac{1}{2}}. \tag{2.11}$$

Moreover by definition,

$$r^{N+1}_{\varepsilon} = \varepsilon^{N+1}u^{N+1} + \varepsilon^{N+2}u^{N+2} + r^{N+3}_{\varepsilon}. \tag{2.12}$$

Since for every integer n, $\left\|u^n\right\|_{1,\Omega_{\mathrm{int}}} = \mathcal{O}(1)$ and $\left\|u^n\right\|_{1,\Omega^\varepsilon_{\mathrm{ext}}} = \mathcal{O}(\varepsilon^{-\frac{1}{2}})$, we obtain the stated result from (2.11) and (2.12). ■

Remark 2.2 The estimate (2.9) is optimal, since u^{N+1} does not vanish, in general. □

Observing the inductive solution of problems (2.6)-(2.7) we can write the relations between its interior terms u^n_{int} without mention of the exterior terms U^n_{ext}. We can also give an expression of U^n_{ext} as a function of the interior terms u^n_{int} only. This is done thanks to the introduction of four series of partial differential operators, according to:

Proposition 2.3 *Let $n \in \mathbb{N}$, $n \geq 1$. The interior solution u^n_{int} of problems* (2.6)-(2.7) *solves the Dirichlet problem* (P_0) *with $f = 0$ and $h = h^n$ where*

$$h^n = \mathsf{g}^n g + \sum_{k+\ell=n} \big(\mathsf{h}^k u^\ell_{\mathrm{int}} + \mathsf{H}^{k,\ell} F^\ell_{\mathrm{ext}}\big)\big|_\Gamma. \tag{2.13}$$

Here g^k is a differential operator in t of order $\leq k-1$, $\mathsf{H}^{k,\ell}$ a differential operator in t of order $\leq k-2-\ell$ (with the convention that $\mathsf{H}^{k,\ell} = 0$ if $k-2-\ell < 0$) and h^k a partial differential operator $\mathsf{h}^k(t;\partial_t,\partial_n)$ of order $\leq k$. The coefficients of the operators are smooth functions on Γ depending on the geometry of Γ. The first terms are given by $\mathsf{g}^0 = 0$, $\mathsf{g}^1 = \mathbf{I}$, $\mathsf{g}^2 = -\frac{1}{2}c(t)\mathbf{I}$,

$$\mathsf{h}^0 = 0,\ \ \mathsf{h}^1 = -\alpha\partial_n,\ \ \mathsf{h}^2 = \tfrac{\alpha}{2}\,c(t)\partial_n, \quad \textit{and} \quad \mathsf{H}^{0,0} = \mathsf{H}^{1,0} = 0,\ \ \mathsf{H}^{2,0} = -\tfrac{1}{2}\mathbf{I}. \tag{2.14}$$

The exterior part U^n_{ext} is given by a similar formula as (2.13)*, with operators g^k, h^k, and $\mathsf{H}^{k,\ell}$ replaced by operators a^k, b^k, and $\mathsf{B}^{k,\ell}$ which are polynomial of degree $\leq k$ in the variable S:*

$$U^n_{\mathrm{ext}} = \mathsf{a}^n g + \sum_{k+\ell=n} \mathsf{b}^k u^\ell_{\mathrm{int}} + \mathsf{B}^{k,\ell} F^\ell_{\mathrm{ext}}. \tag{2.15}$$

The first terms are given by $\mathsf{a}^0 = 0$, $\mathsf{a}^1 = (1-S)\mathbf{I}$, $\mathsf{a}^2 = \frac{1}{2}c(t)(S^2-1)\mathbf{I}$,

$$\mathsf{b}^0 = 0,\ \ \mathsf{b}^1 = (S-1)\alpha\partial_n,\ \ \mathsf{b}^2 = -\tfrac{1}{2}\,c(t)(S^2-1)\alpha\partial_n, \tag{2.16}$$

and $\mathsf{B}^{0,0} = \mathsf{B}^{1,0} = 0$, $\mathsf{B}^{2,0} = \frac{1}{2}(S^2-1)\mathbf{I}$

As practical consequences of the above formulas we obtain:

Corollary 2.4
(i) If $f_{\mathrm{int}} \equiv 0$, $f_{\mathrm{ext}} \equiv 0$, and $g \neq 0$, the series (2.8) *starts with εu^1.*
(ii) If $f_{\mathrm{int}} \equiv 0$, $f_{\mathrm{ext}} \neq 0$, and $g \equiv 0$, the series (2.8) *starts in general with $\varepsilon^2 u^2$.*
(iii) More precisely, if $f_{\mathrm{int}} \equiv 0$, $g \equiv 0$, and $\partial^k_n f_{\mathrm{ext}}|_\Gamma \equiv 0$ for $k = 0,\ldots,\ell-1$ with $\partial^\ell_n f_{\mathrm{ext}}|_\Gamma$ non identically 0, the series (2.8) *starts with $\varepsilon^{\ell+2}u^{\ell+2}$.*

This result, in particular *(iii)*, is fundamental. It will be used in the proof of Lemma 4.4, on which the construction of the asymptotic expansion is based.

2.2 Neumann external b.c.

If we consider the boundary condition $\partial_n u_{\varepsilon,\mathrm{ext}} = 0$ on $\Gamma^\varepsilon_{\mathrm{ext}}$ in problem (P$_\varepsilon$), a similar algorithmic construction can be done. Due to compatibility conditions, the situation is more complex than in the Dirichlet case.

The compatibility conditions (1.11) in the exterior part can be written as

$$0 = \int_0^\varepsilon \int_\Gamma \big[1 + sc(x)\big] f_{\mathrm{ext}}(x + s\mathbf{n}(x))\,\mathrm{d}x\,\mathrm{d}s = \varepsilon \int_0^1 \int_\Gamma \big[1 + \varepsilon S c(t)\big] F^\varepsilon_{\mathrm{ext}}(t,S)\,\mathrm{d}t\,\mathrm{d}S, \quad (2.17)$$

where $c(t)$ denotes the curvature of Γ at the point of arclength t and $\mathbf{n}(x)$ the unitary outer normal to Ω_{int}; see (2.4) for the behavior of $F^\varepsilon_{\mathrm{ext}}$ with respect to ε. Since we want (2.17) to be satisfied for every $\varepsilon > 0$, we shall assume

$$\forall \ell \geq 0 \quad \int_\Gamma \big[F^\ell_{\mathrm{ext}}(t) + c(t) F^{\ell-1}_{\mathrm{ext}}(t)\big]\,\mathrm{d}t = 0 \qquad \text{(with the convention } F^{-1}_{\mathrm{ext}} = 0\text{)}. \quad (2.18)$$

Note that for analytic F_{ext}, relation (2.18) is a consequence of (2.17).

We now explain the construction of the first terms in the iterative procedure. Starting from the same Ansatz (2.5), we get again problems (2.6) (whose third line is replaced by the Neumann condition $\partial_n U^n_{\mathrm{ext}} = 0$) and (2.7). At step $n = 0$, $U^0_{\mathrm{ext}}(t,\cdot)$ solves a totally homogeneous one-dimensional Neumann problem, hence $U^0_{\mathrm{ext}}(t,S)$ is a function of the arc length t, denoted by $\beta_0(t)$ which cannot be determined at this stage.

For $n = 1$, we get (note that $\mathbf{A}_1 U^0_{\mathrm{ext}} = c(t)\partial_S \beta_0(t) = 0$)

$$\begin{cases} \partial^2_S U^1_{\mathrm{ext}} &= 0 & \text{for } 0 < S < 1, \\ \partial_S U^1_{\mathrm{ext}} &= \alpha \partial_n u^0_{\mathrm{int}} - g & \text{for } S = 0, \\ \partial_S U^1_{\mathrm{ext}} &= 0 & \text{for } S = 1, \end{cases}$$

which is solvable if $\alpha \partial_n u^0_{\mathrm{int}} = g$ on Γ. Thus, let u^0_{int} be solution of the Neumann problem: $\alpha \Delta u^0_{\mathrm{int}} = f_{\mathrm{int}}$ in Ω_{int} and $\alpha \partial_n u^0_{\mathrm{int}} = g$ on Γ (whose data satisfies the compatibility condition (1.11)). Then $\beta_0(t)$ is determined as $u^0_{\mathrm{int}}|_\Gamma$, thanks to the continuity condition across Γ.

Let us now present the general construction: Let us assume that the terms U^k_{ext} and u^k_{int} were built for $k < n$, satisfying the condition on Γ:

$$\forall t \in [0, \ell_\Gamma), \quad \alpha \partial_n u^{n-1}_{\mathrm{int}}(t) = \Phi_{n-1}(t) \qquad (\mathcal{H}_{n-1})$$

where Φ_{n-1} is defined as

$$\Phi_{n-1}(t) := g\,\delta^n_0 - \int_0^1 \Big(F^{n-2}_{\mathrm{ext}}(t) S^{n-2} - \sum_{\ell+p=n} \mathbf{A}_\ell U^p_{\mathrm{ext}}(t,S)\Big)\,\mathrm{d}S.$$

The construction of U^n_{ext} and u^n_{int} consists of three steps.

• *Step 1. Definition of U^n_{ext} up to a constant.* Thanks to assumption ($\mathcal{H}_{n-1}$), the problem

$$\begin{cases} \partial^2_S U^n_{\mathrm{ext}} &= F^{n-2}_{\mathrm{ext}}(t) S^{n-2} - \sum_{\ell+p=n} \mathbf{A}_\ell U^p_{\mathrm{ext}} & \text{for } 0 < S < 1, \\ \partial_S U^n_{\mathrm{ext}} &= \alpha \partial_n u^{n-1}_{\mathrm{int}} - g\,\delta^n_0 & \text{for } S = 0, \\ \partial_S U^n_{\mathrm{ext}} &= 0 & \text{for } S = 1 \end{cases} \quad (2.19)$$

satisfies the compatibility condition. Thus, U^n_{ext} can be determined up to a constant (of S) $\beta^n(t)$.

• *Step 2. Compatibility condition for* U^{n+1}_{ext} *and construction of* u^n_{int}. Let us consider problem (2.19) at rank $n+1$. The right-hand side

$$F^{n-1}_{\text{ext}}(t)S^{n-1} - \sum_{\ell+p=n+1} \mathsf{A}_\ell U^p_{\text{ext}}$$

is well defined since $\mathsf{A}_1\beta^n(t) = 0$ (remember $\mathsf{A}_1 = c(t)\partial_S$). The compatibility condition is nothing but ($\mathcal{H}_n$): It reads $\alpha\partial_n u^n_{\text{int}} = \Phi_n$.

If we insert the previous condition ($\mathcal{H}_n$) into the interior problem at rank n, we obtain

$$\left\{\begin{array}{rcll} \alpha\Delta u^n_{\text{int}} & = & f_{\text{int}}\delta^n_0 & \text{in } \Omega_{\text{int}}, \\ \alpha\partial_n u^n_{\text{int}} & = & \Phi_n & \text{on } \Gamma. \end{array}\right. \tag{2.20}$$

Therefore, we can uniquely determine u^n_{int} with the condition $\int_{\Omega_{\text{int}}} u^n_{\text{int}} = 0$, provided the compatibility condition for this Neumann problem is fulfilled:

Lemma 2.5 *The interior Neumann problem* (2.20) *is compatible.*

Proof: For $n = 0$, $\Phi_n = g$ and it directly follows from the compatibility condition for problem (P_ε), see (1.11). For $n \geq 1$, we must show that the integral of Φ_n over Γ vanishes. Thus, the condition to be satisfied is the following:

$$-\int_\Gamma \Phi_n(t)\,\mathrm{d}t = \int_\Gamma\int_0^1 \Big[F^{n-1}_{\text{ext}}(t)S^{n-1} - \sum_{\ell+p=n+1} \mathsf{A}_\ell U^p_{\text{ext}}(t,S)\Big]\,\mathrm{d}S\,\mathrm{d}t = 0. \tag{2.21}$$

In the sum, we isolate the term corresponding to $\ell = 1$ and $p = n$; integrating the first equation of (2.19), we obtain an expression for $\partial_S U^n_{\text{ext}}$ which can be used to obtain

$$\int_\Gamma\int_0^1 \mathsf{A}_1 U^n_{\text{ext}}(t,S)\,\mathrm{d}S\,\mathrm{d}t = \int_\Gamma\int_0^1\int_S^1 c(t)\Big[-F^{n-2}_{\text{ext}}(t)Y^{n-2} + \sum_{\ell+p=n} \mathsf{A}_\ell U^p_{\text{ext}}(t,Y)\Big]\,\mathrm{d}Y\,\mathrm{d}S\,\mathrm{d}t. \tag{2.22}$$

Inverting the integrals in S and Y yields

$$\int_\Gamma\int_0^1 \mathsf{A}_1 U^n_{\text{ext}}(t,S)\,\mathrm{d}S\,\mathrm{d}t = \int_\Gamma\int_0^1 \Big[-c(t)F^{n-2}_{\text{ext}}(t)Y^{n-1} + \sum_{\ell+p=n} Yc(t)\mathsf{A}_\ell U^p_{\text{ext}}(t,Y)\Big]\,\mathrm{d}Y\,\mathrm{d}t. \tag{2.23}$$

Using equality (2.18), we can deduce from (2.23) the compatibility condition (2.21) if

$$\sum_{\ell+p=n}\int_0^1\int_\Gamma \big[Sc(t)\mathsf{A}_\ell + \mathsf{A}_{\ell+1}\big]U^p_{\text{ext}}(t,S)\,\mathrm{d}t\,\mathrm{d}S = 0. \tag{2.24}$$

From (2.1) and (2.2), it follows that $B_\ell = Sc(t)\mathsf{A}_{\ell-1} + \mathsf{A}_\ell$ is nothing but the operator of rank ℓ in the formal expansion

$$T_\varepsilon := [1 + \varepsilon Sc(t)][\varepsilon^2\Delta_{\text{ext}} - \partial^2_S] - \varepsilon c(t)\partial_S = \sum_{\ell\geq 2}\varepsilon^\ell B_\ell.$$

But for any smooth function φ defined on Γ, (2.1) gives

$$\int_\Gamma T_\varepsilon\varphi(t)\,\mathrm{d}t = \varepsilon^2 \int_\Gamma \partial_t\left[(1+\varepsilon Sc(t))^{-1}\partial_t\varphi\right]\,\mathrm{d}t = 0,$$

since Γ is a closed curve. Therefore $\int_\Gamma B_\ell\varphi = 0$ for every $\ell \geq 2$ and every smooth function φ. This implies (2.24). ■

• *Step 3. Complete determination of* U^n_{ext}. The continuity requirement $U^n_{\mathrm{ext}} = u^n_{\mathrm{int}}$ determines $\beta^n(t) = u^n_{\mathrm{int}}|_\Gamma$.

We have just shown that the construction of the terms U^n_{ext} and u^n_{int} can be achieved by induction. We can obtain a similar result as Theorem 2.1:

Theorem 2.6 *Let* $f_{\mathrm{int}} \in \mathcal{C}^\infty(\overline{\Omega}_{\mathrm{int}})$, $f_{\mathrm{ext}} \in \mathcal{C}^\infty(\overline{\Omega}^{\varepsilon_0}_{\mathrm{ext}})$ *for an* $\varepsilon_0 > 0$, *and* $g \in \mathcal{C}^\infty(\Gamma)$ *satisfying the assumptions* (1.11). *The solution* u_ε *of* (P$_\varepsilon$) *with external Neumann b.c. determined by* $\int_{\Omega_{\mathrm{int}}} u_{\varepsilon,\mathrm{int}}\,\mathrm{d}x = 0$ *has a two-scale expansion which can be written for each* $N \in \mathbb{N}$ *in the form*

$$u_\varepsilon = \sum_{n=0}^{N} \varepsilon^n u^n + r_\varepsilon^{N+1}, \quad \textit{with } u^n|_{\Omega_{\mathrm{int}}} = u^n_{\mathrm{int}} \quad \textit{and} \quad u^n|_{\Omega^\varepsilon_{\mathrm{ext}}}(t,s) = U^n_{\mathrm{ext}}(t, \varepsilon^{-1}s).$$

The remainders satisfy, with a constant C_N *independent of* $\varepsilon \leq \varepsilon_0$:

$$\left\|r_\varepsilon^{N+1}\right\|_{1,\Omega_{\mathrm{int}}} + \sqrt{\varepsilon}\left\|r_\varepsilon^{N+1}\right\|_{1,\Omega^\varepsilon_{\mathrm{ext}}} \leq C_N\,\varepsilon^{N+1}. \tag{2.25}$$

Remark 2.7 For external Neumann boundary conditions we also have a statement like Proposition 2.3, with the following distinctive feature: If $f_{\mathrm{int}} \equiv 0$, $g \equiv 0$, and $f_{\mathrm{ext}} \neq 0$, the series (2.8) starts in general with εu^1 instead of $\varepsilon^2 u^2$ for external Dirichlet b.c., and more precisely, if $\partial_n^k f_{\mathrm{ext}}|_\Gamma \equiv 0$ for $k = 0, \ldots, \ell-1$ and $\partial_n^\ell f_{\mathrm{ext}}|_\Gamma \neq 0$, then (2.8) starts with $\varepsilon^{\ell+1}u^{\ell+1}$. □

2.3 Uniform a priori estimates

Since the transmission problem (P$_\varepsilon$) is elliptic, the solution u_ε has an optimal piecewise regularity depending on the regularity of the data and satisfies correspondingly a priori estimates. In fact, it is even possible to prove that such estimates are *uniform* with respect to ε. Using techniques of differential quotients like in [1] or [2] we prove in the appendix the following local estimates: We assume that Ω_{int} is a smooth domain or a corner domain as introduced in §1.1. We fix a point $A \in \Gamma$, $A \neq O$ if O is the corner of Ω_{int}. Let B_R be the ball of center A and radius R. We choose R small enough, so that in particular, $O \notin \overline{B}_R$. Let ρ be fixed, $0 < \rho < R$.

The following result applies both to Dirichlet and Neumann boundary conditions:

Theorem 2.8 *With the above assumption on* R *and* ρ, *let* $m \geq 1$ *be an integer. For* ε *small enough, we consider the solution* u_ε *of problem* (P$_\varepsilon$) *with a right-hand side satisfying the regularity assumptions* $f_{\mathrm{int}} \in \mathrm{H}^{m-1}(\Omega_{\mathrm{int}} \cap B_R)$, $f_{\mathrm{ext}} \in \mathrm{H}^{m-1}(\Omega^\varepsilon_{\mathrm{ext}} \cap B_R)$, *and* $g \in \mathrm{H}^{m-\frac{1}{2}}(\Gamma \cap B_R)$. *Then*

$$u_{\varepsilon,\mathrm{int}} \in \mathrm{H}^{m+1}(\Omega_{\mathrm{int}} \cap B_\rho) \quad \textit{and} \quad u_{\varepsilon,\mathrm{ext}} \in \mathrm{H}^{m+1}(\Omega^\varepsilon_{\mathrm{ext}} \cap B_\rho).$$

Moreover, there exists a constant C, independent of ε, f, and g such that

$$\left\|u_{\varepsilon,\mathrm{int}}\right\|_{m+1,\Omega_{\mathrm{int}}\cap B_\rho} + \left\|u_{\varepsilon,\mathrm{ext}}\right\|_{m+1,\Omega^\varepsilon_{\mathrm{ext}}\cap B_\rho} \;\leq\; C\Big[\left\|f_{\mathrm{int}}\right\|_{m-1,\Omega_{\mathrm{int}}\cap B_R} + \left\|f_{\mathrm{ext}}\right\|_{m-1,\Omega^\varepsilon_{\mathrm{ext}}\cap B_R} + \left\|g\right\|_{m-\frac12,\Gamma\cap B_R} + \left\|u_\varepsilon\right\|_{0,\Omega^\varepsilon\cap B_R}\Big]. \tag{2.26}$$

As a consequence, for a smooth domain Ω_{int} there holds the following global estimate for the solution $u_\varepsilon \in \mathrm{H}^1(\Omega^\varepsilon)$ of problem (P_ε) with a right-hand side satisfying the regularity assumptions $f_{\mathrm{int}} \in \mathrm{H}^{m-1}(\Omega_{\mathrm{int}})$, $f_{\mathrm{ext}} \in \mathrm{H}^{m-1}(\Omega^\varepsilon_{\mathrm{ext}})$, and $g \in \mathrm{H}^{m-\frac12}(\Gamma)$:

$$u_{\varepsilon,\mathrm{int}} \in \mathrm{H}^{m+1}(\Omega_{\mathrm{int}}) \quad \text{and} \quad u_{\varepsilon,\mathrm{ext}} \in \mathrm{H}^{m+1}(\Omega^\varepsilon_{\mathrm{ext}}).$$

Moreover, there exists a constant C independent of ε such that

$$\left\|u_{\varepsilon,\mathrm{int}}\right\|_{m+1,\Omega_{\mathrm{int}}} + \left\|u_{\varepsilon,\mathrm{ext}}\right\|_{m+1,\Omega^\varepsilon_{\mathrm{ext}}} \leq C\Big[\left\|f_{\mathrm{int}}\right\|_{m-1,\Omega_{\mathrm{int}}} + \left\|f_{\mathrm{ext}}\right\|_{m-1,\Omega^\varepsilon_{\mathrm{ext}}} + \left\|g\right\|_{m-\frac12,\Gamma} + \left\|u_\varepsilon\right\|_{0,\Omega^\varepsilon}\Big]. \tag{2.27}$$

For external Dirichlet b.c., one can remove the contribution $\left\|u_\varepsilon\right\|_{0,\Omega^\varepsilon}$ in the right hand side of (2.27).

When comparing (2.27) with the expansions given in Theorems 2.1 and 2.6, we can remark that uniform estimates are corroborated by the fact that the degree in $S = \frac{s}{\varepsilon}$ inside the exterior stretched part U^n_{ext} is less than n, see Proposition 2.3.

3 Corner singularities and profiles at infinity

From now on we consider the corner case. In this section, we prepare for the special treatment needed by the corner point O of Ω_{int}. The solution u_ε has singular parts, not only at O, but also at the external vertex O^ε. We refer to [17], [11], or [8] for singularities of elliptic boundary value problems and to [23] for interface problems.

Examining problems (2.6)-(2.7) and their solution via Proposition 2.3 we see that the singularities of problem (P_0) are of importance: The application of formula (2.13) presupposes that the traces of $\mathbf{h}^k u^\ell_{\mathrm{int}}$ on Γ are at least in $\mathrm{H}^{1/2}(\Gamma)$. Since the operator $\mathbf{h}^k$ is of degree k in general, u^ℓ_{int} should belong to $\mathrm{H}^{k+1}(\Omega_{\mathrm{int}})$. But the presence of singularities stops the regularity at the level of $\mathrm{H}^{1+\frac{\pi}{\omega}}$, in general.

We propose the following strategy in order to overcome this: We use the standard splitting of u^0_{int} into regular and singular parts, and *replace* the singular parts by *profiles* suitably constructed, so as to solve the whole transmission problem in a neighborhood of O.

3.1 Dirichlet and Neumann corner singularities

Before constructing and investigating these profiles, we describe the singularities of the interior problem (P_0), see [11]. We first introduce the following notations.

Definition 3.1 *(i) The set of singular exponents for the Dirichlet problem* (P_0) *is*

$$\mathfrak{S} = \left\{ \tfrac{q\pi}{\omega}\ ;\ q \in \mathbb{Z},\ q \neq 0 \right\}. \tag{3.1}$$

The singular function associated with the Dirichlet problem corresponding to $\lambda \in \mathfrak{S}$ *is*

$$\mathfrak{s}^\lambda = \begin{cases} r^\lambda \cos(\lambda\theta) & \text{if } \lambda = \frac{q\pi}{\omega} \text{ with } q \text{ odd}, \\ r^\lambda \sin(\lambda\theta) & \text{if } \lambda = \frac{q\pi}{\omega} \text{ with } q \text{ even}, \end{cases} \tag{3.2}$$

where (r, θ) *are polar coordinates centered in* O *such that the plane sector* $-\frac{\omega}{2} \leq \theta \leq \frac{\omega}{2}$ *coincides with* Ω_{int} *in a neighborhood of* O.
(ii) The set of singular exponents for the Neumann problem (2.20) *is* $\mathfrak{S} \cup \{0\}$. *The singular function associated with the Neumann problem corresponding to* $\lambda \in \mathfrak{S}$ *is*

$$\mathfrak{s}^\lambda = \begin{cases} r^\lambda \sin(\lambda\theta) & \text{if } \lambda = \frac{q\pi}{\omega} \text{ with } q \text{ odd}, \\ r^\lambda \cos(\lambda\theta) & \text{if } \lambda = \frac{q\pi}{\omega} \text{ with } q \text{ even}. \end{cases} \tag{3.3}$$

The singularity associated with $\lambda = 0$ *is* $\mathfrak{s}^0 = \log r$.
(iii) For any positive number K *let* $\mathfrak{S}(K)$ *denote the finite set* $\mathfrak{S} \cap (0, K)$.

