Polynomials in the Sobolev World

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Introduction

The origin of this manuscript is a very simple question which arose nearly twenty years ago in the context of spectral methods. It concerns the space $\mathbb{P}_N(\Lambda)$ of restrictions to a bounded interval Λ of polynomials with degree smaller than N and can be stated as follows.

> For $s_1 \geq s_2 \geq 0$, the interpolate space of index θ, $0 < \theta < 1$, between $\mathbb{P}_N(\Lambda)$ provided with the norm of $H^{s_1}(\Lambda)$ and this same space provided with the norm of $H^{s_2}(\Lambda)$ is clearly the space $\mathbb{P}_N(\Lambda)$ provided with a norm which is equivalent to that of $H^{(1-\theta)s_1+\theta s_2}(\Lambda)$.
>
> **But are the equivalence constants independent of N?**

This result was intended to derive some inverse inequalities for polynomials, which are easy to establish in Sobolev norms of integral order but not otherwise, and are useful for handling nonlinear terms in the framework of spectral discretizations.

Several unfruitful attempts led us to work with the interpolation theory between Banach spaces in the sense of traces, according to [23, Chap. 1] for instance. This of course required the construction of a lifting of polynomial traces on one edge of a rectangle. In this direction, preliminary results have been proved in [4] concerning the lifting of traces on one edge of a triangle when all the angles of the triangle were $< \frac{\pi}{2}$. Extending this to a rectangle suggested us that the lifting of polynomial traces on all faces of a cube is also an important problem which has not been investigated before, though having interesting applications such as the treatment of nonhomogeneous Dirichlet boundary conditions for spectral discretizations. This problem appeared to be related to the question of compatibility conditions satisfied by traces on the edges and vertices of a polyhedron, question which was not yet addressed. And all of this is the origin of the three chapters of this manuscript: Trace theorems, lifting of polynomial traces, polynomial inverse inequalities. A positive answer to the initial question is derived from these trace results in Chapter II.

In Chapter I, we first introduce the Sobolev spaces on intervals, polygons and polyhedra: The spaces of integral order are defined in the usual way and those of non integral order are defined via intrinsic norms. Indeed, in a second step, we use this definition to prove a trace theorem which is not standard since it concerns traces of any order in non-smooth domains such as polygons or polyhedra. In a further step, as a generalization of [17], we characterize the compatibility conditions satisfied by these traces at the vertices of a polygon so that they admit a lifting of given regularity. The number of these conditions at each vertex depends on the smoothness of the traces but in any case is smaller than the square of the number of traces. We prove the existence of a lifting operator for the traces satisfying these conditions. Moreover we derive its continuity for appropriate norms of the traces, which is well-known in the non-limit cases but less standard in the limit cases. Finally, we also prove analogous results for the traces on the faces of a polyhedron: Now compatibility conditions concern both the vertices and the edges of the polyhedron and their number highly depends on the geometry at vertices, as announced in [8]. These results are finally extended to the case of weighted Sobolev spaces in the square and the cube, which are the basic geometries for spectral methods.

Chapter II is devoted to the continuous lifting of polynomial traces into polynomials, and the degree of the image as a function of the degree of the traces is also taken into account. Preliminary results concern the lifting from a line to a strip and from an edge to a triangle. Next, we consider the case of the square. By using a very simple transformation that maps a part of a triangle onto the square, we first construct a lifting operator from traces of one edge of a square into polynomials in the square. A modified operator maps polynomials that vanish at the endpoints of edges up to a fixed order into polynomials vanishing on the three other edges of the square up to the same order. These operators are then used to construct the full lifting operator acting on traces on each of the four edges of the square which satisfy all the compatibility conditions at the four vertices of the square and with values in a polynomial space, as announced in [6]. Analogous results are then stated and briefly proved for the cube, since the arguments are the same. As previously and for the same reasons, the extension to the case of weighted Sobolev norms is presented.

Relying on these results, in particular on the weighted case, we prove a positive answer to the initial question: The equivalence constants that appear in the interpolation of polynomial spaces are independent of the degree of polynomials.

In Chapter III, we work with weighted spaces on the interval $]-1,1[$ and present the Jacobi polynomials that form a family of orthogonal functions for weighted measures. As the spaces of polynomials can be provided with norms of different orders, exponents and weights, we thus derive inverse inequalities of several types on the spaces of polynomials of fixed maximal degree in the weighted norms. In a final step, we prove that the inverse inequalities that we have established cannot be improved by exhibiting appropriate counter-examples based on Jacobi polynomials.

The main applications of these results concern the numerical analysis of spectral methods, for instance for the treatment of nonhomogeneous Dirichlet type boundary data, and also of mortar spectral element methods (see [12] or [13] for the definition and basic analysis of these methods). Some of these applications will be presented in a future work.

Chapter I

Sobolev spaces and trace theorems

We first make precise some notation about the different types of domains which will be considered throughout this manuscript. We then recall the definitions, and basic properties of the standard Sobolev spaces. Next, we prove some specific properties of the Sobolev spaces, concerning the existence of traces and the structures of the spaces, on the model geometry of an interval. Even if these facts are already known, the idea is that they give rise to further results about the Sobolev spaces in a sector or in a polyhedral cone, and also that very similar techniques are needed for weighted Sobolev spaces.

At the end of the chapter (Sections 5 and 6), we give a more original version of the trace theorem, for both a polygon or a polyhedron. Indeed, most often, the classical trace theorem states that the range of Sobolev spaces on a domain by the trace operator is a Sobolev space on the whole boundary of the domain, and, when the global domain is not smooth, this is limited to low order trace operators. In the case of a polygon or a polyhedron we have rather work with local traces on the smooth parts of the boundaries, which are edges for a polygon and faces for a polyhedron. However, the trace operator is not onto the product of appropriate spaces on edges or faces: Some compatibility conditions are necessary at the vertices and also on the edges in the case of a polyhedron. For each given regularity and each number of traces, we exhibit the necessary and sufficient conditions in an explicit way. These results extend those in [17, Thm 1.5.2.8] for polygons and have been announced in [8] for polyhedra.

Finally, the analogous properties for some weighted spaces are presented, we refer to [7] for more detailed proofs.

1. About the geometry of domains.

Let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^d$, $d = 1$, 2 or 3, with a Lipschitz–continuous boundary. The generic point in $\mathcal{O}$ is denoted by x or sometimes ζ in dimension $d = 1$, by $\boldsymbol{x} = (x, y)$ in the dimension $d = 2$ and by $\boldsymbol{x} = (x, y, z)$ in dimension $d = 3$. As usual, $\boldsymbol{n}$ stands for the unit outward normal vector to $\mathcal{O}$ on $\partial\mathcal{O}$. According to the dimension, we are interested in more specific domains that we now describe.

- Case of dimension $d = 1$

The key one-dimensional domain is the interval $\Lambda =]-1, 1[$, for which the generic point is ζ. However, for technical reasons, we also consider the model interval $\mathcal{I} = \,]0,1[\,$.

Figure 1.1

- Case of dimension $d = 2$

In this case, the general polygon is denoted by Ω and its edges by Γ_ℓ, $1 \le \ell \le L$. We also introduce its vertices $\boldsymbol{a}_i$, $1 \le i \le I$ (of course, I is equal to L in dimension 2). Among polygons, we consider more specifically:
– the square $\Theta = \Lambda^2$. We denote by $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ and $\boldsymbol{a}_4$ the vertices with coordinates $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$, respectively. With the convention $\boldsymbol{a}_0 = \boldsymbol{a}_4$, each edge $\boldsymbol{a}_{\ell-1}\boldsymbol{a}_\ell$, $1 \le \ell \le 4$, is denoted by Γ_ℓ, see also Figure 5.3.
– the rectangle $\Delta = \Lambda \times \mathcal{I}$.
– the sector $\mathcal{S}$ with vertex $\boldsymbol{a}$ and aperture κ, $0 < \kappa < 2\pi$, which is contained in the circle with centre $\boldsymbol{a}$ and radius equal to 1. We denote by Γ_1 and Γ_2 its edges such that κ is the internal angle to $\mathcal{S}$ when going from Γ_2 to Γ_1 and turning counterclockwise; without loss of generality, we assume that Γ_2 coincides with the edge $\{\boldsymbol{x} = (x, 0);\ 0 < x < 1\}$ when needed, see also Figure 5.1.

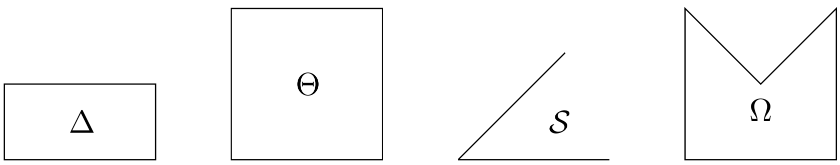

Figure 1.2

On each Γ_ℓ, we denote the unit outward normal vector to the domain by $\boldsymbol{n}_\ell$ and the tangential unit vector to Γ_ℓ directly orthogonal to $\boldsymbol{n}_\ell$ by $\boldsymbol{\tau}_\ell$.

- Case of dimension $d = 3$

We consider a general polyhedron Π with a Lipschitz–continuous boundary. Its faces are denoted by Ω_j, $1 \le j \le J$, its edges by Γ_ℓ, $1 \le \ell \le L$, and its vertices by $\boldsymbol{a}_i$, $1 \le i \le I$. We also introduce a polyhedral cone $\mathcal{C}$ with vertex $\boldsymbol{a}$, contained in the sphere with center $\boldsymbol{a}$ and radius equal to 1, and the cube $\Xi = \Lambda^3$.

Here, $\boldsymbol{n}_j$ stands for the unit outward normal vector to the domain on each Ω_j.

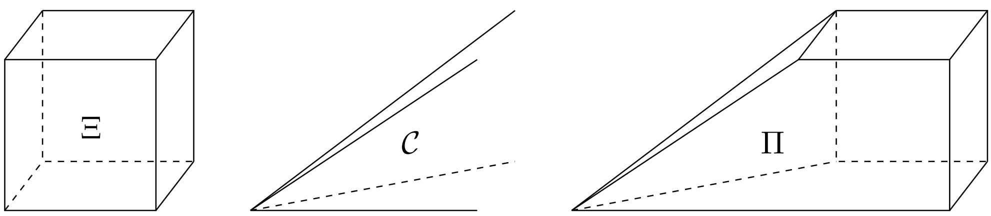

Figure 1.3

2. Definitions and standard properties.

We first give the definitions of the spaces of scalar-valued functions, and we recall two main properties: The Sobolev imbedding theorem, the general interpolation theorem. Next, we extend the definitions to vector-valued functions in view of the applications to tensorized spaces.

2.1. Spaces of scalar functions.

For each p, $1 \leq p < +\infty$, we first introduce the space

$$L^p(\mathcal{O}) = \big\{ v : \mathcal{O} \to \mathbb{R} \text{ measurable; } \|v\|_{L^p(\mathcal{O})} = \big(\int_{\mathcal{O}} |v(\boldsymbol{x})|^p \, d\boldsymbol{x} \big)^{\frac{1}{p}} < +\infty \big\}. \tag{2.1}$$

The space $L^\infty(\mathcal{O})$ associated with the Lebesgue measure, here denoted by μ, is the space of measurable functions v such that

$$\|v\|_{L^\infty(\mathcal{O})} = \sup\big\{ t \geq 0;\ \mu(\{|v| \geq t\}) \neq 0 \big\} < +\infty. \tag{2.2}$$

Next, let $\mathscr{D}(\mathcal{O})$ stand for the space of infinitely differentiable functions with a compact support in $\mathcal{O}$. Its dual space $\mathscr{D}'(\mathcal{O})$ is the space of distributions on $\mathcal{O}$. If $\boldsymbol{k}$ denotes a d–tuple $(k_1, \dots, k_d)$ of nonnegative integers, we use the notation $\partial^{\boldsymbol{k}} v$ for the partial derivative (in the distribution sense) of a function v of total order $|\boldsymbol{k}| = k_1 + \cdots + k_d$ and partial order k_i with respect to the i-th variable. For any nonnegative integer k, we also denote by ∂_x^k, ∂_y^k and ∂_z^k the partial derivatives of order k with respect to x, y and z, respectively.

Next, for each nonnegative integer m and each p, $1 \leq p \leq +\infty$, we consider the space $W^{m,p}(\mathcal{O})$ of functions such that all their partial derivatives of total order $\leq m$ belong to $L^p(\mathcal{O})$, namely

$$W^{m,p}(\mathcal{O}) = \big\{ v \in \mathscr{D}'(\mathcal{O});\ \|v\|_{W^{m,p}(\mathcal{O})} = \big(\sum_{|\boldsymbol{k}| \leq m} \|\partial^{\boldsymbol{k}} v\|_{L^p(\mathcal{O})}^p \big)^{\frac{1}{p}} < +\infty \big\}, \tag{2.3}$$

with the obvious modification for $p = \infty$

$$W^{m,\infty}(\mathcal{O}) = \big\{ v \in \mathscr{D}'(\mathcal{O});\ \|v\|_{W^{m,\infty}(\mathcal{O})} = \sup_{|\boldsymbol{k}| \leq m} \|\partial^{\boldsymbol{k}} v\|_{L^\infty(\mathcal{O})} < +\infty \big\}. \tag{2.4}$$

We recall [1, Chap. III] that $W^{m,p}(\mathcal{O})$ is a Banach space (a Hilbert space for $p = 2$) and also that, for $1 \leq p < +\infty$, the space $\mathscr{C}^\infty(\overline{\mathcal{O}})$ of infinitely differentiable functions on $\overline{\mathcal{O}}$ is dense in $W^{m,p}(\mathcal{O})$.

The Sobolev spaces of non integral order can be defined in several ways, for instance by interpolation methods [1, Chap. VII], however we have rather define them by the way of an intrinsic norm. We need a first notation.

Notation 2.1. For any positive real number τ and for any function v defined a.e. on $\mathcal{O}$, $q_\tau[v]$ is defined a.e. on $\mathcal{O}\times\mathcal{O}$ by

$$q_\tau[v](\boldsymbol{x},\boldsymbol{x}') = \frac{|v(\boldsymbol{x}) - v(\boldsymbol{x}')|}{|\boldsymbol{x}-\boldsymbol{x}'|^\tau}. \tag{2.5}$$

Next, any positive real number s which is not an integer can be written $[s]+\sigma$, where $[s]$ denotes its integral part while its fractional part σ satisfies $0<\sigma<1$. Then, for any p, $1\le p<\infty$, the space $W^{s,p}(\mathcal{O})$ is the space of distributions v in $\mathscr{D}'(\mathcal{O})$ such that

$$\|v\|_{W^{s,p}(\mathcal{O})} = \Big(\|v\|^p_{W^{[s],p}(\mathcal{O})} + \sum_{|\boldsymbol{k}|=[s]} \|q_{\sigma+\frac{d}{p}}[\partial^{\boldsymbol{k}} v]\|^p_{L^p(\mathcal{O}\times\mathcal{O})}\Big)^{\frac{1}{p}} < +\infty. \tag{2.6}$$

Obviously, the double integral on $\mathcal{O}\times\mathcal{O}$ that appears in this definition

$$\|q_{\sigma+\frac{d}{p}}[\partial^{\boldsymbol{k}} v]\|^p_{L^p(\mathcal{O}\times\mathcal{O})} = \int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|\partial^{\boldsymbol{k}} v(\boldsymbol{x}) - \partial^{\boldsymbol{k}} v(\boldsymbol{x}')|^p}{|\boldsymbol{x}-\boldsymbol{x}'|^{\sigma p+d}}\, d\boldsymbol{x}\, d\boldsymbol{x}',$$

can be equivalently replaced with the integral on any neighbourhood of the diagonal of $\mathcal{O}\times\mathcal{O}$, for instance on

$$\Delta_a = \big\{(\boldsymbol{x},\boldsymbol{x}')\in\mathcal{O}\times\mathcal{O};\ |\boldsymbol{x}-\boldsymbol{x}'|\le a\big\},$$

for a fixed positive real number a. There also, $W^{s,p}(\mathcal{O})$ is a Banach space (a Hilbert space for $p=2$) and, for $1\le p<+\infty$, the space $\mathscr{C}^\infty(\overline{\mathcal{O}})$ is dense in $W^{s,p}(\mathcal{O})$.

Finally, for $1\le p<+\infty$ and for any nonnegative real number s, we define the subspace $\mathring{W}^{s,p}(\mathcal{O})$ of $W^{s,p}(\mathcal{O})$ as the closure of $\mathscr{D}(\mathcal{O})$ in $W^{s,p}(\mathcal{O})$. However, we have rather postpone the characterization of this new space to the next sections.

2.2. Sobolev imbeddings and interpolation theorems.

We now recall some further properties of these spaces. We begin with the Sobolev embedding theorem (see [32, § 4.6.1 & 4.10.2] or [1, Thm 7.57]), in a rather general form. For any real number $s\ge 0$, we denote by $\mathscr{C}^s(\overline{\mathcal{O}})$ the space of functions
– which are continuously differentiable up to the order s if s is an integer,
– which are continuously differentiable up to the order $[s]$ and such that all their

partial derivatives of order $[s]$ are Hölderian with exponent σ if $s = [s] + \sigma$ is not an integer.

Theorem 2.2. *Let p and q be such that $1 < p, q < \infty$, and let s and t be two real numbers such that $0 \leq t \leq s$.*
(i) If $t - \frac{d}{q} \leq s - \frac{d}{p}$, the following embedding holds:

$$W^{s,p}(\mathcal{O}) \subset W^{t,q}(\mathcal{O}). \tag{2.7}$$

(ii) If t is not an integer and $t \leq s - \frac{d}{p}$, the following embedding holds:

$$W^{s,p}(\mathcal{O}) \subset \mathscr{C}^t(\overline{\mathcal{O}}). \tag{2.8}$$

(iii) If $t < s$ and $t - \frac{d}{q} < s - \frac{d}{p}$, the embedding (2.7) is compact.
(iv) If $t < s - \frac{d}{p}$, the embedding (2.8) is compact.

The following results are closely linked to the theory of interpolation between Banach spaces, we refer to [5], [23], [24] and [32] for the general theory. Here, if X_0 and X_1 are two Banach spaces such that X_1 is contained in X_0 with a continuous and dense embedding, we denote by $[X_1, X_0]_{\theta,p}$ the interpolate space with index (θ, p) defined by the K–method, $0 < \theta < 1$ and $1 \leq p < +\infty$. We recall (see [5, §3.12] or [23, Chap. 1, Thm 10.1] or [32, §1.8.1]) that this space can be equivalently defined as the set of traces $v(0)$ of measurable functions v in $]0,1[$ with values in X_1 which satisfy

$$\int_0^1 \|v(t)\|_{X_1}^p \, t^{p\theta} \, \frac{dt}{t} < +\infty \quad \text{and} \quad \int_0^1 \|v'(t)\|_{X_0}^p \, t^{p\theta} \, \frac{dt}{t} < +\infty. \tag{2.9}$$

The idea is now to apply this theory to the previously introduced Sobolev spaces. We refer to [32, §4.3] for the next theorem.

Theorem 2.3. *Let p be such that $1 < p < \infty$ and let s_0, s_1 and s be nonnegative real numbers such that $s_0 < s < s_1$ and s is not an integer when $p \neq 2$. The following interpolation result holds:*

$$W^{s,p}(\mathcal{O}) = \Big[W^{s_1,p}(\mathcal{O}), W^{s_0,p}(\mathcal{O})\Big]_{\theta,p} \qquad \text{with} \quad \theta = \frac{s_1 - s}{s_1 - s_0}. \tag{2.10}$$

2.3. Spaces of vector functions.

Let X be a separable Banach space with norm $\|\cdot\|_X$. We introduce the space $\mathscr{D}(\mathcal{O}; X)$ of infinitely differentiable functions with a compact support in $\mathcal{O}$ and values in X, together with its dual space $\mathscr{D}'(\mathcal{O}; X)$.

For $1 \leq p < +\infty$, the space $L^p(\mathcal{O}; X)$ is still defined by (2.1), with the norm $\|\cdot\|_{L^p(\mathcal{O})}$ replaced with

$$\|v\|_{L^p(\mathcal{O};X)} = \Big(\int_{\mathcal{O}} \|v(\boldsymbol{x})\|_X^p \, d\boldsymbol{x}\Big)^{\frac{1}{p}}. \tag{2.11}$$

Then, for any nonnegative integer m, the space $W^{m,p}(\mathcal{O}; X)$ is still defined by (2.3), with $\mathscr{D}'(\mathcal{O})$ replaced with $\mathscr{D}'(\mathcal{O}; X)$ and the norm $\|\cdot\|_{W^{m,p}(\mathcal{O})}$ replaced with

$$\|v\|_{W^{m,p}(\mathcal{O};X)} = \Big(\sum_{|\boldsymbol{k}|\leq m} \|\partial^{\boldsymbol{k}}\varphi\|_{L^p(\mathcal{O};X)}^p\Big)^{\frac{1}{p}}. \tag{2.12}$$

And, finally, for any nonnegative real number $s = [s] + \sigma$, the space $W^{s,p}(\mathcal{O}; X)$ is defined similarly as the space $W^{s,p}(\mathcal{O})$, with the norm $\|\cdot\|_{W^{s,p}(\mathcal{O})}$ replaced with

$$\|v\|_{W^{s,p}(\mathcal{O};X)} = \Big(\|v\|_{W^{[s],p}(\mathcal{O};X)}^p + \sum_{|\boldsymbol{k}|=s} \|q_{\sigma+\frac{d}{p},X}[d^{\boldsymbol{k}}v]\|_{L^p(\mathcal{O}\times\mathcal{O})}^p\Big)^{\frac{1}{p}}, \tag{2.13}$$

where the extended ratio $q_{\tau,X}[\,\cdot\,]$ is defined by

$$q_{\tau,X}[v](\boldsymbol{x}, \boldsymbol{x}') = \frac{\|v(\boldsymbol{x}) - v(\boldsymbol{x}')\|_X}{|\boldsymbol{x} - \boldsymbol{x}'|^\tau}. \tag{2.14}$$

As for scalar functions, the space $\mathscr{C}^\infty(\overline{\mathcal{O}}; X)$ of infinitely differentiable functions on $\overline{\mathcal{O}}$ with values in X is dense in $W^{s,p}(\mathcal{O}; X)$ when $p < +\infty$.

3. Traces and structure properties for an interval.

The aim of this section is to present a proof of the trace theorem for the model interval $\Lambda =]-1, 1[$. This theorem characterizes not only the existence of a trace but also the kernel of the trace operator, which yields results about the structure of the Sobolev spaces near the boundary. For simplicity, we begin by working on the interval $\mathcal{I} =]0, 1[$ and only consider traces in zero. We also take p such that $1 < p < +\infty$.

3.1. Traces for the half of the interval.

In order to give a description of the kernel of trace operators, we first introduce the following "weighted" space. For any nonnegative integer m, we define $V^{m,p}(\mathcal{I})$ by

$$V^{m,p}(\mathcal{I}) = \big\{\varphi \in \mathscr{D}'(\mathcal{I});\ \|\varphi\|_{V^{m,p}(\mathcal{I})} < +\infty\big\}, \tag{3.1}$$

where

$$\|\varphi\|_{V^{m,p}(\mathcal{I})} = \Big(\sum_{k=0}^{m} \int_0^1 |d^k\varphi|^p \, x^{(k-m)p} \, dx\Big)^{\frac{1}{p}}. \tag{3.2}$$

We recall from [32, Chap. 3] that, for any nonnegative integer m and any value of p, the space $\mathscr{D}(]0,1])$ of infinitely differentiable functions with a compact support in $]0,1]$ is dense in $V^{m,p}(\mathcal{I})$.

Next, for any positive real number $s = [s] + \sigma$, we introduce the norm

$$\|\varphi\|_{V^{s,p}(\mathcal{I})} = \Big(\int_0^1 \big(\sum_{k=0}^{[s]} |d^k\varphi|^p \, x^{(k-s)p}\big) \, dx + \|q_{\sigma+\frac{1}{p}}[d^k\varphi]\|^p_{L^p(\mathcal{I}\times\mathcal{I})}\Big)^{\frac{1}{p}}. \tag{3.3}$$

Thus, in the usual way, we set

$$V^{s,p}(\mathcal{I}) = \big\{\varphi \in \mathscr{D}'(\mathcal{I});\ \|\varphi\|_{V^{s,p}(\mathcal{I})} < +\infty\big\}. \tag{3.4}$$

For simplicity, the space $V^{s,2}(\mathcal{I})$ is also denoted by $V^s(\mathcal{I})$.

The main results of this section are stated in the next theorem. We refer to [32, §2.9] for a different proof.

Theorem 3.1. *Let p be such that $1 < p < +\infty$ and s be a positive real number.*
(i) If $sp < 1$, then $\mathscr{D}(]0,1])$ is dense in $W^{s,p}(\mathcal{I})$ and the spaces $W^{s,p}(\mathcal{I})$ and $V^{s,p}(\mathcal{I})$ coincide.
(ii) If $s - \frac{1}{p}$ is not an integer and K denotes the integral part of $s - \frac{1}{p}$, for each integer k, $0 \le k \le K$, the trace mapping γ_k defined on $\mathscr{C}^\infty(\overline{\mathcal{I}})$ by

$$\gamma_k(\varphi) = d^k\varphi(0)$$

is continuous on $W^{s,p}(\mathcal{I})$. Moreover, $V^{s,p}(\mathcal{I})$ is the closure of $\mathscr{D}(]0,1])$ in $W^{s,p}(\mathcal{I})$ and the following characterization holds:

$$V^{s,p}(\mathcal{I}) = \big\{\varphi \in W^{s,p}(\mathcal{I});\ \forall k, 0 \le k \le K,\ \gamma_k(\varphi) = 0\big\}. \tag{3.5}$$

We prove this theorem in three steps. First, we treat the simple case $s = 1$. Next, we explain how to reduce the general statement to the case $0 < s \le 1$. Finally, we prove the result in the case $0 < s < 1$.

Proof of Theorem 3.1. Part I: case $s = 1$.
When s is equal to 1, (2.8) yields the continuity of the trace operator γ_0, and the characterization of $V^{1,p}(\mathcal{I})$ follows from the standard Hardy inequality [32, §3.2.6, Rem. 1] which we now recall.

Lemma 3.2. *For any p such that $1 < p < +\infty$ and for any real number $\beta \neq -1$, the following inequalities hold:*
(i) when β is < -1, for any function $\varphi \in \mathscr{C}_0^\infty(\,]0,1])$,

$$\int_0^1 |\varphi(x)|^p\, x^\beta\, dx \le \big(\frac{p}{|\beta+1|}\big)^p \int_0^1 |\varphi'(x)|^p\, x^{\beta+p}\, dx, \tag{3.6}$$

(ii) when β is > -1, for any function $\varphi \in \mathscr{C}^\infty([0,1])$,

$$\int_0^1 |\varphi(x)|^p\, x^\beta\, dx \le \big(\frac{p}{|\beta+1|}\big)^p \Big(\int_0^1 |\varphi'(x)|^p\, x^{\beta+p}\, dx + \int_0^1 |\varphi(x)|^p\, x^{\beta+p}\, dx\Big). \tag{3.7}$$

Proof of Theorem 3.1. Part II: reductions.
Assume that Theorem 3.1 is valid for $0 < s \le 1$. When s is > 1, the result then follows by an induction argument relying on the characterizations

$$W^{s,p}(\mathcal{I}) = \big\{\varphi \in W^{1,p}(\mathcal{I});\ \varphi' \in W^{s-1,p}(\mathcal{I})\big\},$$

and

$$V^{s,p}(\mathcal{I}) = \big\{\varphi;\ \varphi\, x^{-s} \in L^p(\mathcal{I}) \text{ and } \varphi' \in V^{s-1,p}(\mathcal{I})\big\},$$

together with inequality (3.6).

So, it remains to consider the case $0 < s < 1$. There, we introduce the partition of $\mathcal{I}$ into dyadic intervals $\mathcal{I}_j =]2^{-(j+1)}, 2^{-j}[$, where j runs through the set $\mathbb{N}$ of nonnegative integers, as illustrated in Figure 3.1. Next, for any such j, we associate with any integrable function φ its mean value $m_j(\varphi)$ on $\mathcal{I}_j$:

$$m_j(\varphi) = 2^{j+1} \int_{2^{-(j+1)}}^{2^{-j}} \varphi(x)\, dx. \tag{3.8}$$

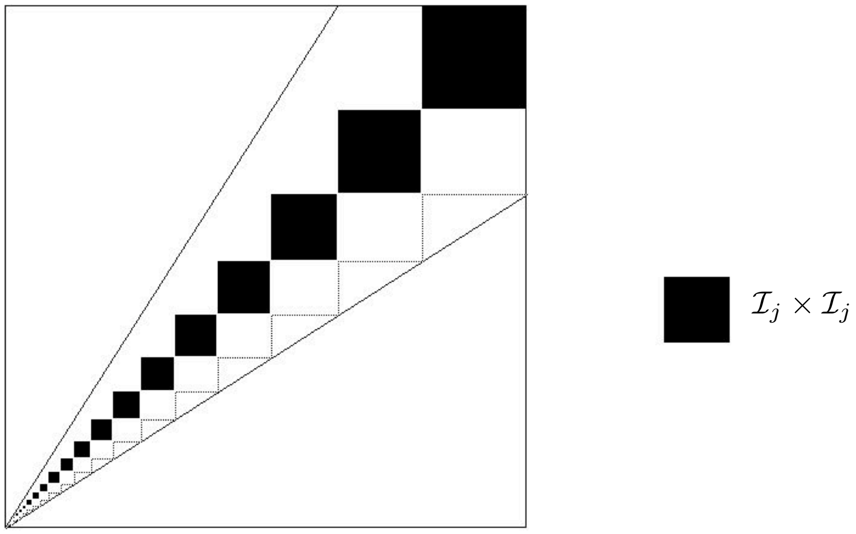

Figure 3.1

We begin with a technical lemma.

Lemma 3.3. *Let V and V' be two subsets of $\mathcal{I}$, with lengths $\mu(V)$ and $\mu(V')$. Let s be any nonnegative real number and p be such that $1 \leq p < +\infty$. For any function φ in $W^{s,p}(I)$, if $m(\varphi)$ and $m'(\varphi)$ denote the mean values of φ on V and V', respectively, the following inequality holds*

$$|m(\varphi) - m'(\varphi)| \leq \Big(\sup_{x \in V, x' \in V'} |x - x'|\Big)^{s+\frac{1}{p}} \mu(V)^{-\frac{1}{p}} \mu(V')^{-\frac{1}{p}} \, \|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(V \times V')}. \quad (3.9)$$

Proof. We start from the formula

$$|m(\varphi) - m'(\varphi)| \leq \mu(V)^{-1} \mu(V')^{-1} \, \Big| \int_V \int_{V'} \big(\varphi(x) - \varphi(x')\big) \, dx \, dx' \Big|.$$

Using Hölder's inequality on $V \times V'$ yields

$$|m(\varphi) - m'(\varphi)| \leq \mu(V)^{-\frac{1}{p}} \mu(V')^{-\frac{1}{p}} \Big(\int_V \int_{V'} |\varphi(x) - \varphi(x')|^p \, dx \, dx'\Big)^{\frac{1}{p}}.$$

So the desired inequality follows by multiplying and dividing by $|x - x'|^{s+\frac{1}{p}}$.

The key arguments to conclude the proof of Theorem 3.1 are presented in the next lemma, however its proof is rather complex.

