Interpolation of nullspaces for polynomial approximation of divergence-free functions in a cube

CHRISTINE BERNARDI
Analyse Numérique — C.N.R.S. et Université Pierre et Marie Curie — Tour 55–65, 5ème étage, 4 place Jussieu F–75252 Paris Cedex 05, France.

MONIQUE DAUGE
U.R.A. 758 du C.N.R.S. — Mathématiques — Université de Nantes, 2 rue de la Houssinière F–44072 Nantes Cedex 03, France.

YVON MADAY
Analyse Numérique — C.N.R.S. et Université Pierre et Marie Curie — Tour 55–65, 5ème étage, 4 place Jussieu F–75252 Paris Cedex 05, France.

Abstract: The aim of this paper is the interpolation between nullspaces of a fixed partial differential operator in different Sobolev spaces, for example the subspaces of divergence-free functions. We present several methods of proofs, which allow for handling various operators in different geometries, with or without boundary conditions. Our application is the optimal approximation of divergence-free functions in a cube by high degree polynomials, which is useful for the numerical analysis of the spectral discretization of the Stokes problem.

1 INTRODUCTION

In this paper, we intend to interpolate the nullspaces of a fixed partial differential operator in different Sobolev spaces. Let us make precise the framework: if X and Y are Hilbert spaces such that X is dense in Y, for any θ, $0 < \theta < 1$, the interpolate space

$[X,Y]_\theta$ is well-defined, independently of the Hilbertian interpolation method which can be chosen among: the K–method (with $p=2$), the interpolation by domains of operators, the complex interpolation and the trace method. Next, let A be a linear operator defined on Y. We have the natural associated nullspaces of A:

$$\mathscr{N}_A^X = \{u \in X \mid \ Au = 0\} \quad \text{and} \quad \mathscr{N}_A^Y = \{u \in Y \mid \ Au = 0\}.$$

On the other hand, the interpolate space $[\mathscr{N}_A^X, \mathscr{N}_A^Y]_\theta$ is well-defined too. We are interested in conditions insuring that:

$$[\mathscr{N}_A^X, \mathscr{N}_A^Y]_\theta = \{u \in [X,Y]_\theta \mid \ \ Au = 0\}. \tag{1.1}$$

When the operator A has a finite-dimensional range, (1.1) holds (*cf* [14, Chap. 1, Th. 13.3] for instance). However, this result cannot be extended to operators with infinite-dimensional range in general.

We are going to investigate the situation where:
(i) the operator A is a partial differential operator acting on scalar or vectorial functions (our main purpose is the application to the Laplace and divergence operators),
(ii) the spaces X and Y are standard Sobolev spaces on a domain $\Omega \subset \mathbb{R}^d$ or weighted Sobolev spaces in an hypercube, with or without boundary conditions (these spaces appear naturally in the numerical analysis of spectral methods).

In a first step, the result is proven in Sobolev spaces without boundary conditions. We present two methods:
(i) for homogeneous operators with constant coefficients in star-shaped domains: the idea is to use the interpolation by the trace method, and the main tool of the proof of the interpolation result is the construction of a "stable" lifting operator which maps functions φ defined on Ω and satisfying $A\varphi = 0$ into functions $\Phi(\cdot, t)$ satisfying for each t, $A\Phi(\cdot, t) = 0$;
(ii) for strongly elliptic operators in Lipschitz–continuous domains: the proof relies on a result of interpolation of subspaces [14, Chap. 1, Th. 14.3] and requires the construction of a right-inverse for A. We also apply this method for the divergence operator.

In Section 2, we construct a lifting of trace and in Section 3 we present the two interpolation methods.

In a second step, we introduce the Dirichlet boundary conditions: we interpolate subspaces of functions φ satisfying $A\varphi = 0$ in Ω and $\varphi = 0$ on the boundary $\partial\Omega$ of Ω. In view of our application, we limit ourselves to the case of the divergence. Once again, we present two different methods (each of them can be extended to other situations). Both of them rely on [14, Chap. 1, Th. 14.3] already quoted:
(i) starting from an interpolation result for functions with a null trace on $\partial\Omega$, we construct a right-inverse for the divergence operator in spaces of functions satisfying the Dirichlet condition;
(ii) starting from our interpolation result for divergence-free functions, we construct a right-inverse for the first trace on $\partial\Omega$ in spaces of divergence-free functions.

In contrast to the situation without boundary conditions, the presence of singular points on the boundary of Ω introduces technical difficulties due to the limited smoothness of solutions of an elliptic boundary value problem. That is why, in Section 4, we study successively the cases of a smooth domain, of an orthogonal cylinder, and of a cube.

An application to spectral methods is described in Section 5: the numerical analysis of the spectral discretization of the Stokes problem relies on optimal results of approximation of divergence-free functions by divergence-free polynomials on a cube. These results are proven in [17] only for functions such that all their third order derivatives are square integrable. However the solutions of the Stokes problem do not satisfy this regularity property in general (see [7] for instance), even for smooth data. The interpolation property which is proven in this paper allows for extending the results of [17] to less smooth functions on a cube. As a consequence, the convergence of the spectral method is established for any variational solution of the Stokes problem. This result is useful for the numerical analysis of the spectral discretization of the nonlinear Navier–Stokes equations.

2 CONSTRUCTION OF A LIFTING OPERATOR

2.1 Sobolev spaces

Let Ω be an open set in $\mathbb{R}^d$, where d is an integer ≥ 1; the generic point in Ω is denoted by $\boldsymbol{x}$. On this domain, we shall use the space $L^2(\Omega)$ of square-integrable functions for the Lebesgue measure $d\boldsymbol{x}$, the standard Sobolev spaces $H^s(\Omega)$ for any real number $s \geq 0$, together with the spaces $H_0^s(\Omega)$ defined as the closure in $H^s(\Omega)$ of the subspace of infinitely differentiable functions with a compact support. A separable Banach space X being given, we also consider the scale of Sobolev spaces $H^s(\Omega; X)$ of functions with values in X.

We introduce the cylinder $\widetilde{\Omega} = \Omega \times]0, \frac{1}{2}[$. The generic point in this cylinder is denoted by $(\boldsymbol{x}, t)$. Then, the Lebesgue measure in the cylinder is the tensor product $d\boldsymbol{x}\, dt$. We are going to define weighted Sobolev spaces on the cylinder.

On the interval $I =]0, \frac{1}{2}[$ and for a real parameter $\beta > -1$, we define successively:
(i) the space $L^2_\beta(I)$ of measurable functions φ on I such that $\int_0^{\frac{1}{2}} \varphi^2(t)\, t^\beta\, dt < +\infty$,
(ii) for any positive integer m, the space $H^m_\beta(I)$ of functions in $L^2_\beta(I)$ such that their derivatives up to the order m also belong to $L^2_\beta(I)$,
(iii) for any real number $s \geq 0$ which is not an integer, the space $H^s_\beta(I)$ as the interpolation space of index $s - [s]$ between $H^{[s]+1}_\beta(I)$ and $H^{[s]}_\beta(I)$ (where $[s]$ stands for the integral part of s).

Finally, for any real number $s \geq 0$, we define the space $H^s_\beta(\widetilde{\Omega})$ by the formula

$$H^s_\beta(\widetilde{\Omega}) = L^2\big(\Omega; H^s_\beta(I)\big) \cap H^s\big(\Omega; L^2_\beta(I)\big).$$

Next, we take the domain Ω equal to the hypercube $]-1, 1[^d$, which is the elementary domain for spectral methods. Let us recall that spectral methods are of two types: Legendre and Chebyshev. The Legendre type techniques involve tensorized bases of Legendre polynomials and they rely on a variational formulation of the problem in standard Sobolev spaces. In Chebyshev type techniques, Chebyshev polynomials are involved, which are orthogonal in the interval $]-1, 1[$ for the measure $(1 - \zeta^2)^{-\frac{1}{2}}\, d\zeta$. Weighted Sobolev spaces are therefore useful. To have a unified presentation, we deal with the general case of the Jacobi weight where the power $-\frac{1}{2}$ is replaced by a general parameter α.

Let α be a real number > -1. We firstly define the weight

$$\varpi_\alpha(\boldsymbol{x}) = \prod_{j=1}^{d} (1 - x_j^2)^\alpha, \tag{2.1}$$

where $\boldsymbol{x}$ with coordinates $x_1, \ldots, x_d$, is the generic point. Next, we introduce the space $L^2_\alpha(\Omega)$ of measurable functions φ on Ω such that

$$\int_\Omega \varphi^2(\boldsymbol{x})\, \varpi_\alpha(\boldsymbol{x})\, d\boldsymbol{x} < +\infty.$$

Then, for any positive integer m, $H^m_\alpha(\Omega)$ stands for the space of functions in $L^2_\alpha(\Omega)$ such that all their partial derivatives of total order $\leq m$ also belong to $L^2_\alpha(\Omega)$. For any real number $s \geq 0$ which is not an integer, the space $H^s_\alpha(\Omega)$ is the interpolation space of index $s - [s]$ between $H^{[s]+1}_\alpha(\Omega)$ and $H^{[s]}_\alpha(\Omega)$. We refer to [3] and [4, Chap. 4] for the main properties of these spaces. These definitions are extended to the spaces $H^s_\alpha(\Omega; X)$ for any separable Banach space X in a natural way.