We recall the result of splitting into singular and regular part of the solutions of the Dirichlet problem (P_0), in the situation where the data are *"flat"* in O, i.e. belong to some weighted spaces of Kondrat'ev type, see [17]:

Definition 3.2 *Let* $\gamma \in \mathbb{R}$ *and* $m \in \mathbb{N}$. *Let*

$$\mathrm{H}^m_\gamma(\Omega_{\text{int}}) = \{ v \in \mathrm{L}^2_{\text{loc}}(\Omega_{\text{int}})\ ;\ r^{\gamma+|\beta|}\partial^\beta v \in \mathrm{L}^2(\Omega_{\text{int}}),\ |\beta| \leq m \}.$$

We denote by $\mathrm{H}^{m-1/2}_{\gamma+1/2}(\Gamma)$ *the trace space of* $\mathrm{H}^m_\gamma(\Omega_{\text{int}})$. *Finally* H^∞_γ *is defined as* $\bigcap_{m\in\mathbb{N}} \mathrm{H}^m_\gamma$.

Theorem 3.3 *Let* $m \in \mathbb{N}$ *and* $K \geq 0$ *be a real number such that* $K \notin \mathfrak{S}$, *and let the data satisfy*

$$f_{\text{int}} \in \mathrm{H}^{m-1}_{-K+1}(\Omega_{\text{int}}) \quad \text{and} \quad h \in \mathrm{H}^{m+1/2}_{-K-1/2}(\Gamma).$$

Then the solution $u^0_{\text{int}} \in \mathrm{H}^1(\Omega_{\text{int}})$ *of the Dirichlet problem* (P_0) *admits the following decomposition:*

$$u^0_{\text{int}} = u^{0,K} + \chi \sum_{\lambda\in\mathfrak{S}(K)} c_\lambda\, \mathfrak{s}^\lambda \quad \text{with} \quad u^{0,K} \in \mathrm{H}^{m+1}_{-K-1}(\Omega_{\text{int}}) \quad \text{and} \quad c_\lambda \in \mathbb{R}. \tag{3.4}$$

Here χ *is a smooth cut-off function as introduced in Definition 1.1.*

Remark 3.4 *(i)* If $m \geq 1$, the regular part $u^{0,K}$ is a $\mathcal{O}(r^K)$.
(ii) For the Neumann problem there holds a similar decomposition like (3.4) with an extra constant term corresponding to $\lambda = 0$. In fact there are two "singular" functions associated with $\lambda = 0$, namely 1 and $\log r$. The latter does not belong to $\mathrm{H}^1(\Omega_{\text{int}})$. However, we will have to take it into account as far as singularities at infinity will be concerned. □

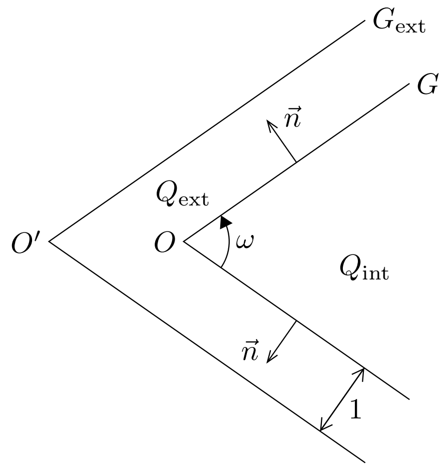

Figure 2: The infinite domain Q.

3.2 Introduction to the profile analysis

As already mentioned, the solution algorithm of Proposition 2.3 does not apply because of the singularities in the splitting (3.4). An essential ingredient to obtain an ε-expansion for problem (P_ε) in this case is the construction of *profiles* solving an associated problem on an infinite domain, see [6] or [7].

Focusing on the corner point O, we perform the dilatation $x \mapsto X = \frac{x}{\varepsilon}$. When ε goes to 0, the domain Ω^ε becomes an infinite sector Q (see Figure 2): Q consists of an interior plane sector Q_{int} of opening ω and of a straight layer Q_{ext} of thickness 1. Let G_{ext} be the external boundary of Q and G denote the common boundary of Q_{int} and Q_{ext}.

A standard feature of the singularities $\mathfrak{s}^\lambda$ is to solve the Dirichlet (or Neumann) problem on the sector Q_{int} of opening ω with zero data, and to be homogeneous of degree λ. The associated profiles $\mathfrak{K}^\lambda$ are solution of complete transmission problem (P_∞)

$$\begin{cases} \alpha\Delta\mathfrak{K}_{\text{int}} &= \mathfrak{f}_{\text{int}} \quad \text{in } Q_{\text{int}},\\ \Delta\mathfrak{K}_{\text{ext}} &= \mathfrak{f}_{\text{ext}} \quad \text{in } Q_{\text{ext}},\\ \mathfrak{K}_{\text{int}} - \mathfrak{K}_{\text{ext}} &= 0 \quad \text{on } G,\\ \alpha\partial_n\mathfrak{K}_{\text{int}} - \partial_n\mathfrak{K}_{\text{ext}} &= \mathfrak{g} \quad \text{on } G,\\ \text{external b.c.} & \quad\quad \text{on } G_{\text{ext}}, \end{cases} \tag{P_∞}$$

for zero data $\mathfrak{f}_{\text{int}}$, $\mathfrak{f}_{\text{ext}}$ and $\mathfrak{g}$. The external b.c. is of course $\mathfrak{K}_{\text{ext}} = 0$ for Dirichlet and $\partial_n\mathfrak{K}_{\text{ext}} = 0$ for Neumann. Moreover, $\mathfrak{K}^\lambda$ has to imitate $\mathfrak{s}^\lambda$ at infinity, namely

$$\mathfrak{K}^\lambda(X) - \mathfrak{s}^\lambda(X) = \mathcal{O}(R^\lambda), \quad R = |X| \to \infty. \tag{3.5}$$

In this §3, we prove the existence of $\mathfrak{K}^\lambda$ solving the homogeneous (P_∞) problem together with condition (3.5) for external Dirichlet and Neumann conditions. For each case, this requires three steps:

(i) An algorithmic part providing an asymptotic series $\breve{\mathfrak{K}}^\lambda$, solution of a model transmission problem ($\breve{\mathrm{P}}_\infty$) with zero data, see 3.3.2,

(ii) Truncating this asymptotic series solution, we define the function $\mathfrak{K}^\lambda$ on the infinite sector Q thanks to a variational formulation, see 3.3.3,

(iii) The expansion of the latter solution at infinity, see 3.4.

Throughout this section we use the following cut-off "at infinity":

Definition 3.5 *Let ρ_0 be the distance OO' between the internal and external corners of Q. Let ψ be a smooth cut-off function equal to 1 for $|X| \geq 2\rho_0$ and 0 for $|X| \leq \rho_0$.*

3.3 Existence of Dirichlet profiles

3.3.1 Variational formulation

We need a variational framework for problem (P_∞). Our variational space $\mathfrak{V}$ is defined as

$$\mathfrak{V} = \left\{ \mathfrak{v}\ ;\ \nabla\mathfrak{v} \in \mathrm{L}^2(Q),\ \frac{\mathfrak{v}}{\langle X\rangle} \in \mathrm{L}^2(Q) \text{ and } \mathfrak{v}|_{G_{\text{ext}}} = 0 \right\}, \tag{3.6}$$

endowed with the natural norm

$$\|\mathfrak{v}\|_{\mathfrak{V}}^2 = \|\nabla\mathfrak{v}\|_{0,Q}^2 + \|\langle X\rangle^{-1}\mathfrak{v}\|_{0,Q}^2,$$

where the weight is $\langle X\rangle := (|X|^2+1)^{1/2}$. This is a standard space for the solution of exterior problems, see [22]. The variational formulation is: Find $\mathfrak{u} \in \mathfrak{V}$ such that

$$\int_{Q_{\text{int}}} \nabla\mathfrak{u}_{\text{int}} \cdot \nabla\mathfrak{v}_{\text{int}}\,\mathrm{d}x + \int_{Q_{\text{ext}}} \nabla\mathfrak{u}_{\text{ext}} \cdot \nabla\mathfrak{v}_{\text{ext}}\,\mathrm{d}x = \int_{Q_{\text{int}}} \mathfrak{f}_{\text{int}}\mathfrak{v}_{\text{int}}\,\mathrm{d}x + \int_{Q_{\text{ext}}} \mathfrak{f}_{\text{ext}}\mathfrak{v}_{\text{ext}}\,\mathrm{d}x + \int_G \mathfrak{g}\mathfrak{v}\,\mathrm{d}\sigma, \quad \forall\mathfrak{v}\in\mathfrak{V}. \tag{3.7}$$

Proposition 3.6 *If $\langle X\rangle\,\mathfrak{f} \in \mathrm{L}^2(Q)$ and $\langle X\rangle^{\frac{1}{2}}\mathfrak{g} \in \mathrm{L}^2(G)$, then problem (P_∞) admits a unique solution $\mathfrak{v} \in \mathfrak{V}$.*

Proof: The bilinear form a associated with the variational formulation of (P_∞) is obviously continuous on $\mathfrak{V}$. For the ellipticity, we use the polar coordinates centered in O' (see Figure 2), denoted by (ρ,φ). Thanks to the Dirichlet conditions in G_{ext}, we can write a Poincaré inequality in the variable φ: There exists a constant C independent of ρ and $\mathfrak{v}$ such that

$$\int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} |\mathfrak{v}(\rho,\varphi)|^2\,\mathrm{d}\varphi \leq C^2 \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} |\partial_\varphi\mathfrak{v}(\rho,\varphi)|^2\,\mathrm{d}\varphi.$$

Multiplying this inequality by ρ^{-1} and integrating, we get $\|\rho^{-1}\mathfrak{v}\|_{0,Q} \leq C\|\nabla\mathfrak{v}\|_{0,Q}$, which gives the coercivity of the bilinear form on $\mathfrak{V}$.

The same technique shows that the prescribed conditions for $\mathfrak{f}$ and $\mathfrak{g}$ ensure the continuity of the linear form associated with the right-hand side. ∎

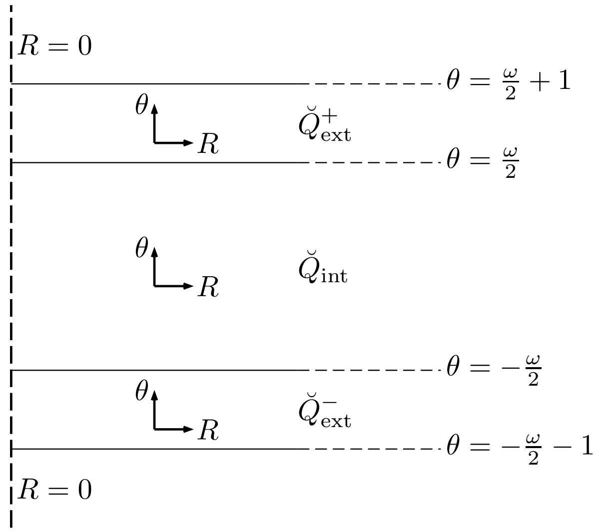

Figure 3: Definition of (R,θ) coordinates, after polar transformation in the interior domain.

3.3.2 Algorithmic construction of kernel elements

We recall that for any fixed $\lambda > 0$ in $\mathfrak{S}$, we are looking for a solution $\mathfrak{K}^\lambda$ of (P_∞) with $\mathfrak{f}_{\rm int} = \mathfrak{f}_{\rm ext} = \mathfrak{g} = 0$, behaving at infinity like $\mathfrak{s}^\lambda$. This is possible because $\mathfrak{s}^\lambda$ does not belong to $\mathfrak{V}$. We proceed by constructing a series of terms decreasing more and more at infinity, until they belong to the variational space $\mathfrak{V}$, which allows the determination of $\mathfrak{K}^\lambda$.

The first step involves an algorithmic construction in singular function spaces. It is more canonical to define these spaces on a new domain $\breve{Q}$ instead of Q, see Figure 3:

Definition 3.7 *In $Q_{\rm int}$, we denote by (R,θ) the polar coordinates centered in O. Thus, considering (R,θ) as new variables $Q_{\rm int}$ is transformed into*

$$\breve{Q}_{\rm int} = \{(R,\theta);\ R>0,\ \theta\in(-\tfrac{\omega}{2},\tfrac{\omega}{2})\},$$

and G becomes

$$\breve{G} = \{(R,\theta);\ R>0,\ \theta = \pm\tfrac{\omega}{2}\}.$$

We consider the exterior layer(s) $\breve{Q}_{\rm ext} = \breve{Q}^+_{\rm ext}\cup\breve{Q}^-_{\rm ext}$ around $\breve{Q}_{\rm int}$

$$\breve{Q}^+_{\rm ext} = \{(R,\theta);\ R>0,\ \theta\in(\tfrac{\omega}{2},\tfrac{\omega}{2}+1)\} \quad \textit{and} \quad \breve{Q}^-_{\rm ext} = \{(R,\theta);\ R>0,\ \theta\in(-1-\tfrac{\omega}{2},-\tfrac{\omega}{2})\}.$$

Thus, in the exterior layer, R and θ are the tangential and normal coordinates. For $\lambda\in\mathbb{R}$, we set

$$\begin{aligned} S^\lambda(\breve{Q}_{\rm int}) &= \Big\{\sum_{\ell\ge 0,\ \rm finite} R^\lambda \log^\ell R\, v_\ell(\theta)\ ;\ v_\ell\in\mathcal{C}^\infty[-\tfrac{\omega}{2},\tfrac{\omega}{2}]\Big\},\\ S^\lambda(\breve{G}) &= \Big\{\sum_{\ell\ge 0,\ \rm finite} c^\pm_\ell R^\lambda \log^\ell R\ \text{ for }\ \theta=\pm\tfrac{\omega}{2}\ ;\ c^+_\ell, c^-_\ell\in\mathbb{R}\Big\},\\ S^\lambda(\breve{Q}_{\rm ext}) &= \Big\{\sum_{\ell\ge 0,\ \rm finite} \theta^\ell\,\varphi_\ell(R)\ ;\ \varphi_\ell\in S^\lambda(\breve{G})\Big\}. \end{aligned} \tag{3.8}$$

Let $\breve{Q}$ the union of $\breve{Q}_{\rm int}$, $\breve{G}$, and $\breve{Q}_{\rm ext}$. We denote by $S^\lambda(\breve{Q})$ the space of functions, continuous inside $\breve{Q}$ and whose restrictions to $\breve{Q}_{\rm int}$ and $\breve{Q}_{\rm ext}$ belong to $S^\lambda(\breve{Q}_{\rm int})$ and $S^\lambda(\breve{Q}_{\rm ext})$, respectively.

It is important to note that θ does not represent any more an angular variable in $\breve{Q}_{\rm ext}$. Rather, (R,θ) are cartesian coordinates. The change of variables defined on $\breve{Q}^+_{\rm ext}$ by

$$(R,\theta) \longmapsto X = (R\cos\tfrac{\omega}{2}, R\sin\tfrac{\omega}{2}) + (\theta - \tfrac{\omega}{2})(-\sin\tfrac{\omega}{2}, \cos\tfrac{\omega}{2})$$

and accordingly on $\breve{Q}^-_{\rm ext}$, maps $\breve{Q}_{\rm ext}$ either onto a subset of $Q_{\rm ext}$ (if $\omega < \pi$) or a superset of $Q_{\rm ext}$ (if $\omega > \pi$). Nevertheless, inside the support of ψ, cf. Definition 3.5, this correspondence is one to one. This is the reason why we can introduce:

Definition 3.8 *We assume that the cut-off $\psi = \psi(R)$ in Definition 3.5 does not depend on θ. For $\lambda \in \mathbb{R}$, let $S^\lambda(Q)$ be defined as the space of functions $\mathfrak{u}$ such that*

$$\exists\, \breve{\mathfrak{u}} \in S^\lambda(\breve{Q}), \quad \mathfrak{u}(X) = \psi(R)\, \breve{\mathfrak{u}}(R,\theta).$$

A direct consequence of the definition is:

Lemma 3.9 *For any $\lambda < 0$, the space $S^\lambda(Q)$ is contained in the variational space $\mathfrak{V}$.*

The problem in $\breve{Q}$ corresponding to problem (P_∞) can be written as

$$\left\{\begin{array}{rcll} \alpha\Delta_X \breve{\mathfrak{K}}_{\rm int} &=& \breve{\mathfrak{f}}_{\rm int} & \text{in } \breve{Q}_{\rm int},\\ (\partial_\theta^2 + \partial_R^2)\breve{\mathfrak{K}}_{\rm ext} &=& \breve{\mathfrak{f}}_{\rm ext} & \text{in } \breve{Q}_{\rm ext},\\ \breve{\mathfrak{K}}_{\rm int} - \breve{\mathfrak{K}}_{\rm ext} &=& 0 & \text{on } \breve{G},\\ \frac{\alpha}{R}\partial_\theta \breve{\mathfrak{K}}_{\rm int} - \partial_\theta \breve{\mathfrak{K}}_{\rm ext} &=& \breve{\mathfrak{g}} & \text{on } \breve{G},\\ \breve{\mathfrak{K}}_{\rm ext} &=& 0 & \text{on } \theta = \pm(\frac{\omega}{2}+1), \end{array}\right. \tag{$\breve{\mathrm{P}}_\infty$}$$

Problem $(\breve{\mathrm{P}}_\infty)$ can be solved in the sense of "asymptotic series at infinity":

Proposition 3.10 *Let $\lambda \in \mathfrak{S}$. Let $\mathfrak{s}_0^\lambda$ denote the extension of the singularity $\mathfrak{s}^\lambda$ in* (3.2) *by 0 on $\breve{Q}_{\rm ext}$. The function $\mathfrak{s}_0^\lambda$ belongs to $S^\lambda(\breve{Q})$. We initialize the series $\breve{\mathfrak{K}}^{\lambda,\mu}$ for $\mu = \lambda+2, \lambda+1$, and λ by setting*

$$\breve{\mathfrak{K}}^{\lambda,\lambda+2} = \breve{\mathfrak{K}}^{\lambda,\lambda+1} = 0 \quad \textit{and} \quad \breve{\mathfrak{K}}^{\lambda,\lambda} = \mathfrak{s}_0^\lambda.$$

Then there exists $\breve{\mathfrak{K}}^{\lambda,\lambda-\ell} \in S^{\lambda-\ell}(\breve{Q})$, $\ell = 1,2,\dots$, satisfying the following sequence of equations:

$$\left\{\begin{array}{rcll} \partial_\theta^2 \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_{\rm ext} &=& -\partial_R^2 \breve{\mathfrak{K}}^{\lambda,\lambda-\ell+2}_{\rm ext} & \theta \in \pm(\frac{\omega}{2}, \frac{\omega}{2}+1),\\ \partial_\theta \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_{\rm ext} &=& \frac{\alpha}{R}\partial_\theta \breve{\mathfrak{K}}^{\lambda,\lambda-\ell+1}_{\rm int} & \theta = \pm\frac{\omega}{2},\\ \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_{\rm ext} &=& 0 & \theta = \pm\frac{\omega}{2} \pm 1, \end{array}\right. \tag{3.9}$$

$$\left\{\begin{array}{rcll} \Delta \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_{\rm int} &=& 0 & \textit{in } Q_{\rm int},\\ \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_{\rm int} &=& \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_{\rm ext} & \textit{for } \theta = \pm\frac{\omega}{2}, \end{array}\right. \tag{3.10}$$

for all $\ell \geq 0$. The degree in θ of $\breve{\mathfrak{K}}^{\lambda,\lambda-\ell}$ in $\breve{Q}_{\rm ext}$ is equal to ℓ. For each integer $p \geq 0$ the partial sum $\sum_{\ell=0}^p \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}$ solves $(\breve{\mathrm{P}}_\infty)$ for

$$\breve{\mathfrak{f}}_{\rm int} = 0, \quad \breve{\mathfrak{f}}_{\rm ext} = -\partial_R^2\left[\breve{\mathfrak{K}}^{\lambda,\lambda-p}_{\rm ext} + \breve{\mathfrak{K}}^{\lambda,\lambda-p+1}_{\rm ext}\right], \quad \breve{\mathfrak{g}} = -\alpha\partial_n \breve{\mathfrak{K}}^{\lambda,\lambda-p}_{\rm int}. \tag{3.11}$$

Proof: The terms $\breve{\mathfrak{K}}^{\lambda,\lambda-\ell}$ are built by induction. For $\ell = 0$, the algorithm is initialized with $\breve{\mathfrak{K}}^{\lambda,\lambda}_{\rm ext} = 0$ and $\breve{\mathfrak{K}}^{\lambda,\lambda}_{\rm int} = \mathfrak{s}^\lambda$ solving the homogeneous Dirichlet problem in $Q_{\rm int}$. Then we solve alternatively problems (3.9) and (3.10): If $\breve{\mathfrak{K}}^{\lambda,\lambda-n}$ are constructed for $n = 0,\dots,\ell-1$, the exterior problem (3.9) is a one-dimensional Sturm-Liouville problem with parameter R and we check that it has a unique solution in $S^{\lambda-\ell}(\breve{Q}_{\rm ext})$, whereas the interior Dirichlet problem (3.10) with boundary data from the trace space $S^{\lambda-\ell}(\breve{G})$ of $S^{\lambda-\ell}(\breve{Q}_{\rm ext})$ has a solution in $S^{\lambda-\ell}(\breve{Q}_{\rm int})$, cf. [8, Ch.4]. Then (3.11) is an easy consequence of equations (3.9) and (3.10). ■

Remark 3.11 Since the terms in (3.11) are $\mathcal{O}(R^{\lambda-p-1})$ as $R \to \infty$, we may say that the series

$$\breve{\mathfrak{K}}^\lambda := \sum\nolimits_{\ell\geq 0} \breve{\mathfrak{K}}^{\lambda,\lambda-\ell} \tag{3.12}$$

solves $(\breve{\rm P}_\infty)$ with $\breve{\mathfrak{f}} = \breve{\mathfrak{g}} = 0$ in the sense of “asymptotic series at infinity”. □

Remark 3.12 *(i)* If $\pi/\omega \notin \mathbb{Q}$, the terms $\breve{\mathfrak{K}}^{\lambda,\lambda-\ell}$, $\ell \geq 1$, are unique in $S^{\lambda-\ell}(\breve{Q})$ since $\lambda-\ell \notin \mathfrak{S}$, and as a consequence the kernel of the Dirichlet problem (3.10) in $S^{\lambda-\ell}(\breve{Q})$ is reduced to zero. Moreover, $\breve{\mathfrak{K}}^{\lambda,\lambda-\ell}$ contains no logarithmic term $\log R$.