Lemma 3.4. *Let s be in $]0, 1[$, and set $\eta = s - \frac{1}{p}$, $1 < p < +\infty$. Let φ be any function in $W^{s,p}(\mathcal{I})$.*

(i) If sp < 1, the sequence $\big(2^{j\eta}\, m_j(\varphi)\big)_{j\geq 0}$ *belongs to* $\ell^p(\mathbb{N})$.
(ii) If sp > 1, the sequence $\big(m_j(\varphi)\big)_{j\geq 0}$ *tends towards a limit which satisfies the estimate*

$$|\lim_{j\to+\infty} m_j(\varphi)| \leq c\,\big(\|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})} + \|\varphi\|_{L^p(\mathcal{I}_0)}\big); \tag{3.10}$$

moreover, if this limit is equal to 0, *the sequence* $\big(2^{j\eta}\, m_j(\varphi)\big)_{j\geq 0}$ *belongs to* $\ell^p(\mathbb{N})$.
In (i) and (ii), when the sequence $\big(2^{j\eta}\, m_j(\varphi)\big)_{j\geq 0}$ *belongs to* $\ell^p(\mathbb{N})$, *it satisfies the estimate*

$$\|2^{j\eta}\, m_j(\varphi)\|_{\ell^p(\mathbb{N})} \leq c\,\big(\|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})} + \|\varphi\|_{L^p(\mathcal{I}_0)}\big). \tag{3.11}$$

PROOF. We first consider the case $sp > 1$ and prove (3.10). Next, we prove (3.11) in this case. We conclude with the case $sp < 1$.
1) If $sp > 1$, we must check that the sequence $\big(m_j(\varphi) - m_{j+1}(\varphi)\big)_{j\geq 1}$ belongs to $\ell^1(\mathbb{N})$. We apply Lemma 3.3 with $V = \mathcal{I}_j$ and $V' = \mathcal{I}_{j+1}$. The quantities $\sup_{x\in V, x'\in V'} |x - x'|$, $\mu(V)$ and $\mu(V')$ in this case are all equal to 2^{-j} up to a multiplicative constant, whence

$$\begin{aligned} |m_j(\varphi) - m_{j+1}(\varphi)| &\leq c\,(2^{-j})^{s+\frac{1}{p}-\frac{2}{p}}\, \|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}_j\times\mathcal{I}_{j+1})} \\ &\leq c\, 2^{-j\eta}\, \|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})}. \end{aligned}$$

Since $\eta = s - \frac{1}{p}$ is positive, 2^η is > 1, hence $\big(m_j(\varphi) - m_{j+1}(\varphi)\big)_{j\geq 0}$ belongs to $\ell^1(\mathbb{N})$. Moreover, we have

$$|\lim_{j\to+\infty} m_j(\varphi)| \leq \sum_{j=0}^{+\infty} |m_j(\varphi) - m_{j+1}(\varphi)| + |m_0(\varphi)|,$$

which yields (3.10).
2) Still in the case $sp > 1$, we introduce special disjoint subsets of the intervals $\mathcal{I}_k$, for any $k \geq 2$: For each integer j, $1 \leq j \leq k-1$, let $\mathcal{I}_{j,k-j}$ be an interval such that

$$\mathcal{I}_{j,k-j} \subset \mathcal{I}_k \qquad \text{and} \qquad \mu(\mathcal{I}_{j,k-j}) = \frac{\mu(\mathcal{I}_k)}{2(k-j)^2}.$$

Since $\sum_{j=1}^{k-1} \frac{1}{2(k-j)^2}$ is < 1, we can assume that all the $\mathcal{I}_{j,\ell}$, $j \geq 1$, $\ell \geq 1$, are disjoint. If $m_{j,\ell}(\varphi)$ stands for the mean value of φ on $\mathcal{I}_{j,\ell}$, Lemma 3.3 implies that

$$|m_{j+1}(\varphi) - m_{j,1}(\varphi)| \leq c\, 2^{-j\eta}\, \|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}_{j+1}\times\mathcal{I}_{j+1})}.$$

Since $\cup_{j=1}^{+\infty} \mathcal{I}_{j+1}\times\mathcal{I}_{j+1}$ is the union of disjoint intervals contained in $\mathcal{I}\times\mathcal{I}$, we deduce that

$$\|2^{j\eta}\,\big(m_{j+1}(\varphi) - m_{j,1}(\varphi)\big)\|_{\ell^p(\mathbb{N})} \leq c\, \|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})}. \tag{3.12}$$

Therefore, proving (3.11) reduces to

$$\|2^{j\eta}\, m_{j,1}(\varphi)\|_{\ell^p(\mathbb{N})} \le c\, \|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})}. \tag{3.13}$$

Assuming for a while that j is fixed, we first study $\lim_{\ell\to+\infty} m_{j,\ell}(\varphi)$. By hypothesis, $\lim_{\ell\to+\infty} m_{j+\ell}(\varphi)$ is equal to 0. Moreover, Lemma 3.3 infers that

$$|m_{j,\ell}(\varphi) - m_{j+\ell}(\varphi)| \le c\, 2^{-(j+\ell)\eta} \ell^{\frac{2}{p}}\, \|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})},$$

and, since η is > 0, $2^{-\ell\eta}\ell^{\frac{2}{p}}$ tends to 0 when ℓ tends to $+\infty$. That implies

$$\forall j \ge 1, \qquad \lim_{\ell\to+\infty} m_{j,\ell}(\varphi) = 0.$$

From this, we deduce that

$$m_{j,1}(\varphi) = \sum_{\ell=1}^{+\infty} \big(m_{j,\ell}(\varphi) - m_{j,\ell+1}(\varphi)\big).$$

We set:

$$J_{j,\ell} = \|q_{s+\frac{1}{p}}[\varphi]\|^p_{L^p(\mathcal{I}_{j,\ell}\times\mathcal{I}_{j,\ell+1})}.$$

Lemma 3.3 infers that

$$|m_{j,\ell}(\varphi) - m_{j,\ell+1}(\varphi)| \le c\, 2^{-(j+\ell)\eta}\, \ell^{\frac{4}{p}}\, J_{j,\ell}^{\frac{1}{p}}.$$

Thus

$$2^{j\eta}\, |m_{j,1}(\varphi)| \le c \sum_{\ell=1}^{+\infty} 2^{-\ell\eta}\, \ell^{\frac{4}{p}}\, J_{j,\ell}^{\frac{1}{p}}.$$

Using Hölder's inequality yields, with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$2^{j\eta p}\, |m_{j,1}(\varphi)|^p \le c^p\, \Big(\sum_{\ell=1}^{+\infty} 2^{-\ell\eta p'}\, \ell^{\frac{4p'}{p}}\Big)^{\frac{p}{p'}}\, \Big(\sum_{\ell=1}^{+\infty} J_{j,\ell}\Big).$$

Since η is positive, we finally obtain:

$$\|2^{j\eta}\, m_{j,1}(\varphi)\|_{\ell^p(\mathbb{N})} \le c_0 \Big(\sum_{j=1}^{+\infty}\sum_{\ell=1}^{+\infty} J_{j,\ell}\Big)^{\frac{1}{p}}, \tag{3.14}$$

and, since the $\mathcal{I}_{j,\ell}$ are disjoint, we have

$$\Big(\sum_{j=1}^{+\infty}\sum_{\ell=1}^{+\infty} J_{j,\ell}\Big)^{\frac{1}{p}} \le \|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})}.$$

Hence, estimate (3.14) implies (3.13), which achieves the proof of (3.11).
3) Finally, when $sp < 1$, the proof is similar to the previous step, however we need another family of subintervals. For any integers $k \geq 1$ and $j \geq k+1$, let $\widetilde{\mathcal{I}}_{j,j-k}$ be an interval such that:

$$\widetilde{\mathcal{I}}_{j,j-k} \subset \mathcal{I}_k \qquad \text{and} \qquad \mu(\widetilde{\mathcal{I}}_{j,j-k}) = \frac{\mu(\mathcal{I}_k)}{2(j-k)^2}.$$

Like before, we can choose all the $\widetilde{\mathcal{I}}_{j,\ell}$, $j \geq 1$, $\ell < j$, disjoint. If $\widetilde{m}_{j,\ell}(\varphi)$ denotes the mean value of the function φ on $\widetilde{\mathcal{I}}_{j,\ell}$, we obtain, in a similar way to (3.12),

$$\|2^{j\eta}\left(m_{j-1}(\varphi) - \widetilde{m}_{j,1}(\varphi)\right)\|_{\ell^p(\mathbb{N})} \leq c\,\|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})}. \tag{3.15}$$

So, as in (3.13), it is sufficient to prove:

$$\|2^{j\eta}\,\widetilde{m}_{j,1}(\varphi)\|_{\ell^p(\mathbb{N})} \leq c\,(\|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})} + \|\varphi\|_{L^p(\mathcal{I}_1)}). \tag{3.16}$$

We have:

$$\widetilde{m}_{j,1}(\varphi) = \sum_{\ell=1}^{j-2}\big(\widetilde{m}_{j,\ell}(\varphi) - \widetilde{m}_{j,\ell+1}(\varphi)\big) + \widetilde{m}_{j,j-1}(\varphi). \tag{3.17}$$

We obtain as in (3.14):

$$\|2^{j\eta}\left(\widetilde{m}_{j,1}(\varphi) - \widetilde{m}_{j,j-1}(\varphi)\right)\|_{\ell^p(\mathbb{N})} \leq c_{00}\,\|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})}. \tag{3.18}$$

Now, thanks to Lemma 3.3, we know that:

$$|m_1(\varphi) - \widetilde{m}_{j,j-1}(\varphi)| \leq c\,j^{\frac{2}{p}}\,\|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})}. \tag{3.19}$$

Estimates (3.17), (3.18) and (3.19) yield

$$\begin{aligned}\|2^{j\eta}\,\widetilde{m}_{j,1}(\varphi)\|_{\ell^p(\mathbb{N})} \leq c_{00}\|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})}& \\ + c\,\|2^{j\eta}\,j^{\frac{2}{p}}\|_{\ell^p(\mathbb{N})}\,\|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})} &+ \|2^{j\eta}\|_{\ell^p(\mathbb{N})}\,|m_1(\varphi)|.\end{aligned}$$

Since η is negative, this last inequality gives (3.11), which ends the proof.

Proof of Theorem 3.1. Part III: case $0 < s < 1$.
The key idea is now to define, in the case $sp > 1$, the trace operator γ_0 by

$$\gamma_0\varphi = \lim_{j\to+\infty} m_j(\varphi).$$

Indeed, thanks to Lemma 3.4, it suffices to check that, for any function φ of $W^{s,p}(\mathcal{I})$, if the sequence $\left(2^{-j(-s+\frac{1}{p})}\,m_j(\varphi)\right)_{j\geq 0}$ belongs to $\ell^p(\mathbb{N})$, then the function φ belongs to $V^{s,p}(\mathcal{I})$ and satisfies the estimate:

$$\|\varphi\,x^{-s}\|_{L^p(\mathcal{I})} \leq c\,(\|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}\times\mathcal{I})} + \|2^{-j(-s+\frac{1}{p})}\,m_j(\varphi)\|_{\ell^p(\mathbb{N})}). \tag{3.20}$$

To check this, we start from an estimate on $\mathcal{I}_0$ that can be derived from the Peetre–Tartar lemma (see for instance [14, Chap. I, Thm 2.1]): Any φ in $W^{s,p}(\mathcal{I}_0)$ satisfies

$$\|\varphi\|_{L^p(\mathcal{I}_0)} \leq c\,(\|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}_0\times\mathcal{I}_0)} + |\int_{\mathcal{I}_0} \varphi(x)\,dx|).$$

Together with the scaling from $\mathcal{I}_j$ onto $\mathcal{I}_0$: $x \mapsto \tilde{x} = 2^j x$, this yields

$$2^{\frac{j}{p}}\,\|\varphi\|_{L^p(\mathcal{I}_j)} \leq c\,(2^{-j(s-\frac{1}{p})}\,\|q_{s+\frac{1}{p}}[\varphi]\|_{L^p(\mathcal{I}_j\times\mathcal{I}_j)} + |m_j(\varphi)|),$$

whence

$$\|2^{js}\,\varphi\|^p_{L^p(\mathcal{I}_j)} \leq c\,(\|q_{s+\frac{1}{p}}[\varphi]\|^p_{L^p(\mathcal{I}_j\times\mathcal{I}_j)} + |2^{-j(-s+\frac{1}{p})}\,m_j(\varphi)|^p). \tag{3.21}$$

Since $x/2^{-j}$ and its inverse are bounded on $\mathcal{I}_j$, summing up this last estimate on all integers $j \geq 1$ gives the desired result.

Note that, if the assumptions of part (ii) of Theorem 3.1 hold, $V^{s,p}(\mathcal{I})$ has codimension $K+1$ in $W^{s,p}(\mathcal{I})$. So proving the following corollary is now easy.

Corollary 3.5. *Let p be such that $1 < p < +\infty$ and s be a positive real number. If $s - \frac{1}{p}$ is not an integer and K denotes the integral part of $s - \frac{1}{p}$, the space $W^{s,p}(\mathcal{I})$ is equal to the direct sum of the space $V^{s,p}(\mathcal{I})$ and of the space $\mathbb{P}_K(\mathcal{I})$ of polynomials with degree $\leq K$ on $\mathcal{I}$.*

3.2. Traces for the interval.

All the previous properties can trivially be extended to the interval Λ, where the weight ρ is the product of the distance to the two endpoints of the interval:

$$\rho(\zeta) = 1 - \zeta^2. \tag{3.22}$$

For any nonnegative integer m, we define the space $V^{m,p}(\Lambda)$ by (3.1), with the norm $\|\cdot\|_{V^{m,p}(\mathcal{I})}$ replaced with

$$\|\varphi\|_{V^{m,p}(\Lambda)} = \big(\sum_{k=0}^{m} \int_{-1}^{1} |d^k\varphi|^p\,\rho^{(k-m)p}(\zeta)\,d\zeta\big)^{\frac{1}{p}}. \tag{3.23}$$

Next, for any positive real number $s = [s] + \sigma$, we define the space $V^{s,p}(\Lambda)$ by (3.4), with the norm $\|\cdot\|_{V^{m,p}(\mathcal{I})}$ replaced with

$$\|\varphi\|_{V^{s,p}(\Lambda)} = \Big(\int_{-1}^{1} \big(\sum_{k=0}^{[s]} |d^k\varphi|^p\,\rho^{(k-s)p}(\zeta)\big)\,d\zeta + \|q_{\sigma+\frac{1}{p}}[d^k\varphi]\|^p_{L^p(\Lambda\times\Lambda)}\Big)^{\frac{1}{p}}. \tag{3.24}$$

We also denote by $V^s(\Lambda)$ the space $V^{s,2}(\Lambda)$.

Theorem 3.6. *Let p be such that $1 < p < +\infty$ and s be a positive real number.*
(i) If $sp < 1$, then $\mathscr{D}(\Lambda)$ is dense in $W^{s,p}(\Lambda)$ and the spaces $W^{s,p}(\Lambda)$ and $V^{s,p}(\Lambda)$ coincide.
(ii) If $s - \frac{1}{p}$ is not an integer and K denotes the integral part of $s - \frac{1}{p}$, for each integer k, $0 \le k \le K$, the trace mappings $\gamma_k^{\pm}$ defined on $\mathscr{C}^\infty(\overline{\Lambda})$ by

$$\gamma_k^{\pm}(\varphi) = d^k\varphi(\pm 1)$$

are continuous on $W^{s,p}(\Lambda)$. Moreover, $V^{s,p}(\Lambda)$ is the closure of $\mathscr{D}(\Lambda)$ in $W^{s,p}(\Lambda)$ and the following characterization holds

$$V^{s,p}(\Lambda) = \big\{\varphi \in W^{s,p}(\Lambda);\ \forall k, 0 \le k \le K,\ \gamma_k^{\pm}(\varphi) = 0\big\}. \tag{3.25}$$

The space $W^{s,p}(\Lambda)$ is equal to the direct sum of the space $V^{s,p}(\Lambda)$ and of the space $\mathbb{P}_{2K+1}(\Lambda)$ of polynomials with degree $\le 2K+1$ on Λ.

We finally extract from [32] (without proof) the available results about the *limit cases* which are the situations where $s - \frac{1}{p}$ belongs to $\mathbb{N}$. The next statement involves the spaces $\overset{\circ}{W}{}^{s,p}(\Lambda)$, see the end of Section 2.1.

Corollary 3.7. *Let p be such that $1 < p < +\infty$ and s be a positive real number. If $sp \ge 1$ and if $s - \frac{1}{p}$ is an integer, let K denote $s - \frac{1}{p} - 1$; for each integer k, $0 \le k \le K$, the trace mappings $\gamma_k^{\pm}$ are continuous on $W^{s,p}(\Lambda)$. Moreover the following characterization holds*

$$\overset{\circ}{W}{}^{s,p}(\Lambda) = \big\{\varphi \in W^{s,p}(\Lambda);\ \forall k, 0 \le k \le K,\ \gamma_k^{+}(\varphi) = \gamma_k^{-}(\varphi) = 0\big\}.$$

As a conclusion, note that the space $V^{s,p}(\Lambda)$ and $\overset{\circ}{W}{}^{s,p}(\Lambda)$ coincide when $sp < 1$ (in this case they also coincide with $W^{s,p}(\Lambda)$) or when $s - \frac{1}{p}$ is not an integer. But they do not coincide when $s - \frac{1}{p}$ is an integer. This result is well-known when p is equal to 2: For instance, with the notation of [23, Chap. 1, §11], the space $\overset{\circ}{W}{}^{\frac{1}{2},2}(\Lambda)$ coincides with $H^{\frac{1}{2}}(\Lambda)$ while the space $V^{\frac{1}{2},2}(\Lambda)$ coincides with $H_{00}^{\frac{1}{2}}(\Lambda)$.

4. Traces for a polygon or a polyhedron.

The aim of this section is to establish the first half of the trace theorems, i.e. to prove that the trace operators act from Sobolev spaces on the polygon or the polyhedron into the product of appropriate Sobolev spaces on the edges of the polygon or the faces of the polyhedron. The second part of the trace theorems (see the next sections) consists in providing a full characterization of the range of the trace operator.

With each edge of a polygon, it is possible to associate a trapezium (in the British sense of a quadrilateral with two parallel edges) contained in the polygon such that this edge is contained in the boundary of the trapezium. Moreover, this trapezium can be mapped onto a rectangle by a very simple quadratic mapping that preserves the tangential coordinate on the edge, see Figure 4.1. So, without loss of generality, we now work on the rectangle $\Delta = \Lambda \times \mathcal{I}$ and we are interested in traces on the edge $\Gamma = \Lambda \times \{0\}$.

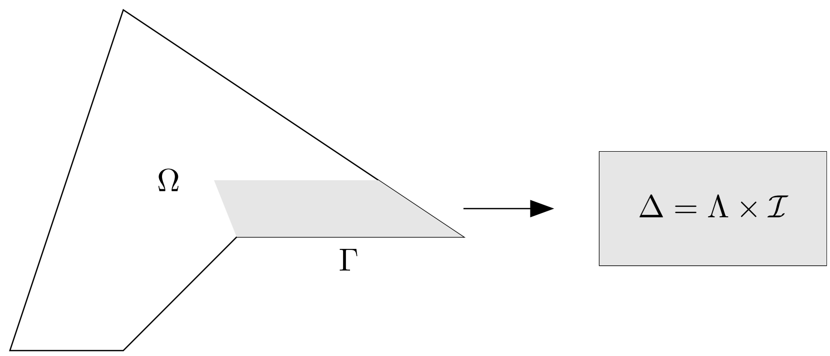

Figure 4.1

Let s be a nonnegative real number and p be such that $1 < p < +\infty$. As in Section 3, we set

$$K = \begin{cases} s - \frac{1}{p} - 1 & \text{if } s - \frac{1}{p} \text{ is an integer,} \\ [s - \frac{1}{p}] & \text{otherwise.} \end{cases} \tag{4.1}$$

4.1. Traces for the rectangle.

The idea is that Sobolev spaces on rectangles can be equivalently defined by tensorization, and these tensorization properties are the key arguments for proving the trace theorem. Using the definitions given in Section 2.3, we are in a position to state the following lemma. Note that the second property when s is an integer is known as the inequality of Aronszajn and Smith [2].

Lemma 4.1. *For any p such that $1 < p < +\infty$ and real nonnegative numbers s and t, $t \leq s$, the following embedding holds*

$$W^{s,p}(\Delta) \subset W^{t,p}(\mathcal{I}; W^{s-t,p}(\Lambda)), \tag{4.2}$$

and the following identity holds

$$W^{s,p}(\Delta) = W^{s,p}(\mathcal{I}; L^p(\Lambda)) \cap L^p(\mathcal{I}; W^{s,p}(\Lambda)). \tag{4.3}$$

Proof. Embedding (4.2) is obvious when both s and t are integers and it follows from [16, Thm 5.1] for $s = 1$. So it suffices to check it for $0 < s < 1$. In this case, we start from the imbeddings with $s = 0$ and $s = 1$:

$$L^p(\Delta) \subset L^p(\mathcal{I}; L^p(\Lambda)) \qquad \text{and} \qquad W^{1,p}(\Delta) \subset W^{t',p}(\mathcal{I}; W^{1-t',p}(\Lambda)),$$

valid for all $t' \leq 1$. Relying on Theorem 2.3 and [23, Chap. 1, Th. 13.1] (note that each $W^{s,p}(\mathcal{I})$, resp. $W^{s,p}(\Lambda)$, is the domain of an operator in $L^p(\mathcal{I})$, resp. $L^p(\Lambda)$, and that these operators act on different variables, hence commute), we derive

$$W^{s,p}(\Delta) \subset W^{st',p}(\mathcal{I}; W^{s-st',p}(\Lambda)),$$

where $t = st'$ runs on all nonnegative real numbers $\leq s$. This concludes the proof of (4.2).
The imbedding of $W^{s,p}(\Delta)$ into $W^{s,p}(\mathcal{I}; L^p(\Lambda)) \cap L^p(\mathcal{I}; W^{s,p}(\Lambda))$ follows from (4.2). The converse imbedding is obvious for $s = 0$ and $s = 1$, and is proven in [32, §4.2.4] for any positive integer s. Moreover, when s is not an integer, applying [23, Chap. 1, Th. 13.1] to interpolate between the imbeddings

$$L^p(\mathcal{I}; L^p(\Lambda)) \subset L^p(\Delta) \qquad \text{and} \qquad W^{m,p}(\mathcal{I}; L^p(\Lambda)) \cap L^p(\mathcal{I}; W^{m,p}(\Lambda)) \subset W^{m,p}(\Delta),$$

for any integer $m \geq s$, yields the desired result.

Thanks to Lemma 4.1, proving the trace theorem is now easy.

Theorem 4.2. *Let p be such that $1 < p < +\infty$ and s be a positive real number.*
(i) If $sp < 1$, then the space $\mathscr{D}(\overline{\Lambda}\times]0,1])$ of infinitely differentiable functions with a compact support in $\overline{\Lambda}\times]0,1]$ is dense in $W^{s,p}(\Delta)$.
(ii) With K defined in (4.1) and for each integer k, $0 \leq k \leq K$, the trace mapping γ_k defined on $\mathscr{C}^\infty(\overline{\Delta})$ by

$$\gamma_k(u)(x) = (\partial_y^k u)(x, 0)$$

is continuous from $W^{s,p}(\Delta)$ onto $W^{s-k-\frac{1}{p},p}(\Gamma)$. There exists a continuous lifting operator of the mapping $\gamma = (\gamma_0, \dots, \gamma_K)$ from $\prod_{k=0}^K W^{s-k-\frac{1}{p},p}(\Gamma)$ into $W^{s,p}(\Delta)$. Moreover, if $s - \frac{1}{p}$ is not an integer, the space $\mathscr{D}(\overline{\Lambda}\times]0,1])$ is dense in the kernel of γ.

Proof. When $sp < 1$, we observe that part (i) of Theorem 3.1 combined with (4.3) implies

$$\Big(\int_{-1}^1 \int_0^1 |v(x,y)|^p \, y^{-sp} \, dx \, dy\Big)^{\frac{1}{p}} \leq c \, \|v\|_{W^{s,p}(\Delta)}.$$

So the product of v by a function χ_n equal to 0 on $]0, \frac{1}{n}[$ and to 1 on $]\frac{1}{n}, 1[$ still belongs to $W^{s,p}(\Delta)$. Using a regularization of $\chi_n v$ by convolution and letting n tend to ∞ gives the desired density. On the other hand, when $sp \geq 1$, we first observe that, for $0 \leq k \leq K$, the existence of the trace operator γ_k on $W^{s,p}(\mathcal{I}; L^p(\Lambda))$

follows from (2.8). Moreover, for $0 \le k \le K-1$, from the imbedding of $W^{s,p}(\Delta)$ into $W^{k,p}(\mathcal{I}; W^{s-k,p}(\Lambda)) \cap W^{k+1,p}(\mathcal{I}; W^{s-k-1,p}(\Lambda))$, we observe that, for any v in $W^{s,p}(\Delta)$, $\partial_y^k v$ satisfies (2.9) with $X_0 = W^{s-k-1,p}(\Lambda)$, $X_1 = W^{s-k,p}(\Lambda)$ and $\theta = \frac{1}{p}$, so that, thanks to Theorem 2.3, the operator γ_k is continuous from $W^{s,p}(\Delta)$ onto $W^{s-k-\frac{1}{p},p}(\Lambda)$. We refer to [32, §2.9.3] for the extension to the case $k = K$.

Note that the arguments used for the proof of this theorem obviously extend to the case of the cylinder $\Omega \times \mathcal{I}$ and its face $\Omega \times \{0\}$, for any polygon Ω.

4.2. Traces for the polygon or the polyhedron.

Let us now consider the case of a general polygon Ω or polyhedron Π, as described in Section 1. From the previous theorem, a trace operator γ_k^ℓ, resp. γ_k^j, can be defined on each edge Γ_ℓ, for $0 \le \ell \le L$, of Ω, respectively on each face Ω_j, $1 \le j \le J$, of Π. This leads to the following corollary.

Corollary 4.3. *Let p be such that $1 < p < +\infty$ and s be a positive real number, and let K be defined in* (4.1).
(i) In the case of the polygon Ω, the global trace operator $(\gamma_0^\ell, \dots, \gamma_K^\ell)_{1 \le \ell \le L}$ is continuous from $W^{s,p}(\Omega)$ into $\prod_{\ell=1}^{L} \prod_{k=0}^{K} W^{s-k-\frac{1}{p},p}(\Gamma_\ell)$. Moreover, if $s - \frac{1}{p}$ is not an integer, the following characterization holds

$$\overset{\circ}{W}{}^{s,p}(\Omega) = \big\{ v \in W^{s,p}(\Omega);\ \forall \ell, 1 \le \ell \le L,\ \forall k, 0 \le k \le K, \gamma_k^\ell v = 0 \big\}. \tag{4.4}$$

(ii) In the case of the polyhedron Π, the global trace operator $(\gamma_0^j, \dots, \gamma_K^j)_{1 \le j \le J}$ is continuous from $W^{s,p}(\Pi)$ into $\prod_{j=1}^{J} \prod_{k=0}^{K} W^{s-k-\frac{1}{p},p}(\Omega_j)$. Moreover, if $s - \frac{1}{p}$ is not an integer, the following characterization holds

$$\overset{\circ}{W}{}^{s,p}(\Pi) = \big\{ v \in W^{s,p}(\Pi);\ \forall j, 1 \le j \le J,\ \forall k, 0 \le k \le K, \gamma_k^j v = 0 \big\}. \tag{4.5}$$

Of course, when $s - \frac{1}{p}$ is larger than $\frac{1}{p}$, traces on the edges Γ_ℓ of the polygon Ω have themselves traces at the endpoints $\boldsymbol{a}_i$ of the edges. The traces on a corner from two neighbouring edges must coincide and this explains why the global trace operator is not onto: Indeed, compatibility conditions must be satisfied at the corners. We describe them in Section 5. Similarly, on the polyhedron Π, compatibility conditions appear at edges and corners, as explained in Section 6.

4.3. A global trace theorem.

In contrast with higher order operators, the first trace operator (i.e. the operator γ_0) can be defined globally on the whole boundary $\partial\mathcal{O}$ for any domain $\mathcal{O}$ with a

Lipschitz–continuous boundary, and not only on regular parts of the boundary. We refer to [20] for an appropriate definition of the space $W^{t,p}(\partial\mathcal{O})$ by local charts and for the global trace theorem that we now state.

Theorem 4.4. *Let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary. Let p be such that $1 < p < +\infty$ and s be a positive real number, $\frac{1}{p} < s < 1+\frac{1}{p}$. The global trace operator γ_0 is continuous from $W^{s,p}(\mathcal{O})$ onto $W^{s-\frac{1}{p},p}(\partial\mathcal{O})$. There exists a continuous inverse of the operator γ_0 from $W^{s-\frac{1}{p}}(\partial\mathcal{O})$ into $W^{s,p}(\mathcal{O})$. Moreover, the following characterization holds*

$$\mathring{W}^{s,p}(\mathcal{O}) = \big\{v \in W^{s,p}(\mathcal{O});\ \gamma_0 v = 0\big\}. \tag{4.6}$$

Note that, for a polygon Ω or a polyhedron Π, the definition of the spaces $W^{s-\frac{1}{p},p}(\partial\Omega)$ and $W^{s-\frac{1}{p},p}(\partial\Pi)$, is obtained thanks to a partition of unity: These spaces inside the edges or faces are defined in Section 1, and elsewhere they are defined by using the simple transformation that maps the two edges of a sector into a straight line, the two faces of a dihedron or all the faces that share a conical point into a plane. It involves compatibility conditions of the traces at the corners and on the edges, as described for general values of m in the next sections.

5. Traces and compatibility conditions for a polygon.

We first identify the compatibility conditions that are satisfied by the traces at the corner $\boldsymbol{a}$ of the sector $\mathcal{S}$ introduced in Section 1 (and illustrated in Figure 5.1). In a second step, we give a full characterization of the range of the trace operator defined on a polygon.

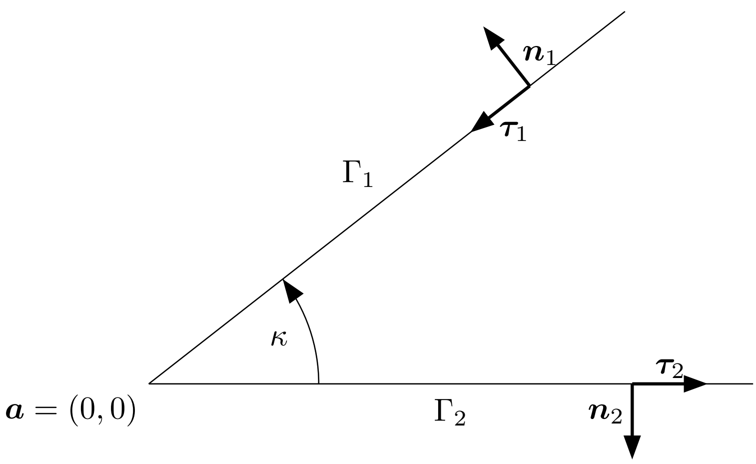

Figure 5.1

From the previous Theorem 4.2 applied on the two edges of the sector $\mathcal{S}$, for any integer m such that $0 \le m-1 < s - \frac{1}{p}$, the trace operator $\gamma^{(m)}$ reduced to the first m traces on Γ_ℓ:

$$\gamma^{(m)} = \Big(\gamma_0^1, \dots, \gamma_{m-1}^1, \gamma_0^2, \dots, \gamma_{m-1}^2\Big).$$

maps $W^{s,p}(\mathcal{S})$ into

$$\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S}) = \prod_{k=0}^{m-1} W^{s-k-\frac{1}{p},p}(\Gamma_1) \times \prod_{k=0}^{m-1} W^{s-k-\frac{1}{p},p}(\Gamma_2),$$

provided with the natural norm $\|\cdot\|_{\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S})}$. The aim of this section is to identify the subspace of $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S})$ which coincides with the range of $\gamma^{(m)}$. We consider successively the case where $s - \frac{2}{p}$ is not an integer, called non-limit case, and the case where it is an integer, which turns out to be a limit case.

5.1. Non-limit case for the sector.

We are going to check that the range of $\gamma^{(m)}$ is closed, has a finite codimension in $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S})$ and is characterized by the cancellation of a finite number of linear forms on $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S})$. These forms are given by linear combinations of derivatives evaluated at the vertex $\boldsymbol{a}$ that we take now equal to $(0,0)$.

The general principle on which the compatibility conditions rely follows from Corollary 3.5: It is sufficient to study the traces of homogeneous polynomials of degree n on $\mathcal{S}$, for $0 \le n \le [s - \frac{2}{p}]$. Let $\overline{\mathcal{P}}_n(\mathbb{R}^2)$ stand for this space. Note that the dual space $\mathcal{E}_n(\mathbb{R}^2)$ of $\overline{\mathcal{P}}_n(\mathbb{R}^2)$ is the space of partial differential operators with constant coefficients which are homogeneous of degree n, and that the duality pairing between $\mathcal{E}_n(\mathbb{R}^2)$ and $\overline{\mathcal{P}}_n(\mathbb{R}^2)$ is given by

$$\langle \mathcal{L}, v\rangle = (\mathcal{L}v)(\boldsymbol{a}).$$

Let us introduce the following subspaces of $\mathcal{E}_n(\mathbb{R}^2)$: For $1 \le \ell \le 2$ and for any positive integer m,

$$\mathcal{E}^\ell_{n,m}(\mathcal{S}) = \big\{\mathcal{L} \in \mathcal{E}_n(\mathbb{R}^2);\ \mathcal{L} = \sum_{k=0}^{\min\{n,m-1\}} c_k\, \partial^k_{n_\ell} \partial^{n-k}_{\tau_\ell}\big\}, \tag{5.1}$$

where, as standard, ∂_{τ_ℓ} and ∂_{n_ℓ} denote the tangential and normal derivatives on Γ_ℓ. Clearly, the image of $\overline{\mathcal{P}}_n(\mathbb{R}^2)$ by the trace operator $(\gamma_0^\ell, \dots, \gamma_{m-1}^\ell)$ is contained in

$\prod_{k=0}^{m-1} \overline{\mathcal{P}}_{n-k}(\Gamma_\ell)$. Its dual space coincides with $\prod_{k=0}^{m-1} \mathcal{E}_{n-k}(\Gamma_\ell)$, where each $\mathcal{E}_n(\Gamma_\ell)$ stands for the space spanned by the operator $\partial_{\tau_\ell}^n$, and the duality pairing is given by

$$\Big\langle (\mathcal{M}_k)_{0\le k\le m-1}, (\varphi_k)_{0\le k\le m-1} \Big\rangle = \sum_{k=0}^{m-1} (\mathcal{M}_k \varphi_k)(\boldsymbol{a}).$$

We now state an algebraic lemma which is trivial but very useful.

Lemma 5.1. *For $1 \le \ell \le 2$, the operator*

$$\pi_\ell : \quad \sum_{k=0}^{m-1} c_k\, \partial_{n_\ell}^k \partial_{\tau_\ell}^{n-k} \quad \mapsto \quad (c_k\, \partial_{\tau_\ell}^{n-k})_{0\le k\le m-1}, \tag{5.2}$$

is an isomorphism from $\mathcal{E}_{n,m}^\ell(\mathcal{S})$ onto $\prod_{k=0}^{m-1} \mathcal{E}_{n-k}(\Gamma_\ell)$.

We are now in a position to prove the main result of this section.

