The lifting of polynomial traces revisited

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Abstract

We construct a lifting operator of polynomial traces on an interval that is stable in appropriate Sobolev norms. Next we extend this result to the case of traces vanishing at the endpoints of the interval. This has two applications, the interpolation of polynomial spaces and the evaluation by discrete formulas of fractional order Sobolev norms on polynomials.

Résumé

Nous construisons un opérateur de relèvement de traces polynomiales sur un intervalle qui est stable par rapport à des normes de Sobolev appropriées. Puis nous étendons ce résultat au cas de traces nulles aux extrémités de l'intervalle. Ceci a deux applications: l'interpolation d'espaces de polynômes, l'évaluation par des formules discrètes de normes de Sobolev d'ordre non entier appliquées à des polynômes.

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1 Introduction

This paper is motivated by the derivation of precise results about the behavior of the fractional order norms $H^{1/2}$ and $H^{1/2}_{00}$ in the set of polynomials with one variable. These norms are the natural candidates for stating stability results on traces of general functions of two variables defined over a bounded, regular enough, domain over its whole boundary or parts of it. They are also the natural measures for deriving stable liftings or extensions for functions defined on (part of) the boundary of a two-dimensional domain, we refer to [14] for an introduction and advanced properties of trace and lifting operators in a general functional framework.

The lifting of polynomial traces into spaces of polynomials has given rise to a large number of works: See the pioneering paper [4] and also, in the same period, [13], [15], [9] for two-dimensional domains, next [5], [16] for three-dimensional domains, and more recently [3]. Indeed, such results are very useful for the treatment of nonhomogeneous Dirichlet boundary conditions both in finite element methods and in spectral discretizations, and also for handling the matching conditions on the interfaces when working with domain decomposition techniques (see [18] or [19] for instance).

For some applications, (sub)optimal estimates which do not make use of fractional order Sobolev spaces, are sufficient, see [13], [9]. For many other applications optimal estimates are required: The trace is estimated in $H^{1/2}$ and the lifting in H^1 (or the trace in L^2 and the lifting in $H^{1/2}$ as in [3]). In comparison with standard lifting operators acting between ordinary functions, the requirement that polynomials should be preserved brings severe difficulties: In particular the localization by means of smooth cut-off functions cannot be employed any more.

The main difficulty is to lift a polynomial trace which is given on an edge (of e.g., a triangle or a rectangle) and which is zero at the ends of this edge, by a polynomial which is zero on both other edges adjacent to the first one. The strategy of [4] (see also [3] for a more global and symmetric implementation of this idea) is to apply a standard regularizing kernel on traces modified by an integral operator acting on the boundary. Another strategy is the division-lifting-multiplication algorithm: It consists in dividing the trace by an elementary polynomial in one variable which is zero at the ends, to lift it by a regularizing kernel, and to multiply by a polynomial in two variables which is zero on the adjacent edges.

This strategy has been mentioned in our note [7], but the complete proof was never published in the literature until now. In this situation, we encounter the limit case of the exponent $1/2$ and the space $H^{1/2}_{00}$ has to be employed. In the present work, we study the construction of two lifting operators from an edge to a rectangle, the first one being classical and the second one extending the nullity conditions to adjacent edges. We prove with full details their continuity on the spaces $H^{1/2}$ and $H^{1/2}_{00}$, respectively. In the second case in particular, our proof requires the use of several types of weighted Sobolev spaces. Nevertheless, this construction by the division-lifting-multiplication algorithm provides an interesting and simple alternative to the constructions of [4] and [3].

The construction of lifting operators which are stable both in the algebraic sense (polynomial traces extended into polynomials in two variables) and in the analytical sense (with respect to the fractional order norms), interesting *per se*, also allows us to derive accurate statements on these norms for polynomials. It is well known that, over finite dimensional spaces (here the space of polynomials with degree $\leq N$), all the norms are equivalent. But the constants arising in the various equivalence may depend on the dimension of these spaces. A main application of the existence of stable liftings is to prove that, in fact, these constants do not depend on N, and this is one of our aims for writing this paper. This result was already briefly announced (see [15]), employed (see [17], [12]), and generalized (see [5]).

There are many possible applications of the properties given here, we quote [12] for recent ones. Note also that the results in [11] can also be made more precise thanks to our analysis. We propose other applications at the end of the paper.

The outline of the paper is as follows.

- In Section 2, we construct a lifting of polynomial traces on one edge of a rectangle into the space of polynomials on this rectangle and prove the continuity of this operator.
- In Section 3, we construct a lifting of polynomial traces on the same edge, but now vanishing at the endpoints of this edge, into the space of polynomials on the rectangle which vanish on its two edges adjacent to the first one, and also prove the continuity of this operator.
- Section 4 is devoted to one of the applications of the previous analysis, i.e., the comparison of the fractional order intrinsic norms on the spaces of polynomials with the interpolation norms.
- In Section 5, we continue the analysis in [11] and [9] and propose a constructive way to evaluate the fractional-order Sobolev norms of the polynomials, see also [12].
- The ability to evaluate fractional order norms is first illustrated in Section 6 on the example of the Legendre polynomials. In addition, based on numerical results, we conjecture an asymptotic behavior for the $H^{1/2}$ operator-norm of the L^2-projection operator onto polynomial spaces.

2 Lifting of polynomial traces

We first present the notation that is used in this section and later on. Next, we describe the different steps that are required for the construction of the lifting operator and, at each step, we prove the corresponding continuity property. We conclude with the final theorem.

2.1 Notation

Let Λ be the open interval $(-1,1)$ and Θ the rectangle $\Lambda \times (0,1)$. The generic points in Λ and Θ are denoted by x and (x,y), respectively. For simplicity, we still use the notation Λ for the edge $\Lambda \times \{0\}$ of Θ. Indeed, we are interested in the lifting of traces on Λ into functions on Θ.

For any one-dimensional interval $\mathcal{I}$ and any nonnegative integer N, let $\mathbb{P}_N(\mathcal{I})$ be the space of restrictions to $\mathcal{I}$ of polynomials with one variable and degree $\leq N$ with respect to this variable. Similarly, for any one-dimensional intervals $\mathcal{I}$ and $\mathcal{J}$ and any nonnegative integer N, let $\mathbb{P}_N(\mathcal{I} \times \mathcal{J})$ be the space of restrictions to $\mathcal{I} \times \mathcal{J}$ of polynomials with two variables and degree $\leq N$ with respect to each variable. In this section, we are interested in the construction of a stable lifting operator which maps $\mathbb{P}_N(\Lambda)$ into $\mathbb{P}_N(\Theta)$ for each positive integer N.

On the one-dimensional interval $\mathcal{I}$, we recall the standard notation

$$L^2(\mathcal{I}) = \big\{\varphi : \mathcal{I} \to \mathbb{R} \text{ measurable};\quad \|\varphi\|_{L^2(\mathcal{I})} = \big(\int_{\mathcal{I}} |\varphi(x)|^2\, dx\big)^{\frac{1}{2}} < +\infty\big\}. \tag{2.1}$$

Next, for any nonnegative integer m, we consider the usual Sobolev space $H^m(\mathcal{I})$ of functions such that all their derivatives of order $\leq m$ belong to $L^2(\mathcal{I})$, namely

$$H^m(\mathcal{I}) = \big\{\varphi \in L^2(\mathcal{I});\ \|v\|_{H^m(\mathcal{I})} = \big(\sum_{k=0}^{m} \|d^k\varphi\|^2_{L^2(\mathcal{I})}\big)^{\frac{1}{2}} < +\infty\big\}, \tag{2.2}$$

where d^k stands for the derivative of order k. In what follows, we also need the seminorm

$$|\varphi|_{H^m(\mathcal{I})} = \|d^m\varphi\|_{L^2(\mathcal{I})}. \tag{2.3}$$

The Sobolev spaces of fractional order can be defined in several ways, for instance by interpolation methods [2, Chap. VII], however we have rather introduce them by the way of an intrinsic norm. For any positive real number

τ and for any function φ defined a.e. on $\mathcal{I}$, let $q_\tau[\varphi]$ be defined a.e. on $\mathcal{I}\times\mathcal{I}$ by

$$q_\tau[\varphi](x,x') = \frac{|\varphi(x)-\varphi(x')|}{|x-x'|^\tau}\,. \tag{2.4}$$

Any positive real number s which is not an integer can be written $\lfloor s\rfloor + \sigma$, where $\lfloor s\rfloor$ denotes its integer part and $0<\sigma<1$; the space $H^s(\mathcal{I})$ is thus defined as the space of functions φ in $L^2(\mathcal{I})$ such that

$$\|\varphi\|_{H^s(\mathcal{I})} = \left(\|\varphi\|^2_{H^{\lfloor s\rfloor}(\mathcal{I})} + \|q_{\sigma+\frac{1}{2}}[d^{\lfloor s\rfloor}\varphi]\|^2_{L^2(\mathcal{I}\times\mathcal{I})}\right)^{\frac{1}{2}} < +\infty. \tag{2.5}$$

Similarly, on any two-dimensional connected domain $\mathcal{O}$ with a Lipschitz-continuous boundary, we recall that

$$L^2(\mathcal{O}) = \big\{v:\mathcal{O}\to\mathbb{R}\text{ measurable};\\ \|v\|_{L^2(\mathcal{O})} = \big(\int_{\mathcal{O}} |v(\boldsymbol{x})|^2\,d\boldsymbol{x}\big)^{\frac{1}{2}} < +\infty\big\}, \tag{2.6}$$

with the notation $\boldsymbol{x}=(x,y)$, and also that

$$H^1(\mathcal{O}) = \big\{v\in L^2(\mathcal{O});\\ \|v\|_{H^1(\mathcal{O})} = \big(\|v\|^2_{L^2(\mathcal{O})} + \|\partial_x v\|^2_{L^2(\mathcal{O})} + \|\partial_y v\|^2_{L^2(\mathcal{O})}\big)^{\frac{1}{2}} < +\infty\big\}. \tag{2.7}$$

The final result of this section involves the spaces $H^1(\Theta)$ and $H^{1/2}(\Lambda)$. Here is the explicit expression of the norm associated with this latter space

$$\|\varphi\|_{H^{\frac{1}{2}}(\Lambda)} = \left(\|\varphi\|^2_{L^2(\Lambda)} + \int_{-1}^{1}\int_{-1}^{1} \frac{|\varphi(X)-\varphi(X')|^2}{|X-X'|^2}\,dXdX'\right)^{\frac{1}{2}}. \tag{2.8}$$

2.2 Construction of the lifting operator

The lifting operator is built in three steps, according to the geometry of the domain in which we lift the traces.