(ii) If $\pi/\omega \in \mathbb{Q}$, for each ℓ such that $\lambda-\ell \in \mathfrak{S}$, a resonance phenomenon may occur, exciting a logarithmic singularity (the degree of $\breve{\mathfrak{K}}^{\lambda,\lambda-\ell}$ as a polynomial in $\log R$ is at most ℓ). In that case, the asymptotic series $\breve{\mathfrak{K}}^\lambda$ contains arbitrary choices. Any other asymptotic series $\breve{\mathfrak{K}}^\lambda_* = \sum_\ell \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_*$ satisfying the sequence of equations in Proposition 3.10 can be compared to the specified one. There exist coefficients (γ^λ_ν) for each $\nu = \lambda-\ell \in \mathfrak{S}$, $\ell \geq 1$, such that

$$\breve{\mathfrak{K}}^\lambda_* = \breve{\mathfrak{K}}^\lambda + \sum\nolimits_{\nu=\lambda-\ell\in\mathfrak{S}} \gamma^\lambda_\nu \breve{\mathfrak{K}}^\nu.$$

□

3.3.3 Effective construction of profiles

Using the asymptotic series $\sum \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}$, we are able to construct genuine solutions for problem $(\rm P_\infty)$ with zero right hand side and asymptotics (3.5) at infinity:

Theorem 3.13 *Let $\lambda \in \mathfrak{S}$, $\lambda > 0$, and let p_λ denote the smallest integer p such that*

$$\lambda - \tfrac{1}{2} < p. \tag{3.13}$$

Recall that ψ is the cut-off function from Definition 3.5. There exists $\mathfrak{u}^{\lambda,p_\lambda}$ in the variational space $\mathfrak{V}$ such that

$$\mathfrak{K}^\lambda := \psi \sum_{\ell=0}^{p_\lambda} \breve{\mathfrak{K}}^{\lambda,\lambda-\ell} + \mathfrak{u}^{\lambda,p_\lambda} \tag{3.14}$$

defines a solution $\mathfrak{K}^\lambda$ of problem $(\rm P_\infty)$ for $\mathfrak{f} = \mathfrak{g} = 0$, such that $\mathfrak{K}^\lambda_{\rm int} \sim \mathfrak{s}^\lambda$ as $R \to \infty$. Moreover for any integer $p \geq p_\lambda$, the function $\mathfrak{u}^{\lambda,p}$ defined as $\mathfrak{K}^\lambda - \psi\sum_{\ell=0}^p \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}$ also belongs to $\mathfrak{V}$:

$$\forall p \geq p_\lambda, \quad \mathfrak{K}^\lambda = \psi \sum_{\ell=0}^{p} \breve{\mathfrak{K}}^{\lambda,\lambda-\ell} + \mathfrak{u}^{\lambda,p}, \quad \textit{with } \mathfrak{u}^{\lambda,p} \in \mathfrak{V}. \tag{3.15}$$

Proof: For any integer q, we define $\mathfrak{v}^{\lambda,q}$ as the sum $-\psi\sum_{\ell=0}^{q}\breve{\mathfrak{K}}^{\lambda,\lambda-\ell}$. By construction, the function $\mathfrak{v}^{\lambda,q}$ solves problem (P_∞) with, compare with (3.15):

$$\mathfrak{f}_{\mathrm{int}} = \varphi_{\mathrm{int}}, \quad \mathfrak{f}_{\mathrm{ext}} = \varphi_{\mathrm{ext}} - \psi\partial_R^2\left[\breve{\mathfrak{K}}_{\mathrm{ext}}^{\lambda,\lambda-q} + \breve{\mathfrak{K}}_{\mathrm{ext}}^{\lambda,\lambda-q+1}\right], \quad \mathfrak{g} = -\psi\tfrac{\alpha}{R}\partial_\theta\breve{\mathfrak{K}}_{\mathrm{int}}^{\lambda,\lambda-q} \tag{3.16}$$

where φ comes from the cut-off: Its support is contained in $\mathrm{supp}(\nabla\psi)$. For q large enough, i.e. $q > \lambda + \frac{1}{2}$, the above right-hand sides satisfy the assumptions of Proposition 3.6. As a consequence, there exists $\mathfrak{u}^{\lambda,q} \in \mathfrak{V}$, solving the same problem as $\mathfrak{v}^{\lambda,q}$. Then $\mathfrak{K}^\lambda = \psi\sum_{\ell=0}^{q}\breve{\mathfrak{K}}^{\lambda,\lambda-\ell} + \mathfrak{u}^{\lambda,q}$ solves problem (P_∞) with $\mathfrak{f} = \mathfrak{g} = 0$. Finally the statement concerning $\mathfrak{u}^{\lambda,p}$ for $p = p_\lambda, p_\lambda+1,\ldots$ follows directly from Lemma 3.9. ∎

3.4 Expansion at infinity of the Dirichlet profiles

Equality (3.14) provides the expansion of $\mathfrak{K}^\lambda$ up to $\mathcal{O}(1)$ as $R\to\infty$. But we need to know the expansion of $\mathfrak{K}^\lambda$ at any order $\mathcal{O}(r^{-P})$ for the construction of the expansion of the solution of problem (P_ε) in Section 4. The theorem below provides the complete expansion of $\mathfrak{K}^\lambda$. For this, the introduction of several sets of indices is useful:

Definition 3.14 *Let $\mathfrak{Q}^-$ be the set of negative exponents defined as*

$$\mathfrak{Q}^- = \left\{-\tfrac{h\pi}{\omega} - q\ ;\ h,q\in\mathbb{N}\ \textit{with}\ h\geq 1,\ q\geq 0\right\}. \tag{3.17}$$

For any $\lambda > 0$ we introduce the infinite set of exponents depending on λ:

$$\mathfrak{Q}^\lambda = \mathfrak{Q}^- \cup \left\{\lambda-1, \lambda-2, \ldots, \lambda-\ell, \ldots\right\} \tag{3.18}$$

and for any number $P > 0$ the finite set $\mathfrak{Q}^\lambda(P) = \mathfrak{Q}^\lambda \cap [-P,\lambda)$.

Theorem 3.15 *Let $\lambda\in\mathfrak{S}$, $\lambda > 0$, and $P > 0$.*
(i) The solution $\mathfrak{K}^\lambda$ of problem (P_∞) introduced in (3.14) has the following expansion at infinity:

$$\forall P > 0, \quad \mathfrak{K}^\lambda = s_0^\lambda + \sum_{\mu\in\mathfrak{Q}^\lambda(P)} \mathfrak{K}^{\lambda,\mu} + \mathcal{O}(R^{-P}), \quad R\to\infty. \tag{3.19}$$

where for any $\mu\in\mathfrak{Q}^\lambda$ the function $\mathfrak{K}^{\lambda,\mu}$ belongs to the space $S^\mu(Q)$ cf. Definition 3.8. The degree of $\mathfrak{K}_{\mathrm{ext}}^{\lambda,\mu}$ as a polynomial in $\theta\in\pm(\frac{\omega}{2},\frac{\omega}{2}+1)$ is less than $\lambda-\mu$. Moreover, one can take derivatives of expansion (3.19), still having estimates on the remainder, see (3.30).
(ii) More precisely, we have the identity between asymptotic series:

$$s_0^\lambda + \sum_{\mu\in\mathfrak{Q}^\lambda}\mathfrak{K}^{\lambda,\mu} \ = \ \breve{\mathfrak{K}}^\lambda + \sum_{\nu=-\frac{h\pi}{\omega}<0} c_\nu^\lambda\breve{\mathfrak{K}}^\nu, \tag{3.20}$$

with the $\breve{\mathfrak{K}}^\nu$ defined by (3.12), and c_ν^λ are real coefficients, characteristic for the domain Q.

The proof of this theorem requires regularity results for the variational terms $\mathfrak{u}^{\lambda,p}$ and uses the Mellin transform. It is performed in the next Sections 3.4.1 and 3.4.2.

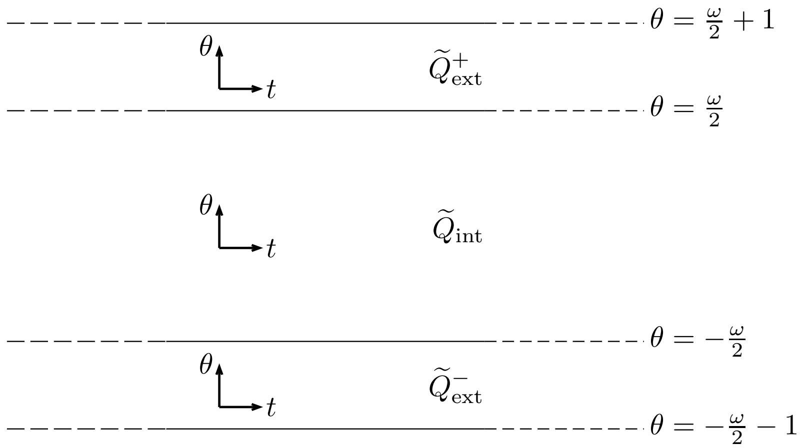

Figure 4: The strip $\widetilde{Q}$.

3.4.1 Regularity of the variational terms in weighted spaces

We are going to study the regularity of the variational terms $\mathfrak{u}^{\lambda,p}$, cf. (3.15), in a scale of weighted Sobolev spaces, as is usual for corner problems, see [17]. Rather than in the sector Q, we work in the strip $\widetilde{Q}$ obtained from $\breve{Q}$ by the change of variable $\mathbb{R}^+ \ni R \mapsto t = \log R \in \mathbb{R}$, see Figure 4.

Let us now introduce the scales of weighted spaces.

Definition 3.16 *(i) Let m be a non-negative integer and γ a real number. The space $\mathrm{K}^m_\gamma(\widetilde{Q}_{\text{int}})$ is defined by*

$$\mathrm{K}^m_\gamma(\widetilde{Q}_{\text{int}}) = \big\{\widetilde{\mathfrak{v}}\ ;\ e^{\gamma t}\,\widetilde{\mathfrak{v}} \in \mathrm{H}^m(\widetilde{Q}_{\text{int}})\big\},$$

endowed with the natural norm $\big\|\widetilde{\mathfrak{v}}\big\|_{\mathrm{K}^m_\gamma(\widetilde{Q}_{\text{int}})} = \big\|e^{\gamma t}\,\widetilde{\mathfrak{v}}\big\|_{m,\widetilde{Q}_{\text{int}}}$. We define similarly

$$\mathrm{K}^m_\gamma(\widetilde{Q}_{\text{ext}}) = \big\{\widetilde{\mathfrak{v}}\ ;\ e^{\gamma t}\,\widetilde{\mathfrak{v}} \in \mathrm{H}^m(\widetilde{Q}_{\text{ext}})\big\} \quad \textit{and} \quad \mathrm{K}^{m-\frac12}_\gamma(\widetilde{G}) = \big\{\widetilde{\mathfrak{v}}\ ;\ e^{\gamma t}\,\widetilde{\mathfrak{v}} \in \mathrm{H}^{m-\frac12}(\widetilde{G})\big\}.$$

(ii) We set $\mathrm{K}^0_{\gamma,\gamma-\frac12}(\widetilde{Q}) = \big\{\widetilde{\mathfrak{v}}\ ;\ \widetilde{\mathfrak{v}}_{\text{int}} \in \mathrm{K}^0_\gamma(\widetilde{Q}_{\text{int}}),\ \ \widetilde{\mathfrak{v}}_{\text{ext}} \in \mathrm{K}^0_{\gamma-\frac12}(\widetilde{Q}_{\text{ext}})\big\}$, and for $m \geq 1$

$$\mathrm{K}^m_{\gamma,\gamma-\frac12}(\widetilde{Q}) = \Big\{\widetilde{\mathfrak{v}}\ ;\ \widetilde{\mathfrak{v}}_{\text{int}} \in \mathrm{K}^m_\gamma(\widetilde{Q}_{\text{int}}),\ \ \widetilde{\mathfrak{v}}_{\text{ext}} \in \mathrm{K}^m_{\gamma-\frac12}(\widetilde{Q}_{\text{ext}})\ \ \textit{and}\ \ \widetilde{\mathfrak{v}}_{\text{int}} = \widetilde{\mathfrak{v}}_{\text{ext}}\ \textit{on}\ \widetilde{G}\Big\}. \tag{3.21}$$

Last, we denote by $\mathrm{K}^{m-\frac12}_{\gamma,\gamma-\frac12}(\widetilde{G})$ the space of traces of $\mathrm{K}^m_{\gamma,\gamma-\frac12}(\widetilde{Q})$ on the interface $\widetilde{G}$.

Remark 3.17 *(i)* The above definitions are inspired by Kondrat'ev spaces, see [17]. Namely, $\mathrm{K}^m_\gamma(\widetilde{Q}_{\text{int}})$ is the image of $\mathrm{H}^m_{\gamma-1}(Q_{\text{int}})$, see Definition 3.2, by the change of variables $X \mapsto (t,\theta)$.

(ii) If $\langle X\rangle^{-1}\mathfrak{u} \in \mathrm{L}^2(Q)$ (and in particular, if $\mathfrak{u} \in \mathfrak{V}$), then $(t,\theta) \mapsto \psi\mathfrak{u}$ belongs to $\mathrm{K}^0_{0,-1/2}(\widetilde{Q})$.

(iii) The natural trace spaces on $\widetilde{G}$ of the spaces $\mathrm{K}^m_\gamma(\widetilde{Q}_{\text{int}})$ and $\mathrm{K}^m_{\gamma-1/2}(\widetilde{Q}_{\text{ext}})$ do not coincide. Thus the transmission condition $\widetilde{\mathfrak{v}}_{\text{int}} = \widetilde{\mathfrak{v}}_{\text{ext}}$ enriches the topology of the space (3.21). □

Using the elliptic regularity away from the corner (see Theorem 2.8), we can prove the following "shift theorem". Note in the following result that more regularity is required for $\widetilde{\mathfrak{f}}_{\text{ext}}$ than for $\widetilde{\mathfrak{f}}_{\text{int}}$ due to the inhomogeneity of the operator in the strips.

Theorem 3.18 *Let $\breve{\mathfrak{u}}$ be solution of problem $(\breve{\mathrm{P}}_\infty)$ with data $\breve{\mathfrak{f}}$ and $\breve{\mathfrak{g}}$. Let $\widetilde{\mathfrak{u}}$, $\tilde{\mathfrak{f}}$, and $\tilde{\mathfrak{g}}$ denote their transforms on $\widetilde{Q}$. We assume the following on the data for some integer $m \geq 2$ and $\gamma \in \mathbb{R}$:*

$$\tilde{\mathfrak{f}}_{\text{int}} \in \mathrm{K}^{m-2}_{\gamma+2}(\widetilde{Q}_{\text{int}} \cap [t>0]), \quad \tilde{\mathfrak{f}}_{\text{ext}} \in \mathrm{K}^{m-2}_{\gamma+m-\frac{1}{2}}(\widetilde{Q}_{\text{ext}} \cap [t>0]), \quad \tilde{\mathfrak{g}} \in \mathrm{K}^{m-\frac{3}{2}}_{\gamma+1}(\widetilde{G} \cap [t>0]).$$

If $\widetilde{\mathfrak{u}}$ belongs to $\mathrm{K}^0_{\gamma,\gamma-\frac{1}{2}}(\widetilde{Q} \cap [t>0])$, then it also belongs to $\mathrm{K}^m_{\gamma,\gamma-\frac{1}{2}}(\widetilde{Q} \cap [t>\eta])$ for all $\eta > 0$.

Proof: In the variables (t, θ), the Laplace operators present in the first two equations of $(\breve{\mathrm{P}}_\infty)$ become

$$T_{\text{int}} = e^{-2t}[\partial_t^2 + \partial_\theta^2] \quad \text{and} \quad T_{\text{ext}} = e^{-2t}\left[\partial_t^2 - \partial_t + e^{2t}\partial_\theta^2\right].$$

Let us fix the real number $\eta > 0$ and consider for some arbitrary $t_0 > 0$ the rectangle $\mathcal{R} := \widetilde{Q} \cap [t_0 + \eta < t < t_0 + 2\eta]$. On such a rectangle, the non-principal parts of the above operators can be neglected and the variable coefficients can be frozen in t_0. Finally we use the following dilatation of the exterior strips:

$$s = \pm\tfrac{\omega}{2} + e^{-t_0}(\theta \mp \tfrac{\omega}{2}) \ \text{ in } \ \widetilde{Q}^{\pm}_{\text{ext}}.$$

As a consequence, the domain $\mathcal{R}$ becomes a rectangle with layers of thickness $\varepsilon = e^{-t_0}$ and the considered operators can be written as

$$T^\varepsilon_{\text{int}} = e^{-2t_0}[\partial_t^2 + \partial_\theta^2] \quad \text{and} \quad T^\varepsilon_{\text{ext}} = e^{-2t_0}\left[\partial_t^2 + \partial_s^2\right],$$

which are nothing but the Laplace operator (multiplied by a constant). Moreover the transmission condition on $\widetilde{G}$ becomes

$$e^{-t_0}\alpha\partial_\theta\widetilde{\mathfrak{u}}_{\text{int}} - e^{-t_0}\partial_s\widetilde{\mathfrak{u}}_{\text{ext}} = \tilde{\mathfrak{g}}.$$

This is the same as in (P_ε), since ∂_θ and ∂_s are the normal derivatives along the transmission boundary. Using Theorem 2.8 and going back to the variables (t, θ), we obtain the estimate, with C independent of t_0 – in the following the derivation multi-indices with respect to the variables t and θ are denoted by $\beta = (\beta_t, \beta_\theta)$:

$$\begin{aligned} &\left\|\widetilde{\mathfrak{u}}_{\text{int}}\right\|_{m,\mathcal{R}_{\text{int}}} + \Big(\sum_{|\beta| \leq m} e^{2\beta_\theta t_0 - t_0}\left\|\partial^\beta\widetilde{\mathfrak{u}}_{\text{ext}}\right\|^2_{0,\mathcal{R}_{\text{ext}}}\Big)^{\frac{1}{2}} \leq \\ &\qquad C\Bigg[e^{2t_0}\left\|\tilde{\mathfrak{f}}_{\text{int}}\right\|_{m-2,\widehat{\mathcal{R}}_{\text{int}}} + e^{2t_0}\Big(\sum_{|\beta| \leq m-2} e^{2\beta_\theta t_0 - t_0}\left\|\partial^\beta\tilde{\mathfrak{f}}_{\text{ext}}\right\|^2_{0,\widehat{\mathcal{R}}_{\text{ext}}}\Big)^{\frac{1}{2}} \\ &\qquad\qquad + e^{t_0}\left\|\tilde{\mathfrak{g}}\right\|_{m-\frac{3}{2},\widehat{\Gamma}} + \left\|\widetilde{\mathfrak{u}}_{\text{int}}\right\|_{0,\widehat{\mathcal{R}}_{\text{int}}} + e^{-t_0/2}\left\|\widetilde{\mathfrak{u}}_{\text{ext}}\right\|_{0,\widehat{\mathcal{R}}_{\text{ext}}}\Bigg], \end{aligned} \tag{3.22}$$

where $\widehat{\mathcal{R}}$ is the rectangle $\widetilde{Q} \cap [t_0 < t < t_0 + 3\eta]$, $\widehat{\Gamma}$ its boundary along $\widetilde{G}$. If we multiply inequality (3.22) by $e^{\gamma t_0}$ and use $0 \leq \beta_\theta \leq m$, we get

$$\begin{aligned} &e^{\gamma t_0}\left\|\widetilde{\mathfrak{u}}_{\text{int}}\right\|_{m,\mathcal{R}_{\text{int}}} + e^{(\gamma-\frac{1}{2})t_0}\left\|\widetilde{\mathfrak{u}}_{\text{ext}}\right\|_{m,\mathcal{R}_{\text{ext}}} \leq \\ &\qquad C\Bigg[e^{(2+\gamma)t_0}\left\|\tilde{\mathfrak{f}}_{\text{int}}\right\|_{m-2,\widehat{\mathcal{R}}_{\text{int}}} + e^{(\gamma+m-\frac{1}{2})t_0}\left\|\tilde{\mathfrak{f}}_{\text{ext}}\right\|_{m-2,\widehat{\mathcal{R}}_{\text{ext}}} + e^{(1+\gamma)t_0}\left\|\tilde{\mathfrak{g}}\right\|_{m-\frac{3}{2},\widehat{\Gamma}} \\ &\qquad\qquad + e^{\gamma t_0}\left\|\widetilde{\mathfrak{u}}_{\text{int}}\right\|_{0,\widehat{\mathcal{R}}_{\text{int}}} + e^{(\gamma-\frac{1}{2})t_0}\left\|\widetilde{\mathfrak{u}}_{\text{ext}}\right\|_{0,\widehat{\mathcal{R}}_{\text{ext}}}\Bigg]. \end{aligned}$$

Since $t \sim t_0$ in the rectangles, we can replace the norms $e^{\delta t_0}\|\mathfrak{v}\|_s$ by $\|e^{\delta t}\mathfrak{v}\|_s$. Summing up all these inequalities for $t_0 \in \eta\mathbb{N}^*$, we get the result. ∎

As a consequence there holds the following result on the regularity of the variational term.

Proposition 3.19 *Let p be an integer, $p \geq p_\lambda$, and let $\widetilde{\mathfrak{u}}^{\lambda,p}$ denote the "variational" function $\psi\mathfrak{u}^{\lambda,p}$, see* (3.15)*, in the variables (t,θ), $t \in \mathbb{R}$, and $\theta \in (-1-\frac{\omega}{2}, \frac{\omega}{2}+1)$. For every integer $m \geq 0$, we have*

$$\widetilde{\mathfrak{u}}^{\lambda,p} \in \mathrm{K}^m_{0,-\frac{1}{2}}(\widetilde{Q}). \tag{3.23}$$

Proof: We apply Theorem 3.18 for $\gamma = 0$. Since $\mathfrak{u}^{\lambda,p} \in \mathfrak{V}$, we have $\widetilde{\mathfrak{u}}^{\lambda,p} \in \mathrm{K}^0_{0,-1/2}(\widetilde{Q})$, cf. Remark 3.17 (ii). It remains to check the assumptions on the right-hand side, which is defined by (3.16). Since it is smooth with compact support, the function φ belongs to every weighted space. On the other hand, thanks to the structure of the functions in $S^\mu(Q)$, we can check that for $p > \lambda + m - 1$,

$$\tilde{\mathfrak{f}}_{\mathrm{ext}} \in \mathrm{K}^{m-2}_m(\widetilde{Q}_{\mathrm{ext}}) \quad \text{and} \quad \tilde{\mathfrak{g}} \in \mathrm{K}_1^{m-\frac{3}{2}}(\widetilde{G}).$$

Theorem 3.18 yields that $\widetilde{\mathfrak{u}} \in \mathrm{K}^m_{0,-1/2}(\widetilde{Q})$ in this case. To examine the situation where p is such that $p_\lambda \leq p \leq \lambda + m - 1$, let us write

$$\mathfrak{u}^{\lambda,p} = \mathfrak{u}^{\lambda,p+m} - \psi \sum_{\ell=p+1}^{p+m} \mathfrak{K}^{\lambda,\lambda-\ell}.$$

Since $p \geq p_\lambda$, we have $p+m > \lambda+m-1$, thus $\widetilde{\mathfrak{u}}^{\lambda,p+m} \in \mathrm{K}^m_{0,-1/2}(\widetilde{Q})$ by the first step. Besides, for all $\ell \geq p+1 \geq p_\lambda+1$, the exponent $\lambda-\ell$ is < 0. The structure of the spaces $S^\mu(Q)$ allows to show that for any $\mu < 0$ they are embedded in $\mathrm{K}^m_{0,0}(\widetilde{Q})$, thus in $\mathrm{K}^m_{0,-1/2}(\widetilde{Q})$, which concludes the proof. ∎

3.4.2 Proof of the expansion of the profiles at infinity

We can now prove the asymptotic expansion (3.19) of the profile $\mathfrak{K}^\lambda$ constructed in Proposition 3.13. The main tool for this study is the Mellin transform, which is a Fourier-Laplace transform in the variable t whose argument is complex, see [17], [8] or [21].

Let $\Lambda \in \mathbb{C}$; if $\widetilde{\mathfrak{v}}_{\mathrm{int}}$ is defined in the strip $\widetilde{Q}_{\mathrm{int}}$, we set, when meaningful

$$\widehat{\mathfrak{v}}_{\mathrm{int}}(\Lambda,\theta) = \int_{\mathbb{R}} e^{-\Lambda t}\,\widetilde{\mathfrak{v}}_{\mathrm{int}}(t,\theta)\,\mathrm{d}t, \quad \theta \in \Theta_{\mathrm{int}} := (-\tfrac{\omega}{2}, \tfrac{\omega}{2}). \tag{3.24}$$

The variable θ is a parameter: If $\Lambda = \xi + i\eta$, $\widehat{\mathfrak{v}}_{\mathrm{int}}(\cdot,\theta)$ is the Fourier transform of $t \mapsto e^{-\xi t}\,\widetilde{\mathfrak{v}}_{\mathrm{int}}(\cdot,\theta)$ evaluated at the point η. Similarly, we define a Mellin transform in the exterior strips:

$$\widehat{\mathfrak{v}}^{\pm}_{\mathrm{ext}}(\Lambda,\theta) = \int_{\mathbb{R}} e^{-\Lambda t}\,\widetilde{\mathfrak{v}}_{\mathrm{ext}}(t,\theta)\,\mathrm{d}t, \quad \theta \in \Theta^{\pm}_{\mathrm{ext}} := \pm(\tfrac{\omega}{2}, \tfrac{\omega}{2}+1). \tag{3.25}$$

The weighted spaces defined above can be characterized by Mellin transform:

$$\|\widetilde{\mathfrak{v}}_{\mathrm{int}}\|^2_{\mathrm{K}^m_\gamma(\widetilde{Q}_{\mathrm{int}})} \simeq \int_{\mathbb{R}} \|\widehat{\mathfrak{v}}_{\mathrm{int}}(-\gamma+i\eta)\|^2_{\mathrm{H}^m(\Theta_{\mathrm{int}},|\eta|+1)}\,\mathrm{d}\eta, \tag{3.26}$$

where $\|g\|^2_{\mathrm{H}^m(\Theta_{\mathrm{int}},\rho)} := \sum_{\beta_t+\beta_\theta=m} \|\rho^{\beta_t}\partial^{\beta_\theta} g\|^2_{0,\Theta_{\mathrm{int}}}$. Conversely, if the integral

$$\int_{\mathbb{R}} \left\|U_{\mathrm{int}}(-\gamma+i\eta)\right\|^2_{\mathrm{H}^m(\Theta_{\mathrm{int}},|\eta|+1)}\,\mathrm{d}\eta$$

is finite, then U_{int} is the Mellin transform of a function $\widetilde{\mathfrak{v}}_{\mathrm{int}} \in \mathrm{K}^m_\gamma(\widetilde{Q}_{\mathrm{int}})$ on the line $\operatorname{Re}\Lambda = -\gamma$. The function $\widetilde{\mathfrak{v}}$ is reconstructed by the inversion formula:

$$\widetilde{\mathfrak{v}}_{\mathrm{int}}(t,\theta) = \mathcal{M}^{-1}_{-\gamma}(U_{\mathrm{int}}) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{(-\gamma+i\eta)t}U_{\mathrm{int}}(-\gamma+i\eta,\theta)\,\mathrm{d}\eta.$$

These results are consequences of the Plancherel identity. The same equivalences hold for the exterior domain $\widetilde{Q}_{\mathrm{ext}}$.

We are ready to study the asymptotics of $\mathfrak{K}^\lambda$. Thanks to equality (3.15), it is sufficient to investigate $\psi\mathfrak{u}^{\lambda,p}$ for $p \geq p_\lambda$:

Proposition 3.20 *Let λ belong to $\mathfrak{S}$ and let p be an integer, $p \geq p_\lambda$. Let κ denote the Mellin transform of the function $\widetilde{\mathfrak{u}}^{\lambda,p}$, cf. Proposition* 3.19. *There holds:*
(i) κ is holomorphic in the half-plane $\operatorname{Re}\Lambda > \frac{1}{2}$.
(ii) Let b be a positive number such that $p > \lambda + b - 1$. The function κ admits a meromorphic extension in the half-plane $\operatorname{Re}\Lambda > -b$. The set of its poles is contained in $\mathfrak{Q}^-$, cf. (3.17).