Theorem 5.2. *Let p be such that $1 < p < +\infty$ and s be a real number $> \frac{1}{p}$ such that $s - \frac{2}{p}$ is not an integer. Let m be an integer such that $0 \le m-1 < s - \frac{1}{p}$. Then, an element $G = (G^1, G^2)$ of $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S})$ is the image of an element v of $W^{s,p}(\mathcal{S})$ by the trace mapping $\gamma^{(m)}$ if and only if the following conditions hold for all n, $0 \le n < s - \frac{2}{p}$, and any pair $(\mathcal{L}_1, \mathcal{L}_2)$ in $\mathcal{E}_{n,m}^1(\mathcal{S}) \times \mathcal{E}_{n,m}^2(\mathcal{S})$ such that $\mathcal{L}_1 + \mathcal{L}_2 = 0$:*

$$([\pi_1 \mathcal{L}_1] G^1)(\boldsymbol{a}) + ([\pi_2 \mathcal{L}_2] G^2)(\boldsymbol{a}) = 0. \tag{5.3}$$

Corollary 5.3. *If the assumptions of Theorem 5.2 hold, the space $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\mathcal{S})$ of all functions in $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S})$ satisfying all conditions (5.3) prescribed in Theorem 5.2 is closed in $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S})$. The operator $\gamma^{(m)}$ is continuous from $W^{s,p}(\mathcal{S})$ onto $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\mathcal{S})$ and admits a continuous inverse from $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\mathcal{S})$ into $W^{s,p}(\mathcal{S})$.*

Proof. Thanks to Corollary 3.5 and with obvious extension of the notation, each m–tuple G^ℓ admits the expansion

$$G^\ell = G_0^\ell + P^\ell,$$

with G_0^ℓ in $\prod_{k=0}^{m-1} V^{s-k-\frac{1}{p},p}(\Gamma_\ell)$ and P^ℓ in $\prod_{k=0}^{m-1} \mathbb{P}^{[s-k-\frac{2}{p}]}(\Gamma_\ell)$. It follows from Theorem 3.1 that each G_0^ℓ is the limit of a sequence in $\mathscr{D}(\Gamma_\ell^*)^m$, where Γ_ℓ^* here means $\overline{\Gamma}_\ell \setminus \{\boldsymbol{a}\}$, so that a lifting of the pair $(G_0^1, 0)$ and $(0, G_0^2)$ is easy to construct. So, it remains to determine, for $0 \le n \le [s - \frac{2}{p}]$, the range of $\gamma^{(m)}$ from $\overline{\mathcal{P}}_n(\mathbb{R}^2)$ into $\prod_{\ell=1}^2 \prod_{k=0}^{m-1} \overline{\mathcal{P}}_{n-k}(\Gamma_\ell)$. This range is the orthogonal of the kernel of the dual operator

$\gamma^{(m)*}$ from $\prod_{\ell=1}^{2}\prod_{k=0}^{m-1}\mathcal{E}_{n-k}(\Gamma_\ell)$ into $\mathcal{E}_n(\mathbb{R}^2)$. Note that, for any $(\mathcal{M}^1,\mathcal{M}^2)$, with each $\mathcal{M}^\ell=(\mathcal{M}^\ell_k)_{0\le k\le m-1}$ in $\prod_{k=0}^{m-1}\mathcal{E}_{n-k}(\Gamma_\ell)$, we have

$$\begin{aligned}\langle\gamma^{(m)*}(\mathcal{M}^1,\mathcal{M}^2),v\rangle &= \langle(\mathcal{M}^1,\mathcal{M}^2),\gamma^{(m)}v\rangle\\ &= \langle\mathcal{M}^1,\gamma^1_m v\rangle+\langle\mathcal{M}^2,\gamma^2_m v\rangle=\langle\pi_1^{-1}\mathcal{M}^1,v\rangle+\langle\pi_2^{-1}\mathcal{M}^2,v\rangle,\end{aligned}$$

which gives

$$\gamma^{(m)*}=\pi_1^{-1}\mathcal{M}^1+\pi_2^{-1}\mathcal{M}^2.$$

Then, we derive that (G_1,G_2) is orthogonal to the kernel of $\gamma^{(m)*}$ if and only if $\langle\mathcal{M}^1,G^1\rangle+\langle\mathcal{M}^2,G^2\rangle$ vanishes for all pairs $(\mathcal{M}^1,\mathcal{M}^2)$ such that $\pi_1^{-1}\mathcal{M}^1+\pi_2^{-1}\mathcal{M}^2=0$. This yields the theorem.

In spite of this rather abstract proof, the compatibility conditions in (5.3) are fully explicit. Their number is easy to compute from the next lemma.

Lemma 5.4. *For any positive integer m and any nonnegative integer n, the dimension of $\mathcal{E}^1_{n,m}(\mathcal{S})\cap\mathcal{E}^2_{n,m}(\mathcal{S})$ is equal to*

$$\begin{cases} n+1 & \text{if } n\le m-1,\\ 2m-1-n & \text{if } m\le n\le 2(m-1),\\ 0 & \text{if } n\ge 2m-1.\end{cases}\tag{5.4}$$

Proof. An operator $\mathcal{L}$ in $\mathcal{E}^2_{n,m}(\mathcal{S})$ can be written as

$$\mathcal{L}=\sum_{k=0}^{\min\{n,m-1\}} c_k\,\partial^k_{n_2}\partial^{n-k}_{\tau_2}.$$

Next, we observe that $\boldsymbol{n}_2$ and $\boldsymbol{\tau}_2$ are linear combinations of $\boldsymbol{n}_1$ and $\boldsymbol{\tau}_1$:

$$\boldsymbol{n}_2=-\cos\kappa\,\boldsymbol{n}_1+\sin\kappa\,\boldsymbol{\tau}_1,\qquad \boldsymbol{\tau}_2=-\sin\kappa\,\boldsymbol{n}_1-\cos\kappa\,\boldsymbol{\tau}_1.$$

So, we have

$$\mathcal{L}=\sum_{k=0}^{\min\{n,m-1\}} c_k\,(-\cos\kappa\,\partial_{n_1}+\sin\kappa\,\partial_{\tau_1})^k(-\sin\kappa\,\partial_{n_1}-\cos\kappa\,\partial_{\tau_1})^{n-k}.$$

Thus it appears that all these operators $\mathcal{L}$ belong to $\mathcal{E}^1_{n,m}(\mathcal{S})$ if $n\le m-1$ while, when $n\ge m$, only those where the sum on k reduces to $n-(m-1)\le k\le m-1$ belong to $\mathcal{E}^1_{n,m}(\mathcal{S})$. This leads to the desired result.

Note that the intersection of $\mathcal{E}^1_{n,m}(\mathcal{S})$ and $\mathcal{E}^2_{n,m}(\mathcal{S})$ is reduced to $\{0\}$ when $n>2(m-1)$. So, the maximal number of compatibility conditions is bounded as a function of m. The total number of conditions in (5.3) is explicitly given by the formula

$$\sum_{n=0}^{\min\{m-1,s-\frac{2}{p}\}} n+1 \quad + \sum_{n=m}^{\min\{2(m-1),s-\frac{2}{p}\}} 2m-1-n. \tag{5.5}$$

It varies from $\frac{m(m+1)}{2}$ when s belongs to $]m-1+\frac{2}{p}, m+\frac{2}{p}[$, to m^2 when $s > 2(m-1)+\frac{2}{p}$.

In the following Tables 5.1 to 5.3, we write the compatibility conditions for $m = 1$, $m = 2$ and $m = 3$. Here, we take each G^ℓ equal to the m-tuple $(g_0^\ell, \dots, g_{m-1}^\ell)$ and denote by $g_k^{\ell\prime}, g_k^{\ell\prime\prime}, \dots$ the successive derivatives of g_k^ℓ with respect to the tangential coordinates associated with $\boldsymbol{\tau}_\ell$. For simplicity, we set: $c = \cos\kappa$, $s = \sin\kappa$.

$m = 1$	$n = 0$	1 condition $$g_0^2(\boldsymbol{a}) = g_0^1(\boldsymbol{a})$$

Table 5.1

$m = 2$	$n = 0$	1 condition $$g_0^2(\boldsymbol{a}) = g_0^1(\boldsymbol{a})$$
	$n = 1$	2 conditions $$g_0^{2\prime}(\boldsymbol{a}) = -c\, g_0^{1\prime}(\boldsymbol{a}) - s\, g_1^1(\boldsymbol{a})$$ $$g_1^2(\boldsymbol{a}) = s\, g_0^{1\prime}(\boldsymbol{a}) - c\, g_1^1(\boldsymbol{a})$$
	$n = 2$	1 condition $$c\, g_0^{2\prime\prime}(\boldsymbol{a}) - s\, g_1^{2\prime}(\boldsymbol{a}) = c\, g_0^{1\prime\prime}(\boldsymbol{a}) + s\, g_1^{1\prime}(\boldsymbol{a})$$

Table 5.2

$m\ =\ 3$	$n=0$	1 condition $$g_0^2(\boldsymbol{a}) = g_0^1(\boldsymbol{a})$$
	$n=1$	2 conditions $$g_0^{2\prime}(\boldsymbol{a}) = -c\,g_0^{1\prime}(\boldsymbol{a}) - s\,g_1^1(\boldsymbol{a})$$ $$g_1^2(\boldsymbol{a}) = s\,g_0^{1\prime}(\boldsymbol{a}) - c\,g_1^1(\boldsymbol{a})$$
	$n=2$	3 conditions $$g_0^{2\prime\prime}(\boldsymbol{a}) = c^2\,g_0^{1\prime\prime}(\boldsymbol{a}) + 2sc\,g_1^{1\prime}(\boldsymbol{a}) + s^2\,g_2^1(\boldsymbol{a})$$ $$g_1^{2\prime}(\boldsymbol{a}) = -sc\,g_0^{1\prime\prime}(\boldsymbol{a}) + (c^2-s^2)g_1^{1\prime}(\boldsymbol{a}) + sc\,g_2^1(\boldsymbol{a})$$ $$g_2^2(\boldsymbol{a}) = s^2\,g_0^{1\prime\prime}(\boldsymbol{a}) - 2sc\,g_1^{1\prime}(\boldsymbol{a}) + c^2\,g_2^1(\boldsymbol{a})$$
	$n=3$	2 conditions $$c\,g_0^{2\prime\prime\prime}(\boldsymbol{a}) - s\,g_1^{2\prime\prime}(\boldsymbol{a}) = -c^2\,g_0^{1\prime\prime\prime}(\boldsymbol{a}) - 2sc\,g_1^{1\prime\prime}(\boldsymbol{a}) - s^2\,g_2^{1\prime}(\boldsymbol{a})$$ $$c^2\,g_0^{2\prime\prime\prime}(\boldsymbol{a}) - 2sc\,g_1^{2\prime\prime}(\boldsymbol{a}) + s^2\,g_2^{2\prime}(\boldsymbol{a}) = -c\,g_0^{1\prime\prime\prime}(\boldsymbol{a}) - s\,g_1^{1\prime\prime}(\boldsymbol{a})$$
	$n=4$	1 condition $$c^2g_0^{2\prime\prime\prime\prime}(\boldsymbol{a}) - 2scg_1^{2\prime\prime\prime}(\boldsymbol{a}) + s^2g_2^{2\prime\prime}(\boldsymbol{a}) = c^2g_0^{1\prime\prime\prime\prime}(\boldsymbol{a}) + 2scg_1^{1\prime\prime\prime}(\boldsymbol{a}) + s^2g_2^{1\prime\prime}(\boldsymbol{a})$$

Table 5.3

In the special case where κ is equal to $\frac{\pi}{2}$, these conditions are much simpler, since $\boldsymbol{\tau}_1$ coincides with $\boldsymbol{n}_2$ and $\boldsymbol{\tau}_2$ with $-\boldsymbol{n}_1$. So we present them in Tables 5.4 and 5.5 respectively for $m = 2$ and $m = 3$ (the condition for $m = 1$ is the same as in Table 5.1).

$m\ =\ 2$	$n=0$	1 condition $$g_0^2(\boldsymbol{a}) = g_0^1(\boldsymbol{a})$$
	$n=1$	2 conditions $$g_0^{2\prime}(\boldsymbol{a}) = -g_1^1(\boldsymbol{a})$$ $$g_1^2(\boldsymbol{a}) = g_0^{1\prime}(\boldsymbol{a})$$
	$n=2$	1 condition $$-g_1^{2\prime}(\boldsymbol{a}) = g_1^{1\prime}(\boldsymbol{a})$$

Table 5.4

$m = 3$	$n = 0$	1 condition	$g_0^2(\boldsymbol{a}) = g_0^1(\boldsymbol{a})$
	$n = 1$	2 conditions	$g_0^{2\prime}(\boldsymbol{a}) = -g_1^1(\boldsymbol{a})$ $g_1^2(\boldsymbol{a}) = g_0^{1\prime}(\boldsymbol{a})$
	$n = 2$	3 conditions	$g_0^{2\prime\prime}(\boldsymbol{a}) = g_2^1(\boldsymbol{a})$ $g_1^{2\prime}(\boldsymbol{a}) = -g_1^{1\prime}(\boldsymbol{a})$ $g_2^2(\boldsymbol{a}) = g_0^{1\prime\prime}(\boldsymbol{a})$
	$n = 3$	2 conditions	$-g_1^{2\prime\prime}(\boldsymbol{a}) = -g_2^{1\prime}(\boldsymbol{a})$ $g_2^{2\prime}(\boldsymbol{a}) = -g_1^{1\prime\prime}(\boldsymbol{a})$
	$n = 4$	1 condition	$g_2^{2\prime\prime}(\boldsymbol{a}) = g_2^{1\prime\prime}(\boldsymbol{a})$

Table 5.5

Remark 5.1. In the case where only m_ℓ traces out of the first m ones are given on each Γ^ℓ in the appropriate spaces $W^{s-k-\frac{1}{p},p}(\Gamma_\ell)$, there exists a continuous lifting of these traces with values in $W^{s,p}(\mathcal{S})$ if, among conditions (5.3), those which only involve the given traces are satisfied (the idea is to replace the missing traces with polynomials such that the other conditions are satisfied). However, the number of conditions in this case can vary according as κ is equal to $\frac{\pi}{2}$ or not. For instance, in the case $m = 2$, a continuous lifting of the traces (g_0^1, g_1^2) in $W^{s-\frac{1}{p},p}(\Gamma_1) \times W^{s-1-\frac{1}{p},p}(\Gamma_2)$ for $s > 2 + \frac{2}{p}$ exists without any condition for $\kappa \neq \frac{\pi}{2}$, with one condition for $\kappa = \frac{\pi}{2}$.

Remark 5.2. Let us for a while consider the case of a sector with a crack ($\kappa = 2\pi$), i.e. the edges Γ_1 and Γ_2 coincide. The boundary $\partial\mathcal{S}$ is no longer Lipschitz–continuous, however Theorem 5.2 is still valid in this case but the spaces $\mathcal{E}_{n,m}^1(\mathcal{S})$ and $\mathcal{E}_{n,m}^2(\mathcal{S})$ coincide, so that the number of compatibility conditions is no longer bounded as a function of m.

5.2. Limit case for the sector.

The limit case occurs when $s - \frac{2}{p}$ is an integer n, together with the condition

$$\mathcal{E}_{n,m}^1(\mathcal{S}) \cap \mathcal{E}_{n,m}^2(\mathcal{S}) \neq \{0\}.$$

We know by Lemma 5.4 that this intersection is not $\{0\}$ only if $n \leq 2m-2$. If we are in this situation, the operators $\pi_1\mathcal{L}_1$ and $\pi_2\mathcal{L}_2$ which appear in (5.3), when applied to the traces in $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S})$, are no longer defined in $\boldsymbol{a}$. The corresponding compatibility conditions in $\boldsymbol{a}$ must be replaced with integral ones. This is stated in the following theorem.

Theorem 5.5. *Let p be such that $1 < p < +\infty$ and s be a real number $\geq \frac{1}{p}$. Let m be an integer such that $0 \leq m-1 < s-\frac{1}{p}$. We assume that $s-\frac{2}{p}$ is an integer. Then, an element $G = (G^1, G^2)$ of $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S})$ is the image of an element v of $W^{s,p}(\mathcal{S})$ by the trace mapping $\gamma^{(m)}$ if and only if properties (i) and (ii) are satisfied:*
(i) For all n, $0 \leq n < s-\frac{2}{p}$, and any pair $(\mathcal{L}_1, \mathcal{L}_2)$ in $\mathcal{E}^1_{n,m}(\mathcal{S}) \times \mathcal{E}^2_{n,m}(\mathcal{S})$ such that $\mathcal{L}_1 + \mathcal{L}_2 = 0$, condition (5.3) holds.
(ii) For $n = s-\frac{2}{p}$, and any pair $(\mathcal{L}_1, \mathcal{L}_2)$ in $\mathcal{E}^1_{n,m}(\mathcal{S}) \times \mathcal{E}^2_{n,m}(\mathcal{S})$ such that $\mathcal{L}_1 + \mathcal{L}_2 = 0$, the following condition holds:

$$\int_0^1 \left|([\pi_1\mathcal{L}_1]G^1)(\boldsymbol{a} - t\,\boldsymbol{\tau}_1) + ([\pi_2\mathcal{L}_2]G^2)(\boldsymbol{a} + t\,\boldsymbol{\tau}_2)\right|^p \frac{dt}{t} < +\infty. \tag{5.6}$$

If $s-\frac{2}{p} \geq 2m-1$ condition (ii) is void.

Corollary 5.6. *If $s-\frac{2}{p}$ belongs to $\{m-1, m, \dots, 2m-2\}$, the space $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\mathcal{S})$ of all functions in $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S})$ satisfying all conditions (5.3) and (5.6) prescribed in Theorem 5.5 is not closed in $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S})$. When this space is provided with the norm, – with $n = s-\frac{2}{p}$,*

$$\left(\|G\|^p_{\mathbb{W}^{s,p\,(m)}(\partial\mathcal{S})} + \sum_{\substack{\mathcal{L}_1+\mathcal{L}_2=0\\ \mathcal{L}_i\in\mathcal{E}^i_{n,m}(\mathcal{S})}} \int_0^1 \left|([\pi_1\mathcal{L}_1]G^1)(\boldsymbol{a}-t\,\boldsymbol{\tau}_1)+([\pi_2\mathcal{L}_2]G^2)(\boldsymbol{a}+t\,\boldsymbol{\tau}_2)\right|^p \frac{dt}{t}\right)^{\frac{1}{p}},$$

the operator $\gamma^{(m)}$ is continuous from $W^{s,p}(\mathcal{S})$ onto $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\mathcal{S})$ and admits a continuous inverse from $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\mathcal{S})$ into $W^{s,p}(\mathcal{S})$.

Remark 5.3. For the lifting of a single trace ($m = 1$), the limit case corresponds to a single value of s, namely $s = \frac{2}{p}$. Nevertheless, when $p = 2$, we find the very important case of integral trace compatibility conditions for the space $H^1(\mathcal{S})$.

Proof of Theorem 5.5. We restrict the proof to the case $s = \frac{2}{p}$, and $m = 1$, since the general argument is similar. Let $G = (g^1_0, g^2_0)$ be any pair in $\mathbb{W}^{\frac{2}{p},p(1)}(\partial\mathcal{S})$. Let us consider the mapping: $g^1 \mapsto \overline{g}^1$, where $\overline{g}^1$ denotes the extension of g^1 to $\overline{\Gamma}_1 \cup \overline{\Gamma}_2$ by parity: More precisely, $\overline{g}^1$ is equal to g^1 on Γ_1 and is defined on Γ_2 by

$$\overline{g}^1(\boldsymbol{a} + t\,\boldsymbol{\tau}_2) = g^1(\boldsymbol{a} - t\,\boldsymbol{\tau}_1), \qquad 0 \leq t \leq 1.$$

This mapping is continuous from $L^p(\Gamma_1)$ into $L^p(\overline{\Gamma}_1 \cup \overline{\Gamma}_2)$ and from $W^{1,p}(\Gamma_1)$ into $W^{1,p}(\overline{\Gamma}_1 \cup \overline{\Gamma}_2)$, hence from $W^{\frac{1}{p},p}(\Gamma_1)$ into $W^{\frac{1}{p},p}(\overline{\Gamma}_1 \cup \overline{\Gamma}_2)$ by interpolation. The idea is to write (with obvious notation) the pair G as

$$(g_0^1, g_0^2) = (\overline{g}_{0\,|\Gamma_1}^1, \overline{g}_{0\,|\Gamma_2}^1) + (0, g_0^2 - \overline{g}_{0\,|\Gamma_2}^1).$$

Indeed, from Theorem 4.4, since $\overline{g}_0^1$ belongs to $W^{\frac{1}{p},p}(\overline{\Gamma}_1 \cup \overline{\Gamma}_2)$, there exists a lifting of the pair $(\overline{g}_{0\,|\Gamma_1}^1, \overline{g}_{0\,|\Gamma_2}^1)$ into $W^{\frac{2}{p},p}(\mathcal{S})$. Moreover, thanks to condition (5.6), the function $g^2 - \overline{g}_{0\,|\Gamma_2}^1$ belongs to $V^{\frac{1}{p},p}(\Gamma_2)$, so that applying once more Theorem 4.4 yields the existence of a lifting of $(0, g_0^2 - \overline{g}_{0\,|\Gamma_2}^1)$ into $W^{\frac{2}{p},p}(\mathcal{S})$. This ends the proof.

5.3. Trace theorem for a polygon.

The construction of a lifting of traces on all edges of a polygon Ω is performed by using an appropriate partition of unity, the support of each function of the partition being a neighbourhood of a corner, indeed the compatibility conditions at each corner are the same as in (5.3) and (5.6). So we only state them in this general case for the sake of completeness.

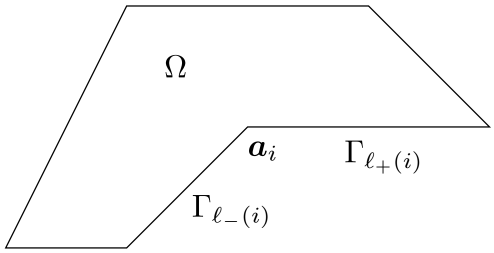

Figure 5.2

For each corner $\boldsymbol{a}_i$, $1 \le i \le I$, we agree to denote by $\Gamma_{\ell_+(i)}$ and $\Gamma_{\ell_-(i)}$ the two edges that contain $\boldsymbol{a}_i$, see Figure 5.2. We also denote by h_i the smallest of the lengths of $\Gamma_{\ell_-(i)}$ and $\Gamma_{\ell_+(i)}$. The space $\mathcal{E}_{n,m}^{\ell}(\Omega)$ and the operator π_ℓ are still defined by (5.1) and (5.2) respectively (with $\mathcal{S}$ replaced with Ω), but now for $1 \le \ell \le L$. We also introduce the space

$$\mathbb{W}^{s,p\,(m)}(\partial\Omega) = \prod_{\ell=1}^{L} \Big(\prod_{k=0}^{m-1} W^{s-k-\frac{1}{p},p}(\Gamma_\ell) \Big).$$

The trace operator $\gamma^{(m)}$ is obviously defined by

$$\gamma^{(m)} = (\gamma_0^\ell, \cdots, \gamma_{m-1}^\ell)_{1 \le \ell \le L}.$$

Theorem 5.7. *Let p be such that $1 < p < +\infty$ and s be a real number $\ge \frac{1}{p}$. Let m be an integer such that $0 \le m-1 < s - \frac{1}{p}$. Then, an element $G = (G^\ell)_{1 \le \ell \le L}$ of*

$\mathbb{W}^{s,p\,(m)}(\partial\Omega)$ *is the image of an element* v *of* $W^{s,p}(\Omega)$ *by the trace mapping* $\gamma^{(m)}$ *if and only if properties (i) and (ii) are satisfied:*
(i) For $1 \le i \le I$*, the following conditions hold for all* n*,* $0 \le n < s - \frac{2}{p}$ *, and any pair* $(\mathcal{L}_-, \mathcal{L}_+)$ *in* $\mathcal{E}_{n,m}^{\ell_-(i)}(\Omega) \times \mathcal{E}_{n,m}^{\ell_+(i)}(\Omega)$ *such that* $\mathcal{L}_- + \mathcal{L}_+ = 0$*:*

$$([\pi_{\ell_-}\mathcal{L}_-]G^{\ell_-(i)})(\boldsymbol{a}_i) + ([\pi_{\ell_+}\mathcal{L}_+]G^{\ell_+(i)})(\boldsymbol{a}_i) = 0, \tag{5.7}$$

(ii) When $s - \frac{2}{p}$ *is an integer, the following conditions hold for* $n = s - \frac{2}{p}$ *, and any pair* $(\mathcal{L}_-, \mathcal{L}_+)$ *in* $\mathcal{E}_{n,m}^{\ell_-(i)}(\Omega) \times \mathcal{E}_{n,m}^{\ell_+(i)}(\Omega)$ *such that* $\mathcal{L}_- + \mathcal{L}_+ = 0$*:*

$$\int_0^{h_i} |([\pi_{\ell_-}\mathcal{L}_-]G^{\ell_-(i)})(\boldsymbol{a}_i - t\,\boldsymbol{\tau}_{\ell_-(i)}) + ([\pi_{\ell_+}\mathcal{L}_+]G^{\ell_+(i)})(\boldsymbol{a}_i + t\,\boldsymbol{\tau}_{\ell_+(i)})|^p \,\frac{dt}{t} < +\infty. \tag{5.8}$$

Corollary 5.8. *If the assumptions of Theorem 5.7 hold, the space* $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\Omega)$ *of all functions in* $\mathbb{W}^{s,p\,(m)}(\partial\Omega)$ *satisfying all conditions* (5.7) *and* (5.8) *prescribed in Theorem 5.7, is closed in* $\mathbb{W}^{s,p\,(m)}(\partial\Omega)$ *if and only if* $s - \frac{2}{p} \notin \{m-1, m, \dots, 2m-2\}$*. Let this space be provided with the norm of* $\mathbb{W}^{s,p\,(m)}(\partial\Omega)$*, augmented with the left-hand sides in* (5.8) *when* $s - \frac{2}{p} \in \{m-1, m, \dots, 2m-2\}$*. Then the operator* $\gamma^{(m)}$ *is continuous from* $W^{s,p}(\Omega)$ *onto* $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\Omega)$ *and admits a continuous inverse from* $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\Omega)$ *into* $W^{s,p}(\Omega)$*.*

Since in the following chapters we are mainly interested in the application of these results for spectral methods, we now give the specific statement in the case of the square Θ. We recall that each vertex $\boldsymbol{a}_i$, $1 \le i \le 4$, is the intersection of the two edges Γ_i et Γ_{i+1} (with the convention $\Gamma_5 = \Gamma_1$). With this notation, it is clear that $\boldsymbol{n}_i$ is equal to $-\boldsymbol{\tau}_{i+1}$ and that $\boldsymbol{n}_{i+1}$ is equal to $\boldsymbol{\tau}_i$.

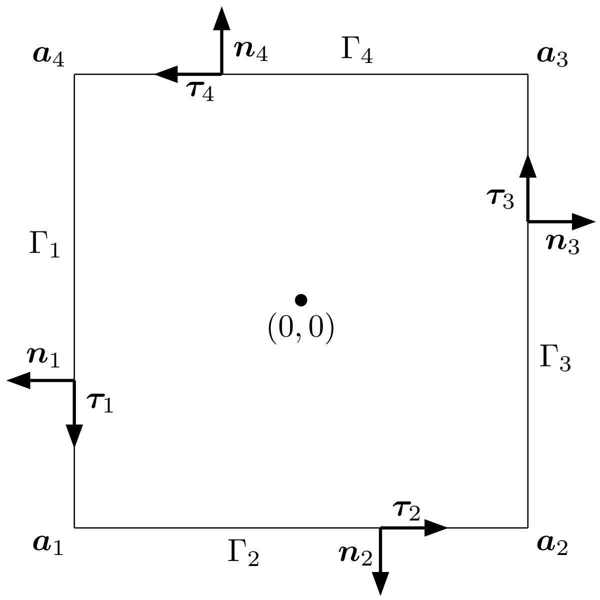

Figure 5.3

Corollary 5.9. *Let p be such that $1 < p < +\infty$ and s be a real number $> \frac{1}{p}$. Let m be an integer such that $0 \le m-1 < s-\frac{1}{p}$. Then, a 4-tuple $(G^1,\dots,G^4)$, with $G^j = (g_0^j,\cdots,g_{m-1}^j)$, in $\mathbb{W}^{s,p\,(m)}(\partial\Theta)$ is the image of a function in $W^{s,p}(\Theta)$ by the trace mapping $\gamma^{(m)}$ if and only if properties (i) and (ii) are satisfied:*
(i) For all i, $1 \le i \le 4$, and for all integers n, $0 \le n \le 2m-2$ with $n < s - \frac{2}{p}$,

$$(\partial_{\tau_i}^{n-k} g_k^i)(\boldsymbol{a}_i) = (-1)^k\,(\partial_{\tau_{i+1}}^{k} g_{n-k}^{i+1})(\boldsymbol{a}_i), \qquad 0 \le k,\ n-k \le m-1, \tag{5.9}$$

(ii) When $s-\frac{2}{p} \in \{m-1, m, \dots, 2m-2\}$: For all i, $1 \le i \le 4$, and with $n = s-\frac{2}{p}$,

$$\int_0^1 \left|(\partial_{\tau_i}^{n-k} g_k^i)(\boldsymbol{a}_i - t\,\boldsymbol{\tau}_i) - (-1)^k\,(\partial_{\tau_{i+1}}^{k} g_{n-k}^{i+1})(\boldsymbol{a}_i + t\,\boldsymbol{\tau}_{i+1})\right|^p \frac{dt}{t} < +\infty, \qquad 0 \le k,\ n-k \le m-1. \tag{5.10}$$

6. Traces and compatibility conditions for a polyhedron.

We work in the polyhedral cone $\mathcal{C}$ with vertex $\boldsymbol{a}$ introduced in Section 1, we assume that it has a Lipschitz–continuous boundary. We begin with the compatibility conditions on the edges. Next, since compatibility conditions at the corners are more complex, we first describe them and prove the result in the simpler case $m = 1$ of one trace per face. Next we state them in the general case, with only a sketch of proof. We conclude this chapter by giving a full characterization of the range of the trace operator defined on a polyhedron.

6.1. Compatibility conditions on the edges for the cone.

Here, we denote by Γ_ℓ, $1 \le \ell \le L$, the edges of $\mathcal{C}$. Let also $\boldsymbol{\sigma}_\ell$ stand for the unit tangential vector to Γ_ℓ, pointing towards $\boldsymbol{a}$. Each Γ_ℓ intersect two faces $\overline{\Omega}_j$, which we denote $\overline{\Omega}_{j_-(\ell)}$ and $\overline{\Omega}_{j_+(\ell)}$. With these two faces, we associate the two orthogonal bases $\{\boldsymbol{n}_{\ell_-}, \boldsymbol{\sigma}_\ell, \boldsymbol{\tau}_{\ell_-}\}$ and $\{\boldsymbol{n}_{\ell_+}, \boldsymbol{\sigma}_\ell, \boldsymbol{\tau}_{\ell_+}\}$, where $\boldsymbol{n}_{\ell_\pm}$ stands for the unit outward normal vector to $\Omega_{j_\pm(\ell)}$ and $\boldsymbol{\tau}_{\ell_\pm}$ for a unit tangential vector to $\Omega_{j_\pm(\ell)}$ which is orthogonal to $\boldsymbol{\sigma}_\ell$.

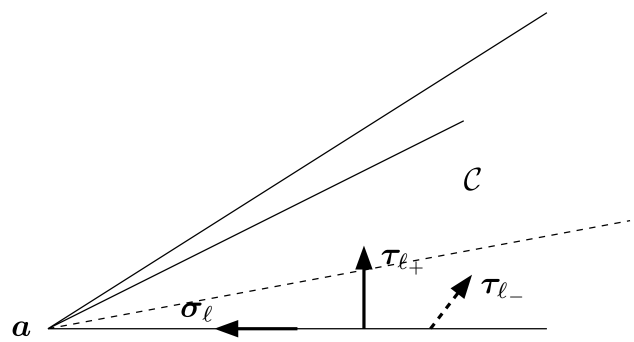

Figure 6.1

Indeed, note that in a neighbourhood of the edge Γ_ℓ, Ω coincides with the tensorized product of Γ_ℓ by the sector $\mathcal{S}_\ell$ with edges parallel to $\boldsymbol{\tau}_{\ell_-}$ and $\boldsymbol{\tau}_{\ell_+}$. So the compatibility conditions on the edge Γ_ℓ are directly derived from Theorems 5.2 and 5.5 by a tensorization argument. Let $\mathcal{E}_n(\mathbb{R}^3)$ be the space of partial differential operators on $\mathbb{R}^3$ homogeneous of degree n with constant coefficients, and let $\mathcal{E}^{\ell-}_{n,m}(\mathcal{C}, \boldsymbol{\sigma}_\ell)$ and $\mathcal{E}^{\ell+}_{n,m}(\mathcal{C}, \boldsymbol{\sigma}_\ell)$ be the spaces

$$\begin{aligned}
\mathcal{E}^{\ell-}_{n,m}(\mathcal{C}, \boldsymbol{\sigma}_\ell) &= \big\{\mathcal{L} \in \mathcal{E}_n(\mathbb{R}^3);\ \mathcal{L} = \sum_{k=0}^{\min\{n,m-1\}} c_k\, \partial^k_{n_{\ell_-}}\, \partial^{n-k}_{\tau_{\ell_-}}\big\},\\
\mathcal{E}^{\ell+}_{n,m}(\mathcal{C}, \boldsymbol{\sigma}_\ell) &= \big\{\mathcal{L} \in \mathcal{E}_n(\mathbb{R}^3);\ \mathcal{L} = \sum_{k=0}^{\min\{n,m-1\}} c_k\, \partial^k_{n_{\ell_+}}\, \partial^{n-k}_{\tau_{\ell_+}}\big\}.
\end{aligned} \tag{6.1}$$

We also introduce the operators $\pi_{\ell_\pm}$ defined respectively by

$$\begin{aligned}
\pi_{\ell_-} :&\quad \sum_{k=0}^{\min\{n,m-1\}} c_k\, \partial^k_{n_{\ell_-}}\, \partial^{n-k}_{\tau_{\ell_-}} \quad\mapsto\quad (c_k\, \partial^{n-k}_{\tau_{\ell_-}})_{0\le k\le m-1},\\
\pi_{\ell_+} :&\quad \sum_{k=0}^{\min\{n,m-1\}} c_k\, \partial^k_{n_{\ell_+}}\, \partial^{n-k}_{\tau_{\ell_+}} \quad\mapsto\quad (c_k\, \partial^{n-k}_{\tau_{\ell_+}})_{0\le k\le m-1}.
\end{aligned} \tag{6.2}$$

So the compatibility conditions on the edge Γ_ℓ, $1 \le \ell \le L$, and for any pair $(G^{j_-(\ell)}, G^{j_+(\ell)})$ in

$$\prod_{k=0}^{m-1} W^{s-k-\frac{1}{p},p}(\Omega_{j_-(\ell)}) \times \prod_{k=0}^{m-1} W^{s-k-\frac{1}{p},p}(\Omega_{j_+(\ell)}),$$

now read:

- for all n, $0 \le n < s - \frac{2}{p}$, and any pair $(\mathcal{L}_-, \mathcal{L}_+)$ in $\mathcal{E}^{\ell-}_{n,m}(\mathcal{C}, \boldsymbol{\sigma}_\ell) \times \mathcal{E}^{\ell+}_{n,m}(\mathcal{C}, \boldsymbol{\sigma}_\ell)$ such that $\mathcal{L}_- + \mathcal{L}_+ = 0$:

$$([\pi_{\ell_-}\mathcal{L}_-]G^{j_-(\ell)})(\boldsymbol{x}) + ([\pi_{\ell_+}\mathcal{L}_+]G^{j_+(\ell)})(\boldsymbol{x}) = 0 \quad \text{for a.e. } \boldsymbol{x} \text{ in } \Gamma_\ell, \tag{6.3}$$

- and if, moreover, when $s - \frac{2}{p}$ is an integer, the following conditions hold for $n = s - \frac{2}{p}$, and any pair $(\mathcal{L}_-, \mathcal{L}_+)$ in $\mathcal{E}^{\ell-}_{n,m}(\mathcal{C}, \boldsymbol{\sigma}_\ell) \times \mathcal{E}^{\ell+}_{n,m}(\mathcal{C}, \boldsymbol{\sigma}_\ell)$ such that $\mathcal{L}_- + \mathcal{L}_+ = 0$:

$$\int_0^{h(\boldsymbol{x})} |([\pi_{\ell_-}\mathcal{L}_-]G^{j_-(\ell)})(\boldsymbol{x} + t\,\boldsymbol{\tau}_{\ell_-}) + [(\pi_{\ell_+}\mathcal{L}_+]G^{j_+(\ell)})(\boldsymbol{x} - t\,\boldsymbol{\tau}_{\ell_+})|^p\, \frac{dt}{t} < +\infty \quad \text{for a.e. } \boldsymbol{x} \text{ in } \Gamma_\ell, \tag{6.4}$$

for an appropriate $h(\boldsymbol{x})$ depending on $\boldsymbol{x}$.