We also need the corresponding spaces on the cylinder $\widetilde{\Omega} = \Omega \times I$: for any parameter $\beta > -1$ and for any real number $s \geq 0$, we define the space

$$H^s_{\alpha\beta}(\widetilde{\Omega}) = L^2_\alpha\big(\Omega; H^s_\beta(I)\big) \cap H^s_\alpha\big(\Omega; L^2_\beta(I)\big).$$

2.2 Definition of the lifting operator

In this section, we assume that Ω is star-shaped with respect to a ball, i.e., there exists a ball B such that, for all $\boldsymbol{x}$ in Ω and $\boldsymbol{y}$ in B, the segment $[\boldsymbol{x}, \boldsymbol{y}]$ is contained in Ω.

Note that, even if Ω has a Lipschitz–continuous boundary, it can be star-shaped with respect to a point without being star-shaped with respect to a ball (see the following figure where the domain on the left is star-shaped with respect to only one point and the domain on the right is star-shaped with respect to a ball without being convex). A convex domain is star-shaped with respect to any ball contained in it.

The standard way to construct a lifting operator is the convolution by a regularizing family. But such a method will *never* produce an operator acting from functions defined on Ω into functions defined on the whole cylinder $\widetilde{\Omega}$. However, the composition of such a convolution with a change of variables (transforming a cone into the cylinder) allows for the definition of a correct lifting. Such a strategy is linked with an idea due to Gagliardo [8] and Lions [13] (which has already been used by Babuška and Suri [1] for the p–version of the finite element method and developed in [3]).

Let χ be a fixed integrable function on $\mathbb{R}^d$ with its integral equal to 1. With any function φ in $L^2(\mathbb{R}^d)$, we associate the function

$$F^\chi(\varphi)(\boldsymbol{x},t) = \int_{\mathbb{R}^d} \varphi\big((1-t)\boldsymbol{x} + t\boldsymbol{y}\big)\,\chi(\boldsymbol{y})\,d\boldsymbol{y}. \tag{2.2}$$

It is readily checked that, for any continuous function φ on $\mathbb{R}^d$, the function $F^\chi(\varphi)$ is continuous on $\mathbb{R}^d \times I$ and satisfies

$$\forall \boldsymbol{x} \in \mathbb{R}^d, \quad F^\chi(\varphi)(\boldsymbol{x},0) = \varphi(\boldsymbol{x}). \tag{2.3}$$

Hence, the operator F^χ is a trace lifting operator and, when the function χ has a compact support in a ball B, it is a lifting operator of traces on any star-shaped domain Ω with respect to B, with values in the corresponding cylinder $\tilde{\Omega}$. A special (and standard) case is obtained when the domain Ω is convex and χ is its characteristic function χ_Ω. We are going to prove some stability properties of the operator F^χ.

REMARK 2.1 If the function φ is a polynomial on $\mathbb{R}^d$, so is $F^\chi(\varphi)$. Moreover, the operator F^χ preserves the total degree of the polynomials and also the degree with respect to each of the first d variables.

2.3 Stability of the lifting operator

We use the Fourier transform on $\mathbb{R}^d$ and denote it by a hat. The first theorem states the basic result.

THEOREM 2.2 *Let s be a real number ≥ 0 and β a real number > -1. Assume that the function χ is integrable and satisfies:*

$$\sup_{\boldsymbol{\omega}\in\mathbb{S}^{d-1}} \|\hat{\chi}(\cdot\,\boldsymbol{\omega})\|_{H^s_\beta(\mathbb{R}_+)} < +\infty. \tag{2.4}$$

Then the operator F^χ defined in (2.2) is continuous from $H^{s-\frac{1+\beta}{2}}(\mathbb{R}^d)$ into $H^s_\beta(\mathbb{R}^d \times I)$.

PROOF: The proof relies on several arguments: change of variable, use of the Fourier transform, homogeneity properties.
1) The change of variables: $(\boldsymbol{x},t) \mapsto (\boldsymbol{X} = (1-t)\boldsymbol{x}, t)$ maps the cylinder $\widetilde{\Omega}$ onto a part of a cone and the band $\mathbb{R}^d \times I$ onto itself. It can also be checked that, for any $s \geq 0$, it induces an isomorphism of $H^s_\beta(\mathbb{R}^d \times I)$ onto itself. Thus, let us set:

$$G^\chi(\varphi)(\boldsymbol{X},t) = \int_{\mathbb{R}^d} \varphi\big(\boldsymbol{X} + t\boldsymbol{y}\big)\,\chi(\boldsymbol{y})\,d\boldsymbol{y} = (-t)^{-d} \int_{\mathbb{R}^d} \varphi\big(\boldsymbol{X} - \boldsymbol{v}\big)\,\chi(-\frac{\boldsymbol{v}}{t})\,d\boldsymbol{v}.$$

Taking: $\chi_t(\boldsymbol{v}) = (-t)^{-d}\,\chi(-\frac{\boldsymbol{v}}{t})$, we observe that $G(\varphi)(\cdot,t)$ is the convolution product with respect to the $\boldsymbol{X}$ variable of the functions φ and χ_t. So, denoting by a hat the Fourier transform with respect to the $\boldsymbol{X}$ variable and by $\boldsymbol{\xi}$ the corresponding Fourier variable, we derive the formula

$$\hat{G}^\chi(\varphi)(\boldsymbol{\xi},t) = \hat{\varphi}(\boldsymbol{\xi})\,\hat{\chi}_t(\boldsymbol{\xi}) = \hat{\varphi}(\boldsymbol{\xi})\,\hat{\chi}(-t\boldsymbol{\xi}).$$

Next, we note that the statement of the theorem reduces to the estimate

$$\int_{\mathbb{R}^d} \Big(\|\hat{G}(\boldsymbol{\xi},.)\|^2_{H^s_\beta(I)} + (1+|\boldsymbol{\xi}|^2)^s \,\|\hat{G}(\boldsymbol{\xi},.)\|^2_{L^2_\beta(I)} \Big)\, d\boldsymbol{\xi} \;\le\; c \int_{\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^{s-\frac{1+\beta}{2}} \,|\hat{\varphi}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi},$$

or equivalently to

$$\begin{aligned} \int_{\mathbb{R}^d} \Big(\|\hat{\chi}(\cdot\,\boldsymbol{\xi})\|^2_{H^s_\beta(I)} + (1+|\boldsymbol{\xi}|^2)^s \,\|\hat{\chi}(\cdot\,\boldsymbol{\xi})\|^2_{L^2_\beta(I)} \Big)\, |\hat{\varphi}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\ \le c \int_{\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^{s-\frac{1+\beta}{2}} \,|\hat{\varphi}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}. \end{aligned} \tag{2.5}$$

2) We deduce (2.5) from the properties (2.4) of $\hat{\chi}$ and from the homogeneity relations (*cf* [3, (2.*a*.5)])

$$\forall \lambda \in \mathbb{R}_+, \qquad \|u(\cdot\,\lambda)\|^2_{H^s_\beta(\mathbb{R}_+)} + (1+\lambda^2)^s \,\|u(\cdot\,\lambda)\|^2_{L^2_\beta(\mathbb{R}_+)} \;\le\; c\,(1+\lambda^2)^{s-\frac{1+\beta}{2}} \,\|u\|^2_{H^s_\beta(\mathbb{R}_+)}$$

that we use for each fixed $\boldsymbol{\omega}$ in $\mathbb{S}^{d-1}$, $\lambda = |\xi|$ and $u(t) = \hat{\chi}(t\,\boldsymbol{\omega})$. ∎

Let us conclude with the case of the star-shaped domain: it is readily checked that any $\mathscr{C}^\infty$ function χ with support in B satisfies (2.4).

THEOREM 2.3 *Let Ω be a Lipschitz-continuous domain, which is star-shaped with respect to a ball B. Let χ denote a $\mathscr{C}^\infty$ function with support in B. Then, for any real number $s \ge 0$ and for any real number $\beta > -1$, the operator F^χ defined in* (2.2) *is continuous from $H^{s-\frac{1+\beta}{2}}(\Omega)$ into $H^s_\beta(\widetilde{\Omega})$.*

REMARK 2.4 When the domain Ω is convex, its characteristic function χ_Ω satisfies (2.4) for any $s \ge 0$ and for $-1 < \beta < 1$. So, when χ is equal to χ_Ω, Theorem 2.3 holds for these values of s and β.

In the case of weighted Sobolev spaces in the hypercube, the analogue of Theorem 2.3 is the following.

THEOREM 2.5 *Let Ω be the hypercube $]-1,1[^d$. Let χ denote a $\mathscr{C}^\infty$ function with support in Ω. For any real numbers $s \ge 0$, $\alpha > -1$ and $\beta > -1$, the operator F^χ is continuous from $H_\alpha^{s-\frac{1+\beta}{2}}(\Omega)$ into $H^s_{\alpha\beta}(\widetilde{\Omega})$.*

The proof of that statement relies on Theorem 2.2 combined with a "dyadic" partition of unity on Ω as used in [3, Thm 3.d.4].

3 INTERPOLATION OF NULLSPACES WITHOUT BOUNDARY CONDITIONS

Here, we present two approaches which allow for proving the interpolation result (1.1) in different frameworks: the differences concern both the geometry of the domain Ω and the properties of the operator A. The first approach allows for treating standard and weighted spaces and also spaces of polynomials, the second one is simpler and provides results in a more general geometry.