Proof: (i) Since by Proposition 3.19 the variational term $\widetilde{\mathfrak{u}}^{\lambda,p}$ belongs to the weighted space $\mathrm{K}^0_{0,-\frac{1}{2}}(\widetilde{Q})$, the equivalence above shows that $\kappa_{\mathrm{int}}(\Lambda,\theta)$ is well defined for $\operatorname{Re}\Lambda \geq 0$ (remember $\widetilde{\mathfrak{u}}^{\lambda,p}$ vanishes near $R=0$) and that, similarly, $\kappa_{\mathrm{ext}}(\Lambda,\theta)$ is defined for $\operatorname{Re}\Lambda \geq \frac{1}{2}$. Therefore, it is clear that $\Lambda \mapsto \kappa(\Lambda,\theta)$ is holomorphic in the domain $\Pi_{\frac{1}{2}}$, where

$$\Pi_a = \{\Lambda \in \mathbb{C}\ ;\ \operatorname{Re}\Lambda > a\}. \tag{3.27}$$

(ii) After Mellin transformation the problem satisfied by $\mathfrak{u}^{\lambda,p}$ becomes

$$\left\{\begin{array}{rcll} (\Lambda^2+\partial^2_\theta)\kappa_{\mathrm{int}}(\Lambda) &=& \hat{\mathfrak{f}}_{\mathrm{int}}(\Lambda-2) & \theta \in (-\frac{\omega}{2},\frac{\omega}{2}),\\ \kappa_{\mathrm{int}}(\Lambda) &=& \kappa_{\mathrm{ext}}(\Lambda) & \theta = \pm\frac{\omega}{2},\\ \partial^2_\theta\kappa_{\mathrm{ext}}(\Lambda) &=& \hat{\mathfrak{f}}_{\mathrm{ext}}(\Lambda)-\Lambda(\Lambda-1)\kappa_{\mathrm{ext}}(\Lambda+2) & \theta \in \pm(\frac{\omega}{2},\frac{\omega}{2}+1),\\ \partial_\theta\kappa_{\mathrm{ext}}(\Lambda) &=& \alpha\partial_\theta\kappa_{\mathrm{int}}(\Lambda+1)-\hat{\mathfrak{g}}(\Lambda) & \theta = \pm\frac{\omega}{2},\\ \kappa_{\mathrm{ext}}(\Lambda) &=& 0 & \theta = \pm(\frac{\omega}{2}+1), \end{array}\right. \tag{3.28}$$

where the terms $\hat{\mathfrak{f}}_{\mathrm{int}}$, $\hat{\mathfrak{f}}_{\mathrm{ext}}$, and $\hat{g}^\pm$ come from the Mellin transform of the terms defined by (3.16) and from the truncation. Since p is sufficiently large ($p > \lambda + b - 1$), this right-hand side is holomorphic for $\operatorname{Re}\Lambda > -b$.

We will build the meromorphic extension of $\kappa(\Lambda)$ in Π_a by descending induction over a, starting from $a = \frac{1}{2}$.

If such an extension is known in the half plane Π_a, we can define $\widetilde{\kappa}^\pm_{\mathrm{ext}}(\Lambda)$ as the unique solution of the last three equations (whose right-hand side is known). As a second step we put

$\widetilde{\kappa}_{\rm ext}^{\pm}(\Lambda)$ in the right-hand side of the second equation of (3.28) and we set $\widetilde{\kappa}_{\rm int}(\Lambda)$ to the solution of the interior problem given by the first two equations in (3.28), which is possible if $\Lambda \notin \mathfrak{S}$.

For $\Lambda \in \Pi_a$, we obviously have $\widetilde{\kappa}(\Lambda) = \kappa(\Lambda)$ since both satisfy problem (3.28), which has a unique solution because it corresponds to the variational problem (P_∞) in the Mellin variables. The function $\widetilde{\kappa}$ is hence an extension of κ. Moreover, $\widetilde{\kappa}$ is meromorphic in Π_{a-1}, the poles being inherited from κ by translation by negative integers and coming from the interior problem (the singular exponents). ■

Thanks to the Mellin inversion formula, we are able to deduce the asymptotic behavior of $\mathfrak{u}^{\lambda,p}$ from meromorphic properties of its Mellin transform.

Proposition 3.21 *Let λ belong to $\mathfrak{S}$ and let p be an integer, $p \geq p_\lambda$. The function $\mathfrak{u}^{\lambda,p}$ is defined through equality* (3.15). *Let P be a positive number such that $p > \lambda + P - 1$. There exist functions $\mathfrak{K}^{\lambda,\mu} \in S^\mu(Q)$ (cf. Definition 3.8), independent of p, such that*

$$\psi \mathfrak{u}^{\lambda,p} = \sum_{\mu \in \mathfrak{Q}^-,\ \mu \geq -P} \mathfrak{K}^{\lambda,\mu} + \mathfrak{u}^{\lambda,p}_{(P)} \quad \textit{where} \quad \mathfrak{u}^{\lambda,p}_{(P)} = \mathcal{O}(R^{-P}) \quad \textit{as } R \to +\infty, \tag{3.29}$$

and the set of indices $\mathfrak{Q}^-$ defined by (3.17). *Moreover the first order derivatives of the remainder satisfy the decay properties*

$$\partial_R\big(\mathfrak{u}^{\lambda,p}_{(P)}\big) = \mathcal{O}(R^{-P-1}) \quad \textit{and} \quad \partial_\theta\big(u^{\lambda,p}_{(P)}\big) = \mathcal{O}(R^{-P}) \quad \textit{as } R \to +\infty, \tag{3.30}$$

Proof: Like in Proposition 3.20 $\kappa(\Lambda)$ is the Mellin transform of $\widetilde{\mathfrak{u}}^{\lambda,p} \simeq \psi\mathfrak{u}^{\lambda,p}$. Let us fix $\alpha, \beta \notin \mathfrak{Q}^-$ such that $\alpha < \beta$ and $p > \lambda - \alpha - 1$. For $\eta > 0$, the boundary of the rectangle

$$\alpha < \operatorname{Re}\Lambda < \beta \quad \text{and} \quad |\operatorname{Im}\Lambda| < \eta$$

will be denoted by G_η. By Cauchy's formula, Proposition 3.20 gives that

$$\int_{G_\eta} e^{t\Lambda}\kappa(\Lambda)\,\mathrm{d}\Lambda = 2i\pi \sum_{\alpha<\mu<\beta} \operatorname*{Res}_{\Lambda=\mu} e^{t\Lambda}\kappa(\Lambda),$$

with residues for $\mu \in \mathfrak{Q}^-$. We let η go to infinity in the above identity. The vertical sides of G_η give inverse Mellin transforms:

$$\int_{-\eta}^{\eta} e^{(\gamma+i\eta)t}\kappa(\gamma+i\eta)i\,\mathrm{d}\eta \longrightarrow 2i\pi\mathcal{M}_\gamma^{-1}[\kappa(\Lambda)], \quad \gamma = \alpha, \beta,$$

where $\mathcal{M}_\gamma^{-1}$ denotes the inverse Mellin transform along the line $\operatorname{Re}\Lambda = \gamma$.

Standard resolvent estimates for the system (3.28) combined with the descending induction argument of the proof of Proposition 3.20 show that $\kappa(\xi + i\eta)$ is rapidly decreasing as $|\eta| \to \infty$. Thus, there is no contribution of the horizontal sides of G_η. In conclusion, we obtain

$$\mathcal{M}_\beta^{-1}[\kappa(\Lambda)] - \mathcal{M}_\alpha^{-1}[\kappa(\Lambda)] = \sum_{\alpha<\mu<\beta} \operatorname*{Res}_{\Lambda=\mu} e^{t\Lambda}\kappa(\Lambda).$$

We can check that, for $\mu \in \mathfrak{Q}^-$, the function $\mathfrak{K}^{\lambda,\mu} := \psi \operatorname{Res}_{\Lambda=\mu} e^{t\Lambda}\kappa(\Lambda)$ belongs to the space $S^\mu(Q)$. The expansion (3.29) is obtained for $\beta = \frac{1}{2}$ and $\alpha = -P - \delta$ for some δ such that $[-P-\delta, -P) \cap \mathfrak{Q}^- = \emptyset$.

It remains to prove that the remainder $\mathfrak{u}^{\lambda,p}_{(P)}$ satisfies the decay properties in (3.29)-(3.30). We set $\widetilde{\mathfrak{u}}^{\lambda,p}_{(P)}(t,\theta) = \psi\mathfrak{u}^{\lambda,p}_{(P)}(X)$. Thus $\widetilde{\mathfrak{u}}^{\lambda,p}_{(P)}$ coincides with $\mathcal{M}_\alpha^{-1}[\kappa(\Lambda)]$ for large t. Since $\kappa(\xi+i\eta)$ is rapidly decreasing as $|\eta|\to\infty$, the norms

$$\int_{\mathbb{R}} \big\|\kappa(\alpha+i\eta)\big\|^2_{\mathrm{H}^m(\Theta,|\eta|+1)}\,\mathrm{d}\eta$$

are finite for any $m>0$. This shows that $\widetilde{\mathfrak{u}}^{\lambda,p}_{(P)}$ belongs to $\mathrm{K}^m_{P+\delta}(\widetilde{Q})$ for any m. For $m>1$, this implies that $\mathfrak{u}^{\lambda,p}_{(P)} = \mathcal{O}(R^{-P})$ as $R\to\infty$, and, for larger values of m, it proves the decay properties (3.30). ∎

Proof of Theorem 3.15: Let us fix $P>0$. Let us take $p\geq\lambda$ such that $\lambda-p\leq -P$. According to Theorem 3.13, there holds

$$\mathfrak{K}^\lambda = \psi\sum_{\ell=0}^{p}\breve{\mathfrak{K}}^{\lambda,\lambda-\ell} + \mathfrak{u}^{\lambda,p}.$$

Proposition 3.21 yields that

$$\mathfrak{u}^{\lambda,p} = \sum_{\mu\in\mathfrak{Q}^-,\ \mu\geq -P}\mathfrak{K}^{\lambda,\mu} + \mathcal{O}(R^{-P}).$$

Therefore we obtain the expansion (3.19) for this P. By virtue of the uniqueness of asymptotic expansions in powers of R at infinity, the terms $\mathfrak{K}^{\lambda,\mu}$ do not depend on P.

The expression of $\mathfrak{K}^\lambda$ as a formal series – see (3.20) – follows again from the Cauchy formula: indeed the terms $(\mathfrak{K}^{\lambda,\nu-\ell})_\ell$ satisfy the equations (3.10) and (3.9).

The assertion about the degree in θ of $\mathfrak{K}^{\lambda,\mu}_{\mathrm{ext}}$ in the layer Q_{ext} results from the equality (3.20): $\mathfrak{K}^{\lambda,\mu}_{\mathrm{ext}}$ is a linear combination of terms of the form $\psi\breve{\mathfrak{K}}^{\nu,\nu-\ell}$, with $\mu=\nu-\ell$ and $\nu\leq\lambda$. According to Proposition 3.10, the degree in θ of $\breve{\mathfrak{K}}^{\nu,\nu-\ell}$ is ℓ, whence $\leq\lambda-\mu$. ∎

3.5 Neumann boundary conditions

In this section, we try to follow the same arguments as before for the Dirichlet boundary conditions. The variational formulation is the same as above, but due to the absence of the Poincaré inequality, the previous variational space cannot be used in this case. Nevertheless, it is possible to find a suitable variational space: Let $\mathfrak{X}$ be defined as

$$\mathfrak{X} = \left\{\mathfrak{v}\ ;\ \nabla\mathfrak{v}\in\mathrm{L}^2(Q)\ \text{and}\ \frac{\mathfrak{v}}{(1+R)\log(2+R)}\in\mathrm{L}^2(Q)\right\}, \tag{3.31}$$

endowed with its natural norm (again R is the distance to the interior corner point O). Since the constant functions belong to $\mathfrak{X}$, we introduce the quotient space $\mathfrak{V}=\mathfrak{X}/\mathbb{R}$. The space $\mathfrak{V}$ is clearly a Hilbert space and we will show that the H^1-seminorm is an equivalent norm for $\mathfrak{V}$:

Proposition 3.22 *The bilinear form $a(\mathfrak{u},\mathfrak{v}) = \int_Q \nabla\mathfrak{u}\cdot\nabla\mathfrak{v}\,\mathrm{d}x$ is continuous and coercive on $\mathfrak{V}$.*

Proof: Only the coercivity needs to be checked. For $R > 0$, let B_R denote the ball of radius R centered in O' (exterior corner point of Q, see Figure 2) and χ a smooth radial cut-off function, supported in B_2 and equal to 1 in B_1.

Let $\mathfrak{v} \in \mathfrak{X}$, we denote by $\langle\mathfrak{v}\rangle$ its mean value on $B_2 \cap Q$:

$$\langle\mathfrak{v}\rangle = \frac{1}{\mathrm{meas}(B_2 \cap Q)} \int_{B_2\cap Q} \mathfrak{v}(x)\,\mathrm{d}x.$$

By the Poincaré-Wirtinger inequality in the bounded domain $B_2 \cap Q$, there exists a constant C such that

$$\big\|\mathfrak{v} - \langle\mathfrak{v}\rangle\big\|_{0,B_2\cap Q} \le C\big\|\nabla\mathfrak{v}\big\|_{0,B_2\cap Q},$$

which gives the following estimate for $\chi(\mathfrak{v} - \langle\mathfrak{v}\rangle)$:

$$\big\|\chi(\mathfrak{v} - \langle\mathfrak{v}\rangle)\big\|_{\mathfrak{X}} \le C\big\|\nabla\mathfrak{v}\big\|_{0,Q}, \tag{3.32}$$

where C is another constant, independent of $\mathfrak{v}$. Let then $\mathfrak{u}$ be defined as $\mathfrak{u} = (1-\chi)(\mathfrak{v} - \langle\mathfrak{v}\rangle)$. If we denote by (ρ,φ) the polar coordinates centered in O', then $\mathfrak{u} = 0$ on the circular arc corresponding to $\rho = 2$. We can use this information to get a Hardy inequality (in this limit case, it corresponds to a “weighted Poincaré inequality”, see [14]): for any $R > 2$,

$$\int_0^\omega \int_2^R \frac{|\mathfrak{u}(\rho,\varphi)|^2}{\rho^2\log^2\rho}\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi \le C \int_0^\omega \int_2^R |\partial_\rho\mathfrak{u}(\rho,\varphi)|^2\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi.$$

Together with (3.32), we obtain the result. ∎

Corollary 3.23 *If* $(1+R)\log(2+R)\mathfrak{f} \in \mathrm{L}^2(Q_{\mathrm{int}})$ *and* $(1+R)^{\frac12}\log(2+R)\mathfrak{g} \in \mathrm{L}^2(G)$*, with the compatibility condition (note that the integrals make sense)*

$$\int_{Q_{\mathrm{int}}} \mathfrak{f}\,\mathrm{d}x + \int_G \mathfrak{g}\,\mathrm{d}\sigma = 0, \tag{3.33}$$

then problem (P_∞) *admits a unique solution* $\mathfrak{v} \in \mathfrak{V}$.

With the space $\mathfrak{V}$, we get a suitable variational framework which allows us to define unique solutions for problem (P_∞) in the case of Neumann boundary conditions. We will continue to use $\mathfrak{X}$ instead of $\mathfrak{V}$, i.e. functions instead of equivalence classes modulo constants, but we have to make sure that elements of the dual space are orthogonal to constants, i.e. satisfy the compatibility condition (3.33).

Similarly to the Dirichlet case, we start from a singularity $\mathfrak{s}^\lambda$ ($\lambda > 0$) of the interior problem (with Neumann condition on Γ this time). Since it does not belong to the variational space $\mathfrak{V}$, we perform a few algorithmic steps in order to decrease the degree in the variable R at infinity.

Proposition 3.24 *Let* $\lambda \in \mathfrak{S} \cup \{0\}$. *Let* $\mathfrak{s}_\star^\lambda$ *denote the extension of* $\mathfrak{s}^\lambda$ (3.3) *in* $\breve{Q}$ *such that*

$$\mathfrak{s}_\star^\lambda(R,\theta) = \mathfrak{s}^\lambda\big|_{\theta=\pm\frac{\omega}{2}}(R) \quad \text{in } \breve{Q}^{\pm}_{\mathrm{ext}}.$$

We set $\breve{\mathfrak{K}}^{\lambda,\lambda} = \mathfrak{s}_\star^\lambda$ *and, for convenience,* $\breve{\mathfrak{K}}^{\lambda,\lambda+1} = \breve{\mathfrak{K}}^{\lambda,\lambda+2} = 0$.

There exist $\breve{\mathfrak{K}}^{\lambda,\lambda-\ell} \in S^{\lambda-\ell}(\breve{Q})$, $\ell = 1, 2, \ldots$, *satisfying the following sequence of equations*

$$\begin{cases} \partial_\theta^2 \breve{\mathfrak{K}}_{\mathrm{ext}}^{\lambda,\lambda-\ell} &= -\partial_R^2 \breve{\mathfrak{K}}_{\mathrm{ext}}^{\lambda,\lambda-\ell+2} & \theta \in \pm(\frac{\omega}{2}, \frac{\omega}{2}+1), \\ \partial_\theta \breve{\mathfrak{K}}_{\mathrm{ext}}^{\lambda,\lambda-\ell} &= \frac{\alpha}{R}\partial_\theta \breve{\mathfrak{K}}_{\mathrm{int}}^{\lambda,\lambda-\ell+1} & \theta = \pm\frac{\omega}{2}, \\ \partial_\theta \breve{\mathfrak{K}}_{\mathrm{ext}}^{\lambda,\lambda-\ell} &= 0 & \theta = \pm\frac{\omega}{2} \pm 1, \end{cases} \tag{3.34}$$

$$\begin{cases} \Delta \breve{\mathfrak{K}}_{\mathrm{int}}^{\lambda,\lambda-\ell} &= 0 & \textit{in } Q_{\mathrm{int}}, \\ \alpha \partial_n \breve{\mathfrak{K}}_{\mathrm{int}}^{\lambda,\lambda-\ell} &= \displaystyle\int_{\pm\frac{\omega}{2}}^{\pm\frac{\omega}{2}\pm 1} \partial_R^2 \breve{\mathfrak{K}}_{\mathrm{ext}}^{\lambda,\lambda-\ell+1}(R,\vartheta)\,\mathrm{d}\vartheta & \textit{for } \theta = \pm\frac{\omega}{2}, \end{cases} \tag{3.35}$$

The exterior part is defined up to a constant, which is determined by the condition $\breve{\mathfrak{K}}_{\mathrm{ext}}^{\lambda,\lambda-\ell} = \breve{\mathfrak{K}}_{\mathrm{int}}^{\lambda,\lambda-\ell}$ *on* Γ.

For each integer $p \geq 0$ *the partial sum* $\sum_{\ell=0}^{p} \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}$ *solves the Neumann problem* $(\breve{\mathrm{P}}_\infty)$ *with*

$$\breve{\mathfrak{f}}_{\mathrm{int}} = 0, \qquad \breve{\mathfrak{f}}_{\mathrm{ext}} = -\partial_R^2 \left[\breve{\mathfrak{K}}_{\mathrm{ext}}^{\lambda,\lambda-p} + \breve{\mathfrak{K}}_{\mathrm{ext}}^{\lambda,\lambda-p+1}\right], \qquad \breve{\mathfrak{g}} = -\alpha\partial_n \breve{\mathfrak{K}}_{\mathrm{int}}^{\lambda,\lambda-p}. \tag{3.36}$$

Proof: Due to the compatibility conditions for Neumann problems, the construction of the terms $\breve{\mathfrak{K}}^{\lambda,\mu}$ is not as straightforward as in the Dirichlet case. Let us give a brief description: If $\breve{\mathfrak{K}}^{\lambda,\lambda-\ell}$ are constructed for $\ell < k$, then consider equations (3.34) for $\ell = k+1$. This is a one-dimensional Neumann problem (with parameter R) whose compatibility condition reads

$$\int_{\pm\frac{\omega}{2}}^{\pm\frac{\omega}{2}\pm 1} \partial_R^2 \breve{\mathfrak{K}}_{\mathrm{ext}}^{\lambda,\lambda-(k-1)}(R,\vartheta)\,\mathrm{d}\vartheta = \frac{\alpha}{R}\partial_\theta \breve{\mathfrak{K}}_{\mathrm{int}}^{\lambda,\lambda-k} = \pm\alpha\partial_n \breve{\mathfrak{K}}_{\mathrm{int}}^{\lambda,\lambda-k},$$

this gives the Neumann data for the interior problem (3.35) for $\ell = k$ (whose compatibility condition is fulfilled). As for the Dirichlet case, the interior boundary value problem with data in $S^{\lambda-k-1}(\breve{G})$ always has a solution in $S^{\lambda-k}(\breve{Q}_{\mathrm{int}})$. We can then define $\breve{\mathfrak{K}}_{\mathrm{int}}^{\lambda,\lambda-k}$; the condition $\breve{\mathfrak{K}}_{\mathrm{ext}}^{\lambda,\lambda-k} = \breve{\mathfrak{K}}_{\mathrm{int}}^{\lambda,\lambda-k}$ on G completely determines the exterior part. ∎

Here is now the analogue of Theorem 3.13 in the Neumann case.

Theorem 3.25 *Let* $\lambda \in \mathfrak{S}$, $\lambda > 0$, *and let* p_λ *be defined by* (3.13). *There exists* $\mathfrak{u}^{\lambda,p_\lambda}$ *in the variational space* $\mathfrak{X}$ *and, if* $\lambda \in \mathbb{N}$, *a constant* I^λ, *such that the sum*

$$\begin{aligned} \mathfrak{K}^\lambda &:= \psi \sum_{\ell=0}^{p_\lambda} \breve{\mathfrak{K}}^{\lambda,\lambda-\ell} + \mathfrak{u}^{\lambda,p_\lambda} && \textit{if } \lambda \notin \mathbb{N} \\ \mathfrak{K}^\lambda &:= \psi \sum_{\ell=0}^{p_\lambda} \breve{\mathfrak{K}}^{\lambda,\lambda-\ell} + I^\lambda \mathfrak{s}_\star^0 + \mathfrak{u}^{\lambda,p_\lambda} && \textit{if } \lambda \in \mathbb{N} \end{aligned} \tag{3.37}$$

defines a solution $\mathfrak{K}^\lambda$ *of problem* (P_∞) *for* $\mathfrak{f} = \mathfrak{g} = 0$, *satisfying* $\mathfrak{K}_{\mathrm{int}}^\lambda \sim \mathfrak{s}^\lambda$ *as* $R \to \infty$.

Proof: For any integer q, we define

$$\mathfrak{v}^{\lambda,q} = -\psi \sum_{\ell=0}^{q} \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}. \tag{3.38}$$

By construction, the function $\mathfrak{v}^{\lambda,q}$ solves problem (P_∞) with

$$\begin{aligned} \mathfrak{f}_{\text{int}} &= \alpha\,\varphi_q, && (3.39)\\ \mathfrak{f}_{\text{ext}} &= \varphi_q - \psi\partial_R^2\left[\breve{\mathfrak{K}}^{\lambda,\lambda-q}_{\text{ext}} + \breve{\mathfrak{K}}^{\lambda,\lambda-q+1}_{\text{ext}}\right], && (3.40)\\ \mathfrak{g} &= -\psi\tfrac{\alpha}{R}\partial_\theta\breve{\mathfrak{K}}^{\lambda,\lambda-q}_{\text{int}}, && (3.41)\end{aligned}$$

where φ_q comes from the cut-off; its support is contained in $\operatorname{supp}(\nabla\psi)$.

For q large enough, i.e. $q > \lambda + \frac{1}{2}$, the above right-hand sides satisfy the assumptions of Corollary 3.23. If we are able to verify the compatibility condition (3.33), we can conclude that there exists $\mathfrak{u}^{\lambda,q} \in \mathfrak{X}$, solving the same problem as $\mathfrak{v}^{\lambda,q}$. Then

$$\mathfrak{K}^\lambda = \psi\sum_{\ell=0}^{q}\breve{\mathfrak{K}}^{\lambda,\lambda-\ell} + \mathfrak{u}^{\lambda,q}$$

solves problem (P_∞) with $\mathfrak{f} = \mathfrak{g} = 0$; the statement concerning $\mathfrak{u}^{\lambda,p}$ directly follows from the inclusion $\mathfrak{S}^\mu \subset \mathfrak{X}$ for $\mu < 0$.