Note that conditions (6.3) are satisfied everywhere when $s-n > \frac{3}{p}$ and also that they can be differentiated in the tangential direction of Γ_ℓ, which leads for all positive ν to

$$\partial^\nu_{\sigma_\ell}([\pi_{\ell_-}\mathcal{L}_-]G^{j-(\ell)} + [\pi_{\ell_+}\mathcal{L}_+]G^{j+(\ell)}) = 0 \quad \text{in the distribution sense on}\,\Gamma_\ell. \tag{6.5}$$

There also these conditions are satisfied a.e. on Γ_ℓ when $s-n-\nu$ is positive and everywhere when it is $> \frac{3}{p}$. Conditions (6.4) can be differentiated similarly.

6.2. The case of a single trace ($m = 1$) for the cone.

We denote by Ω_j, $1 \le j \le J$, the faces of $\partial\mathcal{C}$ and, to avoid useless complexity, we assume that two different faces are not coplanar. We set $m = 1$ and introduce the space

$$\mathbb{W}^{s,p\,(1)}(\partial\mathcal{C}) = \prod_{j=1}^{J} W^{s-\frac{1}{p},p}(\Omega_j),$$

provided with the natural norm, together with the global trace operator of order 1

$$\gamma^{(1)} = (\gamma_0^j)_{1\le j\le J}.$$

Two different faces Ω_j and $\Omega_{j'}$, $1 \le j < j' \le J$, have a unique common tangential direction or, equivalently, the intersection of the two planes P_j and $P_{j'}$ containing Ω_j and $\Omega_{j'}$, respectively, is a line $D_{jj'}$. When $\overline{\Omega}_j$ and $\overline{\Omega}_{j'}$ are adjacent, i.e. share an edge Γ_ℓ, this Γ_ℓ is contained in $D_{jj'}$, otherwise we say that $D_{jj'}$ is a *virtual edge* common to Ω_j and $\Omega_{j'}$. Let $\boldsymbol{\sigma}_{jj'}$ stand for a unit vector in $D_{jj'}$. The key idea here is that the traces of polynomials on Ω_j and $\Omega_{j'}$ are in fact defined on P_j and $P_{j'}$, hence are connected together via their trace on $D_{jj'}$.

Next, we denote by $\Sigma_{\boldsymbol{a}}$ the set

$$\Sigma_{\boldsymbol{a}} = \big\{\boldsymbol{\sigma}_{jj'};\ 1 \le j < j' \le J\big\}. \tag{6.6}$$

For all $\boldsymbol{\sigma}$ in $\Sigma_{\boldsymbol{a}}$, we introduce the set $J(\boldsymbol{\sigma})$ of indices j, $1 \le j \le J$, such that $\boldsymbol{\sigma}$ is tangential to Ω_j. For any j in $J(\boldsymbol{\sigma})$, $\boldsymbol{\tau}_j$ stands for the unit vector tangential to P_j which is directly orthogonal to $\boldsymbol{\sigma}$.

Remark 6.1. We give some examples to illustrate these abstract definitions.
- If the cone has 3 faces, $\Sigma_{\boldsymbol{a}}$ has 3 elements, and no virtual edge.
- If the cone has 4 faces, $\Sigma_{\boldsymbol{a}}$ has 6 elements, and for all $\boldsymbol{\sigma} \in \Sigma_{\boldsymbol{a}}$ the cardinality of $J(\boldsymbol{\sigma})$ is reduced to 2.
- More generally $\Sigma_{\boldsymbol{a}}$ has at most $J(J+1)/2$ elements, and there exist cases, as illustrated in Figure 6.2, where some of the $J(\boldsymbol{\sigma})$ have 3 elements.

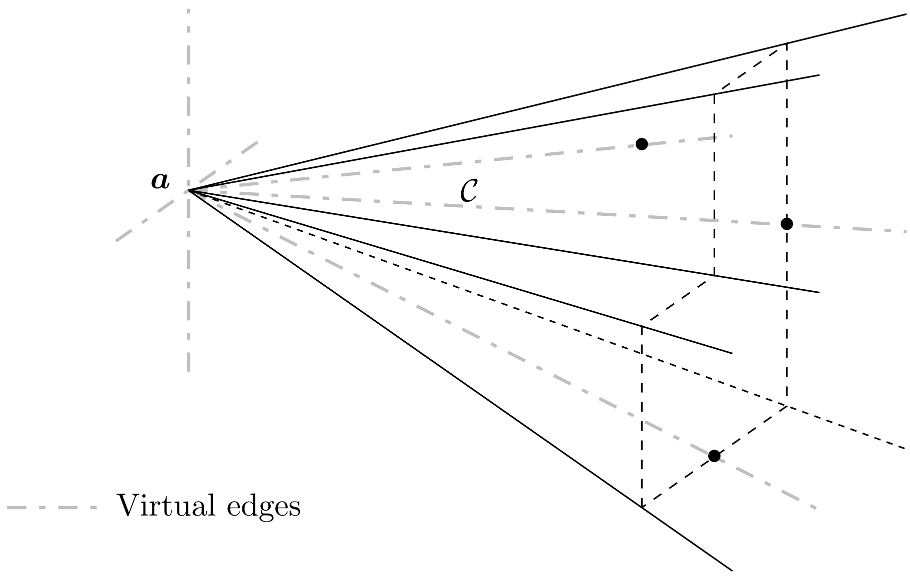

Figure 6.2

We are now in a position to state the main result of this section, in the non-limit case where $s - \frac{3}{p}$ is not an integer. For all nonnegative integers n, we introduce the one-dimensional space of operators $\mathcal{E}_n^j(\mathcal{C}, \boldsymbol{\sigma})$ generated by $\partial_{\tau_j}^n$ (note that this space coincides with the $\mathcal{E}_{n,m}^{\ell_\pm}(\mathcal{C}, \boldsymbol{\sigma}_\ell)$ defined in (6.1) with $m = 1$ and $j = \ell_\pm$).

Theorem 6.1. *Let p be such that $1 < p < +\infty$ and s be a real number $> \frac{1}{p}$ such that $s - \frac{3}{p}$ is not an integer. Then, a J-tuple $(g^1, \dots, g^J)$ in $\mathbb{W}^{s,p\,(1)}(\partial\mathcal{C})$ is the image of a function in $W^{s,p}(\mathcal{C})$ by the trace mapping $\gamma^{(1)}$ if and only if*
(i) For all ℓ, $1 \le \ell \le L$: If $s \neq \frac{2}{p}$,

$$G^{j_-(\ell)}(\boldsymbol{x}) = G^{j_+(\ell)}(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in \Gamma_\ell, \tag{6.7}$$

and, if $s = \frac{2}{p}$,

$$\int_0^{h(\boldsymbol{x})} \left| G^{j_-(\ell)}(\boldsymbol{x} + t\,\boldsymbol{\tau}_{\ell_-}) + G^{j_+(\ell)}(\boldsymbol{x} - t\,\boldsymbol{\tau}_{\ell_+}) \right|^p \frac{dt}{t} < +\infty \quad \text{for a.e. } \boldsymbol{x} \in \Gamma_\ell, \tag{6.8}$$

(ii) For all n, $0 \le n < s - \frac{3}{p}$, for all $\boldsymbol{\sigma} \in \Sigma_{\boldsymbol{a}}$ and any $(\mathcal{L}_j)_{j \in J(\boldsymbol{\sigma})}$ in $\prod_{j \in J(\boldsymbol{\sigma})} \mathcal{E}_n^j(\mathcal{C}, \boldsymbol{\sigma})$ such that $\sum_{j \in J(\boldsymbol{\sigma})} \mathcal{L}_j = 0$, and for all ν, $0 \le \nu < s - \frac{3}{p} - n$, the following conditions hold:

$$\partial_\sigma^\nu \sum_{j \in J(\boldsymbol{\sigma})} (\mathcal{L}_j g^j)(\boldsymbol{a}) = 0. \tag{6.9}$$

Corollary 6.2. *If the assumptions of Theorem 6.1 hold, the space $\widetilde{\mathbb{W}}^{s,p\,(1)}(\partial\mathcal{C})$ of all functions in $\mathbb{W}^{s,p\,(1)}(\partial\mathcal{C})$ satisfying all conditions in (i) and (ii) prescribed in*

Theorem 6.1 *is closed in* $\mathbb{W}^{s,p\,(1)}(\partial\mathcal{C})$ *if and only if* $s \neq \frac{2}{p}$. *Let this space be provided with the norm of* $\mathbb{W}^{s,p\,(1)}(\partial\mathcal{C})$, *augmented with the integral terms in (i) when* $s = \frac{2}{p}$. *Then the operator* $\gamma^{(1)}$ *is continuous from* $W^{s,p}(\mathcal{C})$ *onto* $\widetilde{\mathbb{W}}^{s,p\,(1)}(\partial\mathcal{C})$ *and admits a continuous inverse from* $\widetilde{\mathbb{W}}^{s,p\,(1)}(\partial\mathcal{C})$ *into* $W^{s,p}(\mathcal{C})$.

The proof of this theorem (and of its corollary) is rather complex, it is performed in several steps. We first observe that, for any function v in $W^{s,p}(\mathcal{C})$ with traces g^j, conditions (6.9) are easily derived from the identity $\sum_{j\in J(\boldsymbol{\sigma})} \mathcal{L}_j = 0$. Conversely, let $(g^1, \dots, g^J)$ be any J-tuple in $\mathbb{W}^{s,p\,(1)}(\partial\mathcal{C})$ satisfying all the conditions in the theorem.

Proof of Theorem 6.1. It proceeds in three steps.
1) Reductions.
Let us introduce the spaces

$$V_*^{s,p}(\Omega_j) = \big\{g \in W^{s,p}(\Omega_j);\ \int_{\Omega_j} \partial^{\boldsymbol{k}} g\, r^{|\boldsymbol{k}|-s}\, d\boldsymbol{\tau} < +\infty,\ |\boldsymbol{k}| \leq s\big\},$$

where r denotes the distance to $\boldsymbol{a}$. It can be observed that each g^j admits the decomposition

$$g^j = g_0^j + p^j,$$

with g_0^j in $V_*^{s-\frac{1}{p},p}(\Omega_j)$ and p^j in $\mathcal{P}_{[s-\frac{3}{p}]}(\Omega_j)$, where $\mathcal{P}_n(\Omega_j)$ stands for the space of polynomials with degree $\leq n$ with respect to the tangential variables on Ω_j. By the same arguments as for Theorem 5.2, it suffices to investigate the compatibility conditions for traces of polynomials in the space $\overline{\mathcal{P}}_n(\mathcal{C})$ of homogeneous polynomials of degree n with three variables on $\mathcal{C}$ and for all n, $0 \leq n < s - \frac{3}{p}$. Note that, since polynomials on Ω_j are in fact defined on the plane P_j which contains Ω_j, and since the polynomials $\mathcal{L}_j p^j$ are of degree $< s - \frac{3}{p} - n$, conditions (6.9), which are

$$\partial_\sigma^\nu \sum_{j\in J(\boldsymbol{\sigma})} (\mathcal{L}_j p^j)(\boldsymbol{a}) = 0, \quad 0 \leq \nu < s - \frac{3}{p} - n, \tag{6.10}$$

yield that

$$\sum_{j\in J(\boldsymbol{\sigma})} \mathcal{L}_j p^j = 0 \quad \text{on} \quad D_{\boldsymbol{\sigma}},$$

where $D_{\boldsymbol{\sigma}}$ stands for the line which contains $\boldsymbol{a}$ and is parallel to $\boldsymbol{\sigma}$.
2) Lifting of the polynomial traces.
Let now p_j, $1 \leq j \leq J$, be polynomials in $\mathcal{P}_{[s-\frac{3}{p}]}(\Omega_j)$ satisfying (6.8). The idea is to construct a lifting of the first j traces $(p_1, p_2, \dots, p_j)$, for j running from 1 to J, and the proof proceeds by induction on j.

- There exists a lifting of the trace p_1, so that the result is obvious for $j = 1$.

• Suppose that there exists a lifting u_j of the first j traces $(p_1, \dots, p_j)$. By subtracting this u_j, we want to lift the traces $(0, \cdots, 0, \tilde{p}_{j+1} = p_{j+1} - u_{j\,|\Omega_{j+1}})$. Assume for a while that there exists a polynomial q_{j+1} in $\mathcal{P}_{[s-\frac{3}{p}]}(\Omega_j)$ such that

$$\tilde{p}_{j+1} = \ell_1 \cdots \ell_j \, q_{j+1}, \tag{6.11}$$

where ℓ_i denotes the restriction to $\overline{\Omega}_{j+1}$ of the affine function $\overline{\ell}_i$ such that $\overline{\ell}_i(\boldsymbol{x}) = 0$ is the equation of the plane P_i. Thus, there exists a lifting v_{j+1} of q_{j+1}, and the function $\overline{\ell}_1 \cdots \overline{\ell}_j \, v_{j+1}$ lifts the first $j+1$ traces $(0, \cdots, 0, \tilde{p}_{j+1})$. So it remains to prove (6.11).

3) Proof of (6.11).

Note that the compatibility conditions (6.10) are still satisfied by the $j+1$ traces $(0, \cdots, 0, \tilde{p}_{j+1})$. When the j vectors $\boldsymbol{\sigma}_{i,j+1}$, $1 \leq i \leq j$, are different, the lines $D_{i,j+1}$ are distinct and, since it follows from (6.10) that $\tilde{p}_{j+1}$ vanishes on each line $D_{i,j+1}$, it can be factorized by $\ell_1 \cdots \ell_j$. Now, assume that all the indices j_1, ..., j_q, $j+1$, with $1 \leq j_1 < \cdots < j_q \leq j$, belong to the same $J(\boldsymbol{\sigma})$. We must prove that, if $q < s - \frac{3}{p}$, $\tilde{p}_{j+1}$ vanishes at the order q on $D_{\boldsymbol{\sigma}}$.

• We first check that, for $2 \leq q' \leq q$, the q' partial differential operators $\partial^{q'-1}_{\tau_{j_i}}$, $1 \leq i \leq q'$, are linearly independent. Let (x, y, z) denote a system of Cartesian coordinates such that $\boldsymbol{\sigma}$ is parallel to the z–axis. There exist two constants λ_i and α_i such that

$$\lambda_i \, \partial_{\tau_{j_i}} = \alpha_i \, \partial_x + \partial_y,$$

where the α_i are all different. This yields

$$\lambda_i^{q'-1} \, \partial^{q'-1}_{\tau_{j_i}} = \sum_{k=1}^{q'-1} \begin{pmatrix} q'-1 \\ k \end{pmatrix} \alpha_i^k \, \partial_x^k \, \partial_y^{q'-k-1}.$$

It is readily checked that the determinant made by the $\begin{pmatrix} q'-1 \\ k \end{pmatrix}$ α_i^k is not zero, which prove the independence.

• Next, the operator $\partial^{q'-1}_{\tau_{j+1}}$ belongs to the q'-dimensional space spanned by the $\partial^{q'-1}_{\tau_{j_i}}$, $1 \leq i \leq q'$, so that there exist constants β_i such that

$$\partial^{q'-1}_{\tau_{j+1}} = \sum_{i=1}^{q'} \beta_i \, \partial^{q'-1}_{\tau_{j_i}}.$$

Using the compatibility condition (6.10) associated with this formula and satisfied by $(0, \cdots, 0, \tilde{p}_{j+1})$ gives

$$\partial^{q'-1}_{\tau_{j+1}} \tilde{p}_{j+1} = 0 \quad \text{on } D_{\boldsymbol{\sigma}}, \qquad 2 \leq q' \leq q.$$

This implies (6.11), which concludes the proof of the theorem.

Let us write more explicitly conditions (6.9).

• For $n=0$, they are obvious

$$g^1(\boldsymbol{a}) = \cdots = g^J(\boldsymbol{a}). \tag{6.12}$$

• For $n=1$, and for any pair (j,j'), the previous condition is differentiated in the common direction $\boldsymbol{\sigma}_{jj'}$, which leads to

$$\partial_{\sigma_{jj'}} g^j(\boldsymbol{a}) = \partial_{\sigma_{jj'}} g^{j'}(\boldsymbol{a}), \qquad 1 \le j < j' \le J. \tag{6.13}$$

However, if there exists an $\boldsymbol{\sigma}$ in $\Sigma_{\boldsymbol{a}}$ such that the cardinality of $J(\boldsymbol{\sigma})$ is ≥ 3, for all such $\boldsymbol{\sigma}$ and for any triple (j,j',j'') of $J(\boldsymbol{\sigma})$, $j<j'<j''$, it is readily checked that $\boldsymbol{\tau}_{j''}$ is a linear combination of $\boldsymbol{\tau}_j$ and $\boldsymbol{\tau}_{j'}$, which leads to a further condition of the type

$$\partial_{\tau_{j''}} g^{j''}(\boldsymbol{a}) = \alpha_j\, \partial_{\tau_j}\, g^j(\boldsymbol{a}) + \alpha_{j'}\, \partial_{\tau_{j'}}\, g^{j'}(\boldsymbol{a}), \qquad (j,j',j'') \in J(\boldsymbol{\sigma}), \quad \boldsymbol{\sigma} \in \Sigma_{\boldsymbol{a}}. \tag{6.14}$$

• For $n=2$, conditions (6.13) can be differentiated with respect to $\boldsymbol{\sigma}_{jj'}$ and conditions (6.14) can be differentiated with respect to $\boldsymbol{\sigma}$. However, if there exists an $\boldsymbol{\sigma}$ in $\Sigma_{\boldsymbol{a}}$ such that the cardinality of $J(\boldsymbol{\sigma})$ is ≥ 4, a further condition appears for all such $\boldsymbol{\sigma}$ and for any $4-$tuple (j,j',j'',j''') of $J(\boldsymbol{\sigma})$, $j<j'<j''<j'''$.
And so on... So the maximal number of conditions is only bounded as a function of the regularity coefficient s.

However in a large number of geometries these conditions are simpler.

• When the cardinalities of all $J(\boldsymbol{\sigma})$, $\boldsymbol{\sigma} \in \Sigma_{\boldsymbol{a}}$, are equal to 2, all compatibility conditions (6.9) are obtained from (6.12) by tangential differentiation, they write for $0 \le \nu < s - \frac{3}{p}$

$$\partial^{\nu}_{\sigma_{jj'}} g^j(\boldsymbol{a}) = \partial^{\nu}_{\sigma_{jj'}} g^{j'}(\boldsymbol{a}), \qquad 1 \le j < j' \le J. \tag{6.15}$$

• When the intersection of $\overline{\Omega}_j$ and $\overline{\Omega}_{j'}$ is an edge Γ_ℓ, this one is parallel to $\boldsymbol{\sigma}_{jj'}$ and conditions (6.13) are only a consequence of the compatibility conditions (6.7) or (6.8) on this edge, see (6.5).

• And, finally, when J is equal to 3 (as for instance for an octant), all the compatibility conditions at the corner are derived from those on the edges.

We now state the result in the limit case where $s - \frac{3}{p}$ is an integer. We introduce a plane sector $\mathcal{W}$ with corner in $\mathbf{0}$ and angle and radius small enough in order that, for any j, $1 \le j \le J$, there exists a mapping F_j, equal to the product of a translation and a rotation, such that the sector $F_j(\mathcal{W})$ is contained in Ω_j and that its corner coincides with $\boldsymbol{a}$.

Theorem 6.3. *Let p be such that $1<p<+\infty$ and s be a real number $> \frac{1}{p}$ such that $s - \frac{3}{p}$ is an integer. Then, a J-tuple $(g^1,\dots,g^J)$ in $\mathbb{W}^{s,p\,(1)}(\partial\mathcal{C})$ is the image* *of a function in $W^{s,p}(\mathcal{C})$ by the trace mapping $\gamma^{(1)}$ if and only if all conditions in part (i) and (ii) of Theorem* 6.1 *are satisfied, and if moreover the following conditions hold for $n \le s - \frac{3}{p}$, for all $\boldsymbol{\sigma}$ in $\Sigma_{\boldsymbol{a}}$ and any $(\mathcal{L}_j)_{j\in J(\boldsymbol{\sigma})}$ in $\prod_{j\in J(\boldsymbol{\sigma})} \mathcal{E}_n^j(\mathcal{C},\boldsymbol{\sigma})$ such that $\sum_{j\in J(\boldsymbol{\sigma})} \mathcal{L}_j = 0$, and for $\nu = s - \frac{3}{p} - n$:*

$$\int_{\mathcal{W}} \Big|\partial_\sigma^\nu \sum_{j\in J(\boldsymbol{\sigma})} (\mathcal{L}_j g^j)(F_j(\boldsymbol{t}))\Big|^p \frac{d\boldsymbol{t}}{|\boldsymbol{t}|^2} < +\infty. \tag{6.16}$$

Corollary 6.4. *If the assumptions of Theorem* 6.3 *hold, let $\widetilde{\mathbb{W}}^{s,p\,(1)}(\partial\mathcal{C})$ be the space of all functions in $\mathbb{W}^{s,p\,(1)}(\partial\mathcal{C})$ satisfying all conditions in part (i) and (ii) of Theorem* 6.1 *and also all conditions* (6.16) *prescribed in Theorem* 6.3. *When this space is provided with the norm*

$$\Big(\|g\|^p_{\mathbb{W}^{s,p\,(1)}(\partial\mathcal{C})} + \int_{\mathcal{W}} \Big|\partial_\sigma^\nu \sum_{j\in J(\boldsymbol{\sigma})} (\mathcal{L}_j g^j)(F_j(\boldsymbol{t}))\Big|^p \frac{d\boldsymbol{t}}{|\boldsymbol{t}|^2}\Big)^{\frac{1}{p}},$$

the operator $\gamma^{(1)}$ is continuous from $W^{s,p}(\mathcal{C})$ onto $\widetilde{\mathbb{W}}^{s,p\,(1)}(\partial\mathcal{C})$ and admits a continuous inverse from $\widetilde{\mathbb{W}}^{s,p\,(1)}(\partial\mathcal{C})$ into $W^{s,p}(\mathcal{C})$.

Proof in the case $s = \frac{3}{p}$. We first note that conditions (6.16) in this case write

$$\int_{\mathcal{W}} |g^j(F_j(\boldsymbol{t})) - g^{j'}(F_{j'}(\boldsymbol{t}))|^p \frac{d\boldsymbol{t}}{|\boldsymbol{t}|^2} < +\infty, \quad 1 \le j < j' \le J.$$

We use the polar coordinates on $\mathcal{W}$ and assume that

$$\mathcal{W} = \big\{(r,\theta);\ 0 < r < r_0 \text{ and } 0 < \theta < \theta_0\big\}.$$

Let v be any function in $W^{\frac{3}{p},p}(\mathcal{C})$, with traces on the Ω_j denoted by g^j. Thus, if D_θ denotes the segment $\{(r,\theta);\ 0 < r < r_0\}$, each trace g_j has a trace on $F_j(D_\theta)$ which belongs to $W^{\frac{1}{p},p}(D_\theta)$. Moreover, it follows from Theorem 5.5 that, in the plane sector S_θ with edges $F_j(D_\theta)$ and $F_{j'}(D_\theta)$, $1 \le j < j' \le J$, these traces satisfy

$$\int_0^{r_0} |g^j(F_j(r,\theta)) - g^{j'}(F_{j'}(r,\theta))|^p \frac{dr}{r} \le c\,\|v\|_{W^{\frac{2}{p},p}(S_\theta)} \le c'\,\|v\|_{W^{\frac{3}{p},p}(\mathcal{C})}.$$

Integrating this inequality with respect to θ yields

$$\int_0^{\theta_0}\int_0^{r_0} |g^j(F_j(r,\theta)) - g^{j'}(F_{j'}(r,\theta))|^p \frac{dr\,d\theta}{r} < +\infty.$$

For any $\boldsymbol{t} = (r, \theta)$ in $\mathcal{W}$, we have $d\boldsymbol{t} = r\,dr\,d\theta$ and $|\boldsymbol{t}|^2 = r^2$, so that the g^j satisfy conditions (6.16).
Conversely, let $(g^1, \dots, g^J)$ be a J-tuple in $\mathbb{W}^{\frac{2}{p},p,(1)}(\partial\mathcal{C})$ satisfying (6.16). We introduce a lifting v_1 of the trace g^1 in $W^{\frac{3}{p},p}(\mathcal{C})$, so that it remains to lift the J-tuple $(0, \tilde{g}^2, \dots, \tilde{g}^J)$, with each $\tilde{g}^j$ equal to $g^j - v_{1\,|\Omega_j}$. Conditions (6.16) are satisfied by the functions $\tilde{g}_j$, they now write

$$\int_{\mathcal{W}} |\tilde{g}^j(F_j(\boldsymbol{t}))|^p \, \frac{d\boldsymbol{t}}{|\boldsymbol{t}|^2} < +\infty, \quad 2 \le j \le J.$$

Equivalently, each function $\tilde{g}^j$ belongs to the space of flat functions

$$V^{\frac{2}{p},p}(\Omega_j) = \big\{ g \in V^{\frac{2}{p},p}(\Omega_j);\ \int_{\Omega_j} |g(\boldsymbol{t})|^p \, \frac{d\boldsymbol{t}}{|\boldsymbol{t}|^2} < +\infty \big\}.$$

This yields the existence of a lifting $\tilde{v}$ of $(0, \tilde{g}^2, \dots, \tilde{g}^J)$ in $W^{\frac{3}{p},p}(\mathcal{C})$, which moreover satisfies

$$\int_{\mathcal{C}} |\tilde{v}(\boldsymbol{x})|^p \, \frac{d\boldsymbol{x}}{|\boldsymbol{x}|^3} < +\infty,$$

which concludes the proof.

Remark 6.2. The sector $\mathcal{W}$ is only involved in condition (6.16), and it is readily checked that this condition is independent of the choice of $\mathcal{W}$.

Remark 6.3. When J is equal to 3, we have seen that the pointwise conditions (6.9) are a consequence the conditions (6.7) on the edges. Likewise, the compatibility conditions (6.16) are derived from the conditions (6.7). Thus, $\widetilde{\mathbb{W}}^{s,p\,(1)}(\partial\mathcal{C})$ is a closed subspace of $\mathbb{W}^{s,p\,(1)}(\partial\mathcal{C})$ and the case where $s - \frac{3}{p}$ is an integer is not a limit case. But this is no longer true for $J > 3$.

Remark 6.4. Theorems 6.1 and 6.3, when compared with Theorem 4.4, provide a full characterization of the space $W^{s-\frac{1}{p},p}(\mathcal{G})$, where $\mathcal{G}$ is a neighbourhood of $\boldsymbol{a}$ in $\partial\mathcal{C}$.

Remark 6.5. When two faces are coplanar, conditions (6.7) and (6.8) on the edges remain unchanged. But the conditions on the corner are modified since the intersection of the planes containing these faces is no longer a line. So all tangential derivatives on the coplanar faces must coincide in $\boldsymbol{a}$, and conditions (6.9) and (6.16) must be reformulated.

6.3. The general case for a cone.

We now introduce the general trace space

$$\mathbb{W}^{s,p\,(m)}(\partial\mathcal{C}) = \prod_{j=1}^{J} \Big(\prod_{k=0}^{m-1} W^{s-k-\frac{1}{p},p}(\Omega_j) \Big),$$

provided with the natural norm, and we consider the global trace operator

$$\gamma^{(m)} = (\gamma_0^j, \dots, \gamma_{m-1}^j)_{1\le j\le J}.$$

With any $\boldsymbol{\sigma} \in \Sigma_{\boldsymbol{a}}$ and $j = 1, \dots, J$, we now associate the space $\mathcal{E}^j_{n,m}(\mathcal{C}, \boldsymbol{\sigma})$ spanned by the operators $\partial^k_{n_j}\partial^{n-k}_{\tau_j}$, $0 \le k \le \min\{n, m-1\}$, and the corresponding operator π_j

$$\pi_j : \qquad \sum_{k=0}^{\min\{n,m-1\}} c_k\,\partial^k_{n_j}\partial^{n-k}_{\tau_j} \quad\mapsto\quad (c_k\,\partial^{n-k}_{\tau_j})_{0\le k\le m-1}.$$

There also, we separately state the results depending on $s - \frac{3}{p}$ being an integer or not.

Theorem 6.5. *Let p be such that $1 < p < +\infty$ and s be a real number $> \frac{1}{p}$ such that $s - \frac{3}{p}$ is not an integer. Let m be an integer such that $0 \le m-1 < s - \frac{1}{p}$. Then, a J-tuple $(G^1, \dots, G^J)$ in $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{C})$ is the image of a function in $W^{s,p}(\mathcal{C})$ by the trace mapping $\gamma^{(m)}$ if and only if*
(i) conditions (6.3) hold for all ℓ, $1 \le \ell \le L$, and for all n, $0 \le n < s - \frac{2}{p}$, and, moreover, if $s - \frac{2}{p}$ is an integer, conditions (6.4) hold for all ℓ, $1 \le \ell \le L$, and for $n = s - \frac{2}{p}$,
(ii) the following conditions hold for all n, $0 \le n < s - \frac{3}{p}$, for all $\boldsymbol{\sigma}$ in $\Sigma_{\boldsymbol{a}}$ and any $(\mathcal{L}_j)_{j\in J(\boldsymbol{\sigma})}$ in $\prod_{j\in J(\boldsymbol{\sigma})}\mathcal{E}^j_{n,m}(\mathcal{C},\boldsymbol{\sigma})$ such that $\sum_{j\in J(\boldsymbol{\sigma})}\mathcal{L}_j = 0$, and for all ν, $0 \le \nu \le s - \frac{3}{p} - n$:

$$\partial^\nu_\sigma \sum_{j\in J(\boldsymbol{\sigma})} ([\pi_j\mathcal{L}_j]G^j)(\boldsymbol{a}) = 0. \tag{6.17}$$

Corollary 6.6. *If the assumptions of Theorem 6.5 hold, the space $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\mathcal{C})$ of all functions in $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{C})$ satisfying all conditions (6.3), (6.4) and (6.17) prescribed in Theorem 6.5 is closed in $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{C})$ if and only if $s - \frac{2}{p} \notin \{m-1, m\dots, 2m-2\}$. Let this space be provided with the norm of $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{C})$, augmented with the integral terms in (6.4) when $s - \frac{2}{p} \in \{m-1, m\dots, 2m-2\}$. Then the operator $\gamma^{(m)}$ is continuous from $W^{s,p}(\mathcal{C})$ onto $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\mathcal{C})$ and admits a continuous inverse from $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\mathcal{C})$ into $W^{s,p}(\mathcal{C})$.*

The proof of these theorem and corollary is also rather complex. Since it is very similar to the proof of Theorem 6.1, we only give an abridged version of it.

Sketch of proof. It is performed in four steps.
1) By the same arguments as in the proof of Theorem 6.1, first part, relying on a decomposition of each G^j as the sum of a "flat" function and a polynomial, we must

prove the result with G^j replaced with P^j in $\prod_{k=0}^{m-1} \mathcal{P}_{[s-k-\frac{3}{p}]}(\Omega_j)$.
2) Next, as in the proof of Theorem 6.1, second part, we proceed by induction on j and try to lift the traces $(0, \cdots, 0, \tilde{P}_{j+1})$. The idea is to look for a lifting of these traces of the form $u_{j+1} = (\ell_1 \cdots \ell_j)^m \, v_{j+1}$.
3) A necessary and sufficient condition for the existence of v_{j+1} is that, with the notation $\tilde{P}^{j+1} = (p_0^{j+1}, \ldots, p_{m-1}^{j+1})$, there exist polynomials q_k^{j+1}, $0 \le k \le m-1$, such that

$$\begin{aligned} &p_0^{j+1} = (\ell_1 \cdots \ell_j)^m \, q_0^{j+1} \qquad \text{and} \\ &p_k^{j+1} + r_{k,0}\, q_0^{j+1} + \cdots + r_{k,k-1}\, q_{k-1}^{j+1} = (\ell_1 \cdots \ell_j)^m \, q_k^{j+1}, \quad 1 \le k \le m-1, \end{aligned} \tag{6.18}$$

where the $r_{k,\ell}$ stand for the successive derivatives of $(\ell_1 \cdots \ell_j)^m$ in the direction $\boldsymbol{n}_{j+1}$ multiplied by appropriate constants. Indeed, if these conditions are satisfied, the function v_{j+1} is taken such that: $\partial_{n_{j+1}}^k v_{j+1} = q_k^{j+1}$, $0 \le k \le m-1$.
4) Conditions (6.18) can be derived from conditions (6.17), easily if all the cardinalities of the J are equal to 2, in a more technical way if not.