Step 1: Lifting from $\mathbb{R}$ into a strip

Let $\mathcal{S}$ denote the infinite strip $\mathbb{R}\times(0,2)$. For any nonnegative integer N, we introduce the space $\mathcal{P}_N(\mathcal{S})$ of restrictions to $\mathcal{S}$ of polynomials with two variables and total degree $\le N$. Next, we define the operator $\mathscr{L}_\mathcal{S}$ on integrable functions φ by

$$\text{For a.e. }(x,y)\in\mathcal{S},\qquad (\mathscr{L}_\mathcal{S}\varphi)(x,y) = \frac{1}{y}\int_{x-\frac{y}{2}}^{x+\frac{y}{2}} \varphi(t)\,dt. \tag{2.9}$$

It satisfies the basic property:

$$\text{For a.e. } x \in \mathbb{R}, \quad \lim_{y \to 0} (\mathscr{L}_{\mathcal{S}}\varphi)(x, y) = \varphi(x), \tag{2.10}$$

so that $\mathscr{L}_{\mathcal{S}}$ is a lifting operator.

We omit the proof of the first lemma since it is obvious.

Lemma 2.1 *For any nonnegative integer N, the operator $\mathscr{L}_{\mathcal{S}}$ maps $\mathbb{P}_N(\mathbb{R})$ into $\mathcal{P}_N(\mathcal{S})$.*

In order to prove the next lemma, we write (2.9) in a different form:

$$(\mathscr{L}_{\mathcal{S}}\varphi)(x, y) = \int_{-\infty}^{+\infty} \chi(t)\varphi(x + yt)\, dt, \tag{2.11}$$

where χ stands for the characteristic function of the interval $(-\frac{1}{2}, \frac{1}{2})$. Denoting by a hat the Fourier transform on $\mathbb{R}$, we observe that

$$\hat{\chi}(\xi) = \frac{2}{\sqrt{2\pi}} \frac{\sin(\frac{\xi}{2})}{\xi}, \tag{2.12}$$

so that, in particular, the function $\hat{\chi}$ belongs to $H^1(\mathbb{R})$.

Lemma 2.2 *The operator $\mathscr{L}_{\mathcal{S}}$ is continuous from $H^{1/2}(\mathbb{R})$ into $H^1(\mathcal{S})$.*

Proof. Let φ be any function in $H^{1/2}(\mathbb{R})$. Due to the tensorization property

$$H^1(\mathcal{S}) = H^1(\mathbb{R}; L^2(0, 2)) \cap L^2(\mathbb{R}; H^1(0, 2)),$$

it suffices to check that

$$\int_{-\infty}^{+\infty} \|\widehat{\mathscr{L}_{\mathcal{S}}\varphi}(\xi, \cdot)\|^2_{H^1(\mathcal{I},\xi)}\, d\xi < +\infty,$$

where $\mathcal{I}$ stands for the interval $(0, 2)$ and the parameter-dependent norm $\|\cdot\|_{H^1(\mathcal{I},\xi)}$ is defined as

$$\|v\|^2_{H^1(\mathcal{I},\xi)} = (1 + \xi^2)\, \|v\|^2_{L^2(\mathcal{I})} + |v|^2_{H^1(\mathcal{I})}.$$

Denoting by χ_y the function: $\chi_y(t) = \frac{1}{y}\, \chi(\frac{t}{y})$ we observe that (2.11) can be written equivalently as the convolution

$$\mathscr{L}_{\mathcal{S}}\varphi(\cdot, y) = \varphi * \chi_y,$$

whence (see [20, §2.2.1] for instance)

$$\widehat{\mathscr{L}_{\mathcal{S}}\varphi}(\xi,y) = \sqrt{2\pi}\,\hat{\varphi}(\xi)\,\hat{\chi}_y(\xi) = \sqrt{2\pi}\,\hat{\varphi}(\xi)\,\hat{\chi}(y\xi). \tag{2.13}$$

Thus, we derive

$$\|\widehat{\mathscr{L}_{\mathcal{S}}\varphi}(\xi,\cdot)\|^2_{H^1(\mathcal{I},\xi)} = 2\pi\,|\hat{\varphi}(\xi)|^2\,\|\hat{\chi}(y\xi)\|^2_{H^1(\mathcal{I},\xi)}.$$

First, for $|\xi| \geq 1$, by using the change of variable $z = y\xi$, we obtain

$$\|\widehat{\mathscr{L}_{\mathcal{S}}\varphi}(\xi,\cdot)\|^2_{H^1(\mathcal{I},\xi)} \leq c\,(1+\xi^2)^{\frac{1}{2}}\,|\hat{\varphi}(\xi)|^2\,\|\hat{\chi}\|^2_{H^1(\mathbb{R})}. \tag{2.14}$$

Second, for $|\xi| < 1$, the same change of variable yields

$$\xi^2\,\|\widehat{\mathscr{L}_{\mathcal{S}}\varphi}(\xi,\cdot)\|^2_{L^2(\mathcal{I})} + |\widehat{\mathscr{L}_{\mathcal{S}}\varphi}(\xi,\cdot)|^2_{H^1(\mathcal{I})} \leq c\,|\xi|\,|\hat{\varphi}(\xi)|^2\,\|\hat{\chi}\|^2_{H^1(\mathbb{R})},$$

while it follows from (2.13) that (note that $0 < y < 2$)

$$\|\widehat{\mathscr{L}_{\mathcal{S}}\varphi}(\xi,\cdot)\|^2_{L^2(\mathcal{I})} \leq c\,|\xi|^{-1}|\hat{\varphi}(\xi)|^2\,\|\hat{\chi}\|^2_{L^2(0,2\xi)}.$$

Next, we observe that the quantity $|\xi|^{-1}\,\|\hat{\chi}\|^2_{L^2(0,2\xi)}$ is bounded independently of ξ (see (2.12)), hence inequality (2.14) still holds for $|\xi| < 1$. We also note that the quantity $\|(1+\xi^2)^{\frac{1}{4}}\,\hat{\varphi}\|_{L^2(\mathbb{R})}$ is equivalent to $\|\varphi\|_{H^{\frac{1}{2}}(\mathbb{R})}$ (see [20, §2.3.3]). Combining all this leads to

$$\int_{-\infty}^{+\infty} \|\widehat{\mathscr{L}_{\mathcal{S}}\varphi}(\xi,\cdot)\|^2_{H^1(\mathcal{I},\xi)}\,d\xi \leq c\,\|\varphi\|^2_{H^{\frac{1}{2}}(\mathbb{R})},$$

which is the desired result.

STEP 2: Lifting from Λ into a triangle

Let $\mathcal{T}$ denote the equilateral triangle with vertices $\boldsymbol{a}_- = (-1,0)$, $\boldsymbol{a}_+ = (1,0)$ and $\boldsymbol{a}_0 = (0,\sqrt{3})$. For any nonnegative integer N, we introduce the space $\mathcal{P}_N(\mathcal{T})$ of restrictions to $\mathcal{T}$ of polynomials with two variables and total degree $\leq N$.

It is readily checked from (2.9) (see also Figure 1) that, for any point (x,y) in $\mathcal{T}$, the value of $\mathscr{L}_{\mathcal{S}}\varphi$ at (x,y) only depends on the values of φ on Λ. Therefore, in analogy with (2.9), we define the operator $\mathscr{L}_{\mathcal{T}}$ on functions φ integrable on Λ by

$$\text{For a.e. } (x,y) \in \mathcal{T}, \qquad (\mathscr{L}_{\mathcal{T}}\varphi)(x,y) = \frac{1}{y}\int_{x-\frac{y}{2}}^{x+\frac{y}{2}} \varphi(t)\,dt. \tag{2.15}$$

Thus, this operator satisfies the lifting property:

$$\text{For a.e. } x \in \Lambda, \quad \lim_{y\to 0}\,(\mathscr{L}_{\mathcal{T}}\varphi)(x,y) = \varphi(x). \tag{2.16}$$

The next statement is easily derived from Lemmas 2.1 and 2.2.

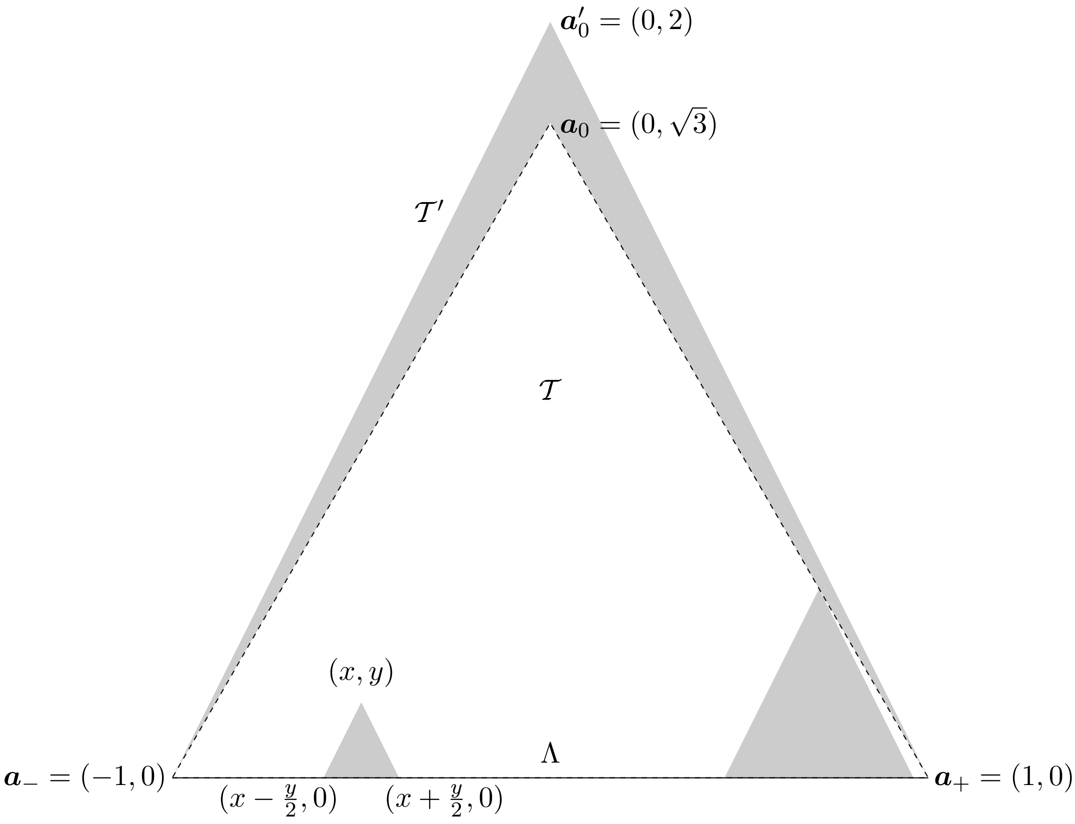

Figure 1: The lifting from Λ to triangles $\mathcal{T}$ and $\mathcal{T}'$

Lemma 2.3 *The operator* $\mathscr{L}_{\mathcal{T}}$
(i) maps $\mathbb{P}_N(\Lambda)$ *into* $\mathcal{P}_N(\mathcal{T})$ *for any nonnegative integer* N*,*
(ii) is continuous from $H^{1/2}(\Lambda)$ *into* $H^1(\mathcal{T})$*.*