3.1 The trace method

Let X and Y be separable Hilbert spaces such that X is dense in Y with a continuous embedding. The following property can be found in [14, Chap. 1, §3.2] for instance: for any θ, $0 < \theta < 1$, the interpolate space $[X, Y]_\theta$ is the space of the traces $u(0)$ in $t = 0$ of the functions

$$t \mapsto u(t) \quad \text{in} \quad L^2_{2k\theta-1}(I; X) \cap H^k_{2k\theta-1}(I; Y). \tag{3.1}$$

Let Ω be a bounded domain with a Lipschitz–continuous boundary. Let r and s be real numbers, $0 \leq r \leq s$. Then for $0 < \theta < 1$, we have

$$[H^s(\Omega), H^r(\Omega)]_\theta = H^{(1-\theta)s+\theta r}(\Omega). \tag{3.2}$$

Now, let A be an homogeneous partial differential operator with constant coefficients, acting on $\mathscr{D}'(\Omega)^\ell$. For any real number s, let us denote by $\mathscr{N}^s_A(\Omega)$ the space

$$\mathscr{N}^s_A(\Omega) = \{u \in H^s(\Omega)^\ell \mid Au = 0\}. \tag{3.3}$$

THEOREM 3.1 *Let Ω be a Lipschitz-continuous bounded domain, which is star-shaped with respect to a ball B. The following interpolation result holds for any real numbers r and s such that $0 \leq r \leq s$ and for any θ, $0 < \theta < 1$:*

$$[\mathscr{N}^s_A(\Omega), \mathscr{N}^r_A(\Omega)]_\theta = \mathscr{N}^{(1-\theta)s+\theta r}_A(\Omega). \tag{3.4}$$

PROOF: Using the reiteration theorem [14, Chap. 1, Th. 6.1], we are reduced to prove the theorem in the case when r is equal to 0 and s is an integer k. We have the obvious inclusion

$$[\mathscr{N}^k_A(\Omega), \mathscr{N}^0_A(\Omega)]_\theta \subset [H^k(\Omega)^\ell, L^2(\Omega)^\ell]_\theta \cap \mathscr{N}^0_A(\Omega) \subset \mathscr{N}^{(1-\theta)k}_A(\Omega).$$

To prove the converse inclusion, let φ be any function in $\mathscr{N}^{(1-\theta)k}_A(\Omega)$. Let us consider Φ defined on the cylinder $\widetilde{\Omega}$ by $\Phi = F^\chi(\varphi)$ where χ is as in Theorem 2.3. We check that

$$\forall t \in [0, \frac{1}{2}], \qquad A\Phi(t) = (1-t)^m \, F^\chi(A\varphi)(t),$$

where m is the degree of the homogeneous operator A. Therefore $A\Phi(t)$ is equal to 0. On the other hand, choosing $\beta = 2k\theta - 1$, we obtain that φ belongs to $H^{k-\frac{1+\beta}{2}}(\Omega)^\ell$, hence Φ belongs to $H^k_\beta(\widetilde{\Omega})^\ell$. But, using the embedding

$$H^k_\beta(\widetilde{\Omega}) \subset L^2_{2k\theta-1}(I; H^k(\Omega)) \cap H^k_{2k\theta-1}(I; L^2(\Omega)),$$

we finally derive that

$$\Phi \in L^2_{2\theta-1}(I; \mathscr{N}^k_A(\Omega)) \cap H^k_{2\theta-1}(I; \mathscr{N}^0_A(\Omega)).$$

Hence we deduce from (3.1) that φ, as the trace of Φ, belongs to the interpolate space $[\mathscr{N}^k_A(\Omega), \mathscr{N}^0_A(\Omega)]_\theta$. ∎

Of course, by the same method, we can extend this result to weighted Sobolev spaces in the case of the hypercube $\Omega =]-1, 1[^d$. Here, we set for any $s \geq 0$ and

$\alpha > -1$:

$$\mathscr{N}^s_{A,\alpha}(\Omega) = \{u \in H^s_\alpha(\Omega)^\ell \mid Au = 0\}. \tag{3.5}$$

The same proof with Theorem 2.3 replaced by Theorem 2.5 leads to the following result.

THEOREM 3.2 *Let α be a real number > -1. In the hypercube $\Omega =]-1,1[^d$, the following interpolation result holds for any real numbers r and s such that $0 \le r \le s$ and for any θ, $0 < \theta < 1$:*

$$[\mathscr{N}^s_{A,\alpha}(\Omega), \mathscr{N}^r_{A,\alpha}(\Omega)]_\theta = \mathscr{N}^{(1-\theta)s+\theta r}_{A,\alpha}(\Omega). \tag{3.6}$$

REMARK 3.3 As noted in Remark 2.1, the operator F^χ preserves the polynomials. As a consequence, if N is a fixed integer, the same arguments imply that, on the hypercube $\Omega =]-1,1[^d$ for $\alpha > -1$ or in any star-shaped domain for $\alpha = 0$, the interpolate of index θ, $0 < \theta < 1$, between the space of polynomials φ with total degree (resp. partial degree with respect to each variable) $\le N$ satisfying $A\varphi = 0$ provided with the norm of $H^s_\alpha(\Omega)$ and this same space provided with the norm of $H^r_\alpha(\Omega)$ is the space of polynomials φ with total degree (resp. partial degree with respect to each variable) $\le N$ satisfying $A\varphi = 0$ provided with the norm of $H^{(1-\theta)s+\theta r}_\alpha(\Omega)$.

3.2 The method of the right-inverse

We recall the result [14, Chap. 1, Th. 14.3] in a simplified framework:

LEMMA 3.4 *Let X and Y be separable Hilbert spaces such that X is dense in Y with a continuous embedding, and let A be a linear operator continuous from X into $\widetilde{X}$ and from Y into $\widetilde{Y}$. If there exists a right-inverse operator $\mathcal{R}$ continuous from $\widetilde{X}$ into X and from $\widetilde{Y}$ into Y such that*

$$\forall g \in \widetilde{Y}, \quad A(\mathcal{R}g) = g, \tag{3.7}$$

then the interpolation formula (1.1) *holds for any θ, $0 < \theta < 1$.*

Let Ω be a bounded Lipschitz-continuous domain and let A be a partial differential operator of degree m with smooth coefficients on Ω, acting on $\mathscr{D}'(\Omega)^\ell$ into $\mathscr{D}'(\Omega)^{\tilde{\ell}}$. For fixed real numbers r and s, $r < s$, we choose $X = H^s(\Omega)^\ell$, $Y = H^r(\Omega)^\ell$ and $\widetilde{X} = H^{s-m}(\Omega)^{\tilde{\ell}}$, $\widetilde{Y} = H^{r-m}(\Omega)^{\tilde{\ell}}$. If A has a right-inverse $\mathcal{R}$ continuous from $H^{\sigma-m}(\Omega)^{\tilde{\ell}}$ into $H^\sigma(\Omega)^\ell$ for all $\sigma > \sigma_0$ for instance, then property (1.1) holds for any r and s such that $\sigma_0 < r < s$.

Such a right-inverse can be constructed in a number of situations. For instance, one can rely on the existence, for a fixed σ_0, of an extension operator $\mathcal{E}$ which is continuous from $H^\sigma(\Omega)$ into $H^\sigma(\mathbb{R}^d)$ for any σ, $0 \le \sigma \le \sigma_0$, and such that $\mathcal{E}u$ coincides with u in Ω (see [18, §4.2.3] or also [10, §1.4.3]). Using this extension allows for working on the whole space $\mathbb{R}^d$, with the help of the Fourier transform for example when A is elliptic with constant coefficients (but not necessarily homogeneous). The extension $\mathcal{E}$ also allows for working in a smooth bounded domain Ω_0 which contains Ω. If A can be extended into an elliptic operator with analytic coefficients on Ω_0, the Fredholm alternative applied to the Dirichlet problem for A on Ω_0 together with the unique extension of functions in the kernel, allows for modifying the operator $\mathcal{E}$ in order to build a right inverse for the operator A.

We can also use this method in the case where the operator A is the divergence operator. From now on, we denote the corresponding nullspace in $H^s(\Omega)^d$ by $\mathscr{N}^s_{\text{div}}(\Omega)$.

THEOREM 3.5 *Let Ω be a Lipschitz-continuous bounded domain. The following interpolation result holds for any real numbers r and s such that $1 \le r \le s$ and for any θ, $0 < \theta < 1$:*

$$[\mathscr{N}^s_{\text{div}}(\Omega), \mathscr{N}^r_{\text{div}}(\Omega)]_\theta = \mathscr{N}^{(1-\theta)s+\theta r}_{\text{div}}(\Omega). \tag{3.8}$$

PROOF: In view of the previously quoted result, it suffices to construct a lifting operator $\mathcal{R}$ which is continuous from $H^s(\Omega)$ into $H^{s+1}(\Omega)^d$, $s \ge 0$, such that div $(\mathcal{R}g)$ is equal to g for any g in $H^s(\Omega)$. Let Ω_0 be an open ball which contains $\overline{\Omega}$. We use the extension operator $\mathcal{E}$ and modify it so that for any g in $H^s(\Omega)$, $\mathcal{E}g$ satisfies $\int_{\Omega_0} \mathcal{E}g = 0$; next we solve the Stokes problem in Ω_0:

$$\begin{cases} -\Delta \boldsymbol{v} + \mathbf{grad}\ q = \mathbf{0} & \text{in } \Omega_0, \\ \operatorname{div} \boldsymbol{v} = \mathcal{E}g & \text{in } \Omega_0, \\ \boldsymbol{v} = \mathbf{0} & \text{on } \partial\Omega_0. \end{cases} \tag{3.9}$$

This problem has a unique solution $(\boldsymbol{v}, q)$ in $H^1_0(\Omega_0)^d \times L^2(\Omega_0)/\mathbb{R}$, and the mapping: $\mathcal{E}g \mapsto \boldsymbol{v}|_{\Omega_0}$ is continuous from $H^s(\Omega)$ into $H^{s+1}(\Omega)^d$. So, we take $\mathcal{R}g$ equal to $\boldsymbol{v}|_\Omega$, and apply Lemma 3.4 to conclude. ■