Theorem 6.7. *Let p be such that $1 < p < +\infty$ and s be a real number $> \frac{1}{p}$ such that $s - \frac{3}{p}$ is an integer. Let m be an integer such that $0 \le m - 1 < s - \frac{1}{p}$. Then, a J-tuple $(G^1, \ldots, G^J)$ in $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{C})$ is the image of a function in $W^{s,p}(\mathcal{C})$ by the trace mapping $\gamma^{(m)}$ if and only if all conditions in parts (i) and (ii) of Theorem 6.5 are satisfied, and if moreover the following conditions hold for $n \le s - \frac{3}{p}$, for all $\boldsymbol{\sigma}$ in $\Sigma_{\boldsymbol{a}}$ and any $(\mathcal{L}_j)_{j\in J(\boldsymbol{\sigma})}$ in $\prod_{j\in J(\boldsymbol{\sigma})} \mathcal{E}^j_{n,m}(\mathcal{C}, \boldsymbol{\sigma})$ such that $\sum_{j\in J(\boldsymbol{\sigma})} \mathcal{L}_j = 0$, and for $\nu = s - \frac{3}{p} - n$:*

$$\int_{\mathcal{W}} |\partial_\sigma^\nu \sum_{j\in J(\boldsymbol{\sigma})} ([\pi_j \mathcal{L}_j] G^j)(F_j(\boldsymbol{t}))|^p \, \frac{d\boldsymbol{t}}{|\boldsymbol{t}|^2} < +\infty. \tag{6.19}$$

Corollary 6.8. *If the assumptions of Theorem 6.7 hold, let $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\mathcal{C})$ be the space of all functions in $\mathbb{W}^{s,p\,(m)}(\partial\mathcal{C})$ satisfying all conditions in part (i) and (ii) of Theorem 6.5 and also all conditions (6.19) prescribed in Theorem 6.7. When this space is provided with the norm*

$$\Big(\|G\|^p_{\mathbb{W}^{s,p\,(m)}(\partial\mathcal{C})} + \int_{\mathcal{W}} \Big| \partial_\sigma^\nu \sum_{j\in J(\boldsymbol{\sigma})} ([\pi_j \mathcal{L}_j] G^j)(F_j(\boldsymbol{t})) \Big|^p \, \frac{d\boldsymbol{t}}{|\boldsymbol{t}|^2} \Big)^{\frac{1}{p}}, \tag{6.20}$$

the operator $\gamma^{(m)}$ is continuous from $W^{s,p}(\mathcal{C})$ onto $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\mathcal{C})$ and admits a continuous inverse from $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\mathcal{C})$ into $W^{s,p}(\mathcal{C})$.

6.4. Trace theorem for a polyhedron.

For the sake of completeness, we now state the result in the case of a general polyhedron Π. The trace operator is defined as previously, with values into the space

$$\mathbb{W}^{s,p\,(m)}(\partial\Pi) = \prod_{j=1}^{J}\Big(\prod_{k=0}^{m-1} W^{s-k-\frac{1}{p},p}(\Omega_j)\Big).$$

The trace operator $\gamma^{(m)}$ is now given by

$$\gamma^{(m)} = (\gamma_0^j, \dots, \gamma_{m-1}^j)_{1\le j\le J}.$$

As in Section 6.1, for each edge Γ_ℓ, $1 \le \ell \le L$, we denote by $\overline{\Omega}_{j_-(\ell)}$ and $\overline{\Omega}_{j_+(\ell)}$ the two faces that intersect Γ_ℓ and, with these faces, we associate the orthogonal bases $\{\boldsymbol{n}_{\ell_\pm}, \boldsymbol{\sigma}_\ell, \boldsymbol{\tau}_{\ell_\pm}\}$, the spaces $\mathcal{E}^{\ell_\pm}_{n,m}(\mathcal{C}, \boldsymbol{\sigma}_\ell)$ defined in (6.1) and the operators $\pi_{\ell_\pm}$ defined in (6.2). As in Section 6.2 and 6.3, with each vertex $\boldsymbol{a}$, we associate the set $\Sigma_{\boldsymbol{a}}$ defined in (6.6), and for each face Ω_j tangential to an $\boldsymbol{\sigma}$ in $\Sigma_{\boldsymbol{a}}$, the orthogonal basis $\{\boldsymbol{n}_j, \boldsymbol{\sigma}, \boldsymbol{\tau}_j\}$, the space $\mathcal{E}^j_{n,m}(\mathcal{C}, \boldsymbol{\sigma})$ spanned by the operators $\partial^k_{n_j}\partial^{n-k}_{\tau_j}$, $0 \le k \le m-1$, and the operator π_j.

Theorem 6.9. *Let p be such that $1 < p < +\infty$ and s be a real number $> \frac{1}{p}$. Let m be an integer such that $0 \le m-1 < s - \frac{1}{p}$. Then, a J-tuple $(G^1, \dots, G^J)$ in $\mathbb{W}^{s,p\,(m)}(\partial\Pi)$ is the image of a function in $W^{s,p}(\Pi)$ by the trace mapping $\gamma^{(m)}$ if and only if*
(i) conditions (6.3) hold for all ℓ, $1 \le \ell \le L$, and for all n, $0 \le n < s - \frac{2}{p}$, and, moreover, if $s - \frac{2}{p}$ is an integer, conditions (6.4) hold for all ℓ, $1 \le \ell \le L$, and for $n = s - \frac{2}{p}$,
(ii) conditions (6.17) hold for all corners $\boldsymbol{a}$ of Π and for all n, $0 \le n < s - \frac{3}{p}$, and, moreover, if $s - \frac{3}{p}$ is an integer, conditions (6.19) hold for all all corners $\boldsymbol{a}$ of Π and for $n = s - \frac{3}{p}$.

Corollary 6.10. *If the assumptions of Theorem 6.9 hold, the space $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\Pi)$ of all functions in $\mathbb{W}^{s,p\,(m)}(\partial\Pi)$ satisfying all the conditions prescribed in this theorem is closed in $\mathbb{W}^{s,p\,(m)}(\partial\Pi)$ when neither $s - \frac{2}{p}$ nor $s - \frac{3}{p}$ is an integer. When this space is provided with the norm of $\mathbb{W}^{s,p\,(m)}(\partial\Pi)$ augmented with appropriate terms (issued from (6.4) and (6.19)) the operator $\gamma^{(m)}$ is continuous from $W^{s,p}(\Pi)$ onto $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\Pi)$ and admits a continuous inverse from $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\Pi)$ into $W^{s,p}(\Pi)$.*

This theorem is rather complex, this is due to the generality of the geometry. In view of the application to spectral methods in the next chapters, we sum up the results in the simple case of the cube Ξ in the next corollary. Here, each edge Γ_ℓ,

$1 \leq \ell \leq 12$, is the intersection of the two faces $\Omega_{j_-(\ell)}$ and $\Omega_{j_+(\ell)}$ and is parallel to the unit vector $\boldsymbol{\sigma}_\ell$. For simplicity, we choose the vectors $\boldsymbol{n}_{\ell_\pm}$ and $\boldsymbol{\tau}_{\ell_\pm}$ such that: $\boldsymbol{n}_{\ell_-} = \boldsymbol{\tau}_{\ell_+}$ and $\boldsymbol{n}_{\ell_-} = \boldsymbol{\tau}_{\ell_+}$.

Corollary 6.11. *Let p be such that $1 < p < +\infty$ and s be a real number $> \frac{1}{p}$. Let m be an integer such that $0 \leq m-1 < s - \frac{1}{p}$. Then, a 6-tuple $(G^1, \ldots, G^6)$, with $G^j = (g_0^j, \cdots, g_{m-1}^j)$, in $\mathbb{W}^{s,p\,(m)}(\partial\Xi)$ is the image of a function in $W^{s,p}(\Xi)$ by the trace mapping $\gamma^{(m)}$ if and only if, properties (i) and (ii) are satisfied:*
(i) For all ℓ, $1 \leq \ell \leq 12$, and for all integers n, $0 \leq n \leq 2m-2$ with $n < s - \frac{2}{p}$,

$$\partial_{\tau_{\ell_-}}^{n-k} g_k^{j_-(\ell)} = \partial_{\tau_{\ell_+}}^{k} g_{n-k}^{j_+(\ell)} \quad \text{on } \Gamma_\ell, \qquad 0 \leq k,\, n-k \leq m-1, \tag{6.21}$$

(ii) When $s - \frac{2}{p} \in \{m-1, m, \ldots, 2m-2\}$: For all ℓ, $1 \leq \ell \leq 12$, and with $n = s - \frac{2}{p}$,

$$\int_0^1 \left| (\partial_{\tau_{\ell_-}}^{n-k} g_k^{j_-(\ell)})(\boldsymbol{x} - t\,\boldsymbol{\tau}_{\ell_-}) - (\partial_{\tau_{\ell_+}}^{k} g_{n-k}^{j_+(\ell)})(\boldsymbol{x} - t\,\boldsymbol{\tau}_{\ell_+}) \right|^p \frac{dt}{t} < +\infty \tag{6.22}$$
$$\text{for a.e. } \boldsymbol{x} \text{ in } \Gamma_\ell, \quad 0 \leq k,\, n-k \leq m-1.$$

The space $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\Xi)$ of all functions in $\mathbb{W}^{s,p\,(m)}(\partial\Xi)$ satisfying all prescribed conditions (6.21) and (6.22) is closed in $\mathbb{W}^{s,p,(m)}(\partial\Xi)$ if and only if $s - \frac{2}{p} \notin \{m-1, m, \ldots, 2m-2\}$. When this space is provided with the norm of $\mathbb{W}^{s,p\,(m)}(\partial\Xi)$, augmented with the left-hand side of (6.22) when $s - \frac{2}{p} \in \{m-1, m, \ldots, 2m-2\}$, the operator $\gamma^{(m)}$ is continuous from $W^{s,p}(\Xi)$ onto $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\Xi)$ and admits a continuous inverse from $\widetilde{\mathbb{W}}^{s,p\,(m)}(\partial\Xi)$ into $W^{s,p}(\Xi)$.

As already hinted, since all the compatibility conditions at the vertices are derived from those on the edges, $\widetilde{\mathbb{W}}^{s,p,(1)}(\partial\Xi)$ is a closed subspace of $\mathbb{W}^{s,p,(1)}(\partial\Xi)$ even in the case where $s - \frac{3}{p}$ is an integer. A similar simple statement holds for any tetrahedron. But this is no longer true for all pentahedra (think of a pyramid with a square basis!).

7. Extension to weighted Sobolev spaces.

The aim of this section is to present the analogues of the previous trace theorems in weighted Sobolev spaces, the main application being the interpolation of polynomial spaces in the sense of traces. For simplicity, we only define the spaces on the intervals $\mathcal{I}$ and Λ, the square Θ and the cube Ξ. In each case, we state the corresponding trace theorems. Since the trace results are local, the extension to more complex weights, for instance the product of the distance to the endpoints, edges and faces up to different powers, is obvious. We refer to [32] and [21], [22] for analogous results.

7.1. Case of the half-interval.

Like in Section 3.1, $\mathcal{I}$ is considered as the half of the reference interval Λ. Let α be a fixed real number. For any p, $1 \leq 1 < +\infty$, $L^p_\alpha(\mathcal{I})$ denotes the space of measurable functions which are L^p-integrable with respect to the measure $x^\alpha\,dx$. In other words,

$$L^p_\alpha(\mathcal{I}) = \big\{\varphi : \mathcal{I} \to \mathbb{R} \text{ measurable; } \|\varphi\|_{L^p_\alpha(\mathcal{I})} = \big(\int_{-1}^{1} |\varphi(x)|^p\, x^\alpha\, dx\big)^{\frac{1}{p}} < +\infty\big\}. \quad (7.1)$$

What is the convenient definition for $L^\infty_\alpha(\mathcal{I})$? Let us recall that the $L^\infty-$space associated with a measure μ is the space of measurable functions φ such that

$$\sup\big\{t \geq 0;\ \mu(\{|\varphi| \geq t\}) \neq 0\big\} < +\infty.$$

As an obvious consequence of the definition, if ν is an absolutely continuous measure with respect to μ, then the $L^\infty-$space associated with μ is continuously embedded in the $L^\infty-$space associated with ν. Since the measures dx and $x^\alpha\,dx$ are mutually absolutely continuous on $\mathcal{I}$, we define

$$L^\infty_\alpha(\mathcal{I}) = L^\infty(\mathcal{I}). \quad (7.2)$$

Remark 7.1. Note that, for $\alpha > -1$, the measure $x^\alpha\,dx$ is finite on $\mathcal{I}$. Hence, as a consequence of [5, Thm 5.1.2], if $1 \leq p < +\infty$ and if α is > -1, then the interpolate space of index θ between $L^\infty(\mathcal{I})$ and $L^p_\alpha(\mathcal{I})$ coincides with $L^q_\alpha(\mathcal{I})$, for $\frac{1}{q} = \frac{\theta}{p}$.

Now, we introduce the Sobolev spaces associated with the measure $x^\alpha\,dx$; for any nonnegative integer m and for any p, $1 \leq p < +\infty$, $W^{m,p}_\alpha(\mathcal{I})$ is defined by

$$W^{m,p}_\alpha(\mathcal{I}) = \big\{\varphi \in \mathscr{D}'(\mathcal{I});\ \|\varphi\|_{W^{m,p}_\alpha(\mathcal{I})} = \big(\sum_{k=0}^{m} \|d^k\varphi\|^p_{L^p_\alpha(\mathcal{I})}\big)^{\frac{1}{p}} < +\infty\big\}. \quad (7.3)$$

That space coincides with Triebel's space $W^m_p(\Omega;\rho_\alpha)$, defined in [32, § 3.2.1]. According to Triebel's terminology, loc. cit., the weight that we consider is of type 3 for $\alpha \geq 0$ and of type 4 for $\alpha \leq 0$.

As for the unweighted case, it is natural to introduce the following other type of weighted space. We define $V^{m,p}_\alpha(\mathcal{I})$ by:

$$V^{m,p}_\alpha(\mathcal{I}) = \big\{\varphi \in \mathscr{D}'(\mathcal{I});\ \|\varphi\|_{V^{m,p}_\alpha(\mathcal{I})} < +\infty\big\}, \quad (7.4)$$

where

$$\|\varphi\|_{V^{m,p}_\alpha(\mathcal{I})} = \big(\sum_{k=0}^{m} \int_{-1}^{1} |d^k\varphi|^p\, x^{\alpha+(k-m)p}\, dx\big)^{\frac{1}{p}}. \quad (7.5)$$

This space coincides with the space $W_p^m(\mathcal{I};\rho_\alpha,\rho_{\alpha-mp})$ of [32]; in the case $p=2$, it is the space denoted $\overset{\circ}{W}{}_\alpha^m$ in [21]. All these spaces were also introduced by Maz'ja and Plamenevskii with α replaced with α/p in the notation of [27].

When s is a positive real number which is not an integer, we define the spaces $W_\alpha^{s,p}(\mathcal{I})$ by intrinsic norms. We consider the neighbourhood of the diagonal of $\mathcal{I}\times\mathcal{I}$

$$\Delta_{\mathcal{I}}=\{(x,y)\in\mathcal{I}\times\mathcal{I};\ \frac{x}{4}<y<4x\}. \tag{7.6}$$

Let α be a real parameter and p a real number, $1\le p<+\infty$. Using Notation 2.1, for any σ, $0<\sigma<1$, we define the semi-norm:

$$|\varphi|_{\sigma,p,\alpha;\mathcal{I}}=\|q_{\sigma+\frac{1}{p}}[\varphi]\,x^{\frac{\alpha}{p}}\|_{L^p(\Delta_{\mathcal{I}})}, \tag{7.7}$$

or, more explicitly,

$$|\varphi|_{\sigma,p,\alpha;\mathcal{I}}=\Big(\int_{\Delta_{\mathcal{I},a}}\frac{|\varphi(x)-\varphi(\eta)|^p}{|x-\eta|^{\sigma p+1}}\,x^\alpha\,dx\,d\eta\Big)^{\frac{1}{p}}.$$

Indeed, for any positive real number s, denoting by $[s]$ the integral part of s and setting $\sigma=s-[s]$, we introduce the norm

$$\|\varphi\|_{V_\alpha^{s,p}(\mathcal{I})}=\big(\|\varphi\|^p_{V^{[s],p}_{\alpha-\sigma p}(\mathcal{I})}+|d^{[s]}\varphi|^p_{\sigma,p,\alpha;\mathcal{I}}\big)^{\frac{1}{p}}, \tag{7.8}$$

or equivalently

$$\|\varphi\|_{V_\alpha^{s,p}(\mathcal{I})}=\Big(\int_{-1}^1(\sum_{k=0}^{[s]}|d^k\varphi|^p\,x^{\alpha+(k-s)p}\,dx+|d^{[s]}\varphi|^p_{\sigma,p,\alpha;\mathcal{I}}\Big)^{\frac{1}{p}}. \tag{7.9}$$

Thus, in the usual way, we set

$$V_\alpha^{s,p}(\mathcal{I})=\{\varphi\in\mathscr{D}'(\mathcal{I});\ \|\varphi\|_{V_\alpha^{s,p}(\mathcal{I})}<+\infty\}. \tag{7.10}$$

Now, we intend to define the spaces $W_\alpha^{s,p}(\mathcal{I})$. As is well-known [15][32], there is a *limit case* when $\sigma-\frac{\alpha}{p}-\frac{1}{p}$ is equal to 0, so we first assume that

$$\sigma-\frac{\alpha}{p}-\frac{1}{p}\neq 0, \tag{7.11}$$

and we define the norm

$$\|\varphi\|_{W_\alpha^{s,p}(\mathcal{I})}=\big(\|\varphi\|^p_{W_\alpha^{[s],p}(\mathcal{I})}+|d^{[s]}\varphi|^p_{\sigma,p,\alpha;\mathcal{I}}\big)^{\frac{1}{p}}. \tag{7.12}$$

Next, we introduce the space $W_\alpha^{s,p}(\mathcal{I})$ by

$$W_\alpha^{s,p}(\mathcal{I}) = \{\varphi \in \mathscr{D}'(\mathcal{I});\ \|\varphi\|_{W_\alpha^{s,p}(\mathcal{I})} < +\infty\} \tag{7.13}$$

(the space $W_\alpha^{s,p}(\mathcal{I})$ for $\sigma - \frac{\alpha}{p} - \frac{1}{p} = 0$ is defined later on by interpolation). We now state the main properties of these spaces, which extend the results of [32, Chap. 3] and [27, Thm 2.1] to the case of non integral exponents.

Theorem 7.1. *Let p be a real number, $1 \le p < +\infty$, and s be a positive real number.*
(i) If α is ≤ -1 or $> sp-1$, then $\mathscr{D}(]0,1])$ is dense in $W_\alpha^{s,p}(\mathcal{I})$ and the spaces $W_\alpha^{s,p}(\mathcal{I})$ and $V_\alpha^{s,p}(\mathcal{I})$ coincide.
(ii) If α is such that $-1 < \alpha < sp-1$ and if $s-\frac{\alpha}{p}-\frac{1}{p}$ is not an integer, then $\mathscr{C}^\infty([0,1])$ is dense in $W_\alpha^{s,p}(\mathcal{I})$. If K denotes the integral part of $s - \frac{\alpha}{p} - \frac{1}{p}$, for each integer k, $0 \le k \le K$, the trace mappings γ_k defined on $\mathscr{C}^\infty([0,1])$ by

$$\gamma_k(\varphi) = d^k\varphi(0)$$

are continuous on $W_\alpha^{s,p}(\mathcal{I})$. Moreover, $V_\alpha^{s,p}(\mathcal{I})$ is the closure of $\mathscr{D}(]0,1])$ in $W_\alpha^{s,p}(\mathcal{I})$ and the following characterization holds:

$$V_\alpha^{s,p}(\mathcal{I}) = \{\varphi \in W_\alpha^{s,p}(\mathcal{I});\ \forall k, 0 \le k \le K,\ \gamma_k(\varphi) = 0\}.$$

The space $W_\alpha^{s,p}(\mathcal{I})$ is equal to the direct sum of the space $V_\alpha^{s,p}(\mathcal{I})$ and of the space $\mathbb{P}_K(\mathcal{I})$ of polynomials with degree $\le K$ on $\mathcal{I}$.

To conclude, we state some further properties of the weighted spaces, first the Sobolev type embeddings, second some interpolation results. We refer to [7, § 1.d & 1.e] for the proofs. For the sake of completeness, before stating these last properties, we give the definition of the space $W_\alpha^{s,p}(\mathcal{I})$ when $s - \frac{\alpha}{p} - \frac{1}{p}$ is an integer.

For any p, $1 \le p < +\infty$, and any positive real number s such that $\sigma - \frac{\alpha}{p} - \frac{1}{p}$ is equal to 0, where σ is the fractional part of s, the space $W_\alpha^{s,p}(\mathcal{I})$ is defined by

$$W_\alpha^{s,p}(\mathcal{I}) = \Big[W_\alpha^{[s]+1,p}(\mathcal{I}), W_\alpha^{[s],p}(\mathcal{I})\Big]_{1-\sigma,p}. \tag{7.14}$$

Theorem 7.2. *Let α and β be two real numbers > -1. Let p and q be real numbers, $1 < p, q < +\infty$, , and let s and t be two real numbers such that $0 \le t \le s$.*
(i) If the next two conditions are satisfied

$$\begin{cases} t - \frac{1}{q} < s - \frac{1}{p} & \\ \textit{or} & \\ t - \frac{1}{q} = s - \frac{1}{p} & \textit{with } p \le q, \end{cases} \tag{7.15}$$

$$\begin{cases} \quad t-\frac{\beta}{q}-\frac{1}{q}<s-\frac{\alpha}{p}-\frac{1}{p} \\ or \\ \quad t-\frac{\beta}{q}-\frac{1}{q}=s-\frac{\alpha}{p}-\frac{1}{p} \quad \text{with } p\le q \text{ and } s-\frac{\alpha}{p}-\frac{1}{p}\notin\mathbb{N}, \end{cases} \tag{7.16}$$

the following embedding holds:

$$W^{s,p}_\alpha(\mathcal{I}) \subset W^{t,q}_\beta(\mathcal{I}). \tag{7.17}$$

(ii) If t is not an integer, if $t \le s-\frac{1}{p}$ and $t\le s-\frac{\alpha}{p}-\frac{1}{p}$, the following embedding holds:

$$W^{s,p}_\alpha(\mathcal{I}) \subset \mathscr{C}^t(\overline{\mathcal{I}}). \tag{7.18}$$

(iii) If $t<s$, if $t-\frac{1}{q}<s-\frac{1}{p}$ and $t-\frac{\beta}{q}-\frac{1}{q}<s-\frac{\alpha}{p}-\frac{1}{p}$, the embedding (7.17) is compact.
(iv) If $t<s-\frac{1}{p}$ and $t<s-\frac{\alpha}{p}-\frac{1}{p}$, the embedding (7.18) is compact.

Theorem 7.3. *Let α be any real number. Let p be a real number, $1<p<+\infty$, and let s_0, s_1 and s be nonnegative real numbers such that*

$$[s]\le s_0<s<s_1\le [s]+1. \tag{7.19}$$

Then the following interpolation result holds:

$$W^{s,p}_\alpha(\mathcal{I}) = \Big[W^{s_1,p}_\alpha(\mathcal{I}), W^{s_0,p}_\alpha(\mathcal{I})\Big]_{\theta,p} \qquad \text{with} \quad \theta=\frac{s_1-s}{s_1-s_0}. \tag{7.20}$$

We only state the general interpolation result in the simpler Hilbertian framework $p=2$ (see [26, Chapter 1, Thm 3.5] for its proof in the special case $\alpha=-\frac{1}{2}$).

Corollary 7.4. *Let α be any real number. Let s_0, s_1 and s be nonnegative real numbers such that $s_0<s<s_1$. Then the following interpolation result holds:*

$$W^{s,2}_\alpha(\mathcal{I}) = \Big[W^{s_1,2}_\alpha(\mathcal{I}), W^{s_0,2}_\alpha(\mathcal{I})\Big]_{\theta,p} \qquad \text{with} \quad \theta=\frac{s_1-s}{s_1-s_0}. \tag{7.21}$$

Similarly, we define the spaces $W^{s,p}_\alpha(\Lambda)$ and $V^{s,p}_\alpha(\Lambda)$ with the weight x^α replaced by the weight $(1-\zeta^2)^\alpha$. All the results in this section are still valid with $\mathcal{I}$ replaced by Λ.

7.2. Case of the square.

We now work on the square Θ and, for simplicity, we only consider the case $p=2$ of Hilbertian Sobolev spaces. Let α and β be fixed real numbers, and let $L^2_{\alpha\beta}(\Theta)$ denote the space

$$\begin{aligned} L^2_{\alpha\beta}(\Theta) = \big\{ & v:\Theta\to\mathbb{R} \text{ measurable;} \\ & \|v\|_{L^2_{\alpha\beta}(\Theta)} = \Big(\int_{-1}^{1}\int_{-1}^{1} |v(x,y)|^2\,(1-x^2)^\alpha(1-y^2)^\beta\,dx\,dy\Big)^{\frac{1}{2}} < +\infty\big\}. \end{aligned} \tag{7.22}$$

Next, for each positive integer m, $H^m_{\alpha\beta}(\Theta)$ stands for the space of functions in $L^2_{\alpha\beta}(\Theta)$ such that all their derivatives of order $\leq m$ belong to $L^2_{\alpha\beta}(\Theta)$. It is provided with the norm

$$\|v\|_{H^m_{\alpha\beta}(\Theta)} = \big(\sum_{|\boldsymbol{k}|\leq m} \|\partial^{\boldsymbol{k}} v\|^2_{L^2_{\alpha\beta}(\Theta)}\big)^{\frac{1}{2}}. \tag{7.23}$$

Finally, for any positive real number s which is not an integer, the space $H^s_{\alpha\beta}(\Theta)$ is simply defined by interpolation: With $s = [s] + \sigma$, $0 < \sigma < 1$,

$$H^s_{\alpha\beta}(\Theta) = \Big[H^{[s]+1}_{\alpha\beta}(\Theta), H^{[s]}_{\alpha\beta}(\Theta)\Big]_{1-\sigma,2}. \tag{7.24}$$

Then, the following tensorization property can be checked

$$H^s_{\alpha\beta}(\Theta) = L^2_\alpha(\Lambda; H^s_\beta(\Lambda)) \cap H^s_\alpha(\Lambda; L^2_\beta(\Lambda)). \tag{7.25}$$

In order to state the trace theorems, we denote by Γ_1, respectively Γ_2, the edge of Θ contained in the line $x = -1$, respectively $y = -1$, see Figure 5.1. The weighted spaces $H^s_\beta(\Gamma_1)$ and $H^s_\alpha(\Gamma_2)$ are obviously defined as the spaces $W^{s,2}_\beta(\Lambda)$ and $W^{s,2}_\alpha(\Lambda)$ of the previous Section 7.1. Here, we skip the limit cases for simplicity.

Theorem 7.5. *Let s be a positive real number.*
(i) If $s - \frac{\alpha}{2} - \frac{1}{2}$ is not an integer and K_1 denotes the integral part of $s - \frac{\alpha}{2} - \frac{1}{2}$, then, for each integer k, $0 \leq k \leq K_1$, the trace mapping γ^1_k defined on $\mathscr{C}^\infty(\overline{\Theta})$ by

$$\gamma^1_k(u)(y) = (\partial^k_x u)(-1, y)$$

is continuous from $H^s_{\alpha\beta}(\Theta)$ onto $H^{s-k-\frac{\alpha}{2}-\frac{1}{2}}_\beta(\Gamma_1)$. There exists a continuous lifting operator of the mapping $\gamma^1 = (\gamma^1_0, \dots, \gamma^1_{K_1})$ from $\prod_{k=0}^{K_1} H^{s-k-\frac{\alpha}{2}-\frac{1}{2}}_\beta(\Gamma_1)$ into $H^s_{\alpha\beta}(\Theta)$. Moreover, the space $\mathscr{D}(]-1,1] \times [-1,1])$ is dense in the kernel of γ^1.
(ii) If $s - \frac{\beta}{2} - \frac{1}{2}$ is not an integer and K_2 denotes the integral part of $s - \frac{\beta}{2} - \frac{1}{2}$, then, for each integer k, $0 \leq k \leq K_2$, the trace mapping γ^2_k defined on $\mathscr{C}^\infty(\overline{\Theta})$ by

$$\gamma^2_k(u)(x) = (\partial^k_y u)(x, -1)$$

is continuous from $H^s_{\alpha\beta}(\Theta)$ onto $H^{s-k-\frac{\beta}{2}-\frac{1}{2}}_\alpha(\Gamma_2)$. There exists a continuous lifting operator of the mapping $\gamma^2 = (\gamma^2_0, \dots, \gamma^2_{K_2})$ from $\prod_{k=0}^{K_2} H^{s-k-\frac{\beta}{2}-\frac{1}{2}}_\alpha(\Gamma_2)$ into $H^s_{\alpha\beta}(\Theta)$. Moreover, the space $\mathscr{D}([-1,1] \times]-1,1])$ is dense in the kernel of γ^2.

Lifting given traces on the edges of the square requires that compatibility conditions at the corners are fulfilled. Fortunately, they are the same as in the unweighted case, and specially simple since the angle in $\boldsymbol{a}$ is equal to $\frac{\pi}{2}$. Here, without restriction, we take $\alpha \le \beta$. We introduce the space

$$\mathbb{H}^{s\,(m)}_{\alpha\beta}(\partial\Theta) = \prod_{k=0}^{m-1} H_\beta^{s-k-\frac{\alpha}{2}-\frac{1}{2}}(\Gamma_1) \times \prod_{k=0}^{m-1} H_\alpha^{s-k-\frac{\beta}{2}-\frac{1}{2}}(\Gamma_2) \times \prod_{k=0}^{m-1} H_\beta^{s-k-\frac{\alpha}{2}-\frac{1}{2}}(\Gamma_3) \times \prod_{k=0}^{m-1} H_\alpha^{s-k-\frac{\beta}{2}-\frac{1}{2}}(\Gamma_4).$$

The trace operator $\gamma^{(m)}$ is still defined by

$$\gamma^{(m)} = (\gamma_0^j, \cdots, \gamma_{m-1}^j)_{1\le j\le 4}.$$

Theorem 7.6. *Assume that $\alpha \le \beta$. Let s be a real number $> \frac{\beta}{2} + \frac{1}{2}$. Let m be an integer such that $0 \le m-1 < s - \frac{\beta}{2} - \frac{1}{2}$. We assume that $s - \frac{\alpha+\beta}{2} \notin \{m, m+1, \dots, 2m-1\}$. Then, an element $G = (G^1, G^2, G^3, G^4)$ of $\mathbb{H}^{s\,(m)}_{\alpha\beta}(\partial\Theta)$, with each G^j equal to $(g_0^j, \dots, g_{m-1}^j)$, is the image of an element v of $H^s_{\alpha\beta}(\Theta)$ by the trace mapping $\gamma^{(m)}$ if and only if the following conditions hold for all $i = 1, 2, 3, 4$ and for all n, $0 \le n < s - \frac{\alpha+\beta}{2} - 1$,*

$$\partial_{\tau_i}^{n-k} g_k^i(\boldsymbol{a}_i) = (-1)^k\, \partial_{\tau_{i+1}}^k g_{n-k}^{i+1}(\boldsymbol{a}_i), \qquad 0 \le k,\ n-k \le m-1. \tag{7.26}$$

7.3. Case of the cube.

The results in the cube Ξ are very similar to their analogues in the square. Let α, β and γ be fixed real numbers. We now introduce the space

$$L^2_{\alpha\beta\gamma}(\Xi) = \big\{v : \Xi \to \mathbb{R} \text{ measurable; } \|v\|_{L^2_{\alpha\beta\gamma}(\Xi)} < +\infty\big\},$$

with

$$\|v\|_{L^2_{\alpha\beta\gamma}(\Xi)} = \Big(\int_{-1}^1\int_{-1}^1\int_{-1}^1 |v(x,y,z)|^2\,(1-x^2)^\alpha(1-y^2)^\beta(1-z^2)^\gamma\,dx\,dy\,dz\Big)^{\frac{1}{2}}. \tag{7.27}$$

For each positive integer m, we denote by $H^m_{\alpha\beta\gamma}(\Xi)$ the space of functions in $L^2_{\alpha\beta\gamma}(\Xi)$ such that all their derivatives of total order $\le m$ belong to $L^2_{\alpha\beta\gamma}(\Xi)$. It is provided with the norm

$$\|v\|_{H^m_{\alpha\beta\gamma}(\Xi)} = \Big(\sum_{|\boldsymbol{k}|\le m} \|\partial^{\boldsymbol{k}} v\|^2_{L^2_{\alpha\beta\gamma}(\Xi)}\Big)^{\frac{1}{2}}. \tag{7.28}$$

For any positive real number s which is not an integer, we define the space $H^s_{\alpha\beta\gamma}(\Xi)$ is by interpolation: With $s = [s] + \sigma$, $0 < \sigma < 1$,

$$H^s_{\alpha\beta\gamma}(\Xi) = \Big[H^{[s]+1}_{\alpha\beta\gamma}(\Xi), H^{[s]}_{\alpha\beta\gamma}(\Xi)\Big]_{1-\sigma,2}. \tag{7.29}$$

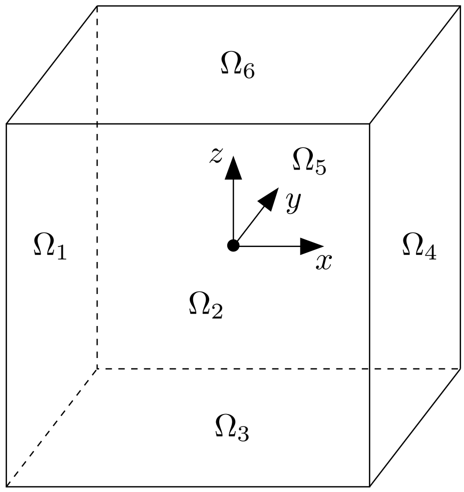

Figure 7.1

Here, we denote by Ω_1, respectively Ω_2, respectively Ω_3, the faces of Ξ contained in the plane $x = -1$, respectively $y = -1$, respectively $z = -1$ (and also by Ω_4, respectively Ω_5, respectively Ω_6, the faces of Ξ contained in the plane $x = 1$, respectively $y = 1$, respectively $z = 1$, see Figure 7.1). Since these faces are squares, the weighted spaces on the faces, for instance the $H^s_{\beta\gamma}(\Omega_1)$, are defined as in Section 7.2.