Let us focus on the compatibility condition (3.33). For $R > 0$, we define Q^R as $Q \cap B_R$, where B_R denotes the ball of radius R, centered in O. Similarly, G^R (resp. G^R_{ext}) denotes $G \cap B_R$ (resp. $G_{\text{ext}} \cap B_R$). With the help of an integration by parts, we get

$$I^\lambda_R := \int_{Q^R}\mathfrak{f}\,\mathrm{d}x + \int_{G^R}\mathfrak{g}\,\mathrm{d}\sigma = -\int_{Q\cap\partial B_R}\partial_n\mathfrak{v}^{\lambda,q}\,\mathrm{d}\sigma, \tag{3.42}$$

the terms on G^R and G^R_{ext} vanishing by construction of $\breve{\mathfrak{K}}^{\lambda,\mu}$. Thanks to definition (3.38) of $\mathfrak{v}^{\lambda,q}$, we get the following expression for the integral I^λ_R:

$$I^\lambda_R = \sum_{m=1}^{M}\sum_{\ell=0}^{L} a_{m\ell}R^{\lambda-m}\,\log^\ell R, \tag{3.43}$$

with unknown coefficients $a_{m\ell}$. For q large enough, expressions (3.39)–(3.41) show that $\mathfrak{f}$ and $\mathfrak{g}$ have finite integrals over Q and G. Hence, I^λ_R has a finite limit I^λ_∞ as $R \to +\infty$, which imposes $a_{m\ell} = 0$ for $\lambda - m > 0$ or ($\lambda = m$ and $\ell > 0$).

If λ is not an integer, we can deduce $I^\lambda_\infty = 0$: This is the expected compatibility condition.

If λ is an integer, I^λ_∞ does not necessarily vanish. But the compatibility condition can be fulfilled with the help of the logarithmic singularity. Indeed, if we apply the same technique as above, starting with $\mathfrak{s}^0 = \log R \notin \mathfrak{X}$, we obtain $I^0_\infty = -1$. Hence, for $\lambda \in \mathbb{N}^*$ we do not know if I^λ_∞ vanishes but

$$\tilde{\mathfrak{v}}^{\lambda,p} = -\psi\Big(\sum_{\ell=0}^{p}\breve{\mathfrak{K}}^{\lambda,\lambda-\ell} + I^\lambda_\infty\mathfrak{s}^0_\star\Big)$$

satisfies the compatibility condition. ∎

Then we can prove by the same tools as in §3.4.1 and §3.4.2 that the Neumann version of the $\mathfrak{K}^\lambda$ satisfies an expansion at infinity like (3.19) with the same set of exponents $\mathfrak{Q}^\lambda$ (3.18). At this stage there is essentially no difference between Dirichlet and Neumann external boundary conditions.

3.6 Non-homogeneous profile problems

The same techniques apply to the non-homogeneous problem (P_∞):

Theorem 3.26 *Let $\lambda \in \mathbb{R}$. Under the following assumptions: $\mathfrak{f}_{\text{int}} = \psi\breve{\mathfrak{f}}_{\text{int}}$, $\mathfrak{f}_{\text{ext}} = \psi\breve{\mathfrak{f}}_{\text{ext}}$, $\mathfrak{g} = \psi\breve{\mathfrak{g}}$ with*

$$\begin{array}{llll} \breve{\mathfrak{f}}_{\text{int}} \in S^{\lambda-2}(\breve{Q}_{\text{int}}), & \breve{\mathfrak{f}}_{\text{ext}} \in S^{\lambda}(\breve{Q}_{\text{ext}}), & \textit{and} \quad \breve{\mathfrak{g}} \in S^{\lambda}(\breve{G}), & \textit{for Dirichlet b.c.} \\ \breve{\mathfrak{f}}_{\text{int}} \in S^{\lambda-2}(\breve{Q}_{\text{int}}), & \breve{\mathfrak{f}}_{\text{ext}} \in S^{\lambda-1}(\breve{Q}_{\text{ext}}), & \textit{and} \quad \breve{\mathfrak{g}} \in S^{\lambda-1}(\breve{G}), & \textit{for Neumann b.c.} \end{array}$$

problem (P_∞) with Dirichlet or Neumann external boundary conditions admits a solution $\mathfrak{W}^\lambda$ which has an asymptotics at infinity of the form

$$\mathfrak{W}^\lambda = \mathfrak{W}^{\lambda,\lambda} + \sum_{\mu \in \mathfrak{Q}^\lambda(P)} \mathfrak{W}^{\lambda,\mu} + \mathcal{O}(r^{-P}) \quad (\forall P \in \mathbb{N}), \tag{3.44}$$

with $\mathfrak{W}^{\lambda,\mu}$ in the space $S^\mu(Q)$ of Definition 3.8, for all $\mu \in \{\lambda\} \cup \mathfrak{Q}^\lambda$.

Proof: We have only to check that the algorithmic construction performed in Proposition 3.10 can be started in the situation of a non-zero right-hand side. We still have to solve the series of problems (3.9)-(3.10) with the initialization $\breve{\mathfrak{W}}^{\lambda,\lambda+1} = \breve{\mathfrak{W}}^{\lambda,\lambda+2} = 0$. For $\ell = 0$ and Dirichlet b.c., problems (3.9)-(3.10) are now:

$$\left\{ \begin{array}{rcll} \partial_\theta^2 \breve{\mathfrak{W}}^{\lambda,\lambda}_{\text{ext}} &=& \breve{\mathfrak{f}}_{\text{ext}} & \theta \in \pm(\frac{\omega}{2}, \frac{\omega}{2}+1), \\ \partial_\theta \breve{\mathfrak{W}}^{\lambda,\lambda}_{\text{ext}} &=& \breve{\mathfrak{g}} & \theta = \pm\frac{\omega}{2}, \\ \breve{\mathfrak{W}}^{\lambda,\lambda}_{\text{ext}} &=& 0 & \theta = \pm\frac{\omega}{2} \pm 1, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcll} \alpha\Delta\breve{\mathfrak{W}}^{\lambda,\lambda}_{\text{int}} &=& \breve{\mathfrak{f}}_{\text{int}} & \text{in } Q_{\text{int}}, \\ \breve{\mathfrak{W}}^{\lambda,\lambda}_{\text{int}} &=& \breve{\mathfrak{W}}^{\lambda,\lambda}_{\text{ext}} & \text{for } \theta = \pm\frac{\omega}{2}, \end{array} \right.$$

The problem in $\breve{Q}_{\text{ext}}$ can be explicitly solved in $S^\lambda(\breve{Q})$. Then the problem in $\breve{Q}_{\text{int}}$ is a Dirichlet problem with boundary data in $S^\lambda(\breve{G})$ and interior data in $S^{\lambda-2}(\breve{Q}_{\text{int}})$. According to [8, Ch.4] for example, it is solvable in $S^\lambda(\breve{Q}_{\text{int}})$.

For Neumann external b.c., we have to take into account the different order in the iterative algorithm, see Proposition 3.24. The right hand sides $\breve{\mathfrak{f}}_{\text{ext}}$ and $\breve{\mathfrak{g}}$ then only appear in the equation for $\breve{\mathfrak{W}}^{\lambda,\lambda-1}$, see also the Remark below.

The whole construction and analysis is then similar to that for $\mathfrak{K}^\lambda$. ∎

Remark 3.27 In the case of external Neumann b.c., if $\breve{\mathfrak{f}}_{\text{ext}}$ and $\breve{\mathfrak{g}}$ satisfy the compatibility condition

$$\forall R, \quad \breve{\mathfrak{g}}(R, \pm\tfrac{\omega}{2}) = \int_{\pm\frac{\omega}{2}}^{\pm\frac{\omega}{2}\pm 1} \breve{\mathfrak{f}}_{\text{ext}}(R, \vartheta)\,\mathrm{d}\vartheta$$

then one can allow $\breve{\mathfrak{f}}_{\text{ext}} \in S^\lambda(\breve{Q}_{\text{ext}})$ and $\breve{\mathfrak{g}} \in S^\lambda(\breve{G})$ in the hypotheses of Theorem 3.26. □

This result will be used for *polynomial right hand sides* $\breve{\mathfrak{f}}_{\text{int}}$, that is why we introduce:

Definition 3.28 *Let $k \in \mathbb{N}$, $k \geq 2$. For any multi-index $\beta = (\beta_1, \beta_2)$ of length $k-2$ we set:*

$$\mathfrak{W}^{k,(\beta)} \textit{ solution of } (\mathrm{P}_\infty) \textit{ for:} \quad \breve{\mathfrak{f}}_{\text{int}} = X^\beta \big(= R^{k-2}\cos^{\beta_1}\theta \sin^{\beta_2}\theta \big), \quad \breve{\mathfrak{f}}_{\text{ext}} \equiv 0, \quad \breve{\mathfrak{g}} \equiv 0.$$

The function $\mathfrak{W}^{k,(\beta)}$ *has the form* (3.44) *with* $\lambda = k$. *The first term* $\mathfrak{W}^{k,k,(\beta)}$ *of its expansion* (3.44) *satisfies*

$$\begin{cases} \partial_\theta^2 \breve{\mathfrak{W}}^{k,k,(\beta)}_{\rm ext} &= 0, \quad \theta \in \pm(\frac{\omega}{2}, \frac{\omega}{2}+1), \\ \partial_\theta \breve{\mathfrak{W}}^{k,k,(\beta)}_{\rm ext} &= 0, \quad \theta = \pm\frac{\omega}{2}, \\ \breve{\mathfrak{W}}^{k,k,(\beta)}_{\rm ext} &= 0, \quad \theta = \pm\frac{\omega}{2} \pm 1, \end{cases} \quad \textit{and} \quad \begin{cases} \alpha\Delta\breve{\mathfrak{W}}^{k,k,(\beta)}_{\rm int} &= X^\beta \quad \textit{in } Q_{\rm int}, \\ \breve{\mathfrak{W}}^{k,k,(\beta)}_{\rm int} &= \breve{\mathfrak{W}}^{k,k,(\beta)}_{\rm ext}, \quad \theta = \pm\frac{\omega}{2}. \end{cases} \tag{3.45}$$

Remark 3.29
(i) For Neumann external boundary conditions, the profiles $\mathfrak{W}^{k,(\beta)}$ are pertaining to the second case in (3.37). Thus a term in $\log R$ may appear in their expansion (3.44) at infinity (even if $\frac{\pi}{\omega} \notin \mathbb{Q}$), together with lower order terms of the form $R^{-j}\log^k R$, $j = 1, 2, \ldots$ and $k \le j$.
(ii) It is also possible to introduce profiles solving polynomial right sides for $\mathfrak{g}$. There we have to take into account the different degrees appearing in the Dirichlet and Neumann cases, cf. Remark 3.27. □

4 ε-Expansion in the coated domain with corner

In this section, we reach our initial aim, that is to build an asymptotic expansion in ε for the solution u_ε of problem (P_ε) with Dirichlet or Neumann external boundary conditions in the case where $\Omega_{\rm int}$ has a corner at the origin O.

4.1 Notations, assumptions, plan

We recall that the Cartesian coordinates with origin O are denoted by x. They are the "slow" variables in $\Omega_{\rm int}$. The polar coordinates centered at O are denoted by (r, θ), the arclength along the interface Γ by t, and the normal coordinate to Γ inside $\Omega^\varepsilon_{\rm ext}$ by s. Note that s is well defined outside an ε-neighborhood of O.

We still need the cut-off function χ introduced in Definition 1.1, which allows a localization independent of ε, in the region where Ω^ε coincides with a sector. In order to avoid non-zero commutators of χ with the normal derivatives ∂_θ and ∂_s on Γ, *we assume for simplicity that*

$$\chi = \chi(r) \text{ in } \Omega_{\rm int} \quad \text{and} \quad \chi = \chi(t) \text{ in } \Omega_{\rm ext}. \tag{4.1}$$

We assume that the data of problem (P_ε) are smooth and, to avoid unnecessary difficulties, that $f_{\rm ext}$ is zero near the corner, that is

$$f_{\rm int} \in \mathcal{C}^\infty(\overline{\Omega}_{\rm int}), \quad g \in \mathcal{C}^\infty(\Gamma), \quad \text{and} \quad f_{\rm ext} \in \mathcal{C}^\infty(\overline{\Omega}^\varepsilon_{\rm ext}), \quad f_{\rm ext}\big|_{\mathcal{V}'} \equiv 0. \tag{4.2}$$

The construction of ε-expansions for the solution u^ε of problem (P_ε) is performed with external Dirichlet boundary conditions. With the results of §3.5 at hand, the Neumann case can be treated similarly.

The study of the Dirichlet case is organized in four parts. First we start like for the smooth case and draw the principles of the special treatment of singularities (§4.2). Then we construct the first terms in the asymptotics (§4.3) before we reach the expression of general terms (§4.4). We end with alternative expansions (§4.5) and the Neumann case (§4.6).

4.2 A recursive approach of the ε-expansion

Let us first try to start with the algorithm we have already used for a smooth domain. It consists in solving equations (2.6) and (2.7) for all integers $n \geq 0$.

For $n = 0$, we find $U^0_{\rm ext} = 0$ and that $u^0_{\rm int}$ solves the homogeneous Dirichlet problem (P_0) with source term $f_{\rm int}$.

For $n = 1$, $U^1_{\rm ext}$ is explicitly given by $U^1_{\rm ext} = (S-1)\left[\alpha\partial_n u^0_{\rm int}|_\Gamma - g\right]$. Its trace on Γ is $g - \alpha\partial_n u^0_{\rm int}$ and has to be inserted as a Dirichlet data into the problem defining $u^1_{\rm int}$. But, due to the corner, we cannot ensure a sufficient regularity: A singularity in $r^{\frac{\pi}{\omega}}$ can arise in $u^0_{\rm int}$, cf. (3.4). Thus $\partial_n u^0_{\rm int}|_\Gamma$ is like $r^{\frac{\pi}{\omega}-1}$, which does not define an $\mathrm{H}^{1/2}(\Gamma)$-function as soon as $\frac{\pi}{\omega} < 1$, and the problem defining $u^1_{\rm int}$ is then not solvable in $\mathrm{H}^1(\Omega_{\rm int})$.

Then our technique consists in splitting $u^0_{\rm int}$ according to (3.4), into a regular and a singular part which are handled separately. The singular part is a linear combination of the singular functions $\mathfrak{s}^\lambda$, see (3.2). Taking advantage of Theorem 3.15, we replace each $\mathfrak{s}^\lambda(x) = \varepsilon^\lambda \mathfrak{s}^\lambda(\frac{x}{\varepsilon})$ by its counterpart $\varepsilon^\lambda \mathfrak{K}^\lambda(\frac{x}{\varepsilon})$ solution of the homogeneous transmission problem.

Thus we are left with a residual transmission problem (P_ε) associated with the regular part of the expansion of u^0 and a finite number of problems (P_ε) generated by the localized differences $\chi(x)\big(\varepsilon^\lambda \mathfrak{K}^\lambda(\frac{x}{\varepsilon}) - \mathfrak{s}^\lambda_0(x)\big)$. The structure of these latter terms is given by the expansion (3.19) of $\mathfrak{K}^\lambda$, resulting in *smaller (in the ε-scale)* right hand sides, smooth and identically zero near the corner. However, the data in $\Omega^\varepsilon_{\rm ext}$ *depend on* ε in a special way. For this reason, we will consider broader assumptions than (4.2) concerning the generic right hand side in $\Omega^\varepsilon_{\rm ext}$, which we now denote by $f_{\varepsilon,\rm ext}$:

$$f_{\varepsilon,\rm ext} = \sum_{\ell\in\mathbb{N},\text{ finite}} \varepsilon^{-2-\ell} f^{-\ell}_{\rm ext} \quad \text{with} \quad \left\{ \begin{array}{l} f^{-\ell}_{\rm ext} \in \mathcal{C}^\infty(\overline{\Omega}^\varepsilon_{\rm ext}),\ \ f^{-\ell}_{\rm ext}\big|_{\mathcal{V}'} \equiv 0, \quad \text{and} \\ \partial^k_n f^{-\ell}_{\rm ext}\big|_\Gamma \equiv 0,\ k = 0,\dots,\ell-1. \end{array} \right. \tag{4.3}$$

Our strategy is constructive: Instead of starting from a general multi-scale Ansatz and trying to identify terms, we construct first terms in such a way that the corresponding remainder is solution of problem (P_ε) with new data $f_{\rm int}$ and $f_{\varepsilon,\rm ext}$ of the form $f_{\rm int} = \sum \varepsilon^\nu f^\nu_{\rm int}$ and $f_{\varepsilon,\rm ext} = \sum \varepsilon^\nu f^\nu_{\varepsilon,\rm ext}$, where ν spans a finite set of *positive exponents*, and $f^\nu_{\rm int}$, $f^\nu_{\varepsilon,\rm ext}$ have the same structure as the data of the initial problem (P_ε). After that point is reached, the general expansion becomes clear.

4.3 The first terms of the ε-expansion in the Dirichlet case

From now on, we assume that the exterior right-hand side $f_{\varepsilon,\rm ext}$ has the structure (4.3), that will show to be stable through the recursive construction. We emphasize that such a $f_{\varepsilon,\rm ext}$ will not lead to negative powers in the expansion of u_ε, see Corollary 2.4.

Additionally, in order to make the exposition slightly simpler[1] we assume that $f_{\rm int}$ and g have a zero Taylor expansion at the corner O:

$$\left\{ \begin{array}{l} f_{\rm int} \in \mathcal{C}^\infty(\overline{\Omega}_{\rm int}),\ \ \partial^\beta_x f(O) = 0\ (\forall\beta\in\mathbb{N}^2) \quad \text{and} \\ g \in \mathcal{C}^\infty(\Gamma),\ \ \partial^j_r g(0, \pm\frac{\omega}{2}) = 0\ (\forall j\in\mathbb{N}). \end{array} \right. \tag{4.4}$$

[1] The general case (4.2) is considered in §4.4.2, by treating separately the Taylor expansion of $f_{\rm int}$ at O. By the same techniques, one could also treat non-vanishing Taylor expansion of g.

Our construction consists, for each term which is a solution of a Dirichlet problem in Ω_{int}, in considering its splitting into (flat) regular and singular parts. As we know from Theorem 3.3, this requires to fix in advance a regularity index K. For each term this regularity index is chosen according to its rank in the ε-expansion. For the beginning we need

- A *maximal regularity index* $K > 0$, $K \notin \mathfrak{S}$.
- A *target precision* $N > 0$ *in* ε, with the aim of constructing an ε-expansion with a remainder of order ε^N,

We will see in the course of the construction that K has eventually to be chosen (at least) larger than $N + \frac{3}{2}$.

4.3.1 Terms of order 0

In the case of exterior data $f_{\varepsilon,\text{ext}}$ satisfying (4.3), we first solve the exterior equation:

$$\begin{cases} \partial_S^2 U_{\text{ext}}^0 &= \sum_\ell \frac{1}{\ell!}\partial_n^\ell f_{\text{ext}}^{-\ell}(t,0)\, S^\ell & \text{for } 0 < S < 1, \\ \partial_S U_{\text{ext}}^0 &= 0 & \text{for } S = 0, \\ U_{\text{ext}}^0 &= 0 & \text{for } S = 1. \end{cases} \tag{4.5}$$

Since the functions $f_{\text{ext}}^{-\ell}$ vanish in a neighborhood of O, the extension by zero of the solution of problem (4.5) uniquely defines a function U_{ext}^0 in the entire layer $\Omega_{\text{ext}}^\varepsilon$. Then u_{int}^0 solves (P$_0$) with $f = f_{\text{int}}$ and $h = U_{\text{ext}}^0|_\Gamma$.

Since f_{int} and $U_{\text{ext}}^0|_\Gamma$ are smooth and infinitely flat near the corner, we can apply Theorem 3.3 to obtain the splitting:

$$u_{\text{int}}^0 = u_{\text{int}}^{0,K} + \chi \sum_{\lambda \in \mathfrak{S}(K)} c_\lambda^0\, \mathfrak{s}^\lambda(r,\theta) \qquad (c_\lambda^0 \in \mathbb{R}), \tag{4.6}$$

where $u_{\text{int}}^{0,K} = \mathcal{O}(r^K)$ near the corner O and, more precisely, $u^{0,K} \in \mathrm{H}^\infty_{-K-1}(\Omega_{\text{int}})$. In $\Omega_{\text{ext}}^\varepsilon$, we do not modify U_{ext}^0 and set $u_{\text{ext}}^{0,K}(t,s) = U_{\text{ext}}^0(t, \frac{s}{\varepsilon})$ – notice here that the equality makes sense since U_{ext}^0 vanishes in a neighborhood of O, see (4.23). Thus we have defined $u^{0,K}$ in the entire domain Ω^ε.

The main idea in our construction is, instead of considering $(u_{\text{int}}^0, U_{\text{ext}}^0)$ as a first term, to modify it by substituting the $\mathfrak{s}^\lambda(x)$ occurring in its singular part with the profiles $\varepsilon^\lambda \mathfrak{K}^\lambda(\frac{x}{\varepsilon})$, defining a new term, $\tilde{u}_\varepsilon^0$:

$$\tilde{u}_\varepsilon^0(x) = u^{0,K}(x) + \chi(x) \sum_{\lambda \in \mathfrak{S}(K)} \varepsilon^\lambda\, c_\lambda^0\, \mathfrak{K}^\lambda\big(\tfrac{x}{\varepsilon}\big) \quad (x \in \Omega^\varepsilon). \tag{4.7}$$

We recall that rapid variables $\frac{x}{\varepsilon}$ are defined by the homothecy centered in the interior corner point O with ratio ε^{-1} and that $\mathfrak{s}_0^\lambda$ denotes the extension of $\mathfrak{s}^\lambda$ by 0 in the exterior part.

The arguments in favor of considering (4.7) instead (4.6) rely on Theorem 3.15:

- Since $\mathfrak{K}^\lambda$ solves the homogeneous problem (P$_\infty$), $\mathfrak{K}^\lambda\big(\frac{x}{\varepsilon}\big)$ solves problem (P$_\varepsilon$) with zero data in the neighborhood $\mathcal{V}$ of O,

- Since $\varepsilon^\lambda \mathfrak{s}^\lambda(\frac{x}{\varepsilon}) = \mathfrak{s}^\lambda(x)$, we find

$$\varepsilon^\lambda\, \mathfrak{K}^\lambda\big(\tfrac{x}{\varepsilon}\big) - \mathfrak{s}_0^\lambda(r,\theta) = \mathcal{O}(\varepsilon) \quad \text{for} \quad x \notin \mathcal{V}'.$$

Thus we take $\tilde{u}_\varepsilon^0$ as our starting point for the expansion of u_ε, and define the first remainder as

$$\tilde{r}_\varepsilon^1 = u_\varepsilon - \tilde{u}_\varepsilon^0 \tag{4.8}$$

4.3.2 Further decomposition of the first remainder

We are going to prove that the first remainder $\tilde{r}_\varepsilon^1$ is solution of problem (P$_\varepsilon$) with data which can be expanded in positive powers of ε:

Let us set $w_\varepsilon^0 = \tilde{u}_\varepsilon^0 - u^0$ in Ω_{int} and $\Omega_{\text{ext}}^\varepsilon$, that is:

$$w_\varepsilon^0 = \chi \sum_{\lambda\in\mathfrak{S}(K)} \varepsilon^\lambda\, c_\lambda^0 \left[\mathfrak{K}^\lambda - \mathfrak{s}_0^\lambda\right]\big(\tfrac{x}{\varepsilon}\big). \tag{4.9}$$

Then $\tilde{r}_\varepsilon^1 = u_\varepsilon - u^0 - w_\varepsilon^0$ and the problem satisfied by $\tilde{r}_\varepsilon^1$ is

$$\left\{\begin{array}{rcll}
\alpha\Delta\tilde{r}^1_{\varepsilon,\text{int}} &=& -\alpha\Delta w^0_{\varepsilon,\text{int}} & \text{in } \Omega_{\text{int}},\\
\Delta\tilde{r}^1_{\varepsilon,\text{ext}} &=& -\Delta w^0_{\varepsilon,\text{ext}} + (f_{\varepsilon,\text{ext}} - f^0_{\varepsilon,\text{ext}}) & \text{in } \Omega^\varepsilon_{\text{ext}},\\
\tilde{r}^1_{\varepsilon,\text{int}} - \tilde{r}^1_{\varepsilon,\text{ext}} &=& 0 & \text{on } \Gamma,\\
\alpha\partial_n\tilde{r}^1_{\varepsilon,\text{int}} - \partial_n\tilde{r}^1_{\varepsilon,\text{ext}} &=& -(\alpha\partial_n w^0_{\varepsilon,\text{int}} - \partial_n w^0_{\varepsilon,\text{ext}}) + g - \alpha\partial_n u^0_{\text{int}} & \text{on } \Gamma,\\
\tilde{r}^1_{\varepsilon,\text{ext}} &=& 0 & \text{on } \Gamma^\varepsilon_{\text{ext}}.
\end{array}\right. \tag{4.10}$$

Here $f^0_{\varepsilon,\text{ext}} = \sum_\ell \varepsilon^{-\ell-2}\frac{1}{\ell!}\partial_n^\ell f_{\text{ext}}^{-\ell}(t,0)\, s^\ell$. Moreover, thanks to (4.1), (4.6) and (4.9) we find that

$$-\left(\alpha\partial_n w^0_{\varepsilon,\text{int}} - \partial_n w^0_{\varepsilon,\text{ext}}\right) + g - \alpha\partial_n u^0 = g - \alpha\partial_n u^{0,K} \quad \text{on} \quad \Gamma. \tag{4.11}$$

Here we have taken advantage of the fact that $\mathfrak{K}^\lambda$ satisfies $\alpha\partial_n\mathfrak{K}^\lambda_{\text{int}} - \partial_n\mathfrak{K}^\lambda_{\text{ext}} = 0$.