Remark 2.4 *Formula* (2.15) *still makes sense for a.e.* (x,y) *in the larger triangle* $\mathcal{T}'$ *with vertices* $\boldsymbol{a}_-, \boldsymbol{a}_+$ *and* $\boldsymbol{a}_0' = (0,2)$*, defining a lifting operator* $\mathscr{L}_{\mathcal{T}'}$ *continuous from* $H^{1/2}(\Lambda)$ *into* $H^1(\mathcal{T}')$*.*

STEP 3: Lifting from Λ into Θ

Finally, let $\mathcal{Z}$ denote the isosceles trapezium (in the British sense of a quadrilateral with two parallel edges)

$$\mathcal{Z} = \big\{(x,y) \in \mathcal{T};\ y < 1\big\}, \tag{2.17}$$

and let $\mathscr{L}_{\mathcal{Z}}$ be the restriction of $\mathscr{L}_{\mathcal{T}}$ to $\mathcal{Z}$:

$$\mathscr{L}_{\mathcal{Z}}\varphi = \mathscr{L}_{\mathcal{T}}\varphi\big|_{\mathcal{Z}}. \tag{2.18}$$

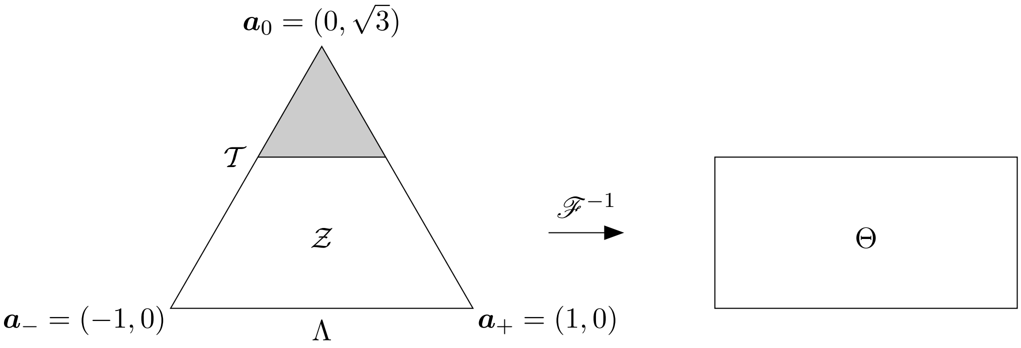

Figure 2: The homographic transformation from $\mathcal{Z}$ onto Θ

The introduction of $\mathcal{Z}$ is motivated by the existence of a one-to-one homography $\mathscr{F}$ which maps the rectangle Θ onto $\mathcal{Z}$, see Figure 2:

$$(x, y) = \mathscr{F}(\text{X}, \text{Y}) = \Big((1 - \frac{\text{Y}}{\sqrt{3}})\text{X}\ ,\ \text{Y}\Big). \tag{2.19}$$

Thus we can define the lifting operator $\mathscr{L}_\Theta$ by

$$\text{For a.e. } (\text{X}, \text{Y}) \in \Theta, \qquad (\mathscr{L}_\Theta \varphi)(\text{X}, \text{Y}) = (\mathscr{L}_\mathcal{Z} \varphi) \circ \mathscr{F}(\text{X}, \text{Y}). \tag{2.20}$$

This of course makes sense because $\mathscr{F}$ maps Θ onto $\mathcal{Z}$. Moreover, $\mathscr{F}$ maps the line $\text{Y} = 0$ onto the line $y = 0$, so that the lifting property (2.16) is still satisfied by the operator $\mathscr{L}_\Theta$.

We need two further properties of the mapping F: $v \mapsto v \circ \mathscr{F}$.

Lemma 2.5 *For any nonnegative integer N, the operator F maps $\mathcal{P}_N(\mathcal{Z})$ into $\mathbb{P}_N(\Theta)$.*

Proof. Any polynomial v_N in $\mathcal{P}_N(\mathcal{Z})$ can be written as

$$v_N(x, y) = \sum_{n=0}^{N} \sum_{\ell=0}^{N-n} \alpha_{n\ell}\, x^n y^\ell,$$

so that

$$(v_N \circ \mathscr{F})(\text{X}, \text{Y}) = \sum_{n=0}^{N} \sum_{\ell=0}^{N-n} \alpha_{n\ell} \Big(1 - \frac{\text{Y}}{\sqrt{3}}\Big)^n \text{X}^n \text{Y}^\ell.$$

This yields the desired result.

Lemma 2.6 *The operator F is continuous from $H^1(\mathcal{Z})$ into $H^1(\Theta)$.*

PROOF. This follows from the fact that the Jacobian matrix of $\mathscr{F}$ is bounded on Θ and has a determinant larger than $1-\frac{1}{\sqrt{3}}$.

2.3 The lifting theorem

The main result of this section now follows from the definition (2.20) and Lemmas 2.3, 2.5 and 2.6.

Theorem 2.7 *The operator $\mathscr{L}_\Theta$ defined in* (2.20)
(i) satisfies the lifting property

$$\text{For a.e. } x \in \Lambda, \quad \lim_{Y \to 0} (\mathscr{L}_\Theta \varphi)(x, y) = \varphi(x); \tag{2.21}$$

(ii) maps $\mathbb{P}_N(\Lambda)$ into $\mathbb{P}_N(\Theta)$ for any nonnegative integer N;
(iii) satisfies the continuity property for a positive constant c

$$\forall \varphi \in H^{\frac{1}{2}}(\Lambda), \quad \|\mathscr{L}_\Theta \varphi\|_{H^1(\Theta)} \le c\, \|\varphi\|_{H^{\frac{1}{2}}(\Lambda)}. \tag{2.22}$$

3 Lifting of flat polynomial traces

By "flat" traces, we mean traces that vanish at the endpoints of the interval Λ. We now intend to construct a lifting operator that preserves this nullity property, i.e., maps these traces onto functions vanishing on the two vertical edges $\{\pm 1\}\times(0,1)$ of the rectangle Θ. Our proof of the continuity of the new lifting operator requires the introduction of some weighted Sobolev spaces.

3.1 Notation

For any integer $N \geq 2$, let $\mathbb{P}_N^0(\Lambda)$ be the space of polynomials in $\mathbb{P}_N(\Lambda)$ which vanish at the endpoints ± 1 of Λ. Similarly, we introduce the space

$$\mathbb{P}_N^\diamond(\Theta) = \big\{ v_N \in \mathbb{P}_N(\Theta);\ v_N(\pm 1, Y) = 0,\ 0 \leq Y \leq 1 \big\}. \tag{3.1}$$

Note that

$$\mathbb{P}_N^0(\Lambda) = (1 - X^2)\mathbb{P}_{N-2}(\Lambda), \qquad \mathbb{P}_N^\diamond(\Theta) = (1 - X^2)\mathbb{P}_{N-2,N}(\Theta), \tag{3.2}$$

where $\mathbb{P}_{N-2,N}(\Theta)$ stands for the space of polynomials with degree $\leq N-2$ with respect to X and $\leq N$ with respect to Y. We are now interested in defining a new stable lifting operator which maps $\mathbb{P}_N^0(\Lambda)$ into $\mathbb{P}_N^\diamond(\Theta)$ for each positive integer N.

On the rectangle Θ, we also introduce the space

$$H^1_\diamond(\Theta) = \big\{ v \in H^1(\Theta);\ v(\pm 1, Y) = 0 \text{ for a.e. } Y,\ 0 \leq Y \leq 1 \big\}, \tag{3.3}$$

and note that $\mathbb{P}_N^\diamond(\Theta)$ is a finite-dimensional subspace of $H^1_\diamond(\Theta)$.

On the interval Λ and for any real number α, we introduce the weighted Sobolev space

$$\mathrm{V}_\alpha^{\frac{1}{2}}(\Lambda) = \big\{ \varphi : \Lambda \to \mathbb{R} \text{ measurable};\ \|\varphi\|_{\mathrm{V}_\alpha^{\frac{1}{2}}(\Lambda)} < +\infty \big\}, \tag{3.4}$$

where the norm $\|\cdot\|_{\mathrm{V}_\alpha^{\frac{1}{2}}(\Lambda)}$ is defined by (see [20, Chap. 3] for analogous definitions)

$$\|\varphi\|_{\mathrm{V}_\alpha^{\frac{1}{2}}(\Lambda)} = \Big(\int_{-1}^{1} |\varphi(X)|^2\, (1 - X^2)^{\alpha - 1}\, dX \\ + \int_{-1}^{1}\int_{-1}^{1} \frac{|\varphi(X)(1 - X^2)^{\frac{\alpha}{2}} - \varphi(X')(1 - X'^2)^{\frac{\alpha}{2}}|^2}{|X - X'|^2}\, dX dX' \Big)^{\frac{1}{2}}. \tag{3.5}$$

Remark 3.1 *In the specific case $\alpha = 0$, when comparing the norm in* (3.5) *with the intrinsic norm on $H^{1/2}_{00}(\Lambda)$ as given in* [14, Chap. I, Th. 11.7] *for instance, we observe that the spaces $\mathrm{V}^{1/2}_0(\Lambda)$ and $H^{1/2}_{00}(\Lambda)$ coincide.*

Similarly, we define weighted Sobolev spaces on the two-dimensional domains introduced in Section 2, for any real number α:

• On the equilateral triangle $\mathcal{T}$, we consider the weight

$$\rho(x, y) = \big((1 - x^2)^2 + y^2\big)^{\frac{1}{2}}. \tag{3.6}$$

Note that this function is equivalent to the distance to the set of corners $\{\boldsymbol{a}_-, \boldsymbol{a}_+\}$ (see Figure 1 for the notation). Then, we define $\mathrm{V}^1_{*,\alpha}(\mathcal{T})$ as the space of measurable functions v on $\mathcal{T}$ such that $\|v\|_{\mathrm{V}^1_{*,\alpha}(\mathcal{T})} < +\infty$, with

$$\|v\|_{\mathrm{V}^1_{*,\alpha}(\mathcal{T}))} = \Big(\int_{\mathcal{T}} |v(x, y)|^2\, \rho(x, y)^{\alpha-2}\, dxdy \\ + \int_{\mathcal{T}} |(\mathbf{grad}\, v)(x, y)|^2\, \rho(x, y)^{\alpha}\, dxdy\Big)^{\frac{1}{2}}. \tag{3.7}$$

The space of restrictions to the trapezium $\mathcal{Z}$ of functions in $\mathrm{V}^1_{*,\alpha}(\mathcal{T})$ is also denoted by $\mathrm{V}^1_{*,\alpha}(\mathcal{Z})$.