Theorem 7.7. *Let s be a positive real number. If $s - \frac{\alpha}{2} - \frac{1}{2}$ is not an integer and K_1 denotes the integral part of $s - \frac{\alpha}{2} - \frac{1}{2}$, then, for each integer k, $0 \le k \le K_1$, the trace mapping γ^1_k defined on $\mathscr{C}^\infty(\overline{\Xi})$ by*

$$\gamma^1_k(u)(y,z) = (\partial^k_x u)(-1, y, z)$$

is continuous from $H^s_{\alpha\beta\gamma}(\Xi)$ onto $H^{s-k-\frac{\alpha}{2}-\frac{1}{2}}_{\beta\gamma}(\Omega_1)$. There exists a continuous lifting operator of the mapping $\gamma^1 = (\gamma^1_0, \dots, \gamma^1_{K_1})$ from $\prod_{k=0}^{K_1} H^{s-k-\frac{\alpha}{2}-\frac{1}{2}}_{\beta\gamma}(\Omega_1)$ into $H^s_{\alpha\beta\gamma}(\Xi)$. Moreover, the space $\mathscr{D}([-1,1]^2 \times]-1,1])$ is dense in the kernel of γ^1.

Of course, the statement of Theorem 7.7 still holds when exchanging the variables x, y and z and the parameters α, β and γ, and the trace operators on Ω_2 and Ω_3 are denoted by γ^2_k and γ^3_k, respectively.

Since only three faces of Ξ share the same vertex, lifting given traces on the faces of the cube only requires compatibility conditions on the edges, in fact the same as in the unweighted case. The trace operator is still defined as $\gamma^{(m)} = \big(\gamma_0^j, \dots, \gamma_{m-1}^j\big)_{j=1,\dots,6}$. and we introduce the space

$$\begin{aligned}\mathbb{H}^{s\,(m)}_{\alpha\beta\gamma}(\partial\Xi) = &\prod_{k=0}^{m-1} H^{s-k-\frac{\alpha}{2}-\frac12}_{\beta\gamma}(\Omega_1) \times \prod_{k=0}^{m-1} H^{s-k-\frac{\beta}{2}-\frac12}_{\alpha\gamma}(\Omega_2) \times \prod_{k=0}^{m-1} H^{s-k-\frac{\gamma}{2}-\frac12}_{\alpha\beta}(\Omega_3)\\ &\times \prod_{k=0}^{m-1} H^{s-k-\frac{\alpha}{2}-\frac12}_{\beta\gamma}(\Omega_4) \times \prod_{k=0}^{m-1} H^{s-k-\frac{\beta}{2}-\frac12}_{\alpha\gamma}(\Omega_5) \times \prod_{k=0}^{m-1} H^{s-k-\frac{\gamma}{2}-\frac12}_{\alpha\beta}(\Omega_6).\end{aligned}$$

Theorem 7.8. *Assume that $\alpha \le \beta \le \gamma$. Let s be a real number $> \frac{\gamma}{2} + \frac12$. Let m be an integer such that $0 \le m-1 < s - \frac{\gamma}{2} - \frac12$. We assume neither $s - \frac{\alpha+\beta}{2}$, nor $s - \frac{\alpha+\gamma}{2}$, nor $s - \frac{\beta+\gamma}{2}$ belongs to $\{m, m+1, \dots, 2m-1\}$. Then, an element $G = (G^1, \dots, G^6)$ of $\mathbb{H}^{s\,(m)}_{\alpha\beta\gamma}(\partial\Xi)$, with each G^j equal to $(g_0^j, \dots, g_{m-1}^j)$, is the image of an element v of $H^s_{\alpha\beta\gamma}(\Xi)$ by the trace mapping $\gamma^{(m)}$ if and only if the following conditions hold*
(i) for all n, $0 \le n < s - \frac{\beta+\gamma}{2} - 1$,

$$\partial_y^{n-k} g_k^1(-1, z) = (-1)^n\, \partial_x^k g_{n-k}^2(-1, z) \quad \text{for a.e. } z \in \Lambda, \qquad 0 \le k,\ n-k \le m-1, \tag{7.30}$$

and similar conditions on the three other edges parallel to the z axis,
(ii) for all n, $0 \le n < s - \frac{\alpha+\gamma}{2} - 1$,

$$\partial_z^{n-k} g_k^1(y, -1) = (-1)^n\, \partial_x^k g_{n-k}^3(-1, y) \quad \text{for a.e. } y \in \Lambda, \qquad 0 \le k,\ n-k \le m-1, \tag{7.31}$$

and similar conditions on the three other edges parallel to the y axis,
(iii) for all n, $0 \le n < s - \frac{\alpha+\beta}{2} - 1$,

$$\partial_z^{n-k} g_k^2(x, -1) = (-1)^n\, \partial_y^k g_{n-k}^3(x, -1) \quad \text{for a.e. } x \in \Lambda, \qquad 0 \le k,\ n-k \le m-1, \tag{7.32}$$

and similar conditions on the three other edges parallel to the x axis.

Remark 7.2. Among the main applications of the weighted spaces on the square and the cube, let us quote the so-called "driven cavity problem" (at least when Navier–Stokes equations are involved): A second-order elliptic equation is set in the square or cube, it is provided with Dirichlet boundary conditions equal to zero except on the top edge (for the square) or top face (for the cube) where they are equal to 1. It is readily checked from the previous results that the solution of such a problem cannot belong to $H^1(\Omega)$. However a variational formulation of this problem can be written in $H^1_{\alpha\beta}(\Theta)$ or $H^1_{\alpha\beta\gamma}(\Xi)$ when α, β and γ are positive.

Chapter II

Polynomial spaces and lifting of polynomial traces

This chapter is devoted to the construction of stable polynomial lifting operators for polynomial traces on the edges of a square or the faces of a cube. The compatibility conditions between traces at the vertices and edges are explicitly written in Chapter I. It can be noted that the construction of a polynomial lifting of traces satisfying these conditions is rather easy. But the lifting operator that we intend to construct has two further properties: It is continuous in appropriate norms, with its norm independent of the degree of the polynomial traces, and it preserves the degree of the polynomials in most cases.

The first results in this direction are due to I. Babuška and M. Suri [4], with applications to the h-$p-$version of the finite element method; as a preliminary step, we generalize their results by constructing a polynomial lifting into a triangle (we also refer to [18] for a pioneering work on this subject). Next we construct the appropriate lifting operators into the square and the cube (we refer to [10] and [25] for related results).

We also state and prove a key result concerning the interpolation of spaces of polynomials by the method of traces, see [23, Chap. 3]. Of course, this result relies on the use of the polynomial lifting operators built and analyzed in the beginning of the chapter.

1. Polynomial spaces.

We first introduce the space of polynomials we work with all along this chapter. We begin with one-dimensional geometries.

Notation 1.1. On any interval Γ of $\mathbb{R}$ and for all nonnegative integers n, we denote by $\mathbb{P}_n(\Gamma)$ the space of restrictions to Γ of polynomials with one variable and degree $\leq n$.

The dimension of $\mathbb{P}_n(\Gamma)$ is $n+1$. Several bases of $\mathbb{P}_n(\Gamma)$ are used in what follows and, among them, the simplest one is made of the polynomials $1, \zeta, \cdots, \zeta^n$. However, in the special case of the interval $\Lambda =]-1,1[$, we also consider the family $(L_m)_{m\geq 0}$ of Legendre polynomials: Each L_m has degree m, is orthogonal to all other ones in $L^2(\Lambda)$ and satisfies $L_m(1) = 1$. We refer to [30] for the properties of these polynomials and only note that the set $\{L_0, \cdots, L_n\}$ is a basis of any $\mathbb{P}_n(\Gamma)$ and an orthogonal basis of $\mathbb{P}_n(\Lambda)$.

Notation 1.2. On any bounded interval Γ of $\mathbb{R}$ and for all nonnegative integers n and positive integers k, we denote by $\mathbb{P}_n^{k,0}(\Gamma)$ the set of polynomials in $\mathbb{P}_n(\Gamma)$ which vanish at the two endpoints of Γ together with their derivatives up to the order $k-1$.

Of course, these spaces are reduced to $\{0\}$ when n is $< 2k$. Otherwise, their dimension is $n + 1 - 2k$.

On the tensorized domains, namely on the square Σ or cube Ξ, we define spaces of polynomials that satisfy analogous tensorization properties as Sobolev spaces, see Lemma I.4.1.

Notation 1.3. On the square $\Sigma = \Lambda^2$, and for all nonnegative integers n, we denote by $\mathbb{P}_n(\Sigma)$ the space of restrictions to Σ of polynomials with two variables and degree $\leq n$ with respect to each variable. On the cube $\Xi = \Lambda^3$, and for all nonnegative integers n, we denote by $\mathbb{P}_n(\Xi)$ the space of restrictions to Ξ of polynomials with three variables and degree $\leq n$ with respect to each variable.

The tensorization properties are described in the next lemma.

Lemma 1.4. *For any nonnegative integer n and any basis $\{\varphi_m;\, 0 \leq m \leq n\}$ of $\mathbb{P}_n(\Lambda)$, a polynomial p belongs to $\mathbb{P}_n(\Sigma)$ if and only if it admits the expansion*

$$p(x, y) = \sum_{m=0}^{n} q_m(x)\, \varphi_m(y), \tag{1.1}$$

where the q_m, $0 \leq m \leq n$, belong to $\mathbb{P}_n(\Lambda)$. For any nonnegative integer n and any basis $\{\varphi_m;\, 0 \leq m \leq n\}$ of $\mathbb{P}_n(\Lambda)$, a polynomial p belongs to $\mathbb{P}_n(\Xi)$ if and only if it admits the expansion

$$p(x, y, z) = \sum_{m=0}^{n} q_m(x, y)\, \varphi_m(z), \tag{1.2}$$

where the q_m, $0 \leq m \leq n$, belong to $\mathbb{P}_n(\Sigma)$.

Notation 1.5. On the square Σ, respectively the cube Ξ, and for all nonnegative integers n and positive integers k, we denote by $\mathbb{P}_n^{k,0}(\Sigma)$, respectively by $\mathbb{P}_n^{k,0}(\Xi)$, the set of polynomials in $\mathbb{P}_n(\Sigma)$, respectively $\mathbb{P}_n(\Xi)$, which vanish on the four edges of Σ, respectively on the six faces of Ξ, together with their normal derivatives up to the order $k - 1$.

There also, the spaces are reduced to zero when n is $< 2k$. The tensorization properties are exactly the same as previously, as stated below.

Lemma 1.6. *For any nonnegative integer n, any positive integer k such that $n \geq 2k$ and any basis $\{\psi_m;\, 0 \leq m \leq n - 2k\}$ of $\mathbb{P}_n^{k,0}(\Lambda)$, a polynomial p belongs to $\mathbb{P}_n^{k,0}(\Sigma)$*

if and only if it admits the expansion

$$p(x,y) = \sum_{m=0}^{n-2k} q_m(x)\,\psi_m(y), \tag{1.3}$$

where the q_m, $0 \le m \le n$, *belong to* $\mathbb{P}_n^{k,0}(\Lambda)$. *For any nonnegative integer* n, *any positive integer* k *such that* $n < 2k$ *and any basis* $\{\psi_m;\ 0 \le m \le n-2k\}$ *of* $\mathbb{P}_n^{k,0}(\Lambda)$, *a polynomial* p *belongs to* $\mathbb{P}_n^{k,0}(\Xi)$ *if and only if it admits the expansion*

$$p(x,y,z) = \sum_{m=0}^{n-2k} q_m(x,y)\,\psi_m(z), \tag{1.4}$$

where the q_m, $0 \le m \le n$, *belong to* $\mathbb{P}_n^{k,0}(\Sigma)$.

An immediate consequence of the previous lemmas is that the dimensions of all the previous spaces $\mathbb{P}_n(\Sigma)$ and $\mathbb{P}_n(\Xi)$, and also of $\mathbb{P}_n^{k,0}(\Sigma)$ and $\mathbb{P}_n^{k,0}(\Xi)$ when n is $\ge 2k$, are respectively

$$(n+1)^2, \quad (n+1)^3, \quad (n+1-2k)^2, \quad (n+1-2k)^3.$$

To conclude, we introduce the following space which is needed for technical arguments.

Notation 1.7. On any domain $\mathcal{T}$ of $\mathbb{R}^2$ and for all nonnegative integers n, we denote by $\mathcal{P}_n(\mathcal{T})$ the space of restrictions to $\mathcal{T}$ of polynomials with two variables and total degree $\le n$.

2. Preliminary lifting results.

The lifting operator is first introduced from the line $\mathbb{R}$ into the strip $\mathbb{R}\times\mathcal{I}$, where $\mathcal{I}$ is the interval $]0,1[$, with the help of Fourier transform. Next, it is defined from one edge of a triangle into the triangle. In each case, its basic properties are established. The construction of these liftings relies on the following key formula: For a function χ with a compact support in $\mathbb{R}$ and such that the integral of χ on $\mathbb{R}$ is equal to 1, the function $F_0(\varphi)$ defined by

$$F_0(\varphi)(x,y) = \int_{-\infty}^{\infty} \chi(v)\varphi(x+yv)\,dv,$$

is a lifting (onto the strip) of the trace φ defined on the line $y=0$.

2.1. Lifting from a line to a strip.

Let χ be a fixed real function of $L^1(\mathbb{R})$, with a compact support. For any nonnegative integer k, we define the operator:

$$F_k(\varphi)(x,y) = \frac{y^k}{k!} \int_{-\infty}^{\infty} \chi(v)\varphi(x+yv)\,dv. \tag{2.1}$$

Remark 2.1. The interest of this operator is that, if φ is a polynomial, so is $F_k(\varphi)$. More precisely, if the degree of φ is $\leq N$, the total degree of $F_k(\varphi)$ is $\leq N+k$ and its degree with respect to the variable x is $\leq N$.

Defining the function χ^y by $\chi^y(v) = \frac{1}{y}\chi(\frac{v}{y})$, we note that the operator F_k can equivalently be written

$$F_k(\varphi)(x,y) = \frac{y^k}{k!}\,(\varphi * \chi^y)(x,y), \tag{2.2}$$

where $*$ denotes the convolution product with respect to the x–variable. Equation (2.2) is equivalent to

$$\widehat{F_k(\varphi)}(\xi,y) = \frac{y^k}{k!}\,\hat{\varphi}(\xi)\hat{\chi}(y\xi). \tag{2.3}$$

where the notation $\hat{v}$ means the Fourier transform of the function v with respect to the x–variable. This last formula allows to derive some properties of the operator F_k, as stated below.

Lemma 2.1. *For any nonnegative integer k and for any real number $s > k + \frac{1}{2}$, if the mapping: $z \mapsto (1+|z|)^k\hat{\chi}(z)$ belongs to $H^s(\mathbb{R})$, the operator F_k is continuous from $H^{s-k-\frac{1}{2}}(\mathbb{R})$ into $H^s(\mathbb{R}\times\mathcal{I})$.*

Proof. Let φ be any function in $H^{s-k-\frac{1}{2}}(\mathbb{R})$. Due to the tensorization properties (see Lemma I.4.1), it suffices to check that

$$\int_{-\infty}^{+\infty} \|\widehat{F_k(\varphi)}(\xi,.)\|^2_{H^s(\mathcal{I},\xi)}\,d\xi < +\infty,$$

where the parameter-dependent norm $\|\cdot\|_{H^s(\mathcal{I},\xi)}$ is defined as

$$\|v\|^2_{H^s(\mathcal{I},\xi)} = (1+\xi^2)^s\,\|v\|^2_{L^2(\mathcal{I})} + |v|^2_{H^s(\mathcal{I})}.$$

From (2.3), we have

$$\|\widehat{F_k(\varphi)}(\xi,.)\|^2_{H^s(\mathcal{I},\xi)} = \frac{1}{k!^2}\,|\hat{\varphi}(\xi)|^2\|y^k\,\hat{\chi}(y\xi)\|^2_{H^s(\mathcal{I},\xi)}.$$

First, for $|\xi| \geq 1$, by using the change of variable $z = y\xi$, we derive

$$\|\widehat{F_k(\varphi)}(\xi,.)\|^2_{H^s(\mathcal{I},\xi)} \leq c\,(1+\xi^2)^{s-k-\frac{1}{2}}\,|\hat{\varphi}(\xi)|^2\|z^k\,\hat{\chi}(z)\|^2_{H^s(\mathbb{R})}.$$

Next, for $|\xi| < 1$, the same change of variables yields

$$\xi^{2s}\,\|\widehat{F_k(\varphi)}(\xi,.)\|^2_{L^2(\mathcal{I})} + \|\widehat{F_k(\varphi)}(\xi,.)\|^2_{H^s(\mathcal{I})} \leq c\,\xi^{2(s-k-\frac{1}{2})}|\hat{\varphi}(\xi)|^2\|z^k\,\hat{\chi}(z)\|^2_{H^s(\mathbb{R})},$$

while (note that y is ≤ 1)

$$\|\widehat{F_k(\varphi)}(\xi,.)\|^2_{L^2(\mathcal{I})} \leq c\,|\hat{\varphi}(\xi)|^2|\xi|^{-1}\,\|\hat{\chi}(z)\|^2_{L^2(-\xi,\xi)}.$$

To bound this last term, we observe that, since the mapping $z \mapsto (1+|z|)^k\hat{\chi}(z)$ belongs to $H^s(\mathbb{R})$, $s > \frac{1}{2}$, it belongs to $L^\infty(-\xi,\xi)$ and the same property holds for the mapping $\hat{\chi}$, so that the quantity $\xi^{-1}\,\|\hat{\chi}(z)\|^2_{L^2(-\xi,\xi)}$ is bounded independently of ξ. Combining all this leads to

$$\int_{-\infty}^{+\infty} \|\widehat{F_k(\varphi)}(\xi,.)\|^2_{H^s(\mathcal{I},\xi)}\,d\xi \leq c\,\|\varphi\|^2_{H^{s-k-\frac{1}{2}}(\mathbb{R})},$$

which is the desired result.

On the other hand, we note that

$$\partial_y^\ell F_k(\varphi)(x,0) = \begin{cases} 0 & \text{if } \ 0 \leq \ell \leq k-1, \\ \frac{\ell!}{k!(\ell-k)!}\Big(\int_{-\infty}^{+\infty} v^{\ell-k}\chi(v)\,dv\Big)\,d_x^{\ell-k}\varphi(x) & \text{if } \ \ell \geq k. \end{cases}$$

Hence, if the following condition is satisfied for a positive integer m:

$$\int_{-\infty}^{+\infty} \chi(v)\,dv = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} v^n\chi(v)\,dv = 0, \quad 1 \leq n \leq m-1, \tag{2.4}$$

we have

$$\partial_y^\ell F_k(\varphi)(x,0) = \delta_{k\ell}\,\varphi(x), \quad 0 \leq \ell \leq k+m-1.$$

This leads to the following statement.

Proposition 2.2. *Let m be any positive real number, and s a real number $> m - \frac{1}{2}$. If the function χ satisfies (2.4) and is such that the mapping: $z \mapsto (1+|z|)^{m-1}\hat{\chi}(z)$ belongs to $H^s(\mathbb{R})$, the operator F defined by*

$$F(g_0,\cdots,g_{m-1}) = \sum_{k=0}^{m-1} F_k(g_k),$$

is continuous from $\prod_{k=0}^{m-1} H^{s-k-\frac{1}{2}}(\mathbb{R})$ *into* $H^s(\mathbb{R}\times\mathcal{I})$ *and satisfies*

$$\forall x \in \mathbb{R}, \quad \partial_y^k\big(F(g_0,\cdots,g_{m-1})\big)(x,0) = g_k(x), \quad 0 \le k \le m-1. \tag{2.5}$$

Moreover, it maps $\mathbb{P}_N(\mathbb{R})^m$ *into* $\mathcal{P}_{N+m-1}(\mathbb{R}\times\mathcal{I})$.

Remark 2.2. We recall that, for any $\ell \ge 0$, each derivative $d^{(\ell)}L_{n+\ell}$ of order ℓ of the Legendre polynomial $L_{n+\ell}$, $n \ge 0$, is orthogonal to $\mathbb{P}_{n-1}(\Lambda)$ for the weighted measure $(1-\zeta^2)^\ell\, d\zeta$ on Λ. So, for any fixed integers $m \ge 1$ and $\ell \ge 0$, coefficients α_k can be computed by induction on k in order that the function $\chi_{\ell m}$ given by

$$\chi_{\ell m}(v) = \begin{cases} (1-v^2)^\ell \sum_{k=0}^{m-1} \alpha_k\, d^{(\ell)}L_{k+\ell}(v) & \text{in } [-1,1], \\ 0 & \text{elsewhere}, \end{cases}$$

satisfies condition (2.4). Moreover, when ℓ is $\ge m-1$, this function $\chi_{\ell m}$ is such that the mapping: $z \mapsto (1+|z|)^{m-1}\hat{\chi}_{\ell m}(z)$ belongs to $H^s(\mathbb{R})$ for all $s \ge 0$.

2.2. Lifting from an edge to a triangle.

We now consider a triangle $\mathcal{T}$ with vertices $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, such that the edge with endpoints $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ is contained in the axis $y=0$ and coincides with the segment $\Lambda = \{(x,y)\in\mathbb{R}^2;\ -1<x<1,\ y=0\}$, see Figure 2.1.

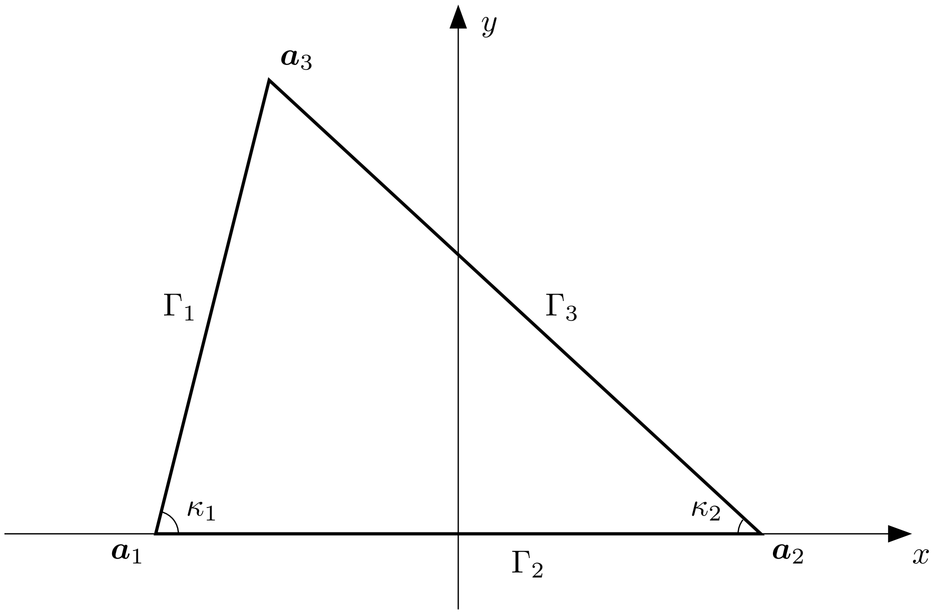

Figure 2.1

Let us denote κ_1 and κ_2 the angles of $\mathcal{T}$ at $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ respectively — they are adjacent to Λ. If the support of the function χ is contained in the interval

$]-\cot\kappa_1, \cot\kappa_2[$ (which is not empty since $\kappa_1+\kappa_2$ is $<\pi$), the values of the function $F_k(\varphi)$ defined in (2.1) on the triangle $\mathcal{T}$ only depends on the values of φ on its edge Λ. So, the operator F_k can be considered as a lifting from Λ to $\mathcal{T}$. Its continuity properties follow from Lemma 2.1.

Lemma 2.3. *For any nonnegative integer k and for any real number $s > k+\frac{1}{2}$, if the function χ has its support contained in $]-\cot\kappa_1, \cot\kappa_2[$ and is such that the mapping: $z \mapsto (1+|z|)^k \hat{\chi}(z)$ belongs to $H^s(\mathbb{R})$, the operator F_k is continuous from $H^{s-k-\frac{1}{2}}(\Lambda)$ into $H^s(\mathcal{T})$.*

So we can now build the general lifting operator in this case.

Proposition 2.4. *Let m be any positive real number, and s a real number $> m-\frac{1}{2}$. If the function χ satisfies (2.4), has its support contained in $]-\cot\kappa_1, \cot\kappa_2[$ and is such that the mapping: $z \mapsto (1+|z|)^{m-1} \hat{\chi}(z)$ belongs to $H^s(\mathbb{R})$, the operator F defined by*

$$F(g_0, \cdots, g_{m-1}) = \sum_{k=0}^{m-1} F_k(g_k),$$

is continuous from $\prod_{k=0}^{m-1} H^{s-k-\frac{1}{2}}(\Lambda)$ into $H^s(\mathcal{T})$ and satisfies

$$\forall x \in \Lambda, \quad \partial_y^k\big(F(g_0, \cdots, g_{m-1})\big)(x,0) = g_k(x), \quad 0 \le k \le m-1. \tag{2.6}$$

Moreover, it maps $\mathbb{P}_N(\Lambda)^m$ into $\mathcal{P}_{N+m-1}(\mathcal{T})$.

We now state two extensions of Lemma 2.3 to weighted spaces: On the triangle, the weight is the distance to the two vertices $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, up to some power. The first lemma involves the following weighted spaces:

• On the edge Λ, in analogy to Definition (I.7.4), for any nonnegative integer m, $V_\alpha^m(\Lambda)$ is the space of distributions φ in $\mathscr{D}'(\Lambda)$ such that $d^k\varphi\,(1-x^2)^{\frac{\alpha}{2}+k-m}$ belongs to $L^2(\Lambda)$ for all k, $0 \le k \le m$. For any positive real number s which is not an integer, $V_\alpha^s(\Lambda)$ is defined by interpolation between $V_\alpha^{[s]+1}(\Lambda)$ and $V_\alpha^{[s]}(\Lambda)$.

• On the triangle $\mathcal{T}$ and in analogy with the spaces $V_*^{s,p}(\Omega_j)$ introduced in the proof of Theorem I.6.1, for any nonnegative integer m, $V_{*\alpha}^m(\mathcal{T})$ is the space of distributions v in $\mathscr{D}'(\mathcal{T})$ such that $\partial^{\boldsymbol{k}} v\, \rho^{\frac{\alpha}{2}+|\boldsymbol{k}|-m}$ belongs to $L^2(\mathcal{T})$ for all $\boldsymbol{k}$, $0 \le |\boldsymbol{k}| \le m$, where ρ denotes the product of the distances to the points $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ (note that its restriction to Λ coincides with $1-x^2$). For any positive real number s which is not an integer, $V_{*\alpha}^s(\mathcal{T})$ is defined by interpolation between $V_{*\alpha}^{[s]+1}(\mathcal{T})$ and $V_{*\alpha}^{[s]}(\mathcal{T})$.

Lemma 2.5. *For any nonnegative integer k, any real number α and for any real number $s > k+\frac{1}{2}$, if the function χ has its support contained in $]-\cot\kappa_1, \cot\kappa_2[$ and is such that the mapping: $z \mapsto (1+|z|)^k \hat{\chi}(z)$ belongs to $H^s(\mathbb{R})$, the operator F_k is continuous from $V_\alpha^{s-k-\frac{1}{2}}(\Lambda)$ into $V_{*\alpha}^s(\mathcal{T})$.*

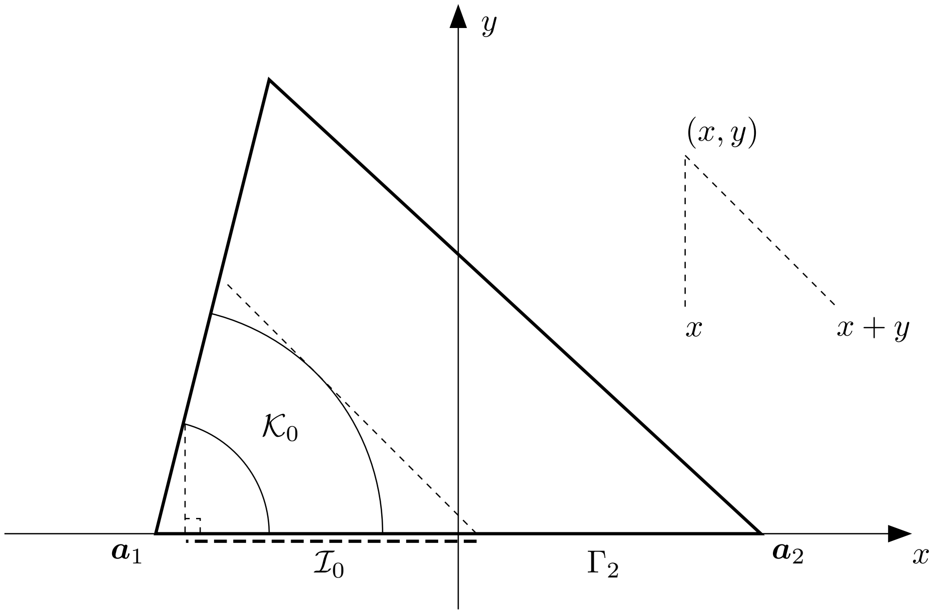

Figure 2.2

PROOF. Let φ be any function in $V_\alpha^{s-k-\frac{1}{2}}(\Lambda)$. As φ belongs to $H^{s-k-\frac{1}{2}}(-1+\delta, 1-\delta)$ for any fixed $\delta > 0$, and due to the assumption on the support of χ, Lemma 2.3 implies that $F_k(\varphi)$ belongs to $H^s(\mathcal{T}_\delta)$, where $\mathcal{T}_\delta$ is the intersection of $\mathcal{T}$ and of another triangle with edge $]-1+\delta, 1-\delta[\times\{0\}$ and angles at the endpoints of this edge larger than κ_1 and κ_2, respectively. So, by symmetry, it remains to prove that $F_k(\varphi)$ belongs to $V_\alpha^s(\mathcal{S})$, where $\mathcal{S}$ denotes the sector with vertex $\boldsymbol{a}_1$ and edges containing the segments $\boldsymbol{a}_1\boldsymbol{a}_2$ and $\boldsymbol{a}_1\boldsymbol{a}_3$, and with radius $\frac{R}{2}$, with R equal to the smallest of the lengths of $\boldsymbol{a}_1\boldsymbol{a}_2$ and $\boldsymbol{a}_1\boldsymbol{a}_3$. Of course the weight on $\mathcal{S}$ is now bounded from above and below by a constant times the distance ρ_1 to $\boldsymbol{a}_1$.
We introduce the domain (see Figure 2.2)

$$\mathcal{K}_0 = \big\{(x, y) \in \mathcal{S};\ \frac{R}{4} < \rho_1(x, y) \le \frac{R}{2}\big\}.$$

Due to the assumption on the support of χ, there exists an interval $\mathcal{I}_0$ contained in $]-1+\varepsilon, 1-\varepsilon[$ for some $\varepsilon > 0$ such that the values of $F_k(\varphi)$ on $\mathcal{K}_0$ only depend on the values of φ on $\mathcal{I}_0$. From Lemma 2.3, we derive the estimate

$$\|F_k(\varphi)\|_{H^s(\mathcal{K}_0)} \le c\, \|\varphi\|_{H^{s-k-\frac{1}{2}}(\mathcal{I}_0)}.$$

Since both weights ρ and $1 - x^2$ are bounded together with their inverses ρ^{-1} and $(1-x^2)^{-1}$ on $\mathcal{K}_0$ and $\mathcal{I}_0$, respectively, we deduce the estimate

$$\|F_k(\varphi)\|_{V_{*\alpha}^s(\mathcal{K}_0)} \le c\, \|\varphi\|_{V_\alpha^{s-k-\frac{1}{2}}(\mathcal{I}_0)}. \tag{2.7}$$

The proof now follows from a dyadic partition argument. For any $j \geq 0$, let Φ_j denote the mapping $(x,y) \mapsto (-1+2^j\,(x+1), 2^j\,y)$ and set

$$\mathcal{K}_j = \Phi_j^{-1}(\mathcal{K}_0), \qquad \mathcal{I}_j = \Phi_j^{-1}(\mathcal{I}_0).$$

Setting $\varphi^j = \varphi \circ \Phi_j^{-1}$, we deduce from (2.7) the uniform estimate

$$\|F_k(\varphi^j)\|_{V^s_{*\alpha}(\mathcal{K}_0)} \leq c\,\|\varphi^j\|_{V_\alpha^{s-k-\frac{1}{2}}(\mathcal{I}_0)}. \tag{2.8}$$

We note that

$$F_k(\varphi^j) = 2^{jk} F_k(\varphi) \circ \Phi_j^{-1},$$

and we derive by change of variables that

$$\|F_k(\varphi)\|_{V^s_{*\alpha}(\mathcal{K}_j)} \leq c\,2^{j(s-\frac{\alpha}{2}-1)}\,\|F_k(\varphi)\circ\Phi_j^{-1}\|_{V^s_{*\alpha}(\mathcal{K}_0)},$$

$$\|\varphi\|_{V_\alpha^{s-k-\frac{1}{2}}(\mathcal{I}_j)} \geq c'\,2^{j(s-k-\frac{\alpha}{2}-1)}\,\|\varphi^j\|_{V_\alpha^{s-k-\frac{1}{2}}(\mathcal{I}_0)},$$

which, combined with (2.8), yields the uniform estimate

$$\|F_k(\varphi)\|_{V^s_{*\alpha}(\mathcal{K}_j)} \leq c\,\|\varphi\|_{V_\alpha^{s-k-\frac{1}{2}}(\mathcal{I}_j)}.$$

Summing up the square of the above inequalities for $j \geq 0$, we obtain that $F_k(\varphi)$ belongs to $V^s_{*\alpha}(\mathcal{S})$, together with the desired continuity property.

The second lemma involves other weighted spaces, defined as follows:

• On the edge Λ, for any nonnegative integer m, $H^m_\alpha(\Lambda)$ is the space of distributions φ in $\mathscr{D}'(\Lambda)$ such that $d^k\varphi\,(1-x^2)^{\frac{\alpha}{2}}$ belongs to $L^2(\Lambda)$ for all k, $0 \leq k \leq m$. For any positive real number s which is not an integer, $H^s_\alpha(\Lambda)$ is defined by interpolation between $H^{[s]+1}_\alpha(\Lambda)$ and $H^{[s]}_\alpha(\Lambda)$.