4 INTERPOLATION OF NULLSPACES WITH BOUNDARY CONDITIONS

In view of our application in §5, we only deal in this section with the divergence operator. But similar methods allow for treating various other situations. Our aim is to interpolate the subspaces $\mathscr{N}^s_{\text{div}}(\Omega) \cap H^1_0(\Omega)^d$ of divergence-free functions in $H^s(\Omega)^d$ which vanish on the boundary of Ω and to investigate whether the following equality holds

$$[\mathscr{N}^s_{\text{div}}(\Omega) \cap H^1_0(\Omega)^d,\ \mathscr{N}^r_{\text{div}}(\Omega) \cap H^1_0(\Omega)^d]_\theta \ =\ \mathscr{N}^{(1-\theta)s+\theta r}_{\text{div}}(\Omega) \cap H^1_0(\Omega)^d. \tag{4.1}$$

Denoting by γ_0 the first trace operator on $\partial\Omega$, we are going to use Lemma 3.4 as above in three different ways:
(i) with $X = H^s(\Omega)^d$, $Y = H^r(\Omega)^d$ and $A = (\text{div}, \gamma_0)$;
(ii) with $X = H^s(\Omega)^d \cap H^1_0(\Omega)^d$, $Y = H^r(\Omega)^d \cap H^1_0(\Omega)^d$ and $A = \text{div}$;
(iii) with $X = \mathscr{N}^s_{\text{div}}(\Omega)$, $Y = \mathscr{N}^r_{\text{div}}(\Omega)$ and $A = \gamma_0$.

4.1 In a smooth domain

The idea here is to start from the interpolation result between Sobolev spaces (3.2) and to construct a right-inverse for the pair (div, γ_0) with the help of the Dirichlet problem for the Stokes operator.

THEOREM 4.1 *Let m be a fixed positive integer, and let Ω be a bounded domain with a $\mathscr{C}^{m-1,1}$ boundary. The interpolation result* (4.1) *holds for any real numbers r and s such that $1 \le r \le s \le m$ and for any θ, $0 < \theta < 1$.*

PROOF: We take $X = H^s(\Omega)^d$ and

$$\widetilde{X} = \Big\{(g, \boldsymbol{h}) \in H^{s-1}(\Omega) \times H^{s-1/2}(\partial\Omega)^d \,;\quad \int_\Omega g(\boldsymbol{x})\, d\boldsymbol{x} - \int_{\partial\Omega} (\boldsymbol{h}\cdot\boldsymbol{n})(\boldsymbol{\sigma})\, d\boldsymbol{\sigma} = 0\Big\} \tag{4.2}$$

and we define $\mathcal{R}(g, \boldsymbol{h})$ as $\boldsymbol{v}$, where the pair $(\boldsymbol{v}, q)$ is the only solution of the Stokes problem:

$$\begin{cases} -\Delta \boldsymbol{v} + \mathbf{grad}\ q = \mathbf{0} & \text{in } \Omega, \\ \mathrm{div}\ \boldsymbol{v} = g & \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{h} & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Due to the regularity property of the domain, the operator $\mathcal{R}$ is continuous from $\widetilde{X}$ into X, for any real number s, $1 \le s \le m$. Of course we take $Y = H^r(\Omega)^d$ and the corresponding space for $\widetilde{Y}$, so that $\mathcal{R}$ has the required properties. ■

REMARK 4.2 It is readily checked that the orthogonal projection operator $\mathcal{P}$ from $H^1(\Omega)^d$ onto $\mathscr{N}^1_{\mathrm{div}}(\Omega) \cap H^1_0(\Omega)^d$ with the gradient norm, associates with any function $\boldsymbol{u}$ in $H^1(\Omega)^d$ the function $\boldsymbol{v}$, where $(\boldsymbol{v}, q)$ is the solution of the Stokes problem

$$\begin{cases} -\Delta \boldsymbol{v} + \mathbf{grad}\ q = -\Delta \boldsymbol{u} & \text{in } \Omega, \\ \mathrm{div}\ \boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

hence it has the same stability properties as the operator $\mathcal{R}$ in the previous proof. As a consequence, relying on Theorem 3.1 or 3.5, we have an alternative proof of Theorem 4.1 since, for $1 \le s \le m$, this operator $\mathcal{P}$ is continuous from $\mathscr{N}^{(1-\theta)s+\theta r}_{\mathrm{div}}(\Omega)$ into $[\mathscr{N}^s_{\mathrm{div}}(\Omega) \cap H^1_0(\Omega)^d,\ \mathscr{N}^r_{\mathrm{div}}(\Omega) \cap H^1_0(\Omega)^d]_\theta$ and its restriction to $\mathscr{N}^1_{\mathrm{div}}(\Omega) \cap H^1_0(\Omega)^d$ coincides with the identity operator.

4.2 In a cylinder

For domains Ω with edges on the boundary like a cylinder for example, we can use a similar method as above, but we have to handle the singularities of solutions along the edges. The geometrical hypotheses are that, for each point $\boldsymbol{x}_0$ on the boundary $\partial\Omega$, there exists a neighbourhood of $\boldsymbol{x}_0$ whose intersection with $\overline{\Omega}$ is diffeomorphic:
(i) to a half-space ($\boldsymbol{x}_0$ is a regular point);
(ii) to a dihedron ($\boldsymbol{x}_0$ is an edge point).
We denote by $\partial_{\mathrm{e}}\Omega$ the set of edge points of Ω. For each $\boldsymbol{x}_0$ in $\partial_{\mathrm{e}}\Omega$, we shall use the following ingredients:
• the plane sector $\Gamma_{\boldsymbol{x}_0}$ which is tangent at $\boldsymbol{x}_0$ to the intersection of Ω with the normal

plane to the edge,

• the polar coordinates (r,θ) in $\Gamma_{\boldsymbol{x}_0}$, centered in $\boldsymbol{x}_0$.

In order to improve the behaviour of solutions along the edges, we are going to consider more general operators than the Stokes operator, namely operators $\boldsymbol{L}$ of the form:

$$\begin{pmatrix} \boldsymbol{v} \\ q \end{pmatrix} \quad \longmapsto \quad \begin{pmatrix} \boldsymbol{f} \\ g \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}\boldsymbol{v} + \mathbf{grad}\ q \\ \operatorname{div} \boldsymbol{v} \end{pmatrix}, \tag{4.4}$$

where $\boldsymbol{M}$ is a $d \times d$ strongly elliptic matrix of operators of degree 2. We introduce, for any $s \geq 1$, the following space of "regular" solutions $(\boldsymbol{v}, q)$ of $\boldsymbol{L}$ on Ω:

$$\mathscr{D}^s(\Omega) = H^s(\Omega)^d \times H^{s-1}(\Omega)/\mathbb{R}. \tag{4.5}$$

For $1 \leq s < 2$, the operator $(\boldsymbol{L}, \gamma_0)$ is continuous from $\mathscr{D}^s(\Omega)$ into the corresponding space of "regular" right-hand sides

$$\begin{aligned}\mathscr{E}^s(\Omega) = \Big\{ &(\boldsymbol{f}, g, \boldsymbol{h}) \in H^{s-2}(\Omega)^d \times H^{s-1}(\Omega) \times H^{s-1/2}(\partial\Omega)^d\ ; \\ &\int_\Omega g(\boldsymbol{x})\ d\boldsymbol{x} - \int_{\partial\Omega} (\boldsymbol{h}\cdot\boldsymbol{n})(\boldsymbol{\sigma})\ d\boldsymbol{\sigma} = 0 \Big\}.\end{aligned} \tag{4.6}$$

We remark that for any $s > 2$, the gradient of $\boldsymbol{v}$ on $\partial_{\mathrm{e}}\Omega$ is completely determined by the trace $\boldsymbol{h}$ of $\boldsymbol{v}$ on $\partial\Omega$. Hence $\operatorname{div} \boldsymbol{h}|_{\partial_{\mathrm{e}}\Omega}$ makes sense and, for $s > 2$, the operator $(\boldsymbol{L}, \gamma_0)$ is continuous from $\mathscr{D}^s(\Omega)$ into:

$$\begin{aligned}\mathscr{E}^s(\Omega) = \Big\{ &(\boldsymbol{f}, g, \boldsymbol{h}) \in H^{s-2}(\Omega)^d \times H^{s-1}(\Omega) \times H^{s-1/2}(\partial\Omega)^d\ ; \\ &\int_\Omega g(\boldsymbol{x})\ d\boldsymbol{x} - \int_{\partial\Omega} (\boldsymbol{h}\cdot\boldsymbol{n})(\boldsymbol{\sigma})\ d\boldsymbol{\sigma} = 0 \\ &(g - \operatorname{div} \boldsymbol{h})|_{\partial_{\mathrm{e}}\Omega} = 0 \Big\}.\end{aligned} \tag{4.7}$$

The same result holds with $s = 2$, the condition $(g - \operatorname{div} \boldsymbol{h})|_{\partial_{\mathrm{e}}\Omega} = 0$ being replaced by an integral condition.

Let us denote by $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_{d-2})$ the tangential coordinates along the edge at $\boldsymbol{x}_0$ and by $\boldsymbol{\nu} = (\nu_1, \nu_2)$ the normal coordinates to the edge. In a first stage we recall the regularity result [6], [7]: it is linked with the spectral properties of the principal part $\check{\boldsymbol{L}}(\boldsymbol{x}_0; 0, \partial_\nu)$ of the operator $\boldsymbol{L}(\boldsymbol{x}_0; 0, \partial_\nu)$ in each point $\boldsymbol{x}_0$ of $\partial_{\mathrm{e}}\Omega$, where $\boldsymbol{L}(\boldsymbol{x}_0; 0, \partial_\nu)$ stands for the operator $\boldsymbol{L}$ written in coordinates $(\boldsymbol{\tau}, \boldsymbol{\nu})$, with its coefficients frozen in $\boldsymbol{x}_0$ and the tangential operator ∂_τ replaced by 0.