Comparing then problem (4.10) with the problem (2.10) satisfied by the standard remainder $u_\varepsilon - u^0$, we find the presence of Δw_ε^0 inside Ω_{int} and $\Omega^\varepsilon_{\text{ext}}$ instead of 0, and $\alpha\partial_n u^{0,K}_{\text{int}}$ instead of $\alpha\partial_n u^0_{\text{int}}$ on Γ. Thus we have gained regularity on Γ, but, in return, have to evaluate Δw_ε^0, see Lemma 4.2. New sets of indices have now to be introduced:

Definition 4.1 *Let $\mathfrak{U}$ be the infinite set of non negative numbers*

$$\mathfrak{U} = \mathbb{N} \cup \left\{\mu = \tfrac{h\pi}{\omega} + p\,;\ p \geq 0, h \geq 2\right\}, \tag{4.12}$$

and for any $N > 0$, let $\mathfrak{U}(N) = \mathfrak{U} \cap [0, N]$.

Moreover we denote the subset of the positive elements of $\mathfrak{U}$ by $\mathfrak{U}^*$:

$$\mathfrak{U}^* = \mathfrak{U} \setminus \{0\} \quad \textit{and} \quad \mathfrak{U}^*(N) = \mathfrak{U}(N) \setminus \{0\}. \tag{4.13}$$

Lemma 4.2 *In Ω_{int} and $\Omega^\varepsilon_{\text{ext}}$, for all number $N > 0$ the residual Δw_ε^0 can be written as*

$$\left\{\begin{array}{rcl}
\Delta w^0_{\varepsilon,\text{int}} &=& \displaystyle\sum_{\nu\,\in\,\mathfrak{U}^*(N)} \varepsilon^\nu k^{0,\nu}_{\varepsilon,\text{int}} + k^0_{\text{rem}}(\varepsilon)\big|_{\Omega_{\text{int}}}\\
\Delta w^0_{\varepsilon,\text{ext}} &=& \displaystyle\sum_{\nu\,\in\,\mathfrak{U}^*(N)} \varepsilon^\nu k^{0,\nu}_{\varepsilon,\text{ext}} + k^0_{\text{rem}}(\varepsilon)\big|_{\Omega_{\text{ext}}}
\end{array}\right. \quad \textit{with} \quad \left\|k^0_{\text{rem}}(\varepsilon)\right\|_{0,\Omega^\varepsilon} = \mathcal{O}(\varepsilon^N). \tag{4.14}$$

The functions $k^{0,\nu}_{\varepsilon,\mathrm{int}}$ *and* $k^{0,\nu}_{\varepsilon,\mathrm{ext}}$ *are* $\mathcal{C}^\infty$ *and vanish near the corner point* O. *Their behavior in* ε *is the following*

$$\begin{cases} k^{0,\nu}_{\varepsilon,\mathrm{int}} &= k^{0,\nu}_{\mathrm{int}}[\log\varepsilon] \quad \textit{i.e. possible polynomial dependence in } \log\varepsilon, \\ k^{0,\nu}_{\varepsilon,\mathrm{ext}} &= \displaystyle\sum_{\ell\in\mathbb{N},\ \mathrm{finite}} \varepsilon^{-2-\ell}\, k^{0,\nu\,;\,-\ell}_{\mathrm{ext}}[\log\varepsilon] \quad \textit{with} \quad \partial^k_n k^{0,\nu\,;\,-\ell}_{\mathrm{ext}}\big|_\Gamma \equiv 0,\ k=0,\ldots,\ell-1. \end{cases} \tag{4.15}$$

Remark 4.3 The degree in $\log\varepsilon$ of $k^{0,\nu}_\varepsilon$ is $\leq \nu$. Moreover, if $\frac{\pi}{\omega}\notin\mathbb{Q}$, no logarithm appears. □

Proof: From the definition (4.9) of w^0_ε, and since, by construction $\Delta\mathfrak{K}^\lambda = \Delta\mathfrak{s}^\lambda_0 = 0$ inside Q_{int} and Q_{ext}, we find inside Ω_{int} and $\Omega^\varepsilon_{\mathrm{ext}}$

$$\Delta w^0_\varepsilon = \sum_{\lambda\in\mathfrak{S}(K)} c^0_\lambda\,\varepsilon^\lambda \Big(2\nabla\chi\cdot\nabla\Big[\big(\mathfrak{K}^\lambda-\mathfrak{s}^\lambda_0\big)\big(\tfrac{x}{\varepsilon}\big)\Big] + \Delta\chi\,\big(\mathfrak{K}^\lambda-\mathfrak{s}^\lambda_0\big)\big(\tfrac{x}{\varepsilon}\big)\Big).$$

We now use the expansion (3.19) of $\mathfrak{K}^\lambda$ given in Theorem 3.15 with $P=N-\lambda$:

$$\Delta w^0_\varepsilon = \sum_{\lambda\in\mathfrak{S}(K)} c^0_\lambda\,\varepsilon^\lambda \sum_{\mu\in\mathfrak{Q}^\lambda(N-\lambda)} \Big(2\nabla\chi\cdot\nabla\Big[\mathfrak{K}^{\lambda,\mu}\big(\tfrac{x}{\varepsilon}\big)\Big] + \Delta\chi\,\mathfrak{K}^{\lambda,\mu}\big(\tfrac{x}{\varepsilon}\big)\Big) + k^0_{\mathrm{rem}}(\varepsilon), \tag{4.16}$$

with a remainder $k^0_{\mathrm{rem}}(\varepsilon)$.

(i) In Ω_{int} each term $\mathfrak{K}^{\lambda,\mu}_{\mathrm{int}}$ satisfies an homogeneity property modulo logarithms, cf. (3.8):

$$\mathfrak{K}^{\lambda,\mu}_{\mathrm{int}}\big(\tfrac{x}{\varepsilon}\big) = \varepsilon^{-\mu}\mathfrak{F}^{\lambda,\mu}[\log\varepsilon](x) \quad\text{and}\quad \nabla\big(\mathfrak{K}^{\lambda,\mu}_{\mathrm{int}}\big(\tfrac{x}{\varepsilon}\big)\big) = \varepsilon^{-\mu}\nabla\mathfrak{F}^{\lambda,\mu}[\log\varepsilon](x),$$

Thus equation (4.16) becomes in Ω_{int}

$$\Delta w^0_\varepsilon = \sum_{\lambda\in\mathfrak{S}(K)}\ \sum_{\mu\in\mathfrak{Q}^\lambda(N-\lambda)} c^0_\lambda\,\varepsilon^{\lambda-\mu}\Big(2\nabla\chi\cdot\nabla\mathfrak{F}^{\lambda,\mu}[\log\varepsilon] + \Delta\chi\,\mathfrak{F}^{\lambda,\mu}[\log\varepsilon]\Big) + k^0_{\mathrm{rem}}(\varepsilon), \tag{4.17}$$

where the remainder $k^0_{\mathrm{rem}}(\varepsilon)$ satisfies, thanks to (3.29)-(3.30) and to assumption (4.1):

$$k^0_{\mathrm{rem}}(\varepsilon) = \sum_{\lambda\in\mathfrak{S}(K)} \varepsilon^\lambda\, c^0_\lambda\Big[2\varepsilon^{-1}\nabla\chi\cdot\vec{F}\big(\tfrac{x}{\varepsilon}\big) + \Delta\chi\,F\big(\tfrac{x}{\varepsilon}\big)\Big], \quad\text{with}$$

$$F(X) = \mathcal{O}\Big(|X|^{\lambda-N}\Big) \quad\text{and}\quad \vec{F}(X) = \mathcal{O}\Big(|X|^{\lambda-N-1}\Big) \quad\text{when}\quad |X|\to+\infty.$$

To estimate the norm of this remainder, we notice that its support is contained in an annulus defined by $0<r_1<|x|<r_2$. Hence

$$\big\|k^0_{\mathrm{rem}}(\varepsilon)\big\|^2_{0,\Omega_{\mathrm{int}}} \leq \mathcal{O}(1)\int_{r_1}^{r_2}\left|\frac{t}{\varepsilon}\right|^{-2N} t\ \mathrm{d}t = \mathcal{O}(\varepsilon^{2N}).$$

Finally, we check that the set of the $\nu=\lambda-\mu$ when $\lambda\in\mathfrak{S}(K)$ and $\mu\in\mathfrak{Q}^\lambda(N-\lambda)$ is contained in the set $\mathfrak{U}^*(N)$. We reorder the sum (4.17) according to the values ν of $\lambda-\mu$, defining the functions $k^{0,\nu}_{\varepsilon,\mathrm{int}}$, and we obtain (4.14) in Ω_{int}.

(ii) In $\Omega_{\rm ext}$ each term $\mathfrak{K}^{\lambda,\mu}_{\rm ext}$ satisfies

$$\mathfrak{K}^{\lambda,\mu}_{\rm ext}\big(\tfrac{x}{\varepsilon}\big) = \varepsilon^{-\mu} \sum_{\ell=0}^{[\lambda-\mu]} \varepsilon^{-\ell}\, \mathfrak{F}^{\lambda,\mu\,;\,\ell}[\log\varepsilon](t)\, s^{\ell}$$

and a similar formula for its gradient. Again, we reorder the sum (4.17) according to the values ν of $\lambda-\mu+2$, defining the functions $k^{0,\nu}_{\varepsilon,\rm ext}$. The above splitting of $\mathfrak{K}^{\lambda,\mu}_{\rm ext}\big(\tfrac{x}{\varepsilon}\big)$ yields expression (4.15) for $k^{0,\nu}_{\varepsilon,\rm ext}$. The estimate of the remainder is similar. ∎

Taking advantage of Lemma 4.2, we come back to problem (4.10) solved by the first remainder $\tilde r^1_\varepsilon$ and prove:

Lemma 4.4 *The first remainder* $\tilde r^1_\varepsilon$ (4.8) *satisfies (for all* $N>0$*)*

$$\tilde r^1_\varepsilon = \sum_{\nu\,\in\,\mathfrak{U}^*(N)} \varepsilon^{\nu} v^{0,\nu}_\varepsilon[\log\varepsilon] + r^{0,1}_\varepsilon + \mathcal{O}(\varepsilon^N), \tag{4.18}$$

where $v^{0,\nu}_\varepsilon[\log\varepsilon]$ *is defined as the solution of the problem* (P_ε) *with data*

$$\begin{array}{lll} f_{\rm int} = \alpha k^{0,\nu}_{\rm int}[\log\varepsilon], & f_{\rm ext} = k^{0,\nu}_{\rm ext}[\log\varepsilon] + \varepsilon^{-1}(f_{\varepsilon,\rm ext} - f^0_{\varepsilon,\rm ext}), & \textit{if } \nu = 1,\\ f_{\rm int} = \alpha k^{0,\nu}_{\rm int}[\log\varepsilon], & f_{\rm ext} = k^{0,\nu}_{\rm ext}[\log\varepsilon], & \textit{if } \nu \neq 1,\end{array}$$

and zero boundary data on Γ*. The new remainder* $r^{0,1}_\varepsilon$ *is solution of*

$$\left\{\begin{array}{rcll} \alpha\Delta r^{0,1}_{\varepsilon,\rm int} &=& 0 & \textit{in } \Omega_{\rm int},\\ \Delta r^{0,1}_{\varepsilon,\rm ext} &=& 0 & \textit{in } \Omega^\varepsilon_{\rm ext},\\ r^{0,1}_{\varepsilon,\rm int} - r^{0,1}_{\varepsilon,\rm ext} &=& 0 & \textit{on } \Gamma,\\ \alpha\partial_n r^{0,1}_{\varepsilon,\rm int} - \partial_n r^{0,1}_{\varepsilon,\rm ext} &=& g - \alpha\partial_n u^{0,K}_{\rm int} & \textit{on } \Gamma,\\ r^{0,1}_{\varepsilon,\rm ext} &=& 0 & \textit{on } \Gamma^\varepsilon_{\rm ext}. \end{array}\right.$$

Proof: The problem solved by $r^{0,1}_{\varepsilon,\rm int}$ directly results from the definitions. We only need to check that the final remainder is $\mathcal{O}(\varepsilon^N)$: it is produced by $k^0_{\rm rem}$, cf. (4.14), whose contribution is of order ε^N thanks to the a priori estimate (1.3). ∎

As a corollary, since $u_\varepsilon = \tilde u^0_\varepsilon + \tilde r^1_\varepsilon$, gathering formulas (4.7) and (4.18), we find

Corollary 4.5 *The solution* u_ε *of problem* (P_ε) *with assumptions* (4.3)-(4.4) *on the data satisfies for all* $N>0$

$$u_\varepsilon = u^{0,K} + \chi \sum_{\lambda\in\mathfrak{S}(K)} \varepsilon^{\lambda}\, c^0_\lambda\, \mathfrak{K}^{\lambda}\big(\tfrac{x}{\varepsilon}\big) + \sum_{\nu\,\in\,\mathfrak{U}^*(N)} \varepsilon^{\nu} v^{0,\nu}_\varepsilon[\log\varepsilon] + r^{0,1}_\varepsilon + \mathcal{O}(\varepsilon^N), \tag{4.19}$$

where the terms $v^{0,\nu}_\varepsilon[\log\varepsilon]$ *and the new remainder* $r^{0,1}_\varepsilon$ *are solutions of problems* (P_ε) *given in Lemma* 4.4.

By construction, $\varepsilon^{-1}(f_{\varepsilon,\mathrm{ext}}-f^0_{\varepsilon,\mathrm{ext}})$ satisfies assumption (4.3), and so do the terms $k^{0,\nu}_{\mathrm{ext}}[\log\varepsilon]$, see (4.15). Thus, each term $v^{0,\nu}$ solves a problem (P_ε) with a right-hand side satisfying the same conditions as the original one, which shows that $v^{0,\nu}$ reproduces the same structure (4.19) as u_ε:

$$\varepsilon^\nu v^{0,\nu}_\varepsilon[\log\varepsilon] = \varepsilon^\nu u^{\nu,K-\nu}[\log\varepsilon] + \chi \sum_{\lambda\in\mathfrak{S}(K-\nu)} \varepsilon^{\nu+\lambda}\, c^\nu_\lambda[\log\varepsilon]\, \mathfrak{K}^\lambda\big(\tfrac{x}{\varepsilon}\big) + \sum_{\nu'\,\in\,\mathfrak{U}^*(N-\nu)} \varepsilon^{\nu+\nu'} v^{\nu,\nu'}_\varepsilon[\log\varepsilon] + \varepsilon^\nu r^{\nu,1}_\varepsilon + \mathcal{O}(\varepsilon^N). \quad (4.20)$$

Note that the equality $\mathfrak{U}+\mathfrak{U}=\mathfrak{U}$ ensures that the exponents generated by $\varepsilon^\nu v^{0,\nu}_\varepsilon$ for $\nu\in\mathfrak{U}$ remain in $\mathfrak{U}$.

4.3.3 Terms of order 1

To continue the expansion construction, the only term we need to study is the new remainder at order 1, $r^{0,1}_\varepsilon$.

To explore the content of $r^{0,1}$, applying the formulas of the smooth case, cf. Proposition 2.3, we define u^1_{int} as the solution of the Dirichlet problem

$$\begin{cases} \alpha\Delta u^1_{\mathrm{int}} = 0 & \text{in } \Omega_{\mathrm{int}},\\ u^1_{\mathrm{int}} = -\alpha\partial_n u^{0,K}_{\mathrm{int}}|_\Gamma + g & \text{on } \Gamma. \end{cases}$$

Since $u^{0,K}_{\mathrm{int}}$ belongs to the weighted space $\mathrm{H}^\infty_{-K-1}(\Omega_{\mathrm{int}})$, the normal trace $\partial_n u^{0,K}$ belongs to $\mathrm{H}^\infty_{-K+1/2}(\Gamma)$, and the above Dirichlet problem in Ω_{int} has a solution which can be itself split according to Theorem 3.3

$$u^1_{\mathrm{int}} = u^{1,K-1}_{\mathrm{int}} + \chi \sum_{\lambda\in\mathfrak{S}(K-1)} c^1_\lambda\, \mathfrak{s}^\lambda(r,\theta) \quad \text{with} \quad u^{1,K-1}_{\mathrm{int}} \in \mathrm{H}^\infty_{-K}(\Omega_{\mathrm{int}}), \qquad (4.21)$$

if we assume that $K-1\notin\mathfrak{S}$. We note that $u^{1,K-1}_{\mathrm{int}} = \mathcal{O}(r^{K-1})$.

According to the formulas for the regular case, we define $U^1_{\mathrm{ext}}(t,S) = U^{1,K-1}_{\mathrm{ext}}(t,S)$ by

$$U^{1,K-1}_{\mathrm{ext}}(t,S) = (S-1)\Big\{\alpha\partial_n u^{0,K}_{\mathrm{int}}|_\Gamma - g\Big\}(t) \quad \text{for} \quad (t,S)\in\Gamma\times[0,1], \qquad (4.22)$$

which does not make sense in the entire layer $\Omega^\varepsilon_{\mathrm{ext}}$. Since $u^{0,K}_{\mathrm{int}}$ does not identically vanish in any neighborhood of O, we have to use the cut-off $x\mapsto\psi\big(\frac{x}{\varepsilon}\big)$, cf. Definition 3.5, to define $u^1_{\mathrm{ext}} = u^{1,K-1}_{\mathrm{ext}}$ in an unambiguous way:

$$u^{1,K-1}_{\mathrm{ext}} = \psi\big(\tfrac{x}{\varepsilon}\big)\, U^{1,K-1}_{\mathrm{ext}}(t,S) = \psi\big(\tfrac{x}{\varepsilon}\big)\,(S-1)\Big\{\alpha\partial_n u^{0,K}_{\mathrm{int}}|_\Gamma - g\Big\}(t). \qquad (4.23)$$

Then, as a continuation of Lemma 4.4, we state

Lemma 4.6 *The remainder $r^{0,1}_\varepsilon$ in* (4.19) *can be split in*

$$r^{0,1}_\varepsilon = \varepsilon u^{1,K-1} + \chi \sum_{\lambda\in\mathfrak{S}(K-1)} \varepsilon^{1+\lambda}\, c^1_\lambda[\log\varepsilon]\, \mathfrak{K}^\lambda\big(\tfrac{x}{\varepsilon}\big) + \sum_{\nu\,\in\,\mathfrak{U}^*(N-1)} \varepsilon^{1+\nu} v^{1,\nu}_\varepsilon[\log\varepsilon] + r^{0,2}_\varepsilon + \mathcal{O}\big(\varepsilon^{\min\{K-1,N\}}\big), \quad (4.24)$$

where the $v_\varepsilon^{1,\nu}[\log\varepsilon]$ *solve problem* (P_ε) *with data satisfying conditions* (4.2)-(4.3) *and the residual term* $r_\varepsilon^{0,2}$ *is solution of:*

$$\begin{cases} \alpha\Delta r^{0,2}_{\varepsilon,\mathrm{int}} &= 0 & \textit{in } \Omega_{\mathrm{int}}, \\ \Delta r^{0,2}_{\varepsilon,\mathrm{ext}} &= -\psi(\frac{x}{\varepsilon})\ \mathbf{R}^1_\varepsilon U^{1,K-1}_{\mathrm{ext}} & \textit{in } \Omega^\varepsilon_{\mathrm{ext}}, \\ r^{0,2}_{\varepsilon,\mathrm{int}} - r^{0,2}_{\varepsilon,\mathrm{ext}} &= 0 & \textit{on } \Gamma, \\ \alpha\partial_n r^{0,2}_{\varepsilon,\mathrm{int}} - \partial_n r^{0,2}_{\varepsilon,\mathrm{ext}} &= -\varepsilon\alpha\partial_n u^{1,K-1}_{\mathrm{int}} & \textit{on } \Gamma, \\ r^{0,2}_{\varepsilon,\mathrm{ext}} &= 0 & \textit{on } \Gamma^\varepsilon_{\mathrm{ext}}, \end{cases}$$

where $\mathbf{R}^1_\varepsilon$ *pertains to the expansion of* Δ *in curvilinear coordinates around* Γ*, see* (2.2).

Proof: The sum of the second and the third block on the right hand side of (4.24) is constructed so as to contribute $\mathcal{O}(\varepsilon^N)$ data for problem (P_ε), therefore generating a remainder of the same order $\mathcal{O}(\varepsilon^N)$. Combining formulas for $r_\varepsilon^{0,1}$, $u^{1,K-1}$ and $r_\varepsilon^{0,2}$, we find that (4.24) holds with an additional term p_ε, solution of the problem

$$\begin{cases} \alpha\Delta p_{\varepsilon,\mathrm{int}} &= 0 & \text{in } \Omega_{\mathrm{int}}, \\ \Delta p_{\varepsilon,\mathrm{ext}} &= -\varepsilon\big[\Delta,\psi(\frac{x}{\varepsilon})\big]\ u^{1,K-1}_{\mathrm{ext}} & \text{in } \Omega^\varepsilon_{\mathrm{ext}}, \\ p_{\varepsilon,\mathrm{int}} - p_{\varepsilon,\mathrm{ext}} &= 0 & \text{on } \Gamma, \\ \alpha\partial_n p_{\varepsilon,\mathrm{int}} - \partial_n p_{\varepsilon,\mathrm{ext}} &= \big(1-\psi(\frac{x}{\varepsilon})\big)(g-\alpha\partial_n u^{0,K}_{\mathrm{int}}) & \text{on } \Gamma, \\ p_{\varepsilon,\mathrm{ext}} &= 0 & \text{on } \Gamma^\varepsilon_{\mathrm{ext}}, \end{cases}$$

where $\big[\Delta,\psi(\frac{x}{\varepsilon})\big]$ denotes the commutator of Δ with the multiplication by $\psi(\frac{x}{\varepsilon})$. Making use of the fact that the support of g does not intersect the support of $1-\psi(\frac{x}{\varepsilon})$ and that $u^{0,K}$ belongs to the weighted space $\mathrm{H}^\infty_{-K-1}(\Omega_{\mathrm{int}})$, ensuring a behavior in $O(r^{K-1})$ for $\partial_n u^{0,K}$, we check:

$$\big\|\varepsilon\big[\Delta,\psi(\tfrac{x}{\varepsilon})\big]\,u^{1,K-1}_{\mathrm{ext}}\big\|_{0,\Omega^\varepsilon_{\mathrm{ext}}} = \mathcal{O}\big(\varepsilon^{K-1}\big) \quad\text{and}\quad \big\|(1-\psi(\tfrac{x}{\varepsilon}))(g-\alpha\partial_n u^{0,K}_{\mathrm{int}})\big\|_{0,\Gamma} = \mathcal{O}(\varepsilon^{K-\frac12}).$$

A priori estimate (1.3) then yields that $\big\|p_\varepsilon\big\|_{1,\Omega^\varepsilon} = \mathcal{O}(\varepsilon^{K-1})$.

We note that the number K can be slightly shifted upwards so that the set $\mathfrak{S}(K)$ remains unchanged, but guaranteeing that $u^{0,K}$ is a little flatter, so that our remainder can be written as $\mathcal{O}(\varepsilon^{K-1})$. ∎

4.4 Complete ε-expansions

4.4.1 Data with zero Taylor expansion at the corner point

The above construction of the first terms in the asymptotic expansion of the solution u_ε of (P_ε) can be extended to any order. Only two kinds of terms appear in this expansion:

- The "flat" terms $u^{\nu,K-\nu}$ which have a similar structure as the terms in the expansion (2.8) of the smooth case. They are linked with each other by the formulas (2.13) and (2.15) of the smooth case. Their exterior parts are functions of the semi-scaled variables $(t,\varepsilon^{-1}s)$ whereas their interior parts are functions in the "slow" variable x. They vanish at the corner O like a $\mathcal{O}(r^{K-\nu})$.

- The profiles $\mathfrak{K}^\lambda$ which take into account the singular behavior of u_ε near the corner point and involve the scaled variable $\frac{x}{\varepsilon}$.

We recall that χ and ψ are cut-off functions respectively equal to 1 and 0 in a neighborhood of the corner point O. The notation $[\log\varepsilon]$ marks a polynomial dependence with respect to $\log\varepsilon$.