• On the triangle $\mathcal{T}$, for any nonnegative integer m, $H^m_{*\alpha}(\mathcal{T})$ is the space of distributions v in $\mathscr{D}'(\mathcal{T})$ such that $\partial^{\boldsymbol{k}} v\,\rho^{\frac{\alpha}{2}}$ belongs to $L^2(\mathcal{T})$ for all $\boldsymbol{k}$, $0 \leq |\boldsymbol{k}| \leq m$, where ρ still denotes the product of the distances to the points $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. For any positive real number s which is not an integer, $H^s_\alpha(\mathcal{T})$ is defined by interpolation between $H^{[s]+1}_{*\alpha}(\mathcal{T})$ and $H^{[s]}_{*\alpha}(\mathcal{T})$.

Lemma 2.6. *For any nonnegative integer k, any real number α and for any real number $s > k + \frac{1}{2}$, if the function χ has its support contained in $]-\cot\kappa_1, \cot\kappa_2[$ and is such that the mapping: $z \mapsto (1+|z|)^k\hat{\chi}(z)$ belongs to $H^s(\mathbb{R})$, the operator F_k is continuous from $H_\alpha^{s-k-\frac{1}{2}}(\Lambda)$ into $H^s_{*\alpha}(\mathcal{T})$.*

Proof. We first consider the case where $s - \frac{\alpha}{2}$ is not an integer. Taking K equal to the integer part of $s - k - \frac{\alpha}{2} - 1$, we observe from an obvious extension of Theorem I.7.1 that $H_\alpha^{s-k-\frac{1}{2}}(\Lambda)$ is the direct sum of $V_\alpha^{s-k-\frac{1}{2}}(\Lambda)$ and of the space $\mathbb{P}_{2K+1}(\Lambda)$. So the result is consequence from Lemma 2.5 and Proposition 2.4. The case where $s - \frac{\alpha}{2}$ is an integer follows by an interpolation argument.

3. Lifting of traces into the square.

The general result consists in lifting a set of polynomial traces on the four edges of the square which satisfy all the compatibility conditions described in Corollary I.5.9. The notation for the vertices $\boldsymbol{a}_i$ and the edges Γ_ℓ of the square Θ are specified in Figure I.5.3. We begin with two preliminary results concerning traces on one edge of the square.

3.1. Lifting on one edge of the square.

We now choose the triangle $\mathcal{T}$ introduced in the previous section to be equilateral, so that the vertex $\boldsymbol{a}_3$ has coordinates $(0, \sqrt{3})$. We then use an homography which maps the trapezium $\mathcal{T}^* = \{(x, y) \in \mathcal{T}; y < 1\}$ onto the square $\Theta = \Lambda^2$: indeed, the change of variables: $x = (1 - \frac{y}{\sqrt{3}})X$ maps $\mathcal{T}^*$ onto the rectangle $]-1, 1[\times]0, 1[$ and the dilatation: $y = \frac{1+Y}{2}$ maps this rectangle onto Θ (see Figure 3.1). We denote by $\mathscr{F}$ the one-to-one mapping from Θ onto $\mathcal{T}^*$:

$$(x, y) = \mathscr{F}(X, Y) = \Big((1 - \frac{1+Y}{2\sqrt{3}})X\ ,\ \frac{1+Y}{2}\Big). \tag{3.1}$$

Since the edge Γ_2 is the image of Λ by $\mathscr{F}^{-1}$, relying on Proposition 2.4, we can build a lifting operator of traces on Γ_2 into Θ. However we have rather use a modification of it, such that its range is contained in a space of polynomials vanishing on the opposite edge Γ_4 at the order $m-1$. Of course, similar results can easily be derived on the other edges by rotation. We now state and prove the existence of such operators.

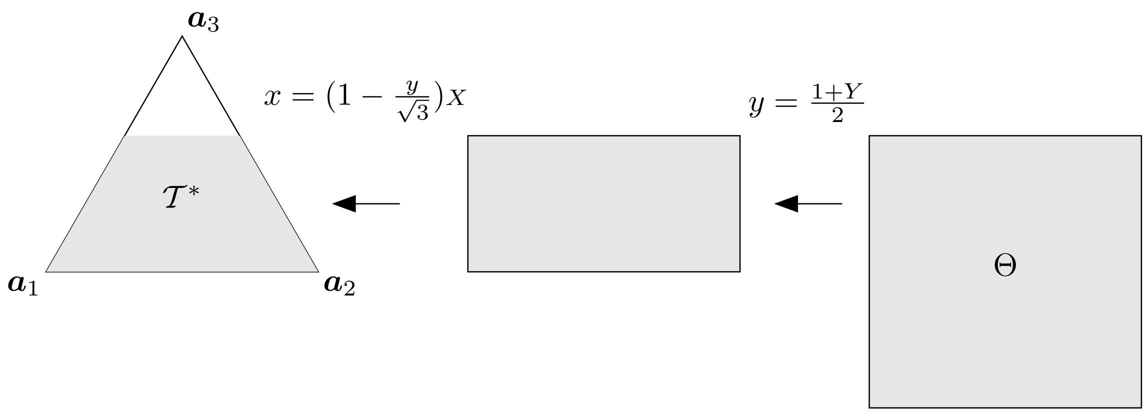

Figure 3.1

Proposition 3.1. *For any edge Γ_ℓ, $1 \le \ell \le 4$, and for any positive integer m, there exists a linear operator $\widetilde{\mathcal{R}}_m^{\Theta,\ell}$:*
(i) which is continuous from $\prod_{k=0}^{m-1} H^{s-k-\frac{1}{2}}(\Gamma_\ell)$ into $H^s(\Theta)$ for any real number

$s > m - \frac{1}{2}$,
(ii) which satisfies

$$\begin{aligned}&\forall G = (g_0, \dots, g_{m-1}) \in \prod_{k=0}^{m-1} H^{s-k-\frac{1}{2}}(\Gamma_\ell), \\ &\qquad\qquad \partial^k_{n_\ell} \widetilde{\mathcal{R}}^{\Theta,\ell}_m(G)\big|_{\Gamma_\ell} = g_k, \quad 0 \le k \le m-1, \\ &\qquad\qquad \partial^k_{n_{\ell+2}} \widetilde{\mathcal{R}}^{\Theta,\ell}_m(G)\big|_{\Gamma_{\ell+2}} = 0, \quad 0 \le k \le m-1,\end{aligned} \tag{3.2}$$

(iii) such that for any nonnegative integer N, the operator $\widetilde{\mathcal{R}}^{\Theta,\ell}_m$ maps the space of polynomials $\mathbb{P}_N(\Gamma_\ell)^m$ into $\mathbb{P}_{N+3m-2}(\Theta)$.

The proof of this proposition requires a preliminary lemma, concerning the continuity of the operator defined on functions in $L^2(\Gamma_2)$ by

$$\breve{F}_k(\varphi)(\mathrm{X}, \mathrm{Y}) = (-2)^k F_k(\varphi) \circ \mathscr{F}(\mathrm{X}, \mathrm{Y}). \tag{3.3}$$

Indeed, the appropriate trace properties of this operator follow from the formula

$$\partial_{n_2} = \frac{\mathrm{X}}{2\sqrt{3}}\,\partial_x - \frac{1}{2}\,\partial_y, \tag{3.4}$$

Lemma 3.2. *For any nonnegative integer k and for any real number $s > k + \frac{1}{2}$, the mapping $\breve{F}_k$ is continuous from $H^{s-k-\frac{1}{2}}(\Gamma_2)$ into $H^s(\Theta)$ and satisfies*

$$\partial^\ell_{n_2} \breve{F}_k(g)\big|_{\Gamma_2} = \delta_{k\ell}\, g, \quad 0 \le \ell \le k. \tag{3.5}$$

Moreover, $\breve{F}_k$ maps $\mathbb{P}_N(\Gamma_2)$ into $\mathbb{P}_{N+k}(\Theta)$ and if φ belongs to $\mathbb{P}_N(\Gamma_2)$, the degree of $\breve{F}_k(\varphi)$ with respect to X is $\le N$.

Proof. The continuity property is obvious when s is an integer and follows by an interpolation argument otherwise. The assertions about the polynomial traces are straightforward consequences of definition (2.1) together with (3.4).

Proof of Proposition 3.1. For any $G = (g_0, \dots, g_{m-1})$ in $\prod_{k=0}^{m-1} H^{s-k-\frac{1}{2}}(\Gamma_2)$, we first set

$$H_k = \breve{F}_k\big(g_k - \sum_{\ell=0}^{k-1} \partial^k_{n_2} H_\ell\big) \quad \text{and} \quad \breve{\mathcal{R}}^{\Theta,2}_m(G) = \sum_{k=0}^{m-1} H_k.$$

Due to the inequality

$$\|\partial^k_{n_2} H_\ell\|_{H^{s-k-\frac{1}{2}}(\Gamma_2)} \le c\|g_\ell\|_{H^s(\Theta)},$$

which is a consequence of the trace Theorem I.4.4, the continuity of the mapping $\breve{\mathcal{R}}_m^{\Theta,2}$ follows from the previous lemma. It remains to achieve the null trace conditions on the opposite side (here Γ_4). To this aim we introduce the unique polynomial χ_m in $\mathbb{P}_{2m-1}(\Lambda)$ such that

$$\begin{cases} \chi_m(-1) = 1 \quad \text{and} \quad \chi_m'(-1) = \chi_m''(-1) = \cdots = d^{m-1}\chi_m(-1) = 0, \\ \chi_m(1) = \chi_m'(1) = \cdots = d^{m-1}\chi_m(1) = 0. \end{cases} \tag{3.6}$$

Then, for G in $\mathbb{P}_N(\Gamma_2)^m$, the function $\widetilde{\mathcal{R}}_m^{\Theta,2}(G)$ defined by

$$\widetilde{\mathcal{R}}_m^{\Theta,2}(G)(\mathrm{x},\mathrm{y}) = \chi_m(\mathrm{y})\breve{\mathcal{R}}_m^{\Theta,2}(G)(\mathrm{x},\mathrm{y})$$

satisfies all the boundary conditions in (3.2) and its degree with respect to x is $\leq N$ and with respect to y less than $N + 3m - 2$. It is also clear that $\widetilde{\mathcal{R}}_m^{\Theta,2}$ acts between Sobolev spaces as stated in part (i) of the proposition.

We now prove, as a consequence of the previous proof, a statement about polynomial trace lifting in weighted spaces, in a form which we will directly use for interpolation of polynomial spaces in Section 4. The following corollary makes use of the weighted spaces $H^s_{0,\beta}(\Lambda \times \mathcal{I})$ defined as, cf. (I.7.25) with $\alpha = 0$,

$$H^s_{0,\beta}(\Lambda \times \mathcal{I}) = L^2(\Lambda; H^s_\beta(\mathcal{I})) \cap H^s(\Lambda; L^2_\beta(\mathcal{I})), \tag{3.7}$$

with the weighted space $H^s_\beta(\mathcal{I})$ associated with the weight t^β, $t \in \mathcal{I}$, cf Section I.7.1.

Corollary 3.3. *Let β be a real number > -1. There exists a linear operator $\breve{\mathcal{R}}^{\Lambda\times\mathcal{I}}$:*
(i) which is continuous from $H^{s-\frac{1+\beta}{2}}(\Lambda)$ into $H^s_{0,\beta}(\Lambda \times \mathcal{I})$ for any $s > \frac{1+\beta}{2}$,
(ii) which satisfies for all $\varphi \in H^{s-\frac{1+\beta}{2}}(\Lambda)$,

$$\breve{\mathcal{R}}^{\Lambda\times\mathcal{I}}(\varphi)\big|_{t=0} = \varphi, \tag{3.8}$$

(iii) which maps, for any nonnegative integer N, the space of polynomials $\mathbb{P}_N(\Lambda)$ into $\mathscr{C}^\infty(\overline{\mathcal{I}}, \mathbb{P}_N(\Lambda))$.

Proof. We define $\breve{\mathcal{R}}^{\Lambda\times\mathcal{I}}$ by a similar formula as (3.3) for $k = 0$:

$$\breve{\mathcal{R}}^{\Lambda\times\mathcal{I}}(\varphi)(\mathrm{x},t) = F_0(\varphi)\Big(\big(1 - \frac{t}{\sqrt{3}}\big)\mathrm{x}, t\Big).$$

We check that $\breve{\mathcal{R}}^{\Lambda\times\mathcal{I}}$ satisfies conditions *(ii)* and *(iii)*. The proof of *(i)* relies on the generalization to weighted spaces of Lemma 2.1: By a similar proof using partial Fourier transformation, we obtain that, under the same assumptions as in Lemma 2.1 on the convolution kernel χ, the operator F_0 is continuous from $H^{s-\frac{1+\beta}{2}}(\mathbb{R})$ into $H^s_{0,\beta}(\mathbb{R} \times \mathcal{I})$, where $H^s_{0,\beta}(\mathbb{R} \times \mathcal{I})$ is defined by (3.7) with $\mathbb{R}$ replacing Λ.

Now, we build a further operator $\mathcal{R}_m^{\Theta,\ell}$ which has the same continuity properties as the mapping $\widetilde{\mathcal{R}}_m^{\Theta,\ell}$, but which maps $\mathbb{P}_N(\Gamma_\ell)^m$ into $\mathbb{P}_N(\Theta)$ instead of $\mathbb{P}_{N+3m-2}(\Theta)$. To obtain this result, we use the "degree reduction" operator introduced in the next lemma.

Lemma 3.4. *For any positive integer M and for any integer $N \geq 2M$, there exists an operator r_N^M from $\mathbb{P}_{N+1}(\Lambda)$ into $\mathbb{P}_N(\Lambda)$ such that*

$$\forall \varphi_N \in \mathbb{P}_{N+1}(\Lambda), \quad d^\ell r_N^M(\varphi_N)(\pm 1) = d^\ell \varphi_N(\pm 1), \quad 0 \leq \ell \leq M-1, \tag{3.9}$$

and that the following uniform continuity property holds: There exists a positive constant c such that, for any integer $N \geq 2M$,

$$\forall \varphi_N \in \mathbb{P}_{N+1}(\Lambda), \quad \|r_N^M(\varphi_N)\|_{H^s(\Lambda)} \leq c\, \|\varphi_N\|_{H^s(\Lambda)}, \quad 0 \leq s \leq M. \tag{3.10}$$

Proof. For any polynomial φ_N of $\mathbb{P}_{N+1}(\Lambda)$, we write the expansion in the basis made by the Legendre polynomials L_n, $0 \leq n \leq N+1$: $\varphi_N = \sum_{n=0}^{N+1} \alpha_n L_n$ and we choose

$$r_N^M(\varphi_N)(\zeta) = \varphi_N(\zeta) - \alpha_{N+1}(-1)^M(1-\zeta^2)^M \frac{k_{N+1}}{k_{N+1-M}\,(N+1-M)(N-M)\dots(N+2-2M)}\, d^{(M)} L_{N+1-M},$$

where k_n denotes the coefficient of ζ^n in $L_n(\zeta)$. It is easy to check that the coefficient of ζ^{N+1} in $r_N^M(\varphi_N)(\zeta)$ is equal to 0, hence $r_N^M(\varphi_N)$ belongs to $\mathbb{P}_N(\Lambda)$. Property (3.9) is also obviously satisfied. To check the stability property (3.10), we first use the formula (see [19]) $k_n = 2^{-n}\frac{(2n)!}{n!^2}$, together with Stirling's formula, to derive that the ratio k_{N+1}/k_{N+1-M} is bounded independently of N. So the desired property follows from the estimates

$$\|\varphi_N\|_{L^2(\Lambda)} \geq c\, N^{-\frac{1}{2}}\, |\alpha_{N+1}| \quad \text{and} \quad |\varphi_N|_{H^M(\Lambda)} \geq c\, N^{M-\frac{1}{2}}\, |\alpha_{N+1}|, \tag{3.11}$$

$$\begin{aligned} N^{-M}\|(1-\zeta^2)^M d^M L_{N+1-M}\|_{L^2(\Lambda)} \leq c\, N^{-\frac{1}{2}} \quad \text{and} \\ N^{-M}|(1-\zeta^2)^M d^M L_{N+1-M}|_{H^M(\Lambda)} \leq c\, N^{M-\frac{1}{2}}. \end{aligned} \tag{3.12}$$

that we prove successively.
1) Since the $(L_n)_{n\geq 0}$ is an orthogonal family in $L^2(\Lambda)$, it is clear that

$$\|\varphi_N\|_{L^2(\Lambda)} \geq |\alpha_{N+1}|\, \|L_{N+1}\|_{L^2(\Lambda)},$$

and the first estimate of (3.10) follows from the formula (see [19] or [11, Thm 3.2]) $\|L_n\|^2_{L^2(\Lambda)} = \frac{2}{2n+1}$. On the other hand, using iteratively the formula in [11, Thm 3.3]

$$L_n = \frac{1}{2n+1}\big(L'_{n+1} - L'_{n-1}\big), \quad n \geq 1, \tag{3.13}$$

we see that $d^M L_{N+1}$ is the sum of $(2N+1)(2N-1)\dots(2N+3-2M)L_{N+1-M}$ and of a polynomial of degree $\le N-M$, hence it follows from the orthogonality property that

$$|\varphi_N|_{H^M(\Lambda)} \ge c\, N^M\, |\alpha_{N+1}| \|L_{N+1-M}\|_{L^2(\Lambda)},$$

whence the second estimate of (3.11).
2) From the differential equation [19]

$$\big((1-\zeta^2)\, L_n'\big)' + n(n+1)\, L_n = 0, \tag{3.14}$$

the following equation is easily derived by induction on M (see [11, form. (3.8)])

$$d^M\big((1-\zeta^2)^M\, d^M L_n\big) + (-1)^{M+1}\,(n-M+1)(n-M+2)\cdots(n+M)L_n = 0.$$

Then, recalling that $\|L_n\|^2_{L^2(\Lambda)}$ is equal to $\frac{2}{2n+1}$, we prove the first estimate in (3.12) by multiplying this equation for $n = N+1-M$ by L_{N+1-M} and integrating M times by parts, the second one by computing the norm in $L^2(\Lambda)$ of the two terms in the equation.

Applying iteratively this lemma gives the next more general result.

Corollary 3.5. *For any positive integers M and K, and for any integer $N \ge 2M$, there exists an operator $r^M_{N,K}$ from $\mathbb{P}_{N+K}(\Lambda)$ into $\mathbb{P}_N(\Lambda)$ such that*

$$\forall \varphi_N \in \mathbb{P}_{N+K}(\Lambda), \quad d^\ell r^M_{N,K}(\varphi_N)(\pm 1) = d^\ell \varphi_N(\pm 1), \quad 0 \le \ell \le M-1, \tag{3.15}$$

and that the following uniform continuity property holds: There exists a positive constant c such that

$$\forall \varphi_N \in \mathbb{P}_{N+K}(\Lambda), \quad \|r^M_{N,K}(\varphi_N)\|_{H^s(\Lambda)} \le c\, \|\varphi_N\|_{H^s(\Lambda)}, \quad 0 \le s \le M. \tag{3.16}$$

We are now in a position to prove the next theorem.

Theorem 3.6. *Let m and M be two positive integers with $m \le M$. For each edge Γ_ℓ, $1 \le \ell \le 4$, and for each integer $N \ge 2M$, there exists a linear operator $\mathcal{R}^{\Theta,\ell}_{N,m}$*
(i) which acts from $\mathbb{P}_N(\Gamma_\ell)^m$ into $\mathbb{P}_N(\Theta)$;
(ii) such that

$$\begin{aligned} &\forall G = (g_0, \dots, g_{m-1}) \in \mathbb{P}_N(\Gamma_\ell)^m, \\ &\qquad\qquad \partial^k_{n_\ell} \mathcal{R}^{\Theta,\ell}_{N,m}(G)\big|_{\Gamma_\ell} = g_k, \quad 0 \le k \le m-1, \\ &\qquad\qquad \partial^k_{n_{\ell+2}} \mathcal{R}^{\Theta,\ell}_{N,m}(G)\big|_{\Gamma_{\ell+2}} = 0, \quad 0 \le k \le m-1, \end{aligned} \tag{3.17}$$

(iii) which satisfies the following uniform continuity property for any real number s such that $m-\frac{1}{2}<s\leq M$:

$$\forall G=(g_0,\dots,g_{m-1})\in\mathbb{P}_N(\Gamma_\ell)^m,\quad \|\mathcal{R}^{\Theta,\ell}_{N,m}(G)\|_{H^s(\Theta)}\leq c\sum_{k=0}^{m-1}\|g_k\|_{H^{s-k-\frac{1}{2}}(\Gamma_\ell)},\tag{3.18}$$

where the constant c is independent of N.

Proof. In the case $\ell=2$, we recall that the operator $\widetilde{\mathcal{R}}^{\Theta,2}_m$ that we have built above satisfies the desired properties of continuity and trace but that it maps the space $\mathbb{P}_N(\Gamma_2)^m$ into $\mathbb{P}_{N+3m-2}(\Theta)$. Moreover, it can be checked from Proposition 3.1 that, for any G in $\mathbb{P}_N(\Gamma_2)^m$, the degree of $\widetilde{\mathcal{R}}^{\Theta,2}_m(G)$ with respect to $\textsc{x}$ is $\leq N$ and its degree with respect to $\textsc{y}$ is $\leq N+3m-2$. That leads us to take

$$\forall \textsc{x}\in\overline{\Lambda},\quad \mathcal{R}^{\Theta,2}_{N,m}(G)(\textsc{x},.)=r^{M\,(Y)}_{N,3m-2}\widetilde{\mathcal{R}}^{\Theta,2}_m(G)(\textsc{x},.)$$

where $r^{M\,(Y)}_{N,3m-2}$ stands for the operator $r^{M}_{N,3m-2}$ applied to the $\textsc{y}$-variable. The stability properties of the new operator $\mathcal{R}^{\Theta,2}_{N,m}$ are a straightforward consequence of Proposition 3.1 and Corollary 3.5, together with the fact that the space $H^s(\Theta)$ coincides with the space $L^2(\Lambda;H^s(\Lambda))\cap H^s(\Lambda;L^2(\Lambda))$ with an equivalent norm, see Lemma I.4.1.

3.2. Lifting of flat traces on one edge of the square.

Here, "flat" traces means polynomials in the space $\mathbb{P}^{m,0}_N(\Gamma_\ell)$ introduced in Notation 1.2, and the idea is to define a lifting operator which preserves these cancellation properties: More precisely, it maps traces in $\mathbb{P}^{m,0}_N(\Gamma_\ell)^m$ into a polynomial in $\mathbb{P}_N(\Theta)$ which has null traces and normal derivatives up to the order $m-1$ on the other edges.

As previously, the lifting operator is built in four steps, for traces on the edge Γ_2 for instance:
(i) We consider the mapping $\mathcal{M}_{-m}$ which associates with any function g on Γ_2 the function $(1-\textsc{x}^2)^{-m}\,g$. Clearly, the image by $\mathcal{M}_{-m}$ of $\mathbb{P}^{m,0}_N(\Gamma_2)$ is $\mathbb{P}_{N-2m}(\Gamma_2)$.
(ii) We apply the operator $\widetilde{\mathcal{R}}^{\Theta,2}_m$ and note that it sends $\mathbb{P}_{N-2m}(\Gamma_2)^m$ into polynomials of degree $\leq N-2m$ with respect to $\textsc{x}$ and $\leq N+m-2$ with respect to $\textsc{y}$.
(iii) So, when m is ≥ 2, we must apply the degree reduction operator $r^{M\,(Y)}_{N,m-2}$ with respect to the $\textsc{y}$-variable.
(iv) We introduce the mapping $\mathcal{M}_m$ which associates with any function v on Θ the function $(1-\textsc{x}^2)^m\,v$ and we apply this operator in order to recover the nullity properties on the two edges Γ_1 and Γ_3 of Θ.

So, on the edge Γ_2, the lifting operator is built from the formula

$$\mathcal{R}^{\Theta,2}_{N,m,0}(G) = (\mathcal{M}_m \circ r^{M\,(Y)}_{N,m-1} \circ \widetilde{\mathcal{R}}^{\Theta,2}_m \circ \mathcal{M}_{-m})(G). \tag{3.19}$$

We now investigate successively the continuity properties of the four operators involved in this definition.

(i) Recalling that the spaces $V^t(\Lambda)$ are introduced in Section I.3.2, see (I.3.24), we consider the spaces, for any positive real number t,

$$Z^{t,m}(\Lambda) = \begin{cases} V^t(\Lambda), & \text{if } t \le m - \frac{1}{2}, \\ \big\{\varphi \in H^t(\Lambda);\ d^k\varphi(\pm 1) = 0,\ 0 \le k \le m-1\big\}, & \text{if } t > m - \frac{1}{2}. \end{cases} \tag{3.20}$$

Lemma 3.7. *For any real number $t > 0$, the operator $\mathcal{M}_{-m}$ is continuous from $Z^{t,m}(\Lambda)$ into $H^t_{2m}(\Lambda)$.*

Proof. We first observe that the derivative of order k of any product $\varphi\,(1-x^2)^{-m}$ is a linear combination of the $d^\ell\varphi\,(1-x^2)^{-m-k+\ell}$, $0 \le \ell \le k$. So $\mathcal{M}_{-m}$ is an isomorphism from $V^t(\Lambda)$ onto $V^t_{2m}(\Lambda)$. We successively consider the cases $t \le m - \frac{1}{2}$ and $t > m - \frac{1}{2}$.
1) When $t \le m - \frac{1}{2}$, $V^t_{2m}(\Lambda)$ and $H^t_{2m}(\Lambda)$ coincide, whence the result.
2) When $t > m - \frac{1}{2}$ and $t - \frac{1}{2}$ is not an integer; it follows from Theorem I.3.6 that $Z^{t,m}(\Lambda)$ is equal to the direct sum of $V^t(\Lambda)$ and of the space of polynomials of degree $< 2[t-\frac{1}{2}]+1$ that vanish in ± 1 together with their derivatives up to the order $m-1$. This yields the result in this case, and an interpolation argument gives the result when $t - \frac{1}{2}$ is an integer.

(ii) We state the properties of the operator $\widetilde{\mathcal{R}}^{\Theta,2}_m$ introduced in Proposition 3.1, now in the weighted spaces. They involve the spaces $H^s_{2m,0}(\Theta)$, i.e. the spaces $H^s_{\alpha\beta}(\Theta)$ with $\alpha = 2m$ and $\beta = 0$, cf Section I.7.2. Let us recall from (I.7.25) that there holds

$$H^s_{2m,0}(\Theta) = L^2_{2m}(\Lambda; H^s(\Lambda)) \cap H^s_{2m}(\Lambda; L^2(\Lambda)). \tag{3.21}$$

Lemma 3.8. *For any positive integer m, the operator $\widetilde{\mathcal{R}}^{\Theta,2}_m$ is continuous from $\prod_{k=0}^{m-1} H^{s-k-\frac{1}{2}}_{2m}(\Lambda)$ into $H^s_{2m,0}(\Theta)$ for any real number $s > m - \frac{1}{2}$.*

Proof. Owing to Lemma 2.6, the operators $\breve{F}_k$ defined in (3.3) are continuous from $H^{s-k-\frac{1}{2}}_{2m}(\Lambda)$ into the space of distributions v in $\mathscr{D}'(\mathcal{T})$ such that $\partial^k v\,\rho^m$ belongs to $L^2(\Theta)$, where ρ is the product of the distances to the two endpoints of Γ_2. Since ρ is larger than the product of the distances to Γ_1 and Γ_3, this space is imbedded in $H^s_{2m,0}(\Theta)$. So the operator $\breve{\mathcal{R}}^{\Theta,2}_m$ introduced in the proof of Proposition 3.1, hence the operator $\widetilde{\mathcal{R}}^{\Theta,2}_m$, satisfy the desired continuity properties.

(iii) The operator $r_{N,m-2}^{M\,(Y)}$ is applied with respect to the Y variable. So its continuity from $H^s_{2m,0}(\Theta)$ into itself follows from the definition of this space and Corollary 3.5.

(iv) Finally, we introduce the spaces, for any real number $s > m - \frac{1}{2}$,

$$Z^{s,m\,(X)}(\Theta) = \big\{v \in H^s(\Theta);\ d_X^k \varphi = 0 \text{ on}\,\Gamma_1 \text{ and}\,\Gamma_3,\ 0 \le k \le m-1\big\}. \tag{3.22}$$

We skip the proof of the next lemma which relies on the same arguments as for Lemma 3.7, combined with tensorization properties.

Lemma 3.9. *For any real number $s > m - \frac{1}{2}$, the operator $\mathcal{M}_m$ is continuous from $H^s_{2m,0}(\Theta)$ into $Z^{s,m\,(X)}(\Theta)$.*

The next proposition is now a consequence of the previous results.

Proposition 3.10. *Let m and M be two positive integers with $m \le M$. For each edge Γ_ℓ, $1 \le \ell \le 4$, and for each integer $N \ge 2M$, there exists a linear operator $\mathcal{R}^{\Theta,\ell}_{N,m,0}$*
(i) which acts from $\mathbb{P}_N^{m,0}(\Gamma_\ell)^m$ into $\mathbb{P}_N(\Theta)$;
(ii) such that

$$\begin{aligned}
&\forall G = (g_0, \dots, g_{m-1}) \in \mathbb{P}_N^{m,0}(\Gamma_\ell)^m,\\
&\qquad\qquad \partial^k_{n_\ell} \mathcal{R}^{\Theta,\ell}_{N,m,0}(G)\big|_{\Gamma_\ell} = g_k, \quad 0 \le k \le m-1,\\
&\qquad\qquad \partial^k_{n_{\ell'}} \mathcal{R}^{\Theta,\ell}_{N,m,0}(G)\big|_{\Gamma_{\ell'}} = 0, \quad 0 \le k \le m-1, \quad \text{for } 1 \le \ell' \le 4, \ell' \ne \ell,
\end{aligned} \tag{3.23}$$

(iii) which satisfies the following continuity property for any real number s such that $m - \frac{1}{2} < s \le M$, when s does not belong to $\{m, m+1, \dots, 2m-1\}$,

$$\forall G = (g_0, \dots, g_{m-1}) \in \mathbb{P}_N(\Gamma_\ell)^m, \quad \|\mathcal{R}^{\Theta,\ell}_{N,m,0}(G)\|_{H^s(\Theta)} \le c \sum_{k=0}^{m-1} \|g_k\|_{H^{s-k-\frac{1}{2}}(\Gamma_\ell)}, \tag{3.24}$$

and when s belongs to $\{m, m+1, \dots, 2m-1\}$,

$$\begin{aligned}
&\forall G = (g_0, \dots, g_{m-1}) \in \mathbb{P}_N(\Gamma_\ell)^m,\\
&\quad \|\mathcal{R}^{\Theta,\ell}_{N,m,0}(G)\|_{H^s(\Theta)} \le c\Big(\sum_{k=0}^{s-m-1} \|g_k\|_{H^{s-k-\frac{1}{2}}(\Gamma_\ell)} + \sum_{k=s-m}^{m-1} \|g_k\|_{V^{s-k-\frac{1}{2}}(\Gamma_\ell)}\Big),
\end{aligned} \tag{3.25}$$

where the constant c is independent of N.

Proof. We assume without restriction that $\ell = 2$ and define $\mathcal{R}^{\Theta,2}_{N,m,0}$ by (3.19). We have obviously (i) and (ii). Applying Lemma 3.7 with t equal to $s - k - \frac{1}{2}$, for $0 \le k \le m-1$ we obtain (3.25) in any case. We deduce (3.24) in the non-limit case, i.e. when $s \notin \{m, m+1, \ldots, 2m-1\}$ since then, by virtue of Theorem I.3.6 the norms $\|\cdot\|_{H^{s-k-\frac{1}{2}}(\Gamma_\ell)}$ and $\|\cdot\|_{V^{s-k-\frac{1}{2}}(\Gamma_\ell)}$ are equivalent on smooth functions vanishing on Γ_ℓ up to the order m.

Note that, however, this equivalence is no longer true in the limit case $s \in \{m, m+1, \ldots, 2m-1\}$.

3.3. The general polynomial lifting onto the square.

We are now in a position to prove the general result of this section, which deals with the lifting of traces on the four edges of the square. From Corollary I.5.9, the existence of a polynomial lifting requires that all the m^2 compatibility conditions must be satisfied by the traces on the edges Γ_i and Γ_{i+1} at each vertex $\boldsymbol{a}_i$, $1 \le i \le 4$ (with the obvious convention $\Gamma_5 = \Gamma_1$). They read

$$(\partial_{\tau_i}^n g_k^i)(\boldsymbol{a}_i) = (-1)^k\, (\partial_{\tau_{i+1}}^k g_n^{i+1})(\boldsymbol{a}_i), \qquad 0 \le k, n \le m-1. \tag{3.26}$$

Let $\mathbb{P}_N^{(m)}(\partial\Theta)$ be the subspace of polynomials in $\prod_{\ell=1}^4 \mathbb{P}_N(\Gamma_\ell)^m$ satisfying these conditions for $1 \le i \le 4$.

Theorem 3.11. *Let m and M be two positive integers with $m \le M$. For each integer $N \ge 2M$, there exists a linear operator $\mathcal{R}_{N,m}^\Theta$:*
(i) which acts from $\mathbb{P}_N^{(m)}(\partial\Theta)$ into $\mathbb{P}_N(\Theta)$;
(ii) such that

$$\begin{aligned} &\forall G = (G^1, G^2, G^3, G^4) \in \mathbb{P}_N^{(m)}(\partial\Theta), \text{ with } G^\ell = (g_0^\ell, \ldots, g_{m-1}^\ell), \\ &\qquad\qquad \partial_{n_\ell}^k \mathcal{R}_{N,m}^\Theta(G)\big|_{\Gamma_\ell} = g_k^\ell, \quad 0 \le k \le m-1, \end{aligned} \tag{3.27}$$

(iii) which satisfies the following continuity property for any real number s such that $m - \frac{1}{2} < s \le M$ and $s \notin \{m, m+1, \ldots, 2m-1\}$,

$$\begin{aligned} &\forall G = (G^1, G^2, G^3, G^4) \in \mathbb{P}_N^{(m)}(\partial\Theta), \text{ with } G^\ell = (g_0^\ell, \ldots, g_{m-1}^\ell), \\ &\qquad\qquad \|\mathcal{R}_{N,m}^\Theta(G)\|_{H^s(\Theta)} \le c \sum_{\ell=1}^4 \sum_{k=0}^{m-1} \|g_k^\ell\|_{H^{s-k-\frac{1}{2}}(\Gamma_\ell)}. \end{aligned} \tag{3.28}$$

Proof. The operator $\mathcal{R}_{N,m}^\Theta$ is simply defined by the following sequence of formulas:

$$H_2 = \mathcal{R}_{N,m}^{\Theta,2}(g_0^2, \ldots, g_{m-1}^2) \quad \text{and} \quad H_4 = \mathcal{R}_{N,m}^{\Theta,4}(g_0^4, \ldots, g_{m-1}^4),$$

where $\mathcal{R}^{\Theta,\ell}_{N,m}$ is the operator of Theorem 3.6,

$$h^1_k = g^1_k - \partial^k_{n_1} H_2 - \partial^k_{n_1} H_4 \quad \text{and} \quad h^3_k = g^3_k - \partial^k_{n_3} H_2 - \partial^k_{n_3} H_4, \quad 0 \le k \le m-1,$$

$$H_1 = \mathcal{R}^{\Theta,1}_{N,m,0}(h^1_0, \dots, h^1_{m-1}) \quad \text{and} \quad H_3 = \mathcal{R}^{\Theta,3}_{N,m,0}(h^3_0, \dots, h^3_{m-1}),$$

where $\mathcal{R}^{\Theta,\ell}_{N,m,0}$ is the operator of Proposition 3.10

$$\mathcal{R}^{\Theta}_{N,m}(\Psi) = H_2 + H_4 + H_1 + H_3.$$

And its properties are ... a straightforward consequence of the previous pages.