THEOREM 4.3 *Let s be a real number ≥ 1 and let $(\boldsymbol{v}, q)$ be any pair in $\mathscr{D}^1(\Omega)$ such that $(\boldsymbol{L}(\boldsymbol{v}, q), \boldsymbol{v}|_{\partial\Omega})$ belongs to $\mathscr{E}^s(\Omega)$. If for any $\boldsymbol{x}_0$ in $\partial_{\mathrm{e}}\Omega$ and any complex number λ with $0 < \operatorname{Re}\lambda \leq s - 1$, the boundary value problem*

$$\begin{cases} \check{\boldsymbol{L}}(\boldsymbol{x}_0; 0, \partial_\nu)\Big(r^\lambda \boldsymbol{u}(\theta), r^{\lambda-1} p(\theta)\Big) = 0 & \text{in } \Gamma_{\boldsymbol{x}_0}, \\ r^\lambda \boldsymbol{u}(\theta) = \mathbf{0} & \text{on } \partial\Gamma_{\boldsymbol{x}_0}, \end{cases} \tag{4.8}$$

has only the trivial solution, then the pair $(\boldsymbol{v}, q)$ belongs to $\mathscr{D}^s(\Omega)$.

The strong ellipticity of the matrix $\boldsymbol{M}$ yields the unique solvability of the Dirichlet problem for $\boldsymbol{L}$, which provides a right-inverse to the operator $(\operatorname{div}, \gamma_0)$. Hence

COROLLARY 4.4 *Let s be a real number ≥ 1. If there exists an operator $\boldsymbol{L}$ of the form* (4.4) *satisfying the regularity condition of Theorem* 4.3*, then the interpolation result* (4.1) *holds for any real number r such that $1 \leq r \leq s$ and for any θ, $0 < \theta < 1$.*

EXAMPLE 4.5 Let us consider the case of the cylinder $\Omega = D \times]-1, 1[$, with D the unit disk in $\mathbb{R}^2$. If we use cylindrical coordinates (ρ, ϑ, z), we can take as tangential coordinates along the edges $\tau = \vartheta$ and normal coordinates (ρ, z). If we take the Stokes system as operator $\boldsymbol{L}$, and if we introduce the new unknowns $(u_\rho, u_\vartheta, u_z)$ by

$$u_\rho = \cos\vartheta\, u_1 + \sin\vartheta\, u_2, \quad u_\vartheta = -\sin\vartheta\, u_1 + \cos\vartheta\, u_2, \quad u_z = u_3,$$

the bi-dimensional problem (4.8) is:

$$\begin{cases} -(\partial_\rho^2 + \partial_z^2)(r^\lambda u_\rho(\theta)) + \partial_\rho(r^{\lambda-1}p(\theta)) = 0 & \text{in } \Gamma_{\boldsymbol{x}_0}, \\ -(\partial_\rho^2 + \partial_z^2)(r^\lambda u_z(\theta)) + \partial_z(r^{\lambda-1}p(\theta)) = 0 & \text{in } \Gamma_{\boldsymbol{x}_0}, \\ -(\partial_\rho^2 + \partial_z^2)(r^\lambda u_\vartheta(\theta)) = 0 & \text{in } \Gamma_{\boldsymbol{x}_0}, \\ \partial_\rho(r^\lambda u_\rho(\theta)) + \partial_z(r^\lambda u_z(\theta)) = 0 & \text{in } \Gamma_{\boldsymbol{x}_0}, \\ r^\lambda \boldsymbol{u}(\theta) = \boldsymbol{0} & \text{on } \partial\Gamma_{\boldsymbol{x}_0} \end{cases} \tag{4.9}$$

and $\Gamma_{\boldsymbol{x}_0}$ is a quadrant. Problem (4.9) is uncoupled into a bi-dimensional Stokes problem and a Laplacian. The "first" non trivial solution of (4.9) is

$$\lambda = 2, \quad u_\vartheta(\theta) = \sin 2\theta, \quad u_\rho = 0, \quad u_z = 0.$$

This allows to prove the interpolation identity (4.1) for $1 \leq r \leq s < 3$, which is not very satisfactory. We obtain a better result with the following operator $\boldsymbol{M}$, where we fix a parameter ω in $]0, \pi/2[$:

$$\boldsymbol{M}\boldsymbol{v} = \begin{pmatrix} (-\Delta - 2z\sin\vartheta\cos\omega\ r\partial_r\partial_z)v_1 \\ (-\Delta + 2z\cos\vartheta\cos\omega\ r\partial_r\partial_z)v_2 \\ -\Delta v_3 \end{pmatrix}$$

(the idea to build this operator is to fold the dihedron in $\boldsymbol{x}_0$ in order to reduce its angle to ω, to consider the standard Laplace operator in the dihedron with angle ω and to use a change of variables which maps it back onto the original dihedron).

Then problem (4.8) is uncoupled into a bi-dimensional Stokes problem and the following Dirichlet problem

$$\begin{cases} -\Big(\partial_\rho^2 + \partial_z^2 - 2z\cos\omega\ \partial_r\partial_z\Big)(r^\lambda u_\vartheta(\theta)) = 0 & \text{in } \Gamma_{\boldsymbol{x}_0}, \\ r^\lambda u_\vartheta(\theta) = 0 & \text{on } \partial\Gamma_{\boldsymbol{x}_0}. \end{cases} \tag{4.10}$$

The "first" non trivial solution of (4.10) is $r^{\pi/\omega}\sin(\theta\pi/\omega)$. The least positive real part of a λ for which the Stokes part has a non-trivial solution, is $\xi_0 \simeq 2.739$. We choose $\omega = \pi/3$ for example and derive the interpolation identity (4.1). ■

Of course, the same argument can be applied in a more general type of cylinder.

COROLLARY 4.6 *Let Ω be the cylinder $\Sigma \times]-1, 1[$, where Σ is a bounded domain in $\mathbb{R}^{d-1}$ with a $\mathscr{C}^{3,1}$ boundary. The interpolation result* (4.1) *holds for any real numbers* *r and s such that* $1 \le r \le s \le 3.739$ *and for any* θ, $0 < \theta < 1$.

4.3 In a cube

In the cube $\Omega =]-1,1[^3$, it is less difficult to treat separately the divergence operator and the trace operator. We present two different proofs of the following theorem: the first argument leads to a slightly higher limit for the order of the spaces while the second one is more constructive and can easily be adapted to treat spaces of functions with a null normal trace on the boundary.

THEOREM 4.7 *On the cube* Ω*, the interpolation result* (4.1) *holds for any real numbers* r *and* s *such that* $1 \le r \le s \le 3.739$ *and for any* θ, $0 < \theta < 1$.

First proof – We firstly recall the interpolation result for the spaces with null traces:

$$[H^s(\Omega) \cap H^1_0(\Omega),\ H^r(\Omega) \cap H^1_0(\Omega)]_\theta \;=\; H^{(1-\theta)s+\theta r}(\Omega) \cap H^1_0(\Omega), \tag{4.11}$$

which can be established by using a lifting of traces, we refer to [3] for this proof in a more general framework.

Relying on (4.11), we need a right-inverse for the divergence operator with values in $H^s(\Omega) \cap H^1_0(\Omega)$. The method is similar as above. If we use the Stokes operator, we are hampered with singularities along the edge, which are not in $H^3(\Omega)$. As this is just the interesting regularity for our application, we avoid the edge singularities by the choice of another operator $\boldsymbol{M}$. We choose, with $\omega = \pi/3$ for example:

$$\boldsymbol{M}\boldsymbol{v} = \begin{pmatrix} (-\Delta + 2x_2\, x_3 \cos\omega\ \partial_2\partial_3)\ v_1 \\ (-\Delta + 2x_3\, x_1 \cos\omega\ \partial_3\partial_1)\ v_2 \\ (-\Delta + 2x_1\, x_2 \cos\omega\ \partial_1\partial_2)\ v_3 \end{pmatrix}.$$

Like for the cylinder above, the corresponding operator $\boldsymbol{L}$ in (4.4) preserves the regularity along the edges for $s \le 3.739$. But now, the corners of Ω may also produce singularities. As opposed to the edge singularities, the space of corner singularities associated with a fixed s is only finite-dimensional. Until now, we constructed our right-inverse $\mathcal{R}g$ as the solution $\boldsymbol{v}$ of the problem $\boldsymbol{L}(\boldsymbol{v}, q) = (\boldsymbol{0}, g)$, cf (4.3). The idea now is to introduce an operator $\boldsymbol{F}$ with suitable continuity properties so that the solution $\boldsymbol{v}$ in $H^1_0(\Omega)^3$ of $\boldsymbol{L}(\boldsymbol{v}, q) = (\boldsymbol{F}(g), g)$ has no singular part at the corners of Ω.