• Next, on $\mathcal{Z}$, we consider the weight

$$\delta(x, y) = \Big(1 - \frac{y}{\sqrt{3}}\Big)^2 - x^2.$$

This function is equivalent to the distance to the union of the two skew edges of $\mathcal{Z}$. Then, we define $\mathrm{V}^1_{\diamond,\alpha}(\mathcal{Z})$ as the space of measurable functions v on $\mathcal{Z}$ such that $\|v\|_{\mathrm{V}^1_{\diamond,\alpha}(\mathcal{Z})} < +\infty$, with

$$\|v\|_{\mathrm{V}^1_{\diamond,\alpha}(\mathcal{Z})} = \Big(\int_{\mathcal{Z}} |v(x, y)|^2\, \delta(x, y)^{\alpha-2}\, dxdy \\ + \int_{\mathcal{Z}} |(\mathbf{grad}\, v)(x, y)|^2\, \delta(x, y)^{\alpha}\, dxdy\Big)^{\frac{1}{2}}. \tag{3.8}$$

• Finally, on the rectangle Θ, we define $\mathrm{V}^1_{\diamond,\alpha}(\Theta)$ as the space of measurable functions v on Θ such that $\|v\|_{\mathrm{V}^1_{\diamond,\alpha}(\Theta)} < +\infty$, with

$$\|v\|_{\mathrm{V}^1_{\diamond,\alpha}(\Theta)} = \Big(\int_{\Theta} |v(X, Y)|^2\, (1 - X^2)^{\alpha-2}\, dXdY \\ + \int_{\Theta} |(\mathbf{grad}\, v)(X, Y)|^2\, (1 - X^2)^{\alpha}\, dXdY\Big)^{\frac{1}{2}}. \tag{3.9}$$

In the case $\alpha = 0$, it follows from the standard Hardy's inequality, see [20, §3.2.6, Rem. 1] for instance, that the space $\mathrm{V}^1_{\diamond,0}(\Theta)$ coincides with $H^1_\diamond(\Theta)$. And our aim is to exhibit a new lifting operator which is continuous from $H^{1/2}_{00}(\Lambda)$ into $H^1_\diamond(\Theta)$.

3.2 Construction of the lifting operator

The lifting operator $\mathscr{L}^0_\Theta$ is constructed from the division-multiplication formula

$$\mathscr{L}^0_\Theta = \mathcal{M}_1 \circ \mathscr{L}_\Theta \circ \mathcal{M}_{-1}, \tag{3.10}$$

where $\mathscr{L}_\Theta$ is the operator constructed in Section 2 (see Theorem 2.7) and, for any real number β, $\mathcal{M}_\beta$ denotes the multiplication by $(1-x^2)^\beta$:

$$\text{For a.e. } x \in \Lambda, \qquad (\mathcal{M}_\beta v)(x) = v(x)\,(1-x^2)^\beta. \tag{3.11}$$

It follows from this definition and the properties of the operator $\mathscr{L}_\Theta$ that the operator $\mathscr{L}^0_\Theta$ still satisfies the lifting property (2.21). Moreover, from (3.2), it is clear that it maps $\mathbb{P}^0_N(\Lambda)$ into $\mathbb{P}^\diamond_N(\Theta)$ (more precisely and with obvious notation into $\mathbb{P}^\diamond_{N,N-2}(\Theta)$). So, it remains to investigate its continuity properties. For this, we study successively the continuity of the three operators which are involved in its definition in appropriate weighted spaces.

Step 1: Continuity of the operator $\mathcal{M}_\beta$ on Λ

The next lemma follows immediately from the definition (3.5) of the norm of the weighted space $V^{1/2}_\alpha(\Lambda)$.

Lemma 3.2 *For any real numbers α and β, the operator $\mathcal{M}_\beta$ is continuous from $V^{1/2}_\alpha(\Lambda)$ into $V^{1/2}_{\alpha-2\beta}(\Lambda)$.*

In what follows, we use this lemma with $\alpha = 0$ and $\beta = -1$.

Step 2: Weighted continuity of the operator $\mathscr{L}_\Theta$

We go back to the different steps leading to the definition (2.20) of the operator $\mathscr{L}_\Theta$. So we first prove the weighted analogue of Lemma 2.3.

Lemma 3.3 *For any real number α, the operator $\mathscr{L}_\mathcal{T}$ defined in (2.15) is continuous from $V^{1/2}_\alpha(\Lambda)$ into $V^1_{*,\alpha}(\mathcal{T})$.*

Proof. Let φ be any function in $V^{1/2}_\alpha(\Lambda)$. As the restriction of φ to the interval $(-1+\delta, 1-\delta)$ for any fixed $\delta > 0$, belongs to $H^{1/2}(-1+\delta, 1-\delta)$, Lemma 2.3 together with Remark 2.4 implies that $\mathscr{L}_\mathcal{T}\varphi$ belongs to $H^1(\mathcal{T}_\delta)$, where, see Figure 3,

$$\mathcal{T}_\delta = \mathcal{T} \cap (1-\delta)\mathcal{T}'.$$

So, by symmetry, it remains to prove that $\mathscr{L}_\mathcal{T}\varphi$ belongs to $V^1_{*,\alpha}(\mathcal{C})$, where

- $\mathcal{C}$ denotes the sector with vertex $\boldsymbol{a}_- = (-1, 0)$, opening $\frac{\pi}{3}$ and radius 1,

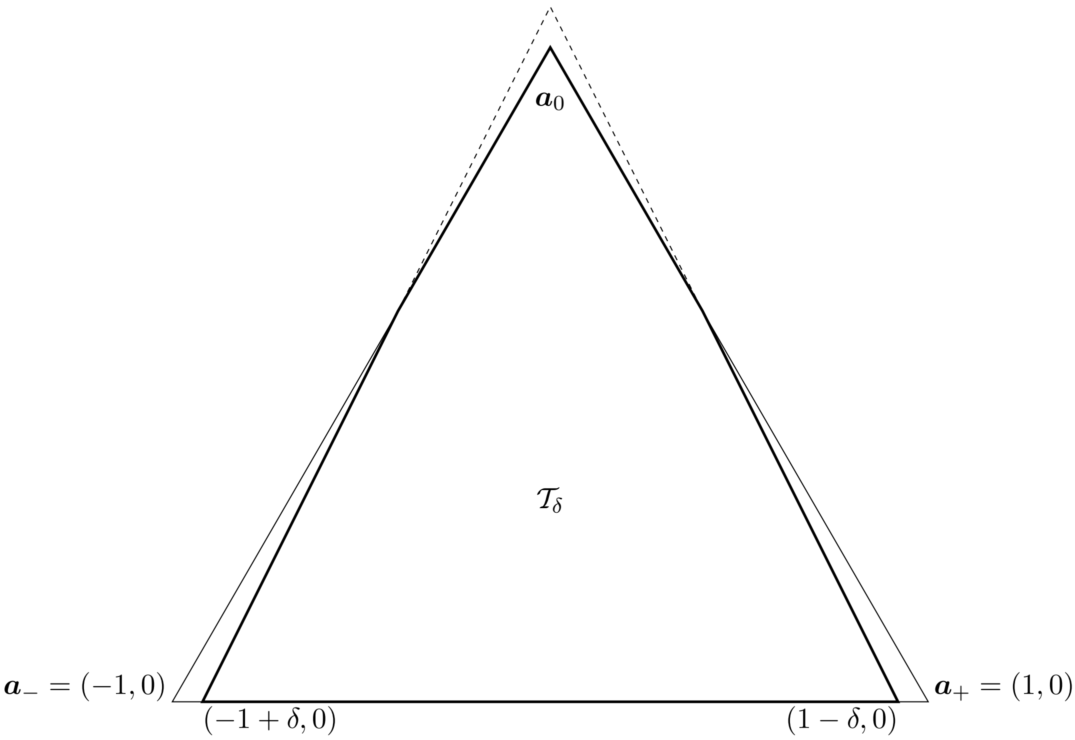

Figure 3: The polygon $\mathcal{T}_\delta$ (with $\delta = 0.08$)

• $V^1_{*,\alpha}(\mathcal{C})$ is obviously defined as the spaces of restrictions to $\mathcal{C}$ of functions in $V^1_{*,\alpha}(\mathcal{T})$; thus the weight on $\mathcal{C}$ is now bounded from above and below by a constant times the distance ρ_- to $\boldsymbol{a}_-$.
We introduce the annulus (see Figure 4)

$$\mathcal{K}_0 = \big\{(x,y) \in \mathcal{C};\ \tfrac{1}{2} < \rho_-(x,y) < 1\big\}.$$

We check that, for the interval $\mathcal{I}_0$

$$\mathcal{I}_0 = \Big(-1 + \frac{2-\sqrt{3}}{8}, -1 + \frac{\sqrt{5}}{2}\Big),$$

the values of $\mathscr{L}_\mathcal{T}\varphi$ on the annulus $\mathcal{K}_0$ only depend on the values of φ on $\mathcal{I}_0$. From Lemma 2.3, we derive the estimate

$$\|\mathscr{L}_\mathcal{T}\varphi\|_{H^1(\mathcal{K}_0)} \le c\,\|\varphi\|_{H^{\frac{1}{2}}(\mathcal{I}_0)}.$$

Since both weights ρ and $1-x^2$ are bounded together with their inverses ρ^{-1} and $(1-x^2)^{-1}$ on $\mathcal{K}_0$ and $\mathcal{I}_0$, respectively, we deduce the estimate, with obvious definition of the new norms by restriction,

$$\|\mathscr{L}_\mathcal{T}\varphi\|_{V^1_{*,\alpha}(\mathcal{K}_0)} \le c\,\|\varphi\|_{V^{\frac{1}{2}}_\alpha(\mathcal{I}_0)}. \tag{3.12}$$