Theorem 4.7 *Let u_ε be solution of* (P_ε) *with data satisfying* (4.3)-(4.4). *Let $K>0$ be a number such that $K, K-1, \dots, K-[K]$ do not belong to the set of singular exponents $\mathfrak{S}$, and $\mathfrak{S}(K)$ denote $\mathfrak{S}\cap(0,K)$. Let $N>0$ be a number such that $N+\frac{3}{2}<K$. We recall from Definition* 4.1 *that $\mathfrak{U}(N)$ denotes $\mathfrak{U}\cap[0,N]$ with $\mathfrak{U}=\mathbb{N}\cup\big\{\mu=\frac{h\pi}{\omega}+p\ ;\ p\geq 0, h\geq 2\big\}$. Then, u_ε admits the following asymptotic expansion:*

$$u_{\varepsilon,\mathrm{int}} \;=\; \sum_{\nu\in\,\mathfrak{U}(N)} \varepsilon^\nu u_{\mathrm{int}}^{\nu,K-\nu}[\log\varepsilon] \;+\; \chi(x)\sum_{\nu\in\,\mathfrak{U}(N)}\;\sum_{\lambda\in\mathfrak{S}(K-\nu)} c_\lambda^\nu[\log\varepsilon]\,\varepsilon^{\nu+\lambda}\mathfrak{K}_{\mathrm{int}}^\lambda\big(\tfrac{x}{\varepsilon}\big) \;+\; r_{\varepsilon,\mathrm{int}}^N \tag{4.25}$$

$$u_{\varepsilon,\mathrm{ext}} \;=\; \psi\big(\tfrac{x}{\varepsilon}\big)\sum_{\nu\in\,\mathfrak{U}(N)} \varepsilon^\nu U_{\mathrm{ext}}^{\nu,K-\nu}\big(t,\tfrac{s}{\varepsilon}\big)\,[\log\varepsilon] \;+\; \chi(x)\sum_{\nu\in\,\mathfrak{U}(N)}\;\sum_{\lambda\in\mathfrak{S}(K-\nu)} c_\lambda^\nu[\log\varepsilon]\,\varepsilon^{\nu+\lambda}\mathfrak{K}_{\mathrm{ext}}^\lambda\big(\tfrac{x}{\varepsilon}\big) \;+\; r_{\varepsilon,\mathrm{ext}}^N \tag{4.26}$$

with a remainder r_ε^N satisfying the estimates

$$\big\|r_\varepsilon^N\big\|_{1,\Omega_{\mathrm{int}}} + \sqrt{\varepsilon}\big\|r_\varepsilon^N\big\|_{1,\Omega_{\mathrm{ext}}} = \mathcal{O}(\varepsilon^N). \tag{4.27}$$

Moreover, $u_{\mathrm{int}}^{\nu,K-\nu}$ and $U_{\mathrm{ext}}^{\nu,K-\nu}$ vanish as $r\to 0$ according to

$$U_{\mathrm{ext}}^{\nu,K-\nu} = \mathcal{O}(r^{K-\nu}) \quad \textit{and} \quad u_{\mathrm{int}}^{\nu,K-\nu} = \mathcal{O}(r^{K-\nu})$$

– more precisely, $u_{\mathrm{int}}^{\nu,K-\nu}\in \mathrm{H}^\infty_{-1-K+\nu}(\Omega_{\mathrm{int}})$. Finally $U_{\mathrm{ext}}^{\nu,K-\nu}$ is polynomial in the variable S.

Proof: We continue the procedure initiated in Lemmas 4.4 and 4.6, that is, we expand $r_\varepsilon^{0,2}$ in (4.24) as $r^{0,1}$ before, but leave the other terms unexpanded, and so on. The successive terms along this "main branch" are given recursively for $n=1,\dots,N+1$ by:

- u_{int}^n is the solution of problem (P_0) with $f_{\mathrm{int}}=0$ and the Dirichlet data

  $$h^n = \mathbf{g}^n g + \mathbf{h}^1 u_{\mathrm{int}}^{n-1,K-n+1}|_\Gamma + \ldots + \mathbf{h}^n u_{\mathrm{int}}^{0,K}|_\Gamma,$$

  compare with (2.13),

- u_{int}^n is split in

  $$u_{\mathrm{int}}^n = u_{\mathrm{int}}^{n,K-n} + \chi\sum_{\lambda\in\mathfrak{S}(K-n)} c_\lambda^n\,\mathfrak{s}^\lambda(r,\theta) \quad \text{with} \quad u_{\mathrm{int}}^{n,K-n}\in \mathrm{H}^\infty_{-1-K+n}(\Omega_{\mathrm{int}}),$$

  defining the "flat" part $u_{\mathrm{int}}^{n,K-n}$

- $u_{\mathrm{ext}}^{n,K-n}$ is defined as

  $$u_{\mathrm{ext}}^{n,K-n} = \psi\big(\tfrac{x}{\varepsilon}\big)\,U_{\mathrm{ext}}^{n,K-n} \quad \text{where} \quad U_{\mathrm{ext}}^{n,K-n} = \mathbf{a}^n g + \mathbf{b}^1 u_{\mathrm{int}}^{n-1,K-n+1} + \ldots + \mathbf{b}^n u_{\mathrm{int}}^{0,K}$$

  compare with (2.15),

- The remainder $r^{0,n+1}$ is solution of:

$$\begin{cases} \alpha\Delta r^{0,n+1}_{\varepsilon,\mathrm{int}} &= 0 & \text{in } \Omega_{\mathrm{int}},\\ \Delta r^{0,n+1}_{\varepsilon,\mathrm{ext}} &= -\psi(\tfrac{x}{\varepsilon})\,\varepsilon^{n-1}\big(\mathbf{R}^1_\varepsilon U^{n,K-n}_{\mathrm{ext}} + \ldots + \mathbf{R}^n_\varepsilon U^{1,K-1}_{\mathrm{ext}}\big) & \text{in } \Omega^\varepsilon_{\mathrm{ext}},\\ r^{0,n+1}_{\varepsilon,\mathrm{int}} - r^{0,n+1}_{\varepsilon,\mathrm{ext}} &= 0 & \text{on } \Gamma,\\ \alpha\partial_n r^{0,n+1}_{\varepsilon,\mathrm{int}} - \partial_n r^{0,n+1}_{\varepsilon,\mathrm{ext}} &= -\varepsilon^n\alpha\partial_n u^{n,K-n}_{\mathrm{int}} & \text{on } \Gamma,\\ r^{0,n+1}_{\varepsilon,\mathrm{ext}} &= 0 & \text{on } \Gamma^\varepsilon_{\mathrm{ext}}, \end{cases}$$

compare with the remainder of the smooth case (2.10).

With these constructions, we obtain expansions of u_ε of the following form:

$$u_\varepsilon = u^{0,K} + \varepsilon u^{1,K-1} + \ldots + \varepsilon^n u^{n,K-n} + \chi \sum_{\ell=0}^{n} \sum_{\lambda\in\mathfrak{S}(K-\ell)} \varepsilon^{\ell+\lambda}\, c^\ell_\lambda[\log\varepsilon]\,\mathfrak{K}^\lambda\big(\tfrac{x}{\varepsilon}\big)$$
$$+ \sum_{\ell=0}^{n} \sum_{\nu\in\mathfrak{U}^*(N-\ell)} \varepsilon^{\ell+\nu} v^{\ell,\nu}_\varepsilon[\log\varepsilon] + r^{0,n}_\varepsilon + \mathcal{O}\big(\varepsilon^{\min\{K-1,N\}}\big). \quad (4.28)$$

We have to estimate the "last" remainder with the help of the a priori estimate (1.3). Like for the smooth case, if we want to have a remainder in $\mathcal{O}(\varepsilon^N)$, we have first to estimate the remainder $r^{0,N+2}$ at the rank $N+2$. Since K is larger than $N+\frac{3}{2}$, the trace of $\partial_n u^{N+1,K-N-1}_{\mathrm{int}}$ on Γ belongs to $\mathrm{L}^2(\Gamma)$. Therefore we can prove like in the smooth case that

$$\big\|r^{0,N+2}_\varepsilon\big\|_{1,\Omega^\varepsilon} \le C\,\varepsilon^{N+\frac{1}{2}}.$$

Each $v^{\ell,\nu}_\varepsilon$ in (4.28) can be expanded in a similar way, thus generating other "branches" successively. Each of these branches starts with a common factor of ε^ν, $\nu > 0$. This shows that this recursive procedure terminates after a finite number of steps. We gather everything and conclude similarly to the smooth case by subsumming into the final remainder r^N_ε all the terms of the asymptotics with powers $\nu > N$ of ε. ■

4.4.2 Data with non-zero Taylor expansion at the corner point

Using the profiles $\mathfrak{W}^{k,(\beta)}$ introduced in Definition 3.28 we may consider general $\mathcal{C}^\infty$ functions for f_{int}, without condition on their Taylor expansion.

Corollary 4.8 *Under the general assumptions* (4.2)*, we still assume that the Taylor expansion of g at O is zero. Let $K>0$ be a non-integer number such that $K, K-1, \ldots, K-[K]$ do not belong to $\mathfrak{S}$. Let $N>0$ be a number such that $N+\frac{3}{2}<K$. Then u_ε, solution of* (P_ε)*, has an expansion similar to* (4.25) *with extra terms due to the Taylor part of degree $[K]-2$ of f_{int}. The interior expansion writes*

$$u_{\varepsilon,\mathrm{int}} = \sum_{\nu\in\mathfrak{T}(N)} \varepsilon^\nu u^{\nu,K-\nu}_{\mathrm{int}}[\log\varepsilon] + \chi(x)\sum_{\nu\in\mathfrak{T}(N)} \sum_{\lambda\in\mathfrak{S}(K-\nu)} c^\nu_\lambda[\log\varepsilon]\,\varepsilon^{\nu+\lambda}\mathfrak{K}^\lambda_{\mathrm{int}}\big(\tfrac{x}{\varepsilon}\big)$$
$$+ \chi(x)\sum_{k=2}^{[K]} \sum_{|\beta|=k-2} \frac{\partial^\beta f_{\mathrm{int}}(O)}{\beta_1!\beta_2!}\,\varepsilon^k\,\mathfrak{W}^{k,(\beta)}_{\mathrm{int}}\big(\tfrac{x}{\varepsilon}\big) + r^N_{\varepsilon,\mathrm{int}}. \quad (4.25')$$

The new index set $\mathfrak{T}(N)$ *is defined as* $\mathfrak{T}\cap[0,N]$ *where*

$$\mathfrak{T}=\mathfrak{U}\cup\left\{\tfrac{\pi}{\omega}+q;\quad q\in\mathbb{N},\ q\geq 1\right\}.$$

The exterior part $u^{\varepsilon}_{\text{ext}}$ *has a structure as in* (4.26)*, with new terms corresponding to those present in* (4.25')*. The remainder* r^N_ε *satisfies the estimates* (4.27)*.*

Proof: We first split f_{int} into a Taylor part at O and a remainder, flat at the order $[K]-2$

$$f_{\text{int}}=\chi(x)\sum_{k=2}^{[K]}\sum_{|\beta|=k-2}\frac{\partial^\beta f_{\text{int}}(O)}{\beta_1!\beta_2!}\,x^{\beta_1}x^{\beta_2}+f^{(K)}_{\text{rem}},\quad\text{with}\quad f^{(K)}_{\text{rem}}\in \mathrm{H}^\infty_{1-K}(\Omega_{\text{int}}),$$

Note that the remainder satisfies the assumption on the right hand side in Theorem 3.3.

Let us denote $\frac{1}{\beta_1!\beta_2!}\partial^\beta f_{\text{int}}(O)$ by d_β for short. Then we define v_ε and w_ε by

$$v_\varepsilon=u_\varepsilon-\chi(x)\sum_{k=2}^{[K]}\sum_{|\beta|=k-2}d_\beta\,\varepsilon^k\,\mathfrak{W}^{k,(\beta)}$$

and, in a similar way to (4.9)

$$w_\varepsilon=\chi(x)\sum_{k=2}^{[K]}\sum_{|\beta|=k-2}d_\beta\,\varepsilon^k\,\left[\mathfrak{W}^{k,(\beta)}-\mathfrak{W}^{k,k,(\beta)}\right]\left(\tfrac{x}{\varepsilon}\right).$$

Using (3.45), we find that the function v_ε solves the following problem of type (P_ε), similar to (4.10):

$$\left\{\begin{array}{rcll}\alpha\Delta v_{\varepsilon,\text{int}}&=&-\alpha\Delta w_{\varepsilon,\text{int}}+f^{(K)}_{\text{rem}}&\text{in }\Omega_{\text{int}},\\ \Delta v_{\varepsilon,\text{ext}}&=&-\Delta w_{\varepsilon,\text{ext}}+f_{\varepsilon,\text{ext}}&\text{in }\Omega^\varepsilon_{\text{ext}},\\ v_{\varepsilon,\text{int}}-v_{\varepsilon,\text{ext}}&=&0&\text{on }\Gamma,\\ \alpha\partial_n v_{\varepsilon,\text{int}}-\partial_n v_{\varepsilon,\text{ext}}&=&g&\text{on }\Gamma,\\ v_{\varepsilon,\text{ext}}&=&0&\text{on }\Gamma^\varepsilon_{\text{ext}}.\end{array}\right.\tag{4.29}$$

The right hand side of (4.29) is the sum of data satisfying (4.2)-(4.3) and of data similar to those investigated in Lemma 4.2: We find for $\Delta w_{\varepsilon,\text{int}}$ and $\Delta w_{\varepsilon,\text{ext}}$ expansions like in (4.14), involving the set of indices $\mathfrak{T}^*(N):=\mathfrak{T}(N)\setminus\{0\}$ instead of $\mathfrak{U}^*(N)$. ■

Remark 4.9 *(i)* If f_{int} vanishes up to the order $[K]-2$ in O, i.e. if

$$\partial^\beta f_{\text{int}}(O)=0,\quad\forall\beta,\ |\beta|\leq[K]-2$$

then expansion (4.25) is still valid.

(ii) We may cut off the "slow" terms $u^{\nu,K-\nu}_{\text{int}}$ in (4.25) or (4.25') by $\psi(\frac{x}{\varepsilon})$. Since $u^{\nu,K-\nu}_{\text{int}}$ is "flat" like $r^{K-\nu}$, we only produce a new contribution of order $\mathcal{O}(\varepsilon^K)$ to the remainder which, thus, still satisfies the estimates (4.27).

(iii) The terms $\breve{\mathfrak{W}}^{k,k-\ell,(\beta)}$ composing the asymptotics at infinity of the profiles $\mathfrak{W}^{k,(\beta)}$ are mainly polynomial functions. They are all polynomial if $k,k-1,\dots,0$ are not in $\mathfrak{S}$. Thus the $\mathfrak{W}^{k,(\beta)}$ take possible Taylor expansion of the solution into account. □

4.5 Alternative ε-expansions

In this section we answer the two questions:

- Is it possible to have $K = N$ in expansions (4.25) or (4.25') ?
- Is it possible to construct an asymptotic expansion independently of a threshold fixed in advance?

To answer (positively) to both questions, we start from expansions (4.25) or (4.25'), we split up some of the terms $\mathfrak{K}^\lambda$ and redistribute their pieces to the terms in slow variables. We base our analysis upon the following definition and result:

Definition 4.10 *Let $\lambda \in \mathfrak{S}$, $\lambda > 0$. Relying on* (3.14)*, we define on Q the profile $\mathfrak{Y}^\lambda$ as*

$$\mathfrak{Y}^\lambda_{\text{int}} = \mathfrak{K}^\lambda_{\text{int}} - \sum_{0 \le \ell < \lambda} \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_{\text{int}} \quad \textit{and} \quad \mathfrak{Y}^\lambda_{\text{ext}} = \mathfrak{K}^\lambda_{\text{ext}} - \psi \sum_{0 \le \ell < \lambda} \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_{\text{ext}}. \tag{4.30}$$

We are going to prove

Proposition 4.11 *Let $\lambda \in \mathfrak{S}$, $\lambda > 0$. The profile $\mathfrak{Y}^\lambda$ satisfies the estimates as $\varepsilon \to 0$*

$$\big\|\chi(x)\,\mathfrak{Y}^\lambda(\tfrac{x}{\varepsilon})\big\|_{1,\Omega_{\text{int}}} + \sqrt{\varepsilon}\big\|\chi(x)\,\mathfrak{Y}^\lambda(\tfrac{x}{\varepsilon})\big\|_{1,\Omega_{\text{ext}}} = \begin{cases} \mathcal{O}(1) & \textit{if } \lambda \notin \mathbb{N} \\ \mathcal{O}(|\log\varepsilon|^\lambda) & \textit{if } \lambda \in \mathbb{N}. \end{cases} \tag{4.31}$$

We prove this proposition as a particular case of the more general statement, which will also yield (1.7) as another particular case:

Lemma 4.12 *Let $\lambda \in \mathfrak{S}$, $\lambda > 0$. For $0 \le \nu \le \lambda$, we set*

$$\mathfrak{Y}^{\lambda,\nu} = \mathfrak{K}^\lambda - \psi \sum_{0 \le \ell < \lambda-\nu} \breve{\mathfrak{K}}^{\lambda,\lambda-\ell}. \tag{4.32}$$

There holds the energy estimate

$$\big\|\chi(x)\,\mathfrak{Y}^{\lambda,\nu}(\tfrac{x}{\varepsilon})\big\|_{1,\Omega_{\text{int}}} + \sqrt{\varepsilon}\big\|\chi(x)\,\mathfrak{Y}^{\lambda,\nu}(\tfrac{x}{\varepsilon})\big\|_{1,\Omega_{\text{ext}}} = \mathcal{O}\big(\varepsilon^{-\nu}|\log\varepsilon|^{[\lambda-\nu]_0}\big), \tag{4.33}$$

where $[\lambda-\nu]_0 = \lambda-\nu$ if $\lambda-\nu \in \mathbb{N}$ and $[\lambda-\nu]_0 = 0$ if not.

Proof: Thanks to (3.19) there holds for all $P > 0$

$$\mathfrak{Y}^{\lambda,\nu} = \psi \sum_{\mu \in \mathfrak{L}^\lambda(P),\ \mu \le \nu} \breve{\mathfrak{K}}^{\lambda,\mu} + \mathfrak{Y}^{\lambda,\nu}_{(P)},$$

where the remainder $\mathfrak{Y}^{\lambda,\nu}_{(P)}$ is a $\mathcal{O}(R^{-P})$ and satisfies also the estimates (3.30), whence

$$\big\|\chi(x)\,\mathfrak{Y}^{\lambda,\nu}_{(P)}(\tfrac{x}{\varepsilon})\big\|_{1,\Omega_{\text{int}}} + \sqrt{\varepsilon}\big\|\chi(x)\,\mathfrak{Y}^{\lambda,\nu}_{(P)}(\tfrac{x}{\varepsilon})\big\|_{1,\Omega_{\text{ext}}} = \mathcal{O}(1).$$

Let us choose $P < \frac{\pi}{\omega}$ and $P < [\lambda] + 1 - \lambda$. Thus $\mathfrak{L}^\lambda(P) \subset [0,\lambda]$, cf. Definition 3.14. The degree of $\breve{\mathfrak{K}}^{\lambda,\mu}$ as a polynomial in $\log R$ is $\le \lambda - \mu$. We check that for $\mu \ge 0$:

$$\big\|\chi(x)\,\psi(\tfrac{x}{\varepsilon})\breve{\mathfrak{K}}^{\lambda,\mu}(\tfrac{x}{\varepsilon})\big\|_{1,\Omega_{\text{int}}} + \sqrt{\varepsilon}\big\|\chi(x)\,\psi(\tfrac{x}{\varepsilon})\breve{\mathfrak{K}}^{\lambda,\mu}(\tfrac{x}{\varepsilon})\big\|_{1,\Omega_{\text{ext}}} = \mathcal{O}(\varepsilon^{-\mu}|\log\varepsilon|^{[\lambda-\mu]}).$$

Then estimate (4.33) is a consequence of the last three equalities. ∎

The proof of Proposition 4.11 is obtained by taking $\nu = 0$ in Lemma 4.12 (note that the absence of the cut-off function $\psi(\frac{x}{\varepsilon})$ in the definition of $\mathfrak{Y}^\lambda_{\rm int}$ does not modify the estimates).

The proof of (1.7) is obtained with $\nu = \lambda$.

Theorem 4.13 *Theorem* 4.7 *holds with* $K = N$*, i.e., we assume properties* (4.3)-(4.4) *on the data and choose a number* $N > 0$ *such that* $N, N-1, \ldots, N-[N]$ *do not belong to* $\mathfrak{S}$*. Then* u_ε*, solution of* (P_ε)*, admits the asymptotic expansion* (4.25) *with* $K = N$ *with the estimate* (4.27) *on the remainder.*

Proof: We start from (4.25) for a $K > N + \frac{3}{2}$. We want to get rid of the profiles $\mathfrak{K}^\lambda$ appearing in (4.25)-(4.26) for $\lambda > N - \nu$. Thus, for each $\nu \in \mathfrak{U}(N)$ and $\lambda \in \mathfrak{S}(K-\nu) \setminus \mathfrak{S}(N-\nu)$ we split $\mathfrak{K}^\lambda$ into two blocks according to

$$\chi(x)\varepsilon^{\nu+\lambda}\mathfrak{K}^\lambda_{\rm int}\big(\tfrac{x}{\varepsilon}\big) = \chi(x)\varepsilon^{\nu+\lambda}\mathfrak{Y}^\lambda_{\rm int}\big(\tfrac{x}{\varepsilon}\big) + \chi(x)\sum_{0\le\ell<\lambda}\varepsilon^{\nu+\lambda}\breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_{\rm int}\big(\tfrac{x}{\varepsilon}\big),$$

in $\Omega_{\rm int}$ and accordingly in $\Omega^\varepsilon_{\rm ext}$, and redistribute them into the remainder and the slow terms, respectively:

1. Since by definition $\nu + \lambda > N$, Proposition 4.11 yields that $\chi\varepsilon^{\nu+\lambda}\mathfrak{Y}^\lambda\big(\frac{x}{\varepsilon}\big)$ contributes to the remainder.

2. Thanks to their quasi-homogeneous structure the $\breve{\mathfrak{K}}^{\lambda,\lambda-\ell}$ can be converted into slow variable functions. We can write:

$$\begin{aligned}
\chi(x)\varepsilon^{\nu+\lambda}\breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_{\rm int}(\tfrac{x}{\varepsilon}) &= \chi(x)\sum_{q\ge0\ \rm finite}\varepsilon^{\nu+\lambda-\lambda+\ell}\log^q\varepsilon\ \mathfrak{s}^{\lambda,\lambda-\ell\,;\,q}_{\rm int}(x) \quad \text{in } \Omega_{\rm int}\\
\chi(x)\psi(\tfrac{x}{\varepsilon})\varepsilon^{\nu+\lambda}\breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_{\rm ext}(\tfrac{x}{\varepsilon}) &= \chi(x)\psi(\tfrac{x}{\varepsilon})\sum_{q\ge0\ \rm finite}\varepsilon^{\nu+\lambda-\lambda+\ell}\log^q\varepsilon\ \mathfrak{s}^{\lambda,\lambda-\ell\,;\,q}_{\rm ext}(t,\tfrac{s}{\varepsilon}) \quad \text{in } \Omega_{\rm ext}.
\end{aligned}$$

   We gather the above terms according to the value of $\nu' = \nu+\ell$ and we add them to $u^{\nu',K-\nu'}$ in order to obtain $u^{\nu',N-\nu'}$. Note that the $\mathfrak{s}^{\lambda,\lambda-\ell\,;\,q}$ are homogeneous of degree $\lambda-\ell$, and since $\lambda > N-\nu$, they are of order $\mathcal{O}(r^{N-\nu'})$ as $r \to 0$.

This ends the proof. ∎

The same splitting of the profiles $\mathfrak{K}^\lambda$, now applied *for all values of* λ, allows to prove the final theorem:

Theorem 4.14 *Let us assume the same hypotheses as in Theorem* 4.13*. We have the expansion*

$$u_{\varepsilon,\rm int} = \sum_{\nu\in\mathfrak{U}(N)}\varepsilon^\nu u^\nu_{\rm int}[\log\varepsilon] + \chi(x)\sum_{\nu\in\mathfrak{U}(N)}\sum_{\lambda\in\mathfrak{S}(N-\nu)}c^\nu_\lambda[\log\varepsilon]\,\varepsilon^{\nu+\lambda}\mathfrak{Y}^\lambda_{\rm int}\big(\tfrac{x}{\varepsilon}\big) + r^N_{\varepsilon,\rm int} \tag{4.34}$$

$$u_{\varepsilon,\rm ext} = \psi\big(\tfrac{x}{\varepsilon}\big)\sum_{\nu\in\mathfrak{U}(N)}\varepsilon^\nu U^\nu_{\rm ext}\big(t,\tfrac{s}{\varepsilon}\big)\,[\log\varepsilon] + \chi(x)\sum_{\nu\in\mathfrak{U}(N)}\sum_{\lambda\in\mathfrak{S}(N-\nu)}c^\nu_\lambda[\log\varepsilon]\,\varepsilon^{\nu+\lambda}\mathfrak{Y}^\lambda_{\rm ext}\big(\tfrac{x}{\varepsilon}\big) + r^N_{\varepsilon,\rm ext} \tag{4.35}$$

with a remainder r^N_ε *satisfying estimate* (4.27) *and with functions (independent of* N*)* $u^\nu_{\rm int}[\log\varepsilon]$ *in* $\mathrm{H}^1(\Omega_{\rm int})$*. Moreover, for any* $k < \frac{\pi}{\omega}$*,* $u^k_{\rm int}$ *is given by the formulas of the smooth case, cf. Proposition 2.3.*

We only have to check that the terms in expansions (4.34) and (4.35) do not depend on N.