Here follows the decomposition result into regular and singular parts. We denote $\boldsymbol{a}_1 = (-1,-1,-1)$, ..., $\boldsymbol{a}_8 = (1,1,1)$ the corners of Ω. Let $\Lambda_{\boldsymbol{L}}$ be the (discrete) set of complex numbers λ with a real part $> -1/2$ such that the boundary value problem on the octant $\Gamma_1 =]-1, +\infty[^3$

$$\begin{cases} \boldsymbol{L}(\boldsymbol{a}_1; \partial_{\boldsymbol{x}})\Big(r^\lambda \boldsymbol{u}(\theta), r^{\lambda-1} p(\theta)\Big) = 0 & \text{in } \Gamma_1, \\ r^\lambda \boldsymbol{u}(\theta) = \boldsymbol{0} & \text{on } \partial\Gamma_1 \end{cases} \tag{4.12}$$

has non-trivial solutions, (r, θ) denoting the spherical coordinates centered in $\boldsymbol{a}_1$ (the corresponding problems for the other corners are equivalent). Relying on [6] (for the asymptotic expansion at corners in the absence of edge singularities) and [5] or [16] (for expressions of the coefficients occuring in these expansions) one can prove:

THEOREM 4.8 *Let s be a real number, $1 \le s \le 3.739$, and let $(\boldsymbol{v}, q)$ be any pair in $\mathscr{D}^1(\Omega)$ such that $\boldsymbol{v}$ vanishes on $\partial\Omega$ and $\boldsymbol{L}(\boldsymbol{v}, q) = (\boldsymbol{f}, g)$ belongs to $\mathscr{E}^s(\Omega)$. If any λ in $\Lambda_{\boldsymbol{L}}$ satisfies* $\mathrm{Re}\lambda \neq s - 3/2$, *then the following decomposition holds*

$$(\boldsymbol{v}, q) = (\boldsymbol{v}_0, q_0) + \sum_{j=1}^{8} (\boldsymbol{v}_j, q_j) \tag{4.13}$$

where $(\boldsymbol{v}_0, q_0)$ belongs to $\mathscr{D}^s(\Omega)$ and each $(\boldsymbol{v}_j, q_j)$, $1 \le j \le 8$, belongs to a finite-dimensional space of singular solutions. There exist functions $(\boldsymbol{V}_k^j, Q_k^j)$ and $(\boldsymbol{V}_k^{*j}, Q_k^{*j})$, $1 \le k \le K$, $1 \le j \le 8$, which depend only on the operator $\boldsymbol{M}$ such that

$$(\boldsymbol{v}_j, q_j) = \sum_{k=1}^{K} \Big(\int_\Omega \boldsymbol{V}_k^{*j} \cdot \boldsymbol{f} + Q_k^{*j}\, g\, d\boldsymbol{x} \Big) \cdot (\boldsymbol{V}_k^j, Q_k^j).$$

These functions $(\boldsymbol{V}_k^j, Q_k^j)$ and $(\boldsymbol{V}_k^{*j}, Q_k^{*j})$ are associated with a λ_k in $\Lambda_{\boldsymbol{L}}$ and have the radial behaviour:

$$\boldsymbol{V}_k^j \sim r^{\lambda_k}, \quad Q_k^j \sim r^{\lambda_k - 1}, \quad \boldsymbol{V}_k^{*j} \sim r^{-\lambda_k - 1}, \quad Q_k^{*j} \sim r^{-\lambda_k - 2}.$$

This statement and a closer look at the structure of the dual singular functions $(\boldsymbol{V}_k^{*j}, Q_k^{*j})$ allow for proving the

LEMMA 4.9 *Let s be a real number, $1 \le s \le 3.739$. There exists an operator $\boldsymbol{F}$, continuous from $H^{s-1}(\Omega)$ into $H^{s-2}(\Omega)^3$ and from $L^2(\Omega)$ into $H^{-1}(\Omega)^3$, such that for any g in $H^{s-1}(\Omega)$, vanishing on $\partial\Omega$ if $s > 2$, the unique solution $(\boldsymbol{v}, q)$ in $\mathscr{D}^1(\Omega)$ of $\boldsymbol{L}(\boldsymbol{v}, q) = (\boldsymbol{F}(g), g)$ belongs to $\mathscr{D}^s(\Omega)$.*

Setting $\mathcal{R}g = \boldsymbol{v}$, the interpolation equality (4.11) and Lemma 3.4 give the result.

REMARK 4.10 The limitation $s \le 3.739$ comes from the method. A better choice of the operator $\boldsymbol{M}$ and/or the direct handling of edge singularities would allow for a weaker limitation.

Second proof – The idea is to construct "by hand" a trace lifting operator from $\mathscr{N}^s_{\mathrm{div}}(\Omega)$ into $\mathscr{N}^s_{\mathrm{div}}(\Omega) \cap H^1_0(\Omega)^3$.

THEOREM 4.11 *Let Ω be the cube $]-1, 1[^3$. There exists a projection operator $\mathcal{Q}$ from $\mathscr{N}^1_{\mathrm{div}}(\Omega)$ onto $\mathscr{N}^1_{\mathrm{div}}(\Omega) \cap H^1_0(\Omega)^3$ which is continuous from $\mathscr{N}^s_{\mathrm{div}}(\Omega)$ into itself for any real number s, $1 \le s < 3.5$.*

PROOF: Let $\boldsymbol{u}$ be any function in $\mathscr{N}^s_{\mathrm{div}}(\Omega)$, $s \ge 1$. We look for a function $\boldsymbol{\psi}$ in $H^{s+1}(\Omega)^3$, depending only on $\gamma_0\boldsymbol{u}$, such that $\boldsymbol{u} - \mathbf{curl}\ \boldsymbol{\psi}$ belongs to $H^1_0(\Omega)^3$. This function is built in two steps. We denote by Γ_j (resp. by Γ_{j+3}), $j = 1, 2, 3$, the face with equation $x_j = 1$ (resp. $x_j = -1$).
1) In order to cancel the normal trace of $\boldsymbol{u}$ on the faces, we firstly remark that, since the integral of $\boldsymbol{u} \cdot \boldsymbol{n}$ on $\partial\Omega$ is equal to 0, there exists a polynomial $\boldsymbol{\chi}$ with degree 2, depending linearly on the quantities $\int_{\Gamma_j} (\boldsymbol{u} \cdot \boldsymbol{n})(\boldsymbol{\sigma})\, d\boldsymbol{\sigma}$, such that the modified function

$\boldsymbol{v} = \boldsymbol{u} - \mathbf{curl}\ \boldsymbol{\chi}$ satisfies

$$\int_{\Gamma_j} (\boldsymbol{v} \cdot \boldsymbol{n})(\boldsymbol{\sigma})\, d\boldsymbol{\sigma} = 0, \quad 1 \le j \le 6.$$

Then, Δ_j denoting the bidimensional Laplace operator on the square Γ_j and $\boldsymbol{n}_j$ the unit outward normal on Γ_j, the problem $-\Delta_j \varphi_j = \boldsymbol{v} \cdot \boldsymbol{n}_j$ provided with homogeneous Neumann boundary conditions, has a unique solution φ_j in $H^1(\Gamma_j)$ with $\int_{\Gamma_j} \varphi_j = 0$. Moreover, it can be checked (see [6] and [10]) that the following regularity result holds if the function $\boldsymbol{u}$, hence $\boldsymbol{v}$, belongs to $H^s(\Omega)^3$, $0 \le s < \frac{7}{2}$:

$$\|\varphi_j\|_{H^{s+\frac{3}{2}}(\Gamma_j)} \le c\, \|\boldsymbol{v}\|_{H^s(\Omega)^3} \le c'\, \|\boldsymbol{u}\|_{H^s(\Omega)^3}. \tag{4.14}$$

Next, we use the trace lifting theorem to construct on Ω three functions ϖ_1, ϖ_2 and ϖ_3 which satisfy

$$\begin{aligned}
&\varpi_1 = -\partial_2 \varphi_i \quad \text{on } \Gamma_i, \quad i = 3, 6, \quad \text{and} \quad \varpi_1 = \partial_3 \varphi_i \quad \text{on } \Gamma_i, \quad i = 2, 5,\\
&\varpi_2 = -\partial_3 \varphi_i \quad \text{on } \Gamma_i, \quad i = 1, 4, \quad \text{and} \quad \varpi_2 = \partial_1 \varphi_i \quad \text{on } \Gamma_i, \quad i = 3, 6,\\
&\varpi_3 = -\partial_1 \varphi_i \quad \text{on } \Gamma_i, \quad i = 2, 5, \quad \text{and} \quad \varpi_3 = \partial_2 \varphi_i \quad \text{on } \Gamma_i, \quad i = 1, 4,
\end{aligned}$$

together with the stability property

$$\sum_{i=1}^{3} \|\varpi_i\|_{H^{s+1}(\Omega)} \le c \sum_{j=1}^{6} \|\varphi_j\|_{H^{s+\frac{3}{2}}(\Gamma_j)}. \tag{4.15}$$