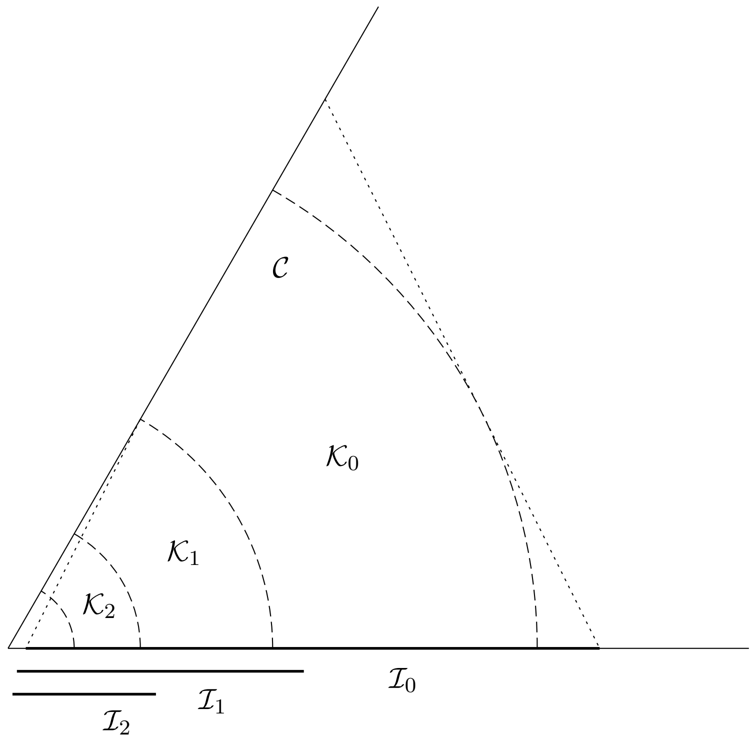

Figure 4: The sector $\mathcal{C}$ and its dyadic partition

The proof now follows from a dyadic partition argument. For any $j \geq 0$, let Φ_j denote the mapping $(x, y) \mapsto (-1 + 2^{-j}\,(x + 1), 2^{-j}\,y)$ and set

$$\mathcal{K}_j = \Phi_j(\mathcal{K}_0), \qquad \mathcal{I}_j = \Phi_j(\mathcal{I}_0).$$

Then, since $\mathscr{L}_{\mathcal{T}}(\varphi \circ \Phi_j) = (\mathscr{L}_{\mathcal{T}}\varphi) \circ \Phi_j$, we deduce from (3.12) that

$$\|(\mathscr{L}_{\mathcal{T}}\varphi) \circ \Phi_j\|_{V^1_{*,\alpha}(\mathcal{K}_0)} \leq c\,\|\varphi \circ \Phi_j\|_{V_\alpha^{\frac{1}{2}}(\mathcal{I}_0)}.$$

Next, we derive by change of variables that

$$\|\mathscr{L}_{\mathcal{T}}\varphi\|^2_{V^1_{*,\alpha}(\mathcal{K}_j)} \leq c\,2^{j\alpha}\,\|(\mathscr{L}_{\mathcal{T}}\varphi) \circ \Phi_j\|^2_{V^1_{*,\alpha}(\mathcal{K}_0)},$$
$$\text{and} \quad 2^{j\alpha}\,\|\varphi \circ \Phi_j\|^2_{V_\alpha^{\frac{1}{2}}(\mathcal{I}_0)} \leq c'\|\varphi\|^2_{V_\alpha^{\frac{1}{2}}(\mathcal{I}_j)}.$$

Combining all this yields the uniform estimate for all integers $j \geq 0$

$$\|\mathscr{L}_{\mathcal{T}}\varphi\|^2_{V^1_{*,\alpha}(\mathcal{K}_j)} \leq c\,\|\varphi\|^2_{V_\alpha^{\frac{1}{2}}(\mathcal{I}_j)}.$$

Summing up the above inequalities on j, we obtain

$$\sum_{j \geq 0} \|\mathscr{L}_{\mathcal{T}}\varphi\|^2_{V^1_{*,\alpha}(\mathcal{K}_j)} \leq c \sum_{j \geq 0} \|\varphi\|^2_{V_\alpha^{\frac{1}{2}}(\mathcal{I}_j)}.$$

Since the union of the $\overline{\mathcal{K}}_j$ is the sector $\mathcal{C}$ and since $\mathscr{L}_{\mathcal{I}}\varphi$ belongs to $H^1_{\rm loc}(\mathcal{C})$, we deduce that

$$\sum_{j\geq 0} \|\mathscr{L}_{\mathcal{I}}\varphi\|^2_{V^1_{*,\alpha}(\mathcal{K}_j)} = \|\mathscr{L}_{\mathcal{I}}\varphi\|^2_{V^1_{*,\alpha}(\mathcal{C})}$$

On the other hand, the union of the intervals $\mathcal{I}_j$ is $(-1,-1+\frac{\sqrt{5}}{2})$, and this union is locally finite: For any $j \geq 0$, $\mathcal{I}_j \cap \mathcal{I}_{j+k} = \emptyset$ if $k \geq 6$. From this we derive

$$\sum_{j\geq 0} \|\varphi\|^2_{V^{\frac12}_\alpha(\mathcal{I}_j)} \leq 6\,\|\varphi\|^2_{V^{\frac12}_\alpha(-1,-1+\frac{\sqrt{5}}{2})}.$$

From the last three formulas, we obtain that $\mathscr{L}_{\mathcal{I}}\varphi$ belongs to $V^1_{*,\alpha}(\mathcal{C})$, together with the desired continuity property.

Next, it is readily checked that the following inequality holds

$$\forall (x,y) \in \mathcal{Z}, \qquad \delta(x,y) \leq \rho(x,y).$$

Thus, for all $\alpha \geq 2$, we have the corresponding inequalities on the weights

$$\forall (x,y) \in \mathcal{Z}, \qquad \delta(x,y)^{\alpha-2} \leq c\,\rho(x,y)^{\alpha-2} \quad \text{and} \quad \delta(x,y)^{\alpha} \leq c\,\rho(x,y)^{\alpha}$$

and we find that the space $V^1_{*,\alpha}(\mathcal{Z})$ is imbedded in $\mathrm{V}^1_{\diamond,\alpha}(\mathcal{Z})$. Hence we deduce from Lemma 3.3 the following lemma.

Lemma 3.4 *For any real number $\alpha \geq 2$, the operator $\mathscr{L}_{\mathcal{Z}}$ defined in (2.18) is continuous from $V^{1/2}_\alpha(\Lambda)$ into $\mathrm{V}^1_{\diamond,\alpha}(\mathcal{Z})$.*

Finally, it follows from the definition (2.19) of $\mathscr{F}$ that

$$\forall (X,Y) \in \Theta, \quad \delta \circ \mathscr{F}(X,Y) = \left(1 - X^2\right)\left(1 - \frac{Y}{\sqrt{3}}\right)^2.$$

Since $1 - \frac{Y}{\sqrt{3}}$ is bounded from below, we have the weighted analogue of Lemma 2.6.

Lemma 3.5 *The operator F is continuous from $\mathrm{V}^1_{\diamond,\alpha}(\mathcal{Z})$ into $\mathrm{V}^1_{\diamond,\alpha}(\Theta)$.*

Thus, the next result follows from the definition (2.20) of $\mathscr{L}_\Theta$ and Lemma 3.4.

Lemma 3.6 *For any real number $\alpha \geq 2$, the operator $\mathscr{L}_\Theta$ defined in (2.20) is continuous from $V^{1/2}_\alpha(\Lambda)$ into $\mathrm{V}^1_{\diamond,\alpha}(\Theta)$.*

STEP 3: Continuity of the operator $\mathcal{M}_\beta$ on Θ

The next property is an obvious consequence of the definition of the spaces $\mathrm{V}^1_{\diamond,\alpha}(\Theta)$.

Lemma 3.7 *For any real numbers α and β, the operator $\mathcal{M}_\beta$ is continuous from $\mathrm{V}^1_{\diamond,\alpha}(\Theta)$ into $\mathrm{V}^1_{\diamond,\alpha-2\beta}(\Theta)$.*

3.3 The lifting theorem

The final lifting result is now an easy consequence of the definition (3.10) and Lemma 3.2 (with $\alpha = 0$, $\beta = -1$), Lemma 3.6 (with $\alpha = 2$) and Lemma 3.7 (with $\alpha = 2$ and $\beta = 1$) and the identity $\mathrm{V}_0^{1/2}(\Lambda) = H_{00}^{1/2}(\Lambda)$ stated in Remark 3.1.

Theorem 3.8 *The operator $\mathscr{L}^0_\Theta$ defined in (3.10)*
(i) satisfies the lifting property

$$\text{For a.e. } \mathrm{x} \in \Lambda, \quad \lim_{Y \to 0} (\mathscr{L}^0_\Theta \varphi)(\mathrm{x}, \mathrm{y}) = \varphi(\mathrm{x}); \tag{3.13}$$

(ii) maps $H^{1/2}_{00}(\Lambda)$ into $H^1_\diamond(\Theta)$ and also $\mathbb{P}^0_N(\Lambda)$ into $\mathbb{P}^\diamond_N(\Theta)$ for any integer $N \geq 2$;
(iii) satisfies the continuity property for a positive constant c

$$\forall \varphi \in H^{\frac{1}{2}}_{00}(\Lambda), \quad \|\mathscr{L}^0_\Theta \varphi\|_{H^1(\Theta)} \leq c\, \|\varphi\|_{H^{\frac{1}{2}}_{00}(\Lambda)}. \tag{3.14}$$

4 Interpolation between polynomial spaces

For the sake of precision, we first recall from [14, Chap. 1, §4.2] the definition of interpolate spaces (in the sense of traces) in the simple case of Hilbert spaces. Note also that there exist several equivalent ways to define these spaces, e.g., the K–method (see [14, Chap. 1, Th. 10.1] or [20, §1.8.1 & 1.8.2] for instance).

Definition 4.1 *If X_0 and X_1 are two Hilbert spaces such that X_1 is contained in X_0 with a continuous and dense embedding, the interpolate space $[X_1, X_0]_\theta$ with index θ, $0 < \theta < 1$, is defined as the set of traces $\varphi = v(0)$ of measurable functions v in $(0,1)$ with values in X_1 such that the quantity*

$$\Big(\int_0^1 \|v(t)\|_{X_1}^2\, t^{2\theta}\, \frac{dt}{t} + \int_0^1 \|v'(t)\|_{X_0}^2\, t^{2\theta}\, \frac{dt}{t}\Big)^{\frac{1}{2}} \tag{4.1}$$

is finite, and its norm is defined as the trace norm, i.e., the infimum of (4.1) on the v such that $\varphi = v(0)$.