This can be proved by using energy estimates as follows. We note that the energy estimates (4.31) can be completed by estimates from below, so that we have for a suitable integer q:

$$\exists c, c' > 0, \quad \forall \varepsilon \in (0, \varepsilon^0], \quad c \le \big\| \chi(x)\, \mathfrak{Y}^{\lambda}(\tfrac{x}{\varepsilon}) \big\|_{1,\Omega_{\text{int}}} \le c' |\log \varepsilon|^q.$$

Likewise, and in an obvious way, as soon as $u^{\nu}_{\text{int}}[\log \varepsilon]$ is not identically zero, there holds

$$\exists q \in \mathbb{N},\ \exists c, c' > 0, \quad \forall \varepsilon \in (0, \varepsilon^0], \quad c \le \big\| u^{\nu}_{\text{int}}[\log \varepsilon] \big\|_{1,\Omega_{\text{int}}} \le c' |\log \varepsilon|^q.$$

From this we can see that the terms in the expansion (4.34) are not modified if N is increased: When going from N to $N+1$, we only add terms

$$\sum_{\nu \in \mathfrak{U}(N+1) \setminus \mathfrak{U}(N)} \varepsilon^{\nu} u^{\nu}_{\text{int}}[\log \varepsilon] \;+\; \chi(x) \sum_{\nu \in \mathfrak{U}(N+1)} \; \sum_{\lambda \in \mathfrak{S}(N+1-\nu) \setminus \mathfrak{S}(N-\nu)} c^{\nu}_{\lambda}[\log \varepsilon]\, \varepsilon^{\nu+\lambda} \mathfrak{Y}^{\lambda}_{\text{int}}\big(\tfrac{x}{\varepsilon}\big),$$

the energy of which is of order $\mathcal{O}(\varepsilon^N)$. Consequently they do not affect the terms in the expansion at order N.

Remark 4.15 *(i)* Introducing in a similar way as (4.30) the layers $\mathfrak{Z}^{k,(\beta)}$ for $k \ge 2$ and $|\beta| = k-2$:

$$\mathfrak{Z}^{k,(\beta)}_{\text{int}} = \mathfrak{W}^{k,(\beta)}_{\text{int}} - \sum_{\ell=0}^{k-3} \breve{\mathfrak{W}}^{k,k-\ell,(\beta)}_{\text{int}} \quad \text{and} \quad \mathfrak{Z}^{k,(\beta)}_{\text{ext}} = \mathfrak{W}^{k,(\beta)}_{\text{ext}} - \psi \sum_{\ell=0}^{k-3} \breve{\mathfrak{W}}^{k,k-\ell,(\beta)}_{\text{ext}},$$

we can easily prove the analogues of Theorems 4.13 and 4.14 in the situation when f_{int} is $\mathcal{C}^{\infty}$ up to the boundary of Ω_{int}.

(ii) A variant of the interior expansion (4.34) is possible. We may multiply the slow terms $u^{\nu}(x)$ by the cut-off $\psi(\frac{x}{\varepsilon})$ but, as opposed to the case of flat terms, see Remark 4.9 *(ii)*, such an operation is not transparent: We have to modify the definition of the corner layers $\mathfrak{Y}^{\lambda}$ and $\mathfrak{Z}^{k,(\beta)}$ accordingly through the multiplication of the terms $\breve{\mathfrak{K}}^{\lambda,\lambda-\ell}_{\text{int}}$ and $\breve{\mathfrak{W}}^{k,k-\ell,(\beta)}_{\text{int}}$ by the *same* cut-off ψ, just like in the layer part. □

4.6 Neumann boundary conditions

The above techniques directly apply to the Neumann case: We still have the splitting of the interior terms into regular and singular parts and the corresponding profiles $\mathfrak{K}^{\lambda}$ are constructed in Theorem 3.25. For integer λ, they may contain a term in $\log R$ in their asymptotics at infinity.

Note that in this case the corner layers $\mathfrak{Y}^{\lambda}$ keep this logarithmic term, see (4.30). Thus they are no more decreasing as $R \to \infty$, but we still have the energy estimate (4.31) above.

5 Concluding remarks

The type of results we have obtained and the techniques we have used evoke the well-known concept of *matched asymptotic expansion* where inner and outer expansions are constructed, see [15]. However, our analysis differs since our different scales coexist in a transition region, as opposed to the inner and outer expansions which contain the rapid and slow scales separately. For a rigorous approach of this method, see [25].

Most of the difficulty of the above analysis is due to the singularities, mainly those of the limit problem, the $\mathfrak{s}^\lambda$. The profiles $\mathfrak{K}^\lambda$ which we have constructed perform the transition between the $\mathfrak{s}^\lambda$ and the behavior near the corner of the solution of the actual problem with ε-layer. Note that the singularities of the transmission problem are different from the $\mathfrak{s}^\lambda$: They are asymptotically contained in the profiles $\mathfrak{K}^\lambda$.

An essential feature of these asymptotics is the possible communications between the terms in slow variables $u^\nu(x)$ and those in rapid variables $\mathfrak{K}^\lambda(\frac{x}{\varepsilon})$, $\mathfrak{W}^k(\frac{x}{\varepsilon})$, $\mathfrak{Y}^\lambda(\frac{x}{\varepsilon})$, or $\mathfrak{Z}^\lambda(\frac{x}{\varepsilon})$. A priori the u^ν and the profiles do not exist in the same world but they are forced to "live" together thanks to cut-off functions $\psi(\frac{x}{\varepsilon})$ for the u^ν and $\chi(x)$ for the profiles. This kind of product form combining rapid and slow variables is an Ansatz of constant use in homogenization, see [24] for instance. Note that such a product Ansatz is not used in [18, 19] where many singular perturbations of a domain (without layer) are investigated. This has to be related with the fact that the presence of $\psi(\frac{x}{\varepsilon})$ *inside* Ω_{int} is optional in our situation.

Nevertheless, in our opinion, the product form Ansatz is more powerful, allowing to take into account more general situations where the interior domain Ω_{int} also depends on ε: The results of this paper can be extended to cases when Ω_{int} presents self-similar structures at scale ε, such as curved corners with curvature radius in $\mathcal{O}(\varepsilon)$. This can be combined with the presence of a layer presenting self-similar structures at scale ε, too. This is the subject of a forthcoming work.

The Helmholtz equation could be treated in a similar way, though new difficulties appear, due to the importance of the zero-th order part of the operator, see for instance [16] where the special Helmholtz features are described in a problem involving a thin structure.

6 Appendix: Elliptic regularity near the boundary

The aim of the appendix is to prove the elliptic regularity result stated in Theorem 2.8. By a classical argument of local mappings, it is sufficient to consider the case of a straight boundary.

For any positive real number a, we define the layered rectangle $\mathcal{R}^{a,\varepsilon} = (-a,a)\times(-a,1+\varepsilon)$, composed of $\mathcal{R}^a_{\text{int}} = (-a,a)\times(-a,1)$ and $\mathcal{R}^{a,\varepsilon}_{\text{ext}} = (-a,a)\times(1,1+\varepsilon)$. We denote by γ^a its interior boundary $(-a,a)\times\{1\}$, by γ^a_{ext} its exterior boundary $(-a,a)\times\{1+\varepsilon\}$, and by γ^a_{D} the set $\partial\mathcal{R}^{a,\varepsilon}\setminus\gamma^a_{\text{ext}}$ (see Figure 5). Clearly $\mathcal{R}^{b,\varepsilon}\subset\mathcal{R}^{a,\varepsilon}$ if $b\leq a$. Let B be the bilinear form

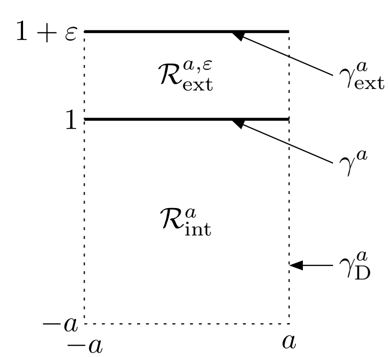

Figure 5: The rectangle $\mathcal{R}^{a,\varepsilon}$.

associated to problem (P_ε) on $\mathcal{R}^{a,\varepsilon}$:

$$B(u,v) = \alpha \int_{\mathcal{R}^a_{\mathrm{int}}} \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\mathcal{R}^{a,\varepsilon}_{\mathrm{int}}} \nabla u \cdot \nabla v \,\mathrm{d}x.$$

We shall use different variational spaces for Dirichlet external b.c. and Neumann external b.c., namely we define

$$\begin{aligned} &V_a = \mathrm{H}^1_0(\mathcal{R}^{a,\varepsilon}) && \text{for Dirichlet external b.c.} \\ &V_a = \{v \in \mathrm{H}^1(\mathcal{R}^{a,\varepsilon})\ ;\ v = 0 \text{ on } \gamma^a_{\mathrm{D}}\} && \text{for Neumann external b.c.} \end{aligned}$$

From the Lax-Milgram lemma, we immediately obtain

Proposition 6.1 *If the linear form F belongs to the dual space V_a' of V_a, then the variational problem*

$$\forall v \in V_a, \quad B(u,v) = \langle F, v\rangle$$

admits a unique solution $u \in V_a$. Moreover, there exists a constant C, independent of ε and u, such that

$$\|u\|_{V_a} \leq C\, \|F\|_{V_a'}. \tag{6.1}$$

We emphasize on the fact that we make no use of the Dirichlet condition on γ^a_{ext} to prove the coercivity of the form B; the condition on γ^a_{D} is enough to get a Poincaré inequality (which consequently also applies for Neumann external b.c.).

Finally we define the linear form F_u by

$$\forall \varphi \in V_a, \quad \langle F_u, \varphi\rangle = -\alpha \int_{\mathcal{R}^a_{\mathrm{int}}} \Delta u_{\mathrm{int}}\, \varphi \,\mathrm{d}x - \int_{\mathcal{R}^{a,\varepsilon}_{\mathrm{ext}}} \Delta u_{\mathrm{ext}}\, \varphi \,\mathrm{d}x + \int_{\gamma^a} (\alpha\partial_n u_{\mathrm{int}} - \partial_n u_{\mathrm{ext}})\varphi \,\mathrm{d}\sigma.$$

We easily check the following lemma:

Lemma 6.2 *If $u \in V_a$ (together with $\partial_n u_{\mathrm{ext}} = 0$ on γ^a_{ext} in the case of Neumann external b.c.) satisfies the assumptions*

$$\Delta u_{\mathrm{int}} \in \mathrm{L}^2(\mathcal{R}^a_{\mathrm{int}}), \quad \Delta u_{\mathrm{ext}} \in \mathrm{L}^2(\mathcal{R}^{a,\varepsilon}_{\mathrm{ext}}) \quad \textit{and} \quad \alpha\partial_n u_{\mathrm{int}} - \partial_n u_{\mathrm{ext}} \in \mathrm{L}^2(\gamma^a), \tag{6.2}$$

then $F_u \in V_a'$ and there exists a constant C independent of ε and u such that

$$\|F_u\|_{V_a'} \leq C \left[\|\Delta u_{\mathrm{int}}\|_{0,\mathcal{R}^a_{\mathrm{int}}} + \|\Delta u_{\mathrm{ext}}\|_{0,\mathcal{R}^{a,\varepsilon}_{\mathrm{ext}}} + \|\alpha\partial_n u_{\mathrm{int}} - \partial_n u_{\mathrm{ext}}\|_{0,\gamma^a} \right]. \tag{6.3}$$

We are now able to prove the first step of Theorem 2.8:

Proposition 6.3 *Let u belong to the space V_a and satisfy* (6.2)*. For any $b < a$, there exists a constant C independent of ε and u such that*

$$\|u\|_{1,\mathcal{R}^{b,\varepsilon}} \leq C \left[\|F_u\|_{V_a'} + \|u\|_{0,\mathcal{R}^{a,\varepsilon}} \right]. \tag{6.4}$$

Proof: Let c be such that $b < c < a$. We introduce a smooth cut-off function χ, defined by $\chi(x) = \chi_1(x_1)\chi_2(x_2)$, with

$$\begin{aligned} \chi_1(x_1) = 1 \text{ if } |x_1| \le b \quad &\text{and} \quad \chi_1(x_1) = 0 \text{ if } |x_1| > c, \\ \chi_2(x_2) = 1 \text{ if } x_2 \ge -b \quad &\text{and} \quad \chi_2(x_2) = 0 \text{ if } x_2 < -c. \end{aligned} \tag{6.5}$$

In particular, $\chi = 1$ on $\mathcal{R}^{b,\varepsilon}$ and $\chi = 0$ on $\mathcal{R}^{a,\varepsilon} \backslash \mathcal{R}^{c,\varepsilon}$.

The truncated function χu belongs to V_a and satisfies for any $v \in V_a$, $B(\chi u, v) = \langle F_{\chi u}, v\rangle$. Thanks to Proposition 6.1, we get

$$\|\chi u\|_{1,\mathcal{R}^{a,\varepsilon}} \le C \|F_{\chi u}\|_{V_a'}. \tag{6.6}$$

We still need to estimate $\|F_{\chi u}\|_{V_a'}$. We write

$$\forall \varphi \in V_a, \quad \langle F_{\chi u}, \varphi\rangle = \langle F_u, \chi\varphi\rangle - \int_{\mathcal{R}^{a,\varepsilon}} \tilde\alpha \left[(\Delta\chi)\, u\, \varphi + 2\, \nabla\chi \cdot \nabla u\, \varphi\right] \mathrm{d}x,$$

with $\tilde\alpha$ the function taking the value α in $\mathcal{R}^a_{\text{int}}$ and 1 in $\mathcal{R}^{a,\varepsilon}_{\text{ext}}$. Thanks to an integration by parts using the tensorial structure of χ, we can estimate the second term and finally obtain

$$|\langle F_{\chi u}, \varphi\rangle| \le C \left[\|F_u\|_{V_a'} + \|u\|_{0,\mathcal{R}^{a,\varepsilon}}\right] \|\varphi\|_{1,\mathcal{R}^{a,\varepsilon}}. \tag{6.7}$$

Since $\chi = 1$ on $\mathcal{R}^{b,\varepsilon}$, we obtain the result from (6.6) and (6.7). ■

Using Nirenberg translations, we prove the following result of elliptic regularity at any order:

Proposition 6.4 *Let d be a positive real number. Let u belong to the space V_d (together with $\partial_n u_{\text{ext}} = 0$ on γ^d_{ext} in the case of Neumann external b.c.) satisfying the following conditions for $m \in \mathbb{N}$,*

$$\Delta u_{\text{int}} \in \mathrm{H}^{m-1}(\mathcal{R}^d_{\text{int}}), \quad \Delta u_{\text{ext}} \in \mathrm{H}^{m-1}(\mathcal{R}^{d,\varepsilon}_{\text{ext}}), \quad \textit{and} \quad \alpha\partial_n u_{\text{int}} - \partial_n u_{\text{ext}} \in \mathrm{H}^{m-\frac12}(\gamma^d).$$

For any $c < d$, u_{int} belongs to $\mathrm{H}^{m+1}(\mathcal{R}^c_{\text{int}})$, u_{ext} to $\mathrm{H}^{m+1}(\mathcal{R}^{c,\varepsilon}_{\text{ext}})$, and there exists a constant C independent of ε and u such that

$$\begin{aligned} \|u_{\text{int}}\|_{m+1,\mathcal{R}^c_{\text{int}}} + \|u_{\text{ext}}\|_{m+1,\mathcal{R}^{c,\varepsilon}_{\text{ext}}} \le C\Big[&\|\Delta u_{\text{int}}\|_{m-1,\mathcal{R}^d_{\text{int}}} + \|\Delta u_{\text{ext}}\|_{m-1,\mathcal{R}^{d,\varepsilon}_{\text{ext}}} \\ &+ \|\alpha\partial_n u_{\text{int}} - \partial_n u_{\text{ext}}\|_{m-\frac12,\gamma^d} + \|u\|_{0,\mathcal{R}^{d,\varepsilon}}\Big]. \end{aligned} \tag{6.8}$$

Proof: We proceed by induction over $m \ge 1$ and make use of the horizontal difference operator D_h defined for any real $h \ne 0$ by

$$\mathrm{D}_h\varphi(x_1, x_2) = \frac{1}{h}\left[\varphi(x_1 + h, x_2) - \varphi(x_1, x_2)\right].$$

Let $\sigma \in \mathbb{R}$ be such that $c < \sigma < d$.

• For $m = 1$, we use a similar cut-off function as in the previous proof, defined by $\chi(x) = \chi_1(x_1)\chi_2(x_2)$ with

$$\begin{aligned} &\chi_1(x_1) = 1 \text{ if } |x_1| \le c \quad \text{and} \quad \chi_1(x_1) = 0 \text{ if } |x_1| > \tfrac{c+\sigma}{2}, \\ &\chi_2(x_2) = 1 \text{ if } x_2 \ge -c \quad \text{and} \quad \chi_2(x_2) = 0 \text{ if } x_2 < -\tfrac{c+\sigma}{2}, \end{aligned}$$

and we apply Proposition 6.3 with $b = c$ and $a = \sigma$ to $u_h = \chi_1 \mathrm{D}_h(\chi_1 u)$, for $|h| \le h_0$ sufficiently small

$$\|u_h\|_{1,\mathcal{R}^{c,\varepsilon}} \le C\left[\|F_{u_h}\|_{V'_\sigma} + \|u_h\|_{0,\mathcal{R}^{\sigma,\varepsilon}}\right]. \tag{6.9}$$

To estimate F_{u_h}, we use the decomposition

$$\begin{aligned} \langle F_{u_h}, \varphi\rangle \quad &= \quad \langle F_{\mathrm{D}_h(\chi_1 u)}, \chi_1\varphi\rangle \;-\; \int_{\mathcal{R}^\sigma_{\text{int}}\cup\mathcal{R}^{\sigma,\varepsilon}_{\text{ext}}} \tilde\alpha\Big[\Delta\chi_1 \mathrm{D}_h(\chi_1 u)\varphi + 2\,\nabla\chi_1\cdot\nabla \mathrm{D}_h(\chi_1 u)\varphi\Big]\,\mathrm{d}x \\ &=: \qquad \boxed{1} \qquad\qquad + \qquad\qquad \boxed{2}, \end{aligned}$$

with $\tilde\alpha$ the function taking the value α in $\mathcal{R}^\sigma_{\text{int}}$ and 1 in $\mathcal{R}^{\sigma,\varepsilon}_{\text{ext}}$. We use the same technique as in the proof of Theorem 6.3. A discrete integration by parts yields

$$\boxed{2} = \int_{\mathcal{R}^\sigma_{\text{int}}\cup\mathcal{R}^{\sigma,\varepsilon}_{\text{ext}}} \tilde\alpha\Big[(\chi_1 u)\mathrm{D}_{-h}(\Delta\chi_1\,\varphi) + 2\,\nabla(\chi_1 u)\cdot \mathrm{D}_{-h}(\nabla\chi_1\,\varphi)\Big]\,\mathrm{d}x,$$

which then gives $|\boxed{2}| \le C\,\|u\|_{1,\mathcal{R}^{\sigma,\varepsilon}}\|\varphi\|_{1,\mathcal{R}^{\sigma,\varepsilon}}$. Similarly for the first part, we get

$$\boxed{1} = \int_{\mathcal{R}^\sigma_{\text{int}}\cup\mathcal{R}^{\sigma,\varepsilon}_{\text{ext}}} \tilde\alpha\,\Delta(\chi_1\,u)\mathrm{D}_{-h}(\chi_1\,\varphi)\,\mathrm{d}x - \int_{\gamma^\sigma} \chi_1(\alpha\partial_n u_{\text{int}} - \partial_n u_{\text{ext}})\mathrm{D}_{-h}(\chi_1\varphi)\,\mathrm{d}\sigma.$$

Since $\chi_1(\alpha\partial_n u_{\text{int}} - \partial_n u_{\text{ext}})$ vanishes at the extremities of γ^σ, we can use the duality $\mathrm{H}^{1/2}_{0\,0}-\mathrm{H}^{-1/2}$ on γ^σ to obtain

$$|\boxed{1}| \le C\,\left[\|\Delta u\|_{0,\mathcal{R}^\sigma_{\text{int}}\cup\mathcal{R}^{\sigma,\varepsilon}_{\text{ext}}} + \|\alpha\partial_n u_{\text{int}} - \partial_n u_{\text{ext}}\|_{\frac12,\gamma^\sigma}\right]\,\|\varphi\|_{1,\mathcal{R}^{\sigma,\varepsilon}}.$$

Together, F_{u_h} can be estimated in the dual of V_σ:

$$\|F_{u_h}\|_{V'_\sigma} \le C\Big[\|\Delta u\|_{0,\mathcal{R}^\sigma_{\text{int}}\cup\mathcal{R}^{\sigma,\varepsilon}_{\text{ext}}} + \|\alpha\partial_n u_{\text{int}} - \partial_n u_{\text{ext}}\|_{\frac12,\gamma^\sigma} + \|u\|_{1,\mathcal{R}^{\sigma,\varepsilon}}\Big]. \tag{6.10}$$

Since $\chi_1 = 1$ on $\mathcal{R}^{c,\varepsilon}$ and $\|u_h\|_{\mathcal{R}^{\sigma,\varepsilon}} \le C\|u\|_{\mathcal{R}^{\sigma,\varepsilon}}$ for h small enough, equations (6.9) and (6.10) lead to

$$\|\mathrm{D}_h u\|_{1,\mathcal{R}^{c,\varepsilon}} \le C\Big[\|\Delta u\|_{0,\mathcal{R}^\sigma_{\text{int}}\cup\mathcal{R}^{\sigma,\varepsilon}_{\text{ext}}} + \|\alpha\partial_n u_{\text{int}} - \partial_n u_{\text{ext}}\|_{\frac12,\gamma^\sigma} + \|u\|_{1,\mathcal{R}^{\sigma,\varepsilon}}\Big].$$

Passing to the limit $h \to 0$, we obtain the same estimate for the second order derivatives $\partial_1^2 u$ and $\partial_1\partial_2 u$. For $\partial_2^2 u$, we obtain the estimate by writing $\partial_2^2 u = \Delta u - \partial_1^2 u$. Then, we get

$$\|u_{\text{int}}\|_{2,\mathcal{R}^c_{\text{int}}} + \|u_{\text{ext}}\|_{2,\mathcal{R}^{c,\varepsilon}_{\text{ext}}} \le C\Big[\|\Delta u\|_{0,\mathcal{R}^\sigma_{\text{int}}\cup\mathcal{R}^{\sigma,\varepsilon}_{\text{ext}}} + \|\alpha\partial_n u_{\text{int}} - \partial_n u_{\text{ext}}\|_{\frac12,\gamma^\sigma} + \|u\|_{1,\mathcal{R}^{\sigma,\varepsilon}}\Big].$$

Using the estimate (6.4) for $b = \sigma$ and $a = \tau$, we conclude

$$\|u_{\text{int}}\|_{2,\mathcal{R}^c_{\text{int}}} + \|u_{\text{ext}}\|_{2,\mathcal{R}^{c,\varepsilon}_{\text{ext}}} \le C\Big[\|\Delta u\|_{0,\mathcal{R}^d_{\text{int}}\cup\mathcal{R}^{d,\varepsilon}_{\text{ext}}} + \|\alpha\partial_n u_{\text{int}} - \partial_n u_{\text{ext}}\|_{\frac12,\gamma^d} + \|u\|_{0,\mathcal{R}^{d,\varepsilon}}\Big].$$

- Suppose the estimation $\mathrm{H}^{m-1} \to \mathrm{H}^{m+1}$ known and apply it to $u_h = \chi_1 \mathrm{D}_h(\chi_1 u)$. With the same techniques as in the case $m = 1$, we can prove

$$\|u_{\rm int}\|_{m+2,\mathcal{R}^c_{\rm int}} + \|u_{\rm ext}\|_{m+2,\mathcal{R}^{c,\varepsilon}_{\rm ext}} \le C\Big[\|\Delta u_{\rm int}\|_{m,\mathcal{R}^\sigma_{\rm int}} + \|\Delta u_{\rm ext}\|_{m,\mathcal{R}^{\sigma,\varepsilon}_{\rm ext}} + \|\alpha\partial_n u_{\rm int} - \partial_n u_{\rm ext}\|_{m+\frac12,\gamma^\sigma} + \|u_{\rm int}\|_{m+1,\mathcal{R}^\sigma_{\rm int}} + \|u_{\rm ext}\|_{m+1,\mathcal{R}^{\sigma,\varepsilon}_{\rm ext}}\Big].$$

Using the induction assumption for u (with σ instead of c), we get the stated result.

$$\|u_{\rm int}\|_{m+2,\mathcal{R}^c_{\rm int}} + \|u_{\rm ext}\|_{m+2,\mathcal{R}^{c,\varepsilon}_{\rm ext}} \le C\Big[\|\Delta u_{\rm int}\|_{m,\mathcal{R}^d_{\rm int}} + \|\Delta u_{\rm ext}\|_{m,\mathcal{R}^{d,\varepsilon}_{\rm ext}} + \|\alpha\partial_n u_{\rm int} - \partial_n u_{\rm ext}\|_{m+\frac12,\gamma^d} + \|u\|_{0,\mathcal{R}^{d,\varepsilon}}\Big].$$

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