The function $\boldsymbol{w} = \boldsymbol{v} - \mathbf{curl}\ (\varpi_1, \varpi_2, \varpi_3)$ has its normal traces equal to 0 on each face Γ_j for $1 \le j \le 6$.
2) In a second step, we want to cancel the tangential value of $\boldsymbol{w}$ on each face without modifying the normal value: this is achieved by subtracting the curl of a function with null tangential traces on the boundary. We begin with introducing a function ψ_0 in $H^{s+1}(\Omega)$ such that

$$\psi_0 = 0 \quad \text{on } \Gamma_j \cup \Gamma_{j+3}, \quad j = 1, 3, \quad \text{and} \quad \partial_3 \psi_0 = -w_1 \quad \text{on } \Gamma_3 \cup \Gamma_6,$$

which defines a new function $\boldsymbol{w}^0 = \boldsymbol{w} - \mathbf{curl}\ (0, \psi_0, 0)$. Next, the standard results about the lifting of traces allow for defining successively three functions ψ_1, ψ_2 and ψ_3 such that

$$\psi_1 = 0 \quad \text{on } \Gamma_j \cup \Gamma_{j+3}, \quad j = 2, 3, \quad \text{and} \quad \psi_2 = \psi_3 = 0 \quad \text{on } \Gamma_j, \quad 1 \le j \le 6,$$

and that

$$\begin{aligned}
&\partial_2 \psi_1 = -w_3^0 \quad \text{on } \Gamma_2 \cup \Gamma_5 \quad \text{and} \quad \partial_3 \psi_1 = w_2^0 \quad \text{on } \Gamma_3 \cup \Gamma_6,\\
&\partial_3 \psi_2 = 0 \qquad\quad \text{on } \Gamma_3 \cup \Gamma_6 \quad \text{and} \quad \partial_1 \psi_2 = w_3^1 \quad \text{on } \Gamma_1 \cup \Gamma_4,\\
&\partial_1 \psi_3 = -w_2^2 \quad \text{on } \Gamma_1 \cup \Gamma_4 \quad \text{and} \quad \partial_2 \psi_3 = w_1^2 \quad \text{on } \Gamma_2 \cup \Gamma_5,
\end{aligned}$$

with $\boldsymbol{w}^1 = \boldsymbol{w} - \mathbf{curl}\ (\psi^1, \psi^0, 0)$, $\boldsymbol{w}^2 = \boldsymbol{w} - \mathbf{curl}\ (\psi^1, \psi^0 + \psi^2, 0)$. Then, it can be checked that the traces of the function $\boldsymbol{z} = \boldsymbol{w} - \mathbf{curl}\ (\psi^1, \psi^0 + \psi^2, \psi^3)$ are equal to 0 on $\partial\Omega$. Moreover, the divergence-free property implies that the right compatibility conditions on the edges are satisfied, so that the function $\boldsymbol{z}$ belongs to the same space

$H^s(\Omega)^3$ as $\boldsymbol{w}$ and satisfies

$$\|\boldsymbol{z}\|_{H^s(\Omega)^3} \leq c\,\|\boldsymbol{w}\|_{H^s(\Omega)^3}. \tag{4.16}$$

As a conclusion, the function $\boldsymbol{z}$ is divergence-free and belongs to $H_0^1(\Omega)^3$. Also, combining the previous estimates implies that, if $\boldsymbol{u}$ belongs to $H^s(\Omega)^3$, $1 \leq s < \frac{7}{2}$,

$$\|\boldsymbol{z}\|_{H^s(\Omega)^3} \leq c\,\|\boldsymbol{u}\|_{H^s(\Omega)^3}. \tag{4.17}$$

Taking $\boldsymbol{\mathcal{Q}u} = \boldsymbol{z}$ proves the theorem. ■

Combining Theorem 4.11 with Lemma 3.4 yields Theorem 4.7 with a slightly more restrictive limit (3.5 instead of 3.739) on the order s of the smaller space. However a better choice of the operator on the faces should improve the limit.

The previous proofs can be extended to the case of the weighted spaces. Here, we denote by $\mathscr{N}^s_{\mathrm{div},\alpha}(\Omega)$ the space $\mathscr{N}^s_{A,\alpha}(\Omega)$ when A is the divergence operator.

COROLLARY 4.12 *Let α be a real number, $-1 < \alpha < 1$. In the cube $\Omega =]-1,1[^3$, the following interpolation result holds for any real numbers r and s such that $1 \leq r \leq s \leq 3.739 + \alpha$ and for any θ, $0 < \theta < 1$:*

$$[\mathscr{N}^s_{\mathrm{div},\alpha}(\Omega) \cap H^1_{\alpha,0}(\Omega)^3, \mathscr{N}^r_{\mathrm{div},\alpha}(\Omega) \cap H^1_{\alpha,0}(\Omega)^3]_\theta = \mathscr{N}^{(1-\theta)s+\theta r}_{\mathrm{div},\alpha}(\Omega) \cap H^1_{\alpha,0}(\Omega)^3. \tag{4.18}$$

REMARK 4.13 Exactly the same arguments as in the first step of the second proof can be applied for the kernels

$$\mathscr{N}^s_{\mathrm{div}\,0}(\Omega) = \{\boldsymbol{v} \in \mathscr{N}^s_{\mathrm{div}}(\Omega) \mid \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Omega\}.$$

Hence, in the cube $\Omega =]-1,1[^3$, the following interpolation result holds for any real numbers r and s, $0 \leq r \leq s < 3.5$, and for any θ, $0 < \theta < 1$:

$$[\mathscr{N}^s_{\mathrm{div}\,0}(\Omega),\ \mathscr{N}^r_{\mathrm{div}\,0}(\Omega)]_\theta \ = \ \mathscr{N}^{(1-\theta)s+\theta r}_{\mathrm{div}\,0}(\Omega).$$

5 SPECTRAL METHODS FOR THE STOKES PROBLEM

In the cube $\Omega =]-1,1[^3$, we consider the Stokes problem with homogeneous boundary conditions:

$$\begin{cases} -\nu\,\Delta\boldsymbol{u} + \mathbf{grad}\ p = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\ \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Here, the unknowns are the velocity $\boldsymbol{u}$ and the pressure p, ν is a positive parameter standing for the kinematic viscosity and $\boldsymbol{f}$ is a density of body forces. As is well-known, for any data $\boldsymbol{f}$ in $H^{-1}(\Omega)^3$, this problem admits the equivalent variational formulation:

find a pair $(\boldsymbol{u}, p)$ *in* $H_0^1(\Omega)^3 \times L_0^2(\Omega)$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v} \in H_0^1(\Omega)^3, \quad a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = < \boldsymbol{f}, \boldsymbol{v} >, \\ &\forall q \in L^2(\Omega), \quad b(\boldsymbol{u}, q) = 0, \end{aligned} \tag{5.2}$$

where $L_0^2(\Omega)$ denotes the space of functions in $L^2(\Omega)$ with a null integral on Ω while the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \nu \int_\Omega \mathbf{grad}\ \boldsymbol{u}(\boldsymbol{x}) \ .\ \mathbf{grad}\ \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x},$$

$$b(\boldsymbol{v}, q) = -\int_\Omega \operatorname{div} \boldsymbol{v}(\boldsymbol{x})\, q(\boldsymbol{x})\, d\boldsymbol{x}.$$

This problem admits a unique solution $(\boldsymbol{u}, p)$ in $H_0^1(\Omega)^3 \times L_0^2(\Omega)$.

5.1 Spectral discretization

Several spectral discretizations of problem (5.2) exist, we only present the simplest one (see [15]). For any integer $n \geq 0$, we denote by $\mathbb{P}_n(\Omega)$ the space of polynomials with three variables and degree $\leq n$ with respect to each variable and by $\mathbb{P}_n^0(\Omega)$ the subspace $\mathbb{P}_n(\Omega) \cap H_0^1(\Omega)$. Next, we fix an integer $N \geq 2$. Using the Galerkin method leads to the following discrete problem: *find a pair* $(\boldsymbol{u}_N, p_N)$ *in* $\mathbb{P}_N^0(\Omega)^3 \times (\mathbb{P}_{N-2} \cap L_0^2)(\Omega)$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v}_N \in \mathbb{P}_N^0(\Omega)^3, \quad a(\boldsymbol{u}_N, \boldsymbol{v}_N) + b(\boldsymbol{v}_N, p_N) = (\boldsymbol{f}, \boldsymbol{v}_N), \\ &\forall q_N \in \mathbb{P}_{N-2}(\Omega), \quad b(\boldsymbol{u}_N, q_N) = 0. \end{aligned} \tag{5.3}$$

REMARK 5.1 In practical situations, the integrals which appear in the forms $a(.,.)$, $b(.,.)$ and $(\boldsymbol{f}, .)$ are replaced by Gauss type formulas, however this does not change the results that follow. So, for the sake of clarity, we limit ourselves to the Galerkin type discretization.

Due to the continuity properties of the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ together with the ellipticity of the form $a(\cdot,\cdot)$, the numerical analysis of problem (5.3) relies on the following inf-sup condition which is proven in [15][4, Thm 25.7]:

$$\beta_N = \inf_{q_N \in (\mathbb{P}_{N-2} \cap L_0^2)(\Omega)} \ \sup_{\boldsymbol{v}_N \in \mathbb{P}_N^0(\Omega)^3} \frac{b(\boldsymbol{v}_N, q_N)}{\|\boldsymbol{v}_N\|_{H^1(\Omega)^3} \|q_N\|_{L^2(\Omega)}} \geq c\, N^{-1}. \tag{5.4}$$

The constant β_N of the inf-sup condition is not independent of N, indeed it is proven (see [4, (25.26)]) that β_N is exactly of order N^{-1}. However, these properties are sufficient to prove that problem (5.3) has a unique solution in $\mathbb{P}_N^0(\Omega)^3 \times (\mathbb{P}_{N-2} \cap L_0^2)(\Omega)$.