From this definition, the interpolate space of index θ between a space of polynomials $\mathbb{X}_N$ provided with the norm $\|\cdot\|_{X_1}$ and this same space provided with the norm $\|\cdot\|_{X_0}$ obviously coincides with $\mathbb{X}_N$. The question is: *Are the equivalence constants between the interpolate norm and the norm $\|\cdot\|_{[X_1,X_0]_\theta}$ independent of N?* The aim of this section is to give an answer in the particular case $\theta = \frac{1}{2}$ for specific spaces X_0 and X_1 and when the spaces $\mathbb{X}_N$ are either the spaces $\mathbb{P}_N(\Lambda)$ or $\mathbb{P}_N^0(\Lambda)$ introduced above.

4.1 First interpolation result

The next result relies on the fact that the interpolate space of index $\frac{1}{2}$ between $H^1(\Lambda)$ and $L^2(\Lambda)$ coincides with the space $H^{1/2}(\Lambda)$, and that the interpolate norm is equivalent to the norm (2.8).

Notation 4.2 *Let $\|\cdot\|_{N,\frac{1}{2}}$ denote the interpolate norm of index $\frac{1}{2}$ between the space $\mathbb{P}_N(\Lambda)$ provided with the norm of $H^1(\Lambda)$ and this same space provided with the norm of $L^2(\Lambda)$.*

Theorem 4.3 *There exist two positive constants c and c' such that, for any nonnegative integer N and for any polynomial φ_N in $\mathbb{P}_N(\Lambda)$, the following inequalities hold*

$$c\,\|\varphi_N\|_{H^{\frac{1}{2}}(\Lambda)} \le \|\varphi_N\|_{N,\frac{1}{2}} \le c'\,\|\varphi_N\|_{H^{\frac{1}{2}}(\Lambda)}. \tag{4.2}$$

Proof. We establish successively the two inequalities.

1) Since the imbeddings of $\mathbb{P}_N(\Lambda)$ endowed with the $H^1(\Lambda)$–norm into $H^1(\Lambda)$ and of $\mathbb{P}_N(\Lambda)$ endowed with the $L^2(\Lambda)$–norm into $L^2(\Lambda)$ both have norm 1, the first inequality is a direct consequence of the principal theorem of interpolation, see [14, Chap. 1, Th. 5.1] (and also [2, Thm 7.17] for a more precise version).

2) Using the Definition 4.1 with $\theta = \frac{1}{2}$, $X_0 = \mathbb{P}_N(\Lambda)$ with $L^2(\Lambda)$-norm and $X_1 = \mathbb{P}_N(\Lambda)$ with $H^1(\Lambda)$-norm, we find that

$$\|\varphi_N\|_{N,\frac{1}{2}} \leq \inf_{\substack{v_N \in \mathbb{P}_N(\Theta) \\ v_N(\cdot,0)=\varphi_N \text{ on } \Lambda}} \|v_N\|_{H^1(\Theta)},$$

whence in particular

$$\|\varphi_N\|_{N,\frac{1}{2}} \leq \|\mathscr{L}_\Theta \varphi_N\|_{H^1(\Theta)}.$$

Thus, the second inequality follows from Theorem 2.7, see (2.22).

Remark 4.4 *The previous lines yield that, when the interpolation norm is used on $H^{1/2}(\Lambda)$ instead of the intrinsic norm (2.8), the first inequality in (4.2) holds with the constant c equal to* 1.

4.2 Second interpolation result

Similarly, the interpolate space of index $\frac{1}{2}$ between $H^1_0(\Lambda)$ and $L^2(\Lambda)$ coincides with the space $H^{1/2}_{00}(\Lambda)$ (see [14, Chap. 1, Th. 11.7] for instance).

Notation 4.5 *Let $\|\cdot\|^0_{N,\frac{1}{2}}$ denote the interpolate norm of index $\frac{1}{2}$ between the space $\mathbb{P}^0_N(\Lambda)$ provided with the norm of $H^1_0(\Lambda)$ and this same space provided with the norm of $L^2(\Lambda)$.*

Theorem 4.6 *There exist two positive constants c and c' such that, for any integer $N \geq 2$ and for any polynomial φ_N in $\mathbb{P}^0_N(\Lambda)$, the following inequalities hold*

$$c\,\|\varphi_N\|_{H^{\frac{1}{2}}_{00}(\Lambda)} \leq \|\varphi_N\|^0_{N,\frac{1}{2}} \leq c'\,\|\varphi_N\|_{H^{\frac{1}{2}}_{00}(\Lambda)}. \tag{4.3}$$

Proof. The first inequality is proved in the same way as before for Theorem 4.3. Concerning the second inequality, we find now that

$$\|\varphi_N\|^0_{N,\frac{1}{2}} \leq \inf_{\substack{v_N \in \mathbb{P}^\diamond_N(\Theta) \\ v_N(\cdot,0)=\varphi_N \text{ on } \Lambda}} \|v_N\|_{H^1(\Theta)},$$

from which we deduce that

$$\|\varphi_N\|^0_{N,\frac{1}{2}} \leq \|\mathscr{L}^0_\Theta \varphi_N\|_{H^1(\Theta)}.$$

Thus, the desired inequality follows from Theorem 3.8.

The results of Theorems 4.3 and 4.6 extend to much more general situations, see [8, §II.4]; however the application that we present in Section 5 only requires these results.

5 Evaluation of fractional-order norms of polynomials

The aim of this section is to evaluate the $H^{1/2}(\Lambda)$–norm of polynomials in $\mathbb{P}_N(\Lambda)$ and the $H_{00}^{1/2}(\Lambda)$–norm of polynomials in $\mathbb{P}_N^0(\Lambda)$ by means of discrete (generalized Fourier) coefficients on suitable polynomial eigenvector bases. The next statements are extensions of the results first presented in [11].

5.1 Constructive evaluation of $H^{1/2}(\Lambda)$-norms of polynomials

For each fixed integer $N \geq 0$, we consider the sequence of discrete Neumann eigenpairs $(\lambda_{N,j}\,, \Phi_{N,j})$, $0 \leq j \leq N$, where the eigenvalues are given in increasing order:

$$\lambda_{N,0} < \lambda_{N,1} < \cdots < \lambda_{N,N}$$

and, for $0 \leq j \leq N$, the eigenvector $\Phi_{N,j}$ belongs to $\mathbb{P}_N(\Lambda)$ and satisfies

$$\forall \varphi_N \in \mathbb{P}_N(\Lambda), \qquad \int_{-1}^{1} \Phi'_{N,j}(x)\,\varphi'_N(x)\,dx = \lambda_{N,j} \int_{-1}^{1} \Phi_{N,j}(x)\,\varphi_N(x)\,dx. \tag{5.1}$$

Obviously both these eigenvalues and the corresponding eigenvectors depend upon the polynomial degree N, whence the index N.

We also assume that all these eigenvectors have been normalized to have a unit $L^2(\Lambda)$–norm,

$$\|\Phi_{N,j}\|_{L^2(\Lambda)} = 1. \tag{5.2}$$

Note that the smallest Neumann eigenvalue $\lambda_{N,0}$ is equal to 0 independently of N, and that the corresponding eigenvector $\Phi_{N,0}$ is constant equal to $\frac{1}{\sqrt{2}}$.

Notation 5.1 *For any integrable function χ on Λ, the quantity $S_{N,\frac{1}{2}}(\chi)$ is defined by*

$$S_{N,\frac{1}{2}}(\chi) = \sum_{j=0}^{N} |\chi_N^j|^2 \big(1 + \lambda_{N,j}\big)^{\frac{1}{2}}, \quad \textit{with} \quad \chi_N^j = \int_{-1}^{1} \chi(x)\,\Phi_{N,j}(x)\,dx. \tag{5.3}$$

Proposition 5.2 *There exist positive constants C and C', independent of the polynomial degree N, such that the following estimates hold for every polynomial χ_N in $\mathbb{P}_N(\Lambda)$*

$$C\,\|\chi_N\|^2_{H^{\frac{1}{2}}(\Lambda)} \leq S_{N,\frac{1}{2}}(\chi_N) \leq C'\,\|\chi_N\|^2_{H^{\frac{1}{2}}(\Lambda)}. \tag{5.4}$$

PROOF. Each polynomial χ_N in $\mathbb{P}_N(\Lambda)$ admits the expansion

$$\chi_N = \sum_{j=0}^{N} \chi_N^j \, \Phi_{N,j} \, ,$$

for the χ_N^j introduced in (5.3). Since the basis $\{\Phi_{N,j}\}_{0 \le j \le N}$ is orthonormal in $L^2(\Lambda)$ and orthogonal in $H^1(\Lambda)$ with norms $\|\Phi_{N,j}\|_{H^1(\Lambda)} = (1+\lambda_{N,j})^{1/2}$, we find that the mapping: $\chi_N \mapsto (\chi_N^j)_{0 \le j \le N}$ is an isometry

(i) from $\mathbb{P}_N(\Lambda)$ provided with the norm $\|\cdot\|_{L^2(\Lambda)}$ onto $\mathbb{R}^{N+1}$ provided with the Euclidean norm

$$\|(\chi_N^j)\|_0 = \Big(\sum_{j=0}^{N} |\chi_N^j|^2 \Big)^{\frac{1}{2}},$$

(ii) from $\mathbb{P}_N(\Lambda)$ provided with the norm $\|\cdot\|_{H^1(\Lambda)}$ onto $\mathbb{R}^{N+1}$ provided with the norm

$$\|(\chi_N^j)\|_1 = \Big(\sum_{j=0}^{N} |\chi_N^j|^2 \big(1+\lambda_{N,j}\big) \Big)^{\frac{1}{2}}.$$

So the desired result follows from an interpolation argument, combined with Theorem 4.3.