5.2 Error estimate

Let us denote by V_N the space of polynomials of degree $\leq N$ with null discrete divergence:

$$V_N = \big\{\boldsymbol{v}_N \in \mathbb{P}_N^0(\Omega)^3;\ \forall q_N \in \mathbb{P}_{N-2}(\Omega),\ b(\boldsymbol{v}_N, q_N) = 0\big\}.$$

Then, standard arguments lead to the following estimate:

$$\|\boldsymbol{u}-\boldsymbol{u}_N\|_{H^1(\Omega)^3} + N^{-1}\,\|p-p_N\|_{L^2(\Omega)} \\ \leq c\Big(\inf_{\boldsymbol{w}_N\in V_N}\|\boldsymbol{u}-\boldsymbol{w}_N\|_{H^1(\Omega)^3} \;+ \inf_{q_N\in(\mathbb{P}_{N-2}\cap L^2_0)(\Omega)}\|p-q_N\|_{L^2(\Omega)}\Big). \tag{5.5}$$

Standard polynomial approximation properties yield (see for instance [4, §7]):

$$\inf_{\boldsymbol{v}_N\in\mathbb{P}^0_N(\Omega)^3}\|\boldsymbol{u}-\boldsymbol{v}_N\|_{H^1(\Omega)^3} \;+ \inf_{q_N\in(\mathbb{P}_{N-2}\cap L^2_0)(\Omega)}\|p-q_N\|_{L^2(\Omega)} \\ \leq c\,N^{1-s}\,\big(\|\boldsymbol{u}\|_{H^s(\Omega)^3}+\|p\|_{H^{s-1}(\Omega)}\big),$$

if $(\boldsymbol{u},p)$ belongs to $H^s(\Omega)^3\times H^{s-1}(\Omega)$ for a real number $s\geq 1$. So, it remains to estimate the quantity $\inf_{\boldsymbol{w}_N\in V_N}\|\boldsymbol{u}-\boldsymbol{w}_N\|_{H^1(\Omega)^3}$.

A general result (see [9, Ch. II, (1.16)]) yields

$$\inf_{\boldsymbol{w}_N\in V_N}\|\boldsymbol{u}-\boldsymbol{w}_N\|_{H^1(\Omega)^3}\leq\frac{c}{\beta_N}\inf_{\boldsymbol{v}_N\in\mathbb{P}^0_N(\Omega)^3}\|\boldsymbol{u}-\boldsymbol{v}_N\|_{H^1(\Omega)^3},$$

where β_N is the inf-sup constant in (5.4). However, since β_N is of order N^{-1}, this cannot lead to an optimal error on the velocity (one power of N is lost). So, a separate study of the approximation error is necessary.

5.3 Approximation error

The idea is now to build directly an approximation of any function $\boldsymbol{u}$ in $\mathscr{N}^1_{\mathrm{div}}(\Omega)\cap H^1_0(\Omega)^3$ by a polynomial in $\mathscr{N}^1_{\mathrm{div}}(\Omega)\cap\mathbb{P}^0_N(\Omega)^3$ (which is strictly contained in V_N). The first result in this direction is due to [17]: for any function $\boldsymbol{w}$ in $\mathscr{N}^s_{\mathrm{div}}(\Omega)\cap H^1_0(\Omega)^3$, with $s\geq 3$,

$$\inf_{\boldsymbol{w}_N\in\mathscr{N}^1_{\mathrm{div}}(\Omega)\cap\mathbb{P}^0_N(\Omega)^3}\|\boldsymbol{w}-\boldsymbol{w}_N\|_{H^1(\Omega)^3}\leq c\,N^{1-s}\,\|\boldsymbol{w}\|_{H^s(\Omega)^3}. \tag{5.6}$$

However, even if this estimate is optimal as far as the power of N is concerned, the condition $s\geq 3$ is rather restrictive since it needs to take $\boldsymbol{f}$ in $H^1(\Omega)^3$ and even then, one does not know whether the velocity $\boldsymbol{u}$ belongs to $H^3(\Omega)^3$. Our aim is to lift the restriction $s\geq 3$ and to replace it by the minimal condition $s\geq 1$.

THEOREM 5.2 *Let s be a real number ≥ 1. Estimate (5.6) holds for any function $\boldsymbol{w}$ in $\mathscr{N}^s_{\mathrm{div}}(\Omega)\cap H^1_0(\Omega)^3$.*

PROOF: Of course, we have

$$\inf_{\boldsymbol{w}_N\in\mathscr{N}^1_{\mathrm{div}}(\Omega)\cap\mathbb{P}^0_N(\Omega)^3}\|\boldsymbol{w}-\boldsymbol{w}_N\|_{H^1(\Omega)^3}=\|\boldsymbol{w}-\Pi_N\boldsymbol{w}\|_{H^1(\Omega)^3},$$

where Π_N is the projection operator from $\mathscr{N}^1_{\mathrm{div}}(\Omega)\cap H^1_0(\Omega)^3$ onto $\mathscr{N}^1_{\mathrm{div}}(\Omega)\cap\mathbb{P}^0_N(\Omega)^3$. Thus the operator $Id-\Pi_N$ is continuous from $\mathscr{N}^1_{\mathrm{div}}(\Omega)\cap H^1_0(\Omega)^3$ into $H^1(\Omega)^3$ with norm ≤ 1 and from $\mathscr{N}^3_{\mathrm{div}}(\Omega)\cap H^1_0(\Omega)^3$ into $H^1(\Omega)^3$ with norm $\leq c\,N^{-2}$ by (5.6). So an interpolation argument relying on Theorem 4.7 leads to the desired result.

REMARK 5.3 The analogue of estimate (5.6) in the case of weighted Sobolev spaces and for $s \geq 3$ is proven in [17] for $\alpha = -\frac{1}{2}$ and can easily be extended to any value of α, $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$. The same arguments as previously, together with Corollary 4.12, lead to the following estimate, which holds for any function $\boldsymbol{w}$ in $\mathscr{N}^s_{\mathrm{div},\alpha}(\Omega) \cap H^1_{\alpha,0}(\Omega)^3$, $s \geq 1$:

$$\inf_{\boldsymbol{w}_N \in \mathscr{N}^1_{\mathrm{div},\alpha}(\Omega) \cap \mathbb{P}^0_N(\Omega)^3} \|\boldsymbol{w} - \boldsymbol{w}_N\|_{H^1_\alpha(\Omega)^3} \leq c\, N^{1-s}\, \|\boldsymbol{w}\|_{H^s_\alpha(\Omega)^3}. \tag{5.7}$$

As a conclusion, we obtain the following error estimate for problem (5.3), if the solution $(\boldsymbol{u}, p)$ of problem (5.1) belongs to $H^s(\Omega)^3 \times H^{s-1}(\Omega)$,

$$\|\boldsymbol{u} - \boldsymbol{u}_N\|_{H^1(\Omega)^3} + N^{-1}\, \|p - p_N\|_{L^2(\Omega)} \leq c\, N^{1-s} \left(\|\boldsymbol{u}\|_{H^s(\Omega)^3} + \|p\|_{H^{s-1}(\Omega)} \right). \tag{5.8}$$

5.4 Convergence of the spectral discretization

We are going to link the error estimate (5.8) with the properties of regularity of the Stokes problem in a cube. Let $\boldsymbol{S}$ denote the Stokes operator in (5.1). We already indicated that we have the regularity along the edges for $s < 3$ and that the regularity at the corners depend on the set $\Lambda_{\boldsymbol{S}}$ of complex numbers λ with real part $> -1/2$ such that the boundary value problem on the octant $\Gamma =]0, +\infty[^3$

$$\begin{cases} \boldsymbol{S}\Big(r^\lambda \boldsymbol{u}(\theta), r^{\lambda-1} p(\theta)\Big) = 0 & \text{in } \Gamma, \\ r^\lambda \boldsymbol{u}(\theta) = \boldsymbol{0} & \text{on } \partial\Gamma \end{cases} \tag{5.9}$$

has solutions satisfying $\boldsymbol{u} \neq \boldsymbol{0}$. The following result is proven in [7]:

THEOREM 5.4 *Let ξ_0 be the least real part of the elements of $\Lambda_{\boldsymbol{S}}$. Let s be a real number, $1 \leq s < \min\{3, \xi_0 + \frac{3}{2}\}$. Then, for any $\boldsymbol{f}$ in $H^{s-2}(\Omega)^3$, the solution $(\boldsymbol{u}, p)$ of problem* (5.1) *belongs to $H^s(\Omega)^3 \times H^{s-1}(\Omega)$.*

It is proven in [7] that ξ_0 is > 1, but, presently, it is not known if ξ_0 is $> \frac{3}{2}$. The set $\Lambda_{\boldsymbol{S}}$ and the associated singular functions are extensively studied in [11] and [12] (for instance, it is proven that, in some strip $1 < \mathrm{Re}\,\lambda < \xi_1$, the elements of $\Lambda_{\boldsymbol{S}}$ are real and that there are no logarithmic singularities).

As a consequence, the value $s_0 = \min\{3, \xi_0 + \frac{3}{2}\}$ is $> \frac{5}{2}$. For any $s < s_0$ and any $\boldsymbol{f}$ in $H^{s-2}(\Omega)^3$, we have the error estimate for the solution $(\boldsymbol{u}, p)$ of problem (5.1):

$$\|\boldsymbol{u} - \boldsymbol{u}_N\|_{H^1(\Omega)^3} + N^{-1}\, \|p - p_N\|_{L^2(\Omega)} \leq c\, N^{1-s}\, \|\boldsymbol{f}\|_{H^{s-2}(\Omega)^3}. \tag{5.10}$$

Also, the standard Aubin–Nitsche argument, combined with Theorems 5.2 and 5.4, gives an estimate for the L^2 norm of the velocity:

$$\|\boldsymbol{u} - \boldsymbol{u}_N\|_{L^2(\Omega)^3} \leq c\, N^{-s}\, \|\boldsymbol{f}\|_{H^{s-2}(\Omega)^3}. \tag{5.11}$$

Combined with a density argument, estimate (5.10) yields the convergence of the discrete velocity $\boldsymbol{u}_N$ towards $\boldsymbol{u}$ for any $\boldsymbol{f}$ in $H^{-1}(\Omega)^3$. The spectral discretization (5.3) can be extended to the full Navier–Stokes equations in a natural way, and the convergence property for the Stokes problem is useful for the numerical analysis of the nonlinear problem.

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