5.2 Constructive evaluation of $H_{00}^{1/2}(\Lambda)$-norms of polynomials

Similarly, for each fixed integer $N \ge 2$, we consider the discrete Dirichlet eigenpairs $(\mu_{N,j} \, , \Psi_{N,j})$, $1 \le i \le N-1$, with eigenvalues

$$\mu_{N,1} < \mu_{N,2} < \cdots < \mu_{N,N-1}$$

and eigenvectors $\Psi_{N,j}$ in $\mathbb{P}_N^0(\Lambda)$ solutions of

$$\forall \psi_N \in \mathbb{P}_N^0(\Lambda), \quad \int_{-1}^{1} \Psi'_{N,j}(x) \, \psi'_N(x) \, dx = \mu_{N,j} \int_{-1}^{1} \Psi_{N,j}(x) \, \psi_N(x) \, dx. \tag{5.5}$$

There also, these eigenvalues and eigenvectors depend upon the polynomial degree N. We still assume that all that these eigenvectors have been normalized to have a unit $L^2(\Lambda)$–norm,

$$\|\Psi_{N,j}\|_{L^2(\Lambda)} = 1. \tag{5.6}$$

Notation 5.3 *For any integrable function χ on Λ, the quantity $S^0_{N,\frac{1}{2}}(\chi)$ is defined by*

$$S^0_{N,\frac{1}{2}}(\chi) = \sum_{j=1}^{N-1} |\chi_N^j|^2 \big(1+\mu_{N,j}\big)^{\frac{1}{2}}, \quad \textit{with} \quad \chi_N^j = \int_{-1}^{1} \chi(x) \, \Psi_{N,j}(x) \, dx. \tag{5.7}$$

We omit the proof of the next statement since it is exactly the same as for Proposition 5.1 when using Theorem 4.6 instead of Theorem 4.3.

Proposition 5.4 *There exist two positive constants C and C', independent of the polynomial degree N, such that the following estimates hold for every polynomial χ_N in $\mathbb{P}_N^0(\Lambda)$*

$$C\,\|\chi_N\|^2_{H^{\frac{1}{2}}_{00}(\Lambda)} \le S^0_{N,\frac{1}{2}}(\chi_N) \le C'\,\|\chi_N\|^2_{H^{\frac{1}{2}}_{00}(\Lambda)}. \tag{5.8}$$

Remark 5.5 *Let us consider the bilinear form*

$$a(\chi_N,\xi_N) = \sum_{j=1}^{N-1} \chi_N^j \xi_N^j \big(1+\mu_{N,j}\big)^{\frac{1}{2}},$$

$$\textit{with}\quad \chi_N^j = \int_{-1}^1 \chi_N(x)\,\Psi_{N,j}(x)\,dx, \quad \xi_N^j = \int_{-1}^1 \xi_N(x)\,\Psi_{N,j}(x)\,dx.$$

Proposition 5.4 yields that it is continuous and elliptic on the space $\mathbb{P}_N^0(\Lambda)$ equipped with the norm $\|\cdot\|_{H^{\frac{1}{2}}_{00}(\Lambda)}$, with norm and ellipticity constant independent of N. So, among other applications, this form could be an efficient tool for the extension to spectral elements of the domain decomposition algorithm recently proposed in [6].

6 Numerical illustrations

We use the results of Section 5 first to evaluate the $H^{1/2}(\Lambda)$-norms of some polynomials, second to evaluate the norms of some projection operators onto polynomial spaces as endomorphisms of $H^{1/2}(\Lambda)$.

6.1 Evaluation of norms of polynomials

Let $(L_n)_n$ denote the family of Legendre polynomials: Each L_n has degree n, is orthogonal to the other ones in $L^2(\Lambda)$ and satisfies $L_n(1) = 1$. It is well known that

$$\|L_n\|_{L^2(\Lambda)} = \sqrt{\frac{2}{2n+1}}, \tag{6.1}$$

and also (see [10, §1 & form. (5.3)])

$$\|L'_n\|_{L^2(\Lambda)} = \sqrt{n(n+1)}. \tag{6.2}$$

Thus the family of Legendre polynomials satisfies the following inverse inequality

$$\|L'_n\|_{L^2(\Lambda)} \le \sqrt{3}\, n^{\frac{3}{2}} \|L_n\|_{L^2(\Lambda)}. \tag{6.3}$$

This is sharper than the general and optimal estimate (see [10, Chap. I, Th. 5.2] for instance): For any integer $N \ge 0$,

$$\forall \varphi_N \in \mathbb{P}_N(\Lambda), \qquad \|\varphi'_N\|_{L^2(\Lambda)} \le \sqrt{3}\, N^2 \|\varphi_N\|_{L^2(\Lambda)}. \tag{6.4}$$

All this naturally leads to the question of the behavior of the $H^{1/2}(\Lambda)$-norm of L_n.

Indeed, it is possible to evaluate the $H^{1/2}(\Lambda)$-norm of L_n analytically by using the explicit definition (2.8) of the norm; we refer to [1] for this very tedious computation that provides the estimates for all $n \ge 2$

$$c\sqrt{\log\, n} \le \|L_n\|_{H^{\frac{1}{2}}(\Lambda)} \le c'\sqrt{\log\, n}, \tag{6.5}$$

with constants c and c' independent of n. We first evaluate numerically these constants. For this, we use formula (2.8) and compute exactly the double integral which appears in it via appropriate quadrature formulas (exact numerical integration is possible since the integrand is a polynomial with two variables and diagonal values for $x = x'$ are equal to the square of the derivative).

Figure 5 presents the quantity $\|L_n\|^2_{H^{\frac{1}{2}}(\Lambda)} / \log\, n$ as a function of n, for n varying from 2 to 40 (left part) and from 2 to 4000 (right part). From this, it appears that c can be taken equal to 2 and c' to 3.11, with a common limit close to 2 when n tends to $+\infty$.

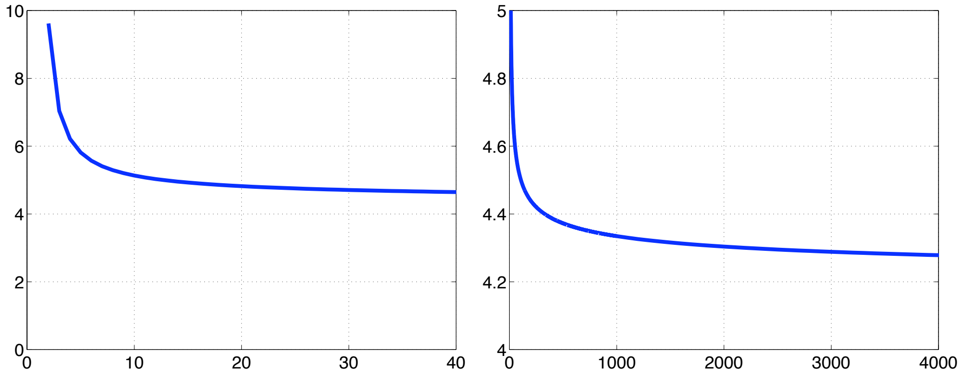

Figure 5: Evaluation of the $H^{1/2}(\Lambda)$-norm of L_n

From (6.5) and (6.1)–(6.2), the following inequalities can be observed:

$$\|L_n\|_{H^1(\Lambda)} \le c\,n\,(\log\,n)^{-\frac{1}{2}}\,\|L_n\|_{H^{\frac{1}{2}}(\Lambda)},$$
$$\|L_n\|_{H^{\frac{1}{2}}(\Lambda)} \le c'\,(n\,\log\,n)^{\frac{1}{2}}\,\|L_n\|_{L^2(\Lambda)}.$$

So, the norm of L_n in $H^s(\Lambda)$, $0 \le s \le 1$, is not evenly distributed as a function of s: It does not behave like $n^{-\frac{1}{2}+\frac{3}{2}s}$, as could be derived from the upper bound

$$\|L_n\|_{H^s(\Lambda)} \le c\,\|L_n\|^{1-s}_{L^2(\Lambda)}\|L_n\|^s_{H^1(\Lambda)}.$$

In a next step we compare the $H^{1/2}(\Lambda)$-norm of the L_n with their discrete evaluation as given in Proposition 5.1. Figure 6 presents the ratio

$$A_{N,n} = \frac{S_{N,\frac{1}{2}}(L_n)}{\|L_n\|^2_{H^{\frac{1}{2}}(\Lambda)}} \tag{6.6}$$

(see Notation 5.1), for even degrees n, $2 \le n \le N$, and for $N = 20$, 40, 60, 80, 100 (left part), for $N = 500$ and 1000 (right part). Note that problem (5.1) is solved by using Matlab routine `eig.m` for computing eigenpairs.

Figure 6 is in good coherence with the results of Proposition 5.1, which states

$$c \le A_{N,n} \le c'.$$

The numerical evidence is that c can be taken equal to 0.15 and c' to 0.3.

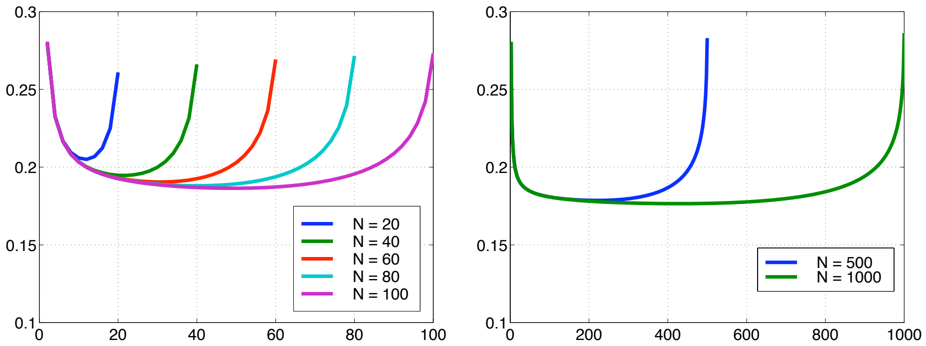

Figure 6: Comparison of the $H^{1/2}(\Lambda)$ -norm of L_n
and of its constructive evaluation

Remark 6.1 *We also observe that, for a fixed n, $A_{N,n}$ tends to a limit $A_{\infty,n}$ when N tends to $+\infty$, and more precisely that*

$$\frac{|A_{N,n} - A_{\infty,n}|}{A_{\infty,n}} < 10^{-2},$$

when $n \leq \frac{N}{2}$, which is useful to have an accurate evaluation of $\|L_n\|_{H^{\frac{1}{2}}(\Lambda)}$.

Remark 6.2 *The computation cost to evaluate the $H^{1/2}(\Lambda)$-norm of a polynomial with degree $\leq \frac{N}{2}$ either by formula (2.8) or by means of $S_{N,\frac{1}{2}}$ is nearly the same. However, when the norms of a large number of polynomials must be evaluated, using the discrete quantity $S_{N,\frac{1}{2}}$ is much more efficient. Indeed, solving problem (5.1) requires a constant times N^3 operations but, once it is done, the quantities $S_{N,\frac{1}{2}}(\varphi)$ for any φ in $\mathbb{P}_N(\Lambda)$ can be computed with a lower complexity.*

6.2 Evaluation of norms of projection operators

The numerical evaluation of the $H^{1/2}(\Lambda)$-norms allows us to answer another interesting question that remained unsolved (at least to our knowledge): It concerns the operator-norm of the $L^2(\Lambda)$-projection operator π_N over the set $\mathbb{P}_N(\Lambda)$ in various norms. By definition, denoting by $\|\cdot\|_{\mathcal{L}(E)}$ the norm of the endomorphisms of any Hilbert space E, we know that

$$\|\pi_N\|_{\mathcal{L}(L^2(\Lambda))} = 1. \tag{6.7}$$

We refer to [10, §II.1] for the following optimal result

$$\|\pi_N\|_{\mathcal{L}(H^1(\Lambda))} \leq C\sqrt{N}, \tag{6.8}$$

from which we derive, by interpolation, the upper bound

$$\|\pi_N\|_{\mathcal{L}(H^{\frac{1}{2}}(\Lambda))} \le C\,N^{\frac{1}{4}}. \tag{6.9}$$

By no means the above equality in the $H^1(\Lambda)$-norm implies that this inequality for the $H^{1/2}(\Lambda)$-norm is optimal.

Remark 6.3 *In order to illustrate the optimality of* (6.8)*, we have solved the eigenvalue problem, for $N_+ = 2N$: Find χ in $\mathbb{P}_{N_+}(\Lambda)/\mathbb{R}$ and ρ in $\mathbb{R}$ such that*

$$\forall \varphi \in \mathbb{P}_{N_+}(\Lambda)/\mathbb{R}, \qquad \int_{-1}^{1} (\pi_{N-1}\chi)'(x)\,(\pi_{N-1}\varphi)'(x)\,dx = \rho \int_{-1}^{1} \chi'(x)\,\varphi'(x)\,dx.$$

Among the $2N$ eigenvalues ρ, $N-1$ are obviously equal to 1 and $N-1$ are equal to 0. But the last two ones behave like N. Note also that the last eigenpair for π_{N-1} coincides with the penultimate eigenpair for π_N. We observe numerically the following asymptotics for these eigenvalues (independently of the choice of $N_+ \ge N+2$):

$$\frac{N}{4} + \frac{5}{8} + \frac{3}{16N} - \frac{3}{32N^2}\cdots, \qquad\qquad \frac{N}{4} + \frac{3}{8} + \frac{3}{16N} + \frac{3}{32N^2}\cdots.$$

This of course corroborates the optimality of (6.8)*, since we have evaluated the quantity*

$$\|\pi_N\|^2_{\mathcal{L}(H^1(\Lambda)/\mathbb{R})} = \max_{\varphi \in H^1(\Lambda)/\mathbb{R}} \frac{\int_{-1}^{1} |(\pi_{N-1}\varphi)'(x)|^2\,dx}{\int_{-1}^{1} |\varphi'(x)|^2\,dx}$$

by means of

$$\max_{\varphi \in \mathbb{P}_{N_+}(\Lambda)/\mathbb{R}} \frac{\int_{-1}^{1} |(\pi_{N-1}\varphi)'(x)|^2, dx}{\int_{-1}^{1} |\varphi'(x)|^2\,dx},$$

which is equal to the maximal eigenvalue ρ.

In order to evaluate $\|\pi_N\|_{\mathcal{L}(H^{\frac{1}{2}}(\Lambda))}$, we again take N_+ equal to $2N$ and compute the quantity

$$B_N = \max_{\varphi \in \mathbb{P}_{N_+}(\Lambda)} \frac{S_{N_{++},\frac{1}{2}}(\pi_N\varphi)}{S_{N_{++},\frac{1}{2}}(\varphi)}, \tag{6.10}$$

(see Notation 5.1) for different choices of $N_{++} \ge N_+$. Indeed, the numerical evaluation of the $H^{1/2}(\Lambda)$-norms is based on N_{++} eigenpairs and, according to Remark 6.1, it seems appropriate to choose N_{++} larger than twice the maximal degree N_+ of the involved polynomials.

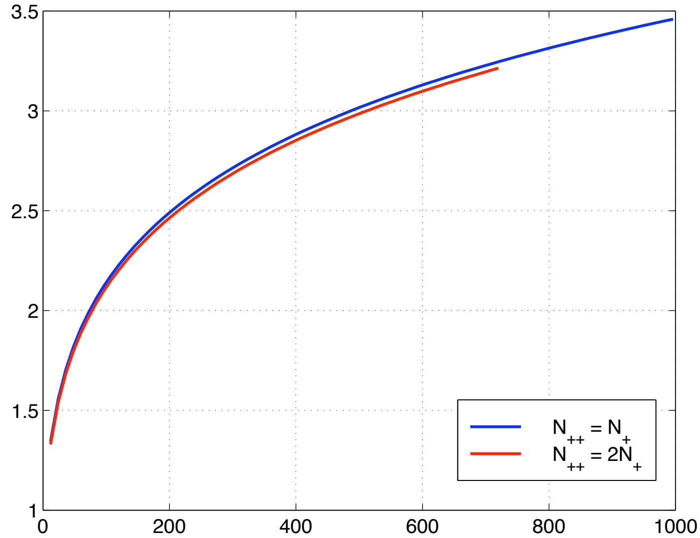

Figure 7: Evaluation of the $H^{1/2}(\Lambda)$ operator-norm of π_N

Figure 7 presents B_N as a function of N running through all multiples of 12, first for $N_{++} = N_+$ and N between 12 and 996, second for $N_{++} = 2N_+$ and N between 12 and 720. It can be observed that the values of B_N are nearly independent of the choice of $N_{++} \geq N_+$, in contrast with our first protective statement; that is why we stop the (extremely time consuming) computation at $N = 720$ for $N_{++} = 2N_+$.

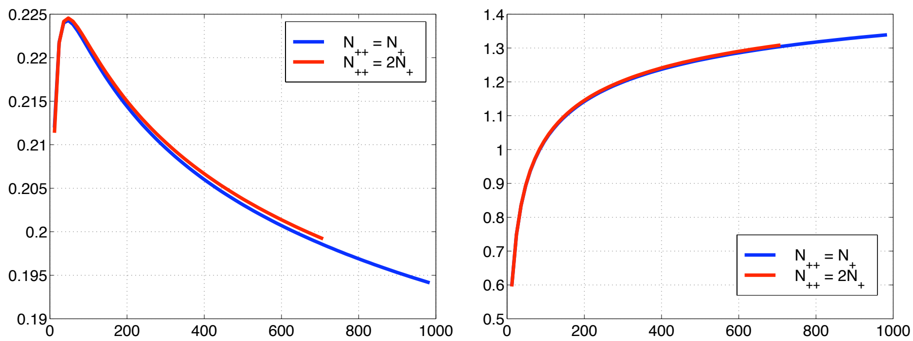

Figure 8: Comparison of the $H^{1/2}(\Lambda)$ operator-norm of π_N with N and $\log N$

In view of the previous computation, we compare the square of the norm of π_N with some power of N or of $\log N$. Figure 8 presents as functions of

N the divided differences

$$\frac{\log B_{N+12} - \log B_N}{\log(N+12) - \log N} \quad \text{(left part)}$$
$$\text{and} \qquad \frac{\log B_{N+12} - \log B_N}{\log\bigl(\log(N+12)\bigr) - \log\bigl(\log N\bigr)} \quad \text{(right part)}, \quad (6.11)$$

for the same values of N as in Figure 7.

We thus observe that the quantity $\|\pi_N\|_{\mathcal{L}(H^{\frac{1}{2}}(\Lambda))}$ satisfies, for all values of N,

$$C\,(\log N)^{1.4} \le \|\pi_N\|^2_{\mathcal{L}(H^{\frac{1}{2}}(\Lambda))} \le C'\,N^{0.2}. \qquad (6.12)$$

So the lack of stability of the operator π_N in $H^{1/2}(\Lambda)$ operator-norm is clearly weaker than what could be deduced from (6.9). Moreover, an extrapolation of the previous numerical results (the left curves of Figure 8 apparently tend to zero and the right curves to 2) leads us to propose the following

Conjecture : $$C\,\log N \le \|\pi_N\|_{\mathcal{L}(H^{\frac{1}{2}}(\Lambda))} \le C'\,\log N.$$

Remark 6.4 *In analogy with Remark 6.3 and again with $N_+ = 2N$, we have solved the following eigenvalue problem: Find χ in $\mathbb{P}_{N_+}(\Lambda)$ and ρ in $\mathbb{R}$ such that*

$$\forall \varphi \in \mathbb{P}_{N_+}(\Lambda), \quad b(\pi_N\chi, \pi_N\varphi) = \rho\, b(\chi, \varphi), \qquad (6.13)$$

for the scalar product $b(\cdot,\cdot)$ defined by

$$b(\psi, \xi) = \sum_{j=0}^{N_{++}} \psi^j \xi^j \bigl(1 + \lambda_{N_{++},j}\bigr)^{\frac{1}{2}},$$
$$\textit{with} \quad \psi^j = \int_{-1}^{1} \psi(x)\,\Phi_{N_{++},j}(x)\,dx, \quad \xi^j = \int_{-1}^{1} \xi(x)\,\Phi_{N_{++},j}(x)\,dx. \qquad (6.14)$$

Figure 9 presents the largest three eigenvalues ρ as functions of N. We observe that the difference between the largest two ones goes decreasing and that the third eigenvalue is close to 1 but slightly increasing with N, which suggests that the situation is more complex than for the $L^2(\Lambda)$- and $H^1(\Lambda)$-norms.

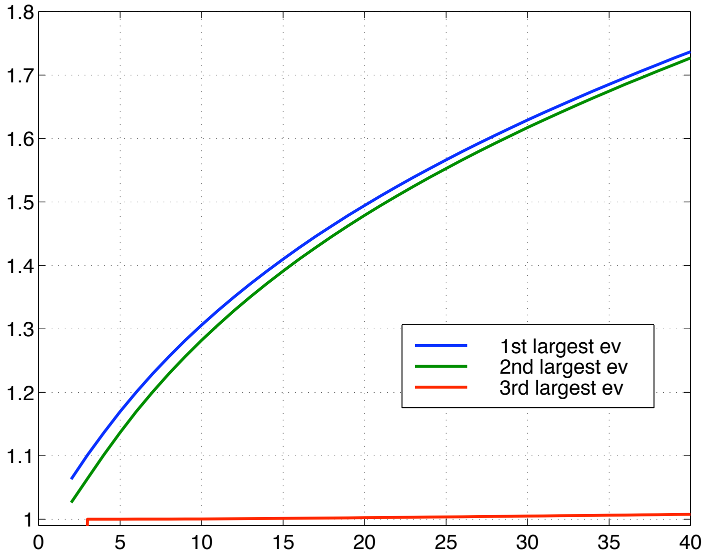

Figure 9: The largest three eigenvalues in problem (6.13)

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