The statement of the continuity property in the limit case $s = m$ is more complex and makes use of the notation in Corollary I.5.9 (see also Figure I.5.3). We leave the other cases to the reader.

Theorem 3.12. *When s is equal to m, the operator $\mathcal{R}^{\Theta}_{N,m}$ introduced in Theorem* 3.11 *satisfies the following continuity property*

$$\forall G = (G^1, G^2, G^3, G^4) \in \mathbb{P}^{(m)}_N(\partial\Theta), \text{ with } G^\ell = (g^\ell_0, \dots, g^\ell_{m-1}),$$
$$\|\mathcal{R}^{\Theta}_{N,m}(G)\|_{H^m(\Theta)} \le c \Bigg(\sum_{\ell=1}^{4} \sum_{k=0}^{m-1} \|g^\ell_k\|_{H^{m-k-\frac{1}{2}}(\Gamma_\ell)}$$
$$+ \sum_{i=1}^{4} \sum_{k=0}^{m-1} \Big(\int_0^1 \Big| (\partial^{m-1-k}_{\tau_i} g^i_k)(\boldsymbol{a}_i - t\,\boldsymbol{\tau}_i) - (-1)^k\, (\partial^k_{\tau_{i+1}} g^{i+1}_{m-1-k})(\boldsymbol{a}_i + t\,\boldsymbol{\tau}_{i+1}) \Big|^2 \frac{dt}{t} \Big)^{\frac{1}{2}} \Bigg). \tag{3.29}$$

Of course, this estimate is much simpler in the case $m = 1$ which is needed later on. It reads

$$\begin{aligned} &\forall g = (g^1, g^2, g^3, g^4) \in \mathbb{P}^{(1)}_N(\partial\Theta), \\ &\qquad \|\mathcal{R}^{\Theta}_{N,1}(g)\|_{H^1(\Theta)} \le c \Bigg(\sum_{\ell=1}^{4} \|g^\ell\|_{H^{\frac{1}{2}}(\Gamma_\ell)} \\ &\qquad\qquad + \sum_{i=1}^{4} \Big(\int_0^1 \Big| g^i(\boldsymbol{a}_i - t\,\boldsymbol{\tau}_i) - g^{i+1}(\boldsymbol{a}_i + t\,\boldsymbol{\tau}_{i+1}) \Big|^2 \frac{dt}{t} \Big)^{\frac{1}{2}} \Bigg). \end{aligned} \tag{3.30}$$

3.4. About the lifting in $\mathcal{P}_N(\Theta)$.

Only for this section, we consider the space $\mathcal{P}_N(\Theta)$ of restrictions to the square Θ of polynomials with total degree $\le N$. We are interested in the action of the trace operator $\gamma^{(1)} = (\gamma^\ell_0)_{1 \le \ell \le 4}$ on this space.

Since the dimension of the space $\mathcal{P}_N(\Theta)$ is equal to $\frac{(N+1)(N+2)}{2}$ and the kernel of the operator $\gamma^{(1)}$ in $\mathcal{P}_N(\Theta)$ coincides with the space $(1-x^2)(1-y^2)\,\mathcal{P}_{N-4}(\Theta)$, it is clear that the dimension of its range $\mathcal{P}_N^{(1)}(\partial\Theta)$ is equal to $2(2N-1)$. On the other hand, we have the imbedding

$$\mathcal{P}_N^{(1)}(\partial\Theta) \subset \prod_{\ell=1}^{4} \mathbb{P}_N(\Gamma_\ell),$$

and the dimension of the space in the right-hand side is $4(N+1)$. Therefore 6 independent compatibility conditions have to be fulfilled.

It follows from Theorem I.5.2 that only 1 compatibility conditions must be enforced at each of the 4 vertices of Θ, in order to have a smooth lifting. However the previous dimension arguments yield that they are not sufficient to obtain a lifting in $\mathcal{P}_N(\Theta)$: 2 extra conditions must be enforced on elements of the product space $\prod_{\ell=1}^4 \mathbb{P}_N(\Gamma_\ell)$ to be the image of a polynomial in $\mathcal{P}_N(\Theta)$ by $\gamma^{(1)}$. These two extra conditions can be easily identified: The leading coefficients of the polynomial traces on two opposite edges must coincide.

Remark 3.13. Extra compatibility conditions are also necessary when Θ is replaced with any polygon with 4 sides or more.

For any 4-tuple in $\prod_{\ell=1}^4 \mathbb{P}_N(\Gamma_\ell)$ satisfying the compatibility conditions at each vertex plus the further 2 conditions, an algebraic lifting can be constructed. This lifting is certainly not unique and we are now interested in exhibiting such a lifting that could be stable in the natural norms. The following lemma states that this quest is hopeless, even in the simplest case $m=1$ we deal with. A posteriori, this is not too surprising since the additional compatibility conditions are no more local.

Lemma 3.14. *For ℓ in $\{1,2,3,4\}$, let $\mathcal{R}^{\Theta,\ell}$ be a mapping:*
(i) which acts from $\mathbb{P}_N(\Gamma_\ell)$ into $\mathcal{P}_N(\Theta)$ for all integers N,
(ii) such that

$$\forall g \in \mathbb{P}_N(\Gamma_\ell), \quad \mathcal{R}^{\Theta,\ell}(g)\big|_{\Gamma_\ell} = g.$$

Then the norm of $\mathcal{R}^{\Theta,\ell}$ from each $\mathbb{P}_N(\Gamma_\ell)$ provided with the norm of $H^{\frac{1}{2}}(\Gamma_\ell)$ into $\mathcal{P}_N(\Theta)$ provided with the norm of $H^1(\Theta)$ is larger than a constant times $N^{\frac{1}{2}}$.

Proof. Let us assume that such an operator $\mathcal{R}^{\Theta,\ell}$ exists, for instance for $\ell = 2$. We take

$$g = \frac{L_N - L_{N-2}}{2N-1},$$

where $(L_n)_n$ denotes the family of Legendre polynomials. and and set: $v = \mathcal{R}^{\Theta,2}(g)$. Since the edge Γ_2 is contained in the line $Y = -1$, we have

$$v(X,Y) = g(X) + (Y+1)\,h(X,Y)$$

with h in $\mathcal{P}_{N-1}(\Theta)$. Thus

$$\partial_X v(X,Y) = g'(X) + (Y+1)\,\partial_X h(X,Y)$$

Let us recall that, by construction, g' is equal to L_{N-1}, hence is orthogonal to any polynomial with degree $< N-1$. This implies that

$$\|\partial_X v\|_{L^2(\Theta)} \geq \|g'\|_{L^2(\Gamma_2)} = \Big(\frac{2}{2N-1}\Big)^{\frac{1}{2}},$$

whence

$$\|v\|_{H^1(\Theta)} \geq \Big(\frac{2}{2N-1}\Big)^{\frac{1}{2}}.$$

On the other hand, a standard calculation gives

$$\|g\|_{H^{\frac{1}{2}}(\Gamma_2)} \leq \frac{c}{N},$$

which concludes the proof of the lemma.

In opposite, a lifting operator from traces on the three edges of a triangle $\mathcal{T}$ satisfying the continuity properties at the vertices into $\mathcal{P}_N(\mathcal{T})$ exists, which is stable into $H^s(\mathcal{T})$, see [4], [18] and [7, Thm 3.e.11].

3.5. Continuity of liftings in weighted Sobolev spaces.

Our aim is now to extend the results of §3.1-3.3 to the general weighted spaces introduced in Section I.7. Let α and β be real numbers > -1. In particular, with the weighted spaces $H^s_{\alpha\beta}(\Theta)$ introduced in §I.7.2, cf (I.7.25), we now wish to build a lifting operator for polynomial traces, analogous to the lifting $\mathcal{R}^{\Theta}_{N,m}$ constructed in Theorem 3.11, which should satisfy conditions *(i)* and *(ii)* of this theorem, plus continuity estimates in $H^s_{\alpha\beta}(\Theta)$. We also want to obtain a generalization of Corollary 3.3 to polynomial spaces with weighted norms $H^{s-(1+\beta)/2}_{\alpha}(\Lambda)$, and to prove a correspondingly statement for spaces of polynomials which vanish at the ends of the interval Λ.

For this we have to review the main statements of Sections 2 and 3.

Concerning Lemmas 2.1 and 2.3, we have already mentioned in the proof of Corollary 3.3 that Lemma 2.1 extends to weighted spaces: The operator F_k is continuous from $H^{s-k-(1+\beta)/2}(\mathbb{R})$ into $H^s_{0,\beta}(\mathbb{R}\times\mathcal{I})$ and, thus, from $H^{s-k-(1+\beta)/2}(\Lambda)$ into $H^s_{0,\beta}(\mathcal{T})$.

Concerning Lemmas 2.5 and 2.6, we have first to define some new weighted spaces on the triangle $\mathcal{T}$: let $V^m_{*\alpha,\beta}(\mathcal{T})$ and $H^m_{*\alpha,\beta}(\mathcal{T})$ be the spaces of functions v

such that $\rho^{\frac{\alpha}{2}+|\boldsymbol{k}|-m}\, y^{\frac{\beta}{2}}\partial^{\boldsymbol{k}}v$ and $\rho^{\frac{\alpha}{2}}\, y^{\frac{\beta}{2}}\partial^{\boldsymbol{k}}v$, respectively, belong to $L^2(\mathcal{T})$ for all $\boldsymbol{k}$, $|\boldsymbol{k}| \leq m$. The spaces $V^s_{*\alpha,\beta}(\mathcal{T})$ and $H^s_{*\alpha,\beta}(\mathcal{T})$ are then defined by interpolation.

We can check by similar arguments to Lemma 2.5 (dyadic partition near the corners) and Lemma 2.6 (splitting into polynomials and flat functions, plus interpolation) that the operator F_k is continuous from $V_\alpha^{s-k-(1+\beta)/2}(\Lambda)$ into $V^s_{*\alpha,\beta}(\mathcal{T})$, and from $H_\alpha^{s-k-(1+\beta)/2}(\Lambda)$ into $H^s_{*\alpha,\beta}(\mathcal{T})$, respectively.

From this last continuity property, we deduce that the lifting operator $\breve{F}_k$ defined on the square Θ by formula (3.3) is continuous from $H_\alpha^{s-k-(1+\beta)/2}(\Lambda)$ into $H^s_{\alpha,\beta}(\Theta)$ when $\alpha \geq 0$ and $\beta \geq 0$, since in this case the change of variables $\mathscr{F}$ defined in (3.1) is continuous from $H^s_{*\alpha,\beta}(\mathcal{T})$ into $H^s_{\alpha,\beta}(\Theta)$. It is also continuous from $H^s_{*\alpha,\beta}(\mathcal{T})$ into the weighted H^s space associated with the weight $(1-\textsc{x}^2)^{\alpha/2}(1+\textsc{y})^{\beta/2}$ on Θ for all $\alpha \geq 0$ and $\beta > -1$.

With this at hand we deduce that the operator $\widetilde{\mathcal{R}}^{\Theta,\ell}_m$ in Proposition 3.1 is continuous from $\prod_{k=0}^{m-1} H_\alpha^{s-k-(1+\beta)/2}(\Gamma_\ell)$ into $H^s_{\alpha,\beta}(\Theta)$ for all $\alpha \geq 0$ and $\beta > -1$.

Likewise, Corollary 3.3 extends to general nonnegative α's: We can check that the lifting operator $\breve{\mathcal{R}}^{\Lambda\times\mathcal{I}}$ on the rectangle $\Lambda\times\mathcal{I}$ is continuous from $H_\alpha^{s-(1+\beta)/2}(\Lambda)$ into $H^s_{\alpha,\beta}(\Lambda\times\mathcal{I})$ for all $\alpha \geq 0$ and $\beta > -1$.

To go further we need a degree reduction operator as exhibited in Lemma 3.4. We skip its proof, since we only have to replace the Legendre polynomials with the Jacobi polynomials $J_n^{\beta,\beta}$, orthogonal on Λ for the measure $(1-\zeta^2)^\beta\, d\zeta$.

Lemma 3.15. *For any real number $\beta > -1$ and any positive integer M and for any integer $N \geq 2M$, there exists an operator $r_N^{M,\beta}$ from $\mathbb{P}_{N+1}(\Lambda)$ into $\mathbb{P}_N(\Lambda)$ such that*

$$\forall \varphi_N \in \mathbb{P}_{N+1}(\Lambda), \quad d^\ell r_N^{M,\beta}(\varphi_N)(\pm 1) = d^\ell \varphi_N(\pm 1), \quad 0 \leq \ell \leq M-1, \tag{3.31}$$

and that the following uniform continuity property holds: There exists a positive constant c such that, for any integer $N \geq 2M$,

$$\forall \varphi_N \in \mathbb{P}_{N+1}(\Lambda), \quad \|r_N^{M,\beta}(\varphi_N)\|_{H^s_\beta(\Lambda)} \leq c\, \|\varphi_N\|_{H^s_\beta(\Lambda)}, \quad 0 \leq s \leq M. \tag{3.32}$$

Using this degree reduction operator we can immediately extend Theorem 3.6 to weighted spaces for any $\alpha \geq 0$ and $\beta > -1$.

Now we prove the analogue of Proposition 3.10 (lifting of flat traces on one edge of the square). Besides the generalization of Proposition 3.10, this statement will serve to settle the case $-1 < \alpha < 0$ which is still pending for the lifting of general traces.

Proposition 3.16. *Let α and β be real numbers > -1. Let m and M be two positive integers with $m \le M$. For each integer $N \ge 2M$, there exists a linear operator $\mathcal{R}^{\Theta,2,\beta}_{N,m,0}$*
(i) which acts from $\mathbb{P}^{m,0}_N(\Gamma_2)^m$ into $\mathbb{P}_N(\Theta)$;
(ii) such that

$$\begin{aligned}\forall G = (g_0, \dots, g_{m-1}) \in \mathbb{P}^{m,0}_N(\Gamma_2)^m,&\\ &\partial^k_{n_2} \mathcal{R}^{\Theta,2,\beta}_{N,m,0}(G)\big|_{\Gamma_2} = g_k, \quad 0 \le k \le m-1,\\ &\partial^k_{n_\ell} \mathcal{R}^{\Theta,2,\beta}_{N,m,0}(G)\big|_{\Gamma_\ell} = 0, \quad 0 \le k \le m-1, \quad \text{for } \ell = 1,3,4\end{aligned} \tag{3.33}$$

(iii) which satisfies the following continuity property for any real number s such that $m - \frac{1}{2} + \frac{\beta}{2} < s \le M$, when $s - \frac{\alpha+\beta}{2}$ does not belong to $\{1, 2, \dots, 2m-1\}$,

$$\forall G = (g_0, \dots, g_{m-1}) \in \mathbb{P}_N(\Gamma_2)^m, \quad \|\mathcal{R}^{\Theta,2,\beta}_{N,m,0}(G)\|_{H^s_{\alpha\beta}(\Theta)} \le c \sum_{k=0}^{m-1} \|g_k\|_{H_\alpha^{s-k-\frac{1+\beta}{2}}(\Gamma_2)}, \tag{3.34}$$

and when $s - \frac{\alpha+\beta}{2}$ belongs to $\{1, 2, \dots, 2m-1\}$,

$$\forall G = (g_0, \dots, g_{m-1}) \in \mathbb{P}_N(\Gamma_2)^m, \quad \|\mathcal{R}^{\Theta,2,\beta}_{N,m,0}(G)\|_{H^s_{\alpha\beta}(\Theta)} \le$$
$$c\Big(\sum_{k=0}^{s-\frac{\alpha+\beta}{2}-m-1} \|g_k\|_{H_\alpha^{s-k-\frac{1+\beta}{2}}(\Gamma_2)} + \sum_{k=\max\{0, s-\frac{\alpha+\beta}{2}-m\}}^{m-1} \|g_k\|_{V_\alpha^{s-k-\frac{1+\beta}{2}}(\Gamma_2)} \Big), \tag{3.35}$$

where the constant c is independent of N. In (3.35) we make the convention that the first sum is empty if $s - \frac{\alpha+\beta}{2} - m - 1 < 0$.

Proof. The operator $\mathcal{R}^{\Theta,2,\beta}_{N,m,0}$ is built by a formula similar to (3.19) with $r^{M\,(Y)}_{N,m-1}$ replaced by its weighted analogue:

$$\mathcal{R}^{\Theta,2,\beta}_{N,m,0}(G) = (\mathcal{M}_m \circ r^{M,\beta\,(Y)}_{N,m-1} \circ \widetilde{\mathcal{R}}^{\Theta,2}_m \circ \mathcal{M}_{-m})(G).$$

We follow the same four steps as in the proof leading to Proposition 3.10.
(i) Introducing, in the same spirit as (3.20)

$$Z^{t,m}_\alpha(\Lambda) = \begin{cases} V^t_\alpha(\Lambda), & \text{if } t \le m - \frac{1}{2} + \frac{\alpha}{2},\\ \big\{\varphi \in H^t_\alpha(\Lambda);\ d^k\varphi(\pm 1) = 0,\ 0 \le k \le m-1\big\}, & \text{if } t > m - \frac{1}{2} + \frac{\alpha}{2}.\end{cases}$$

we check that the operator $\mathcal{M}_{-m}$ maps $Z^{t,m}_\alpha(\Lambda)$ into $H^t_{\alpha+2m}(\Lambda)$, for all $t > 0$.
(ii) Since $\alpha + 2m$ is ≥ 0, as already mentioned, the lifting operator $\widetilde{\mathcal{R}}^{\Theta,2}_m$ is continuous from $\prod_{k=0}^{m-1} H^{s-k-(1+\beta)/2}_{\alpha+2m}(\Gamma_\ell)$ into $H^s_{\alpha+2m,\beta}(\Theta)$.

(iii) The continuity properties of $r_{N,m-1}^{M,\beta\,(Y)}$ are easily derived by applying iteratively Lemma 3.15.
(iv) Setting as in (3.22)

$$Z_{\alpha\beta}^{s,m\,(X)}(\Theta) = \{v \in H_{\alpha\beta}^s(\Theta);\ d_X^k\varphi = 0 \text{ on } \Gamma_1 \text{ and } \Gamma_3,\ 0 \le k \le m-1\},$$

we check that the operator $\mathcal{M}_m$ is continuous from $H_{\alpha+2m,\beta}^s(\Theta)$ into $Z_{\alpha\beta}^{s,m(X)}(\Theta)$.
Finally, we end the proof like for Proposition 3.10, using weighted norms for the estimates of $\|\mathcal{R}_{N,m,0}^{\Theta,2,\beta}(G)\|_{H_{\alpha\beta}^s(\Theta)}$ when necessary.
Note that, in contrast with Proposition 3.10, the condition $s > m - \frac{1}{2} + \frac{\beta}{2}$ does not imply that $s - \frac{\alpha+\beta}{2}$ is larger than m. That is why the situations when $s - \frac{\alpha+\beta}{2}$ belongs to $\{1, \dots, m-1\}$ have to be included in the limit cases.

By a similar proof we obtain the corollary which will be used for interpolating polynomial spaces $\mathbb{P}_N^{m,0}(\Lambda)$ in standard or weighted Sobolev norms:

Corollary 3.17. *Let α, β be two real numbers > -1. For any positive integer m, there exists a linear operator $\breve{\mathcal{R}}_{m,0}^{\Lambda\times\mathcal{I}}$:*
(i) which is continuous, for any real number $s > \frac{1+\beta}{2}$,

$$\begin{cases} \text{from } H_\alpha^{s-\frac{1+\beta}{2}}(\Lambda) \cap \mathbb{P}_N^{m,0}(\Lambda) \text{ into } H_{\alpha\beta}^s(\Lambda\times\mathcal{I}) & \text{if } s - \frac{\alpha+\beta}{2} \notin \{1,\dots,m\}, \\ \text{from } V_\alpha^{s-\frac{1+\beta}{2}}(\Lambda) \text{ into } H_{\alpha\beta}^s(\Lambda\times\mathcal{I}) & \text{if } s - \frac{\alpha+\beta}{2} \in \{1,\dots,m\}. \end{cases}$$

(ii) which satisfies the trace property $\breve{\mathcal{R}}_{m,0}^{\Lambda\times\mathcal{I}}(\varphi)\big|_{t=0} = \varphi$,
(iii) which maps the space of polynomials $\mathbb{P}_N^{m,0}(\Lambda)$ into $\mathscr{C}^\infty(\overline{\mathcal{I}}, \mathbb{P}_N^{m,0}(\Lambda))$.

Proof. With the lifting $\breve{\mathcal{R}}^{\Lambda\times\mathcal{I}}$ introduced in Corollary 3.3, we set

$$\breve{\mathcal{R}}_{m,0}^{\Lambda\times\mathcal{I}}(\varphi) = (\mathcal{M}_m \circ \breve{\mathcal{R}}^{\Lambda\times\mathcal{I}} \circ \mathcal{M}_{-m})(\varphi).$$

The points (ii) and (iii) are obvious. We check the continuity properties in point (i) as in the previous proof.

Now, we can use this corollary with $m = 1$ to treat the case $-1 < \alpha < 0$ for the generalization of Corollary 3.3 to any $\alpha > -1$:

Corollary 3.18. *Let α, β be two real numbers > -1. There exists a linear operator $\breve{\mathcal{R}}^{\Lambda\times\mathcal{I},\alpha}$:*
(i) which is continuous from $H_\alpha^{s-\frac{1+\beta}{2}}(\Lambda)$ into $H_{\alpha\beta}^s(\Lambda\times\mathcal{I})$ for any real number $s > \frac{1+\beta}{2}$ satisfying moreover $s - \frac{\alpha+\beta}{2} \neq 1$ if $\alpha < 0$,
(ii) which satisfies $\breve{\mathcal{R}}^{\Lambda\times\mathcal{I},\alpha}(\varphi)\big|_{t=0} = \varphi$ for all $\varphi \in H_\alpha^{s-\frac{1+\beta}{2}}(\Lambda)$,
(iii) which maps the space of polynomials $\mathbb{P}_N(\Lambda)$ into $\mathscr{C}^\infty(\overline{\mathcal{I}}, \mathbb{P}_N(\Lambda))$.

Proof. There are three cases.
1) If $\alpha \geq 0$, we simply set $\breve{\mathcal{R}}^{\Lambda\times\mathcal{I},\alpha} = \breve{\mathcal{R}}^{\Lambda\times\mathcal{I}}$.
2) In the case $-1 < \alpha < 0$, and if $s - \frac{\alpha+\beta}{2} < 1$, the spaces $H_\alpha^{s-\frac{1+\beta}{2}}(\Lambda)$ and $V_\alpha^{s-\frac{1+\beta}{2}}(\Lambda)$ coincide with each other and we set $\breve{\mathcal{R}}^{\Lambda\times\mathcal{I},\alpha} = \breve{\mathcal{R}}_{1,0}^{\Lambda\times\mathcal{I}}$.
3) In the remaining case $-1 < \alpha < 0$ and $s - \frac{\alpha+\beta}{2} > 1$, using Theorem I.7.1, we write any function φ in $H_\alpha^{s-\frac{1+\beta}{2}}(\Lambda)$ as

$$\varphi(x) = \varphi_0(x) + \varphi(-1)\,\frac{1-x}{2} + \varphi(1)\,\frac{1+x}{2},$$

with $\varphi_0 \in V_\alpha^{s-\frac{1+\beta}{2}}(\Lambda)$. Then the operator $\breve{\mathcal{R}}^{\Lambda\times\mathcal{I},\alpha}$ defined by

$$(\breve{\mathcal{R}}^{\Lambda\times\mathcal{I},\alpha}\varphi)(x,y) = (\breve{\mathcal{R}}_{1,0}^{\Lambda\times\mathcal{I}}\varphi_0)(x,y) + \varphi(-1)\,\frac{1-x}{2} + \varphi(1)\,\frac{1+x}{2},$$

satisfies the desired trace and continuity properties.

We conclude this section with the weighted analogues of Theorems 3.6 and 3.11. For the next statement, just like in the previous proof, we rely on the lifting of flat traces constructed in Proposition 3.16 in order to deal with the case $-1 < \alpha < 0$, the remaining cases being covered by a natural extension of the validity of Theorem 3.6, as already mentioned.

Theorem 3.19. *Let α and β be real numbers > -1. Let m and M be two positive integers with $m \leq M$. For each edge Γ_ℓ, $1 \leq \ell \leq 4$, and for each integer $N \geq 2M$, there exists a linear operator $\mathcal{R}_{N,m}^{\Theta,\ell,\alpha,\beta}$*
(i) which acts from $\mathbb{P}_N(\Gamma_\ell)^m$ into $\mathbb{P}_N(\Theta)$;
(ii) such that for any element $G = (g_0, \dots, g_{m-1})$ of $\mathbb{P}_N(\Gamma_\ell)^m$ the trace properties (3.17) *hold for $\mathcal{R}_{N,m}^{\Theta,\ell,\alpha,\beta}(G)$,*
(iii) which satisfies a continuity property as in (3.34) *for any real number s such that $m - \frac{1}{2} + \frac{\beta}{2} < s \leq M$, with the extra condition that $s - \frac{\alpha+\beta}{2}$ does not belong to $\{1, 2, \dots, 2m-1\}$ if $-1 < \alpha < 0$.*

The next theorem is easily derived by combining Proposition 3.16 and Theorem 3.19. It is the weighted analogue of Theorem 3.11, and makes use of the same polynomial trace space $\mathbb{P}_N^{(m)}(\partial\Theta)$ satisfying the compatibility conditions (3.26) introduced there.

Theorem 3.20. *Let α and β be real numbers > -1, and let m and M be two positive integers with $m \leq M$. For each integer $N \geq 2M$, there exists a linear operator $\mathcal{R}_{N,m}^{\Theta,\alpha,\beta}$*
(i) which acts from $\mathbb{P}_N^{(m)}(\partial\Theta)$ into $\mathbb{P}_N(\Theta)$;
(ii) such that for any element $G = (G^1, G^2, G^3, G^4) \in \mathbb{P}_N^{(m)}(\partial\Theta)$, the trace properties

(3.27) *hold for* $\mathcal{R}_{N,m}^{\Theta,\alpha,\beta}(G)$,
(iii) which satisfies the following continuity property for any real number $s \le M$ *with* $s > m - \frac{1}{2} + \max\{\frac{\alpha}{2}, \frac{\beta}{2}\}$ *and* $s - \frac{\alpha+\beta}{2} \notin \{1, 2, \dots, 2m-1\}$,

$$\begin{aligned}
\forall G = &(G^1, G^2, G^3, G^4) \in \mathbb{P}_N^{(m)}(\partial\Theta), \text{ with } G^\ell = (g_0^\ell, \dots, g_{m-1}^\ell), \\
&\|\mathcal{R}_{N,m}^{\Theta,\alpha\beta}(G)\|_{H^s_{\alpha\beta}(\Theta)} \le c \sum_{k=0}^{m-1} \Big(\|g_k^1\|_{H_\alpha^{s-k-\frac{1+\beta}{2}}(\Gamma_1)} + \|g_k^2\|_{H_\beta^{s-k-\frac{1+\alpha}{2}}(\Gamma_2)} \\
&\qquad\qquad + \|g_k^3\|_{H_\alpha^{s-k-\frac{1+\beta}{2}}(\Gamma_3)} + \|g_k^4\|_{H_\beta^{s-k-\frac{1+\alpha}{2}}(\Gamma_4)} \Big).
\end{aligned} \tag{3.36}$$

4. A general interpolation result for polynomial spaces.

Let k be a positive integer and θ be a real number, $0 < \theta < 1$. On the interval $\Lambda =]-1, 1[$, the interpolate with index θ of the space $\mathbb{P}_N(\Lambda)$ provided with the norm of $H^k(\Lambda)$ and of this same space provided with the norm of $L^2(\Lambda)$, is obviously the space $\mathbb{P}_N(\Lambda)$ provided with a norm equivalent to that of $H^{k(1-\theta)}(\Lambda)$. Nevertheless this argument does not prove that the equivalence constants are independent of N. The approach that we follow here relies on the interpolation of spaces by the method of traces, see [23, Chap. 3]. That is why we use the results of Section 3. We present the results first for the standard Sobolev norms, next for the weighted Sobolev norms introduced in Section I.7.

4.1. Case of standard Sobolev spaces.

Notation 4.1. For any positive real number τ and any integer $k > \tau$, we denote by $\|\cdot\|_{H^\tau(\Lambda)}^{N,k,0}$ the norm of the interpolate space of index $1 - \frac{\tau}{k}$ between $\mathbb{P}_N(\Lambda)$ provided with the norm $\|\cdot\|_{H^k(\Lambda)}$ and this same space provided with the norm $\|\cdot\|_{L^2(\Lambda)}$.

We going to prove that, for any value of the integer k, the norms $\|\cdot\|_{H^\tau(\Lambda)}^{N,k,0}$ and $\|\cdot\|_{H^\tau(\Lambda)}$ are equivalent on $\mathbb{P}_N(\Lambda)$ with equivalence constants independent of N. To do this, we use the following extension of the theory of interpolation by the trace method, for which we refer to [5, §3.12] or [32, Thm 1.8.2]: If X_0 and X_1 are two Banach spaces such that X_1 is contained in X_0 with a continuous and dense embedding, the interpolate space $[X_1, X_0]_{\theta,p}$, $0 < \theta < 1$ and $1 \le p < +\infty$, is the set of traces $v(0)$ of functions v continuous in $[0, 1]$ with values in X_0, and measurable in $]0, 1[$ with values in X_1, which satisfy

$$\int_0^1 \|v(t)\|_{X_1}^p \, t^{kp\theta} \, \frac{dt}{t} < +\infty \quad \text{and} \quad \int_0^1 \|d^k v(t)\|_{X_0}^p \, t^{kp\theta} \, \frac{dt}{t} < +\infty. \tag{4.1}$$

Theorem 4.2. *For any positive real number τ and for any integer $k > \tau$, there exists a positive constant c such that, for any nonnegative integer N and for any polynomial φ_N in $\mathbb{P}_N(\Lambda)$, the following inequalities hold*

$$\|\varphi_N\|_{H^\tau(\Lambda)} \le \|\varphi_N\|_{H^\tau(\Lambda)}^{N,k,0} \le c\,\|\varphi_N\|_{H^\tau(\Lambda)}. \tag{4.2}$$

Proof. The norm of the imbedding of $\mathbb{P}_N(\Lambda)$ provided with the norm $\|\cdot\|_{H^k(\Lambda)}$ and $\|\cdot\|_{L^2(\Lambda)}$ into $H^k(\Lambda)$ and $L^2(\Lambda)$, respectively, being equal to 1, we obtain at once by interpolation

$$\forall \varphi_N \in \mathbb{P}_N(\Lambda), \quad \|\varphi_N\|_{H^\tau(\Lambda)} \le \|\varphi_N\|_{H^\tau(\Lambda)}^{N,k,0}.$$

To prove the converse estimate, we use the characterization of the interpolation norm as a trace norm recalled in (4.1):

$$\|\varphi_N\|_{H^\tau(\Lambda)}^{N,k,0} = \inf\Big\{ \Big(\int_0^1 t^{2(k-\tau)} \Big\{ \|v(.,t)\|_{H^k(\Lambda)}^2 + \|\partial_t^k v(.,t)\|_{L^2(\Lambda)}^2 \Big\}\,\frac{dt}{t}\Big)^{\frac{1}{2}};\quad v \in \mathscr{C}^0(\overline{\mathcal{I}};\mathbb{P}_N(\Lambda)) \text{ and } v(.,0) = \varphi_N \Big\}. \tag{4.3}$$

With $\beta = 2(k-\tau) - 1$, it is readily checked that

$$\Big(\int_0^1 t^{2(k-\tau)} \Big\{ \|v(.,t)\|_{H^k(\Lambda)}^2 + \|\partial_t^k v(.,t)\|_{L^2(\Lambda)}^2 \Big\}\,\frac{dt}{t}\Big)^{\frac{1}{2}} \le \|v\|_{H^k_{0,\beta}(\Lambda\times\mathcal{I})}. \tag{4.4}$$

Let now φ_N be any polynomial in $\mathbb{P}_N(\Lambda)$. Using Corollary 3.3, we derive the existence of a function $v \in \mathscr{C}^\infty(\overline{\mathcal{I}};\mathbb{P}_N(\Lambda))$ such that the trace $v(\cdot,0)$ is equal to φ_N and that

$$\|v\|_{H^k_{0,\beta}(\Lambda\times\mathcal{I})} \le c\|\varphi_N\|_{H^{k-\frac{1+\beta}{2}}(\Lambda)} = c\|\varphi_N\|_{H^\tau(\Lambda)}.$$

The above inequality combined with (4.3) and (4.4) yields that

$$\|\varphi_N\|_{H^\tau(\Lambda)}^{N,k,0} \le c\|\varphi_N\|_{H^\tau(\Lambda)},$$

which ends the proof.

We generalize Notation 4.1 to any triple $\tau_0 < \tau < \tau_1$ of nonnegative numbers:

Notation 4.3. For any nonnegative real numbers τ_0, τ_1 and τ such that $\tau_0 < \tau < \tau_1$, let us denote by $\|\cdot\|_{H^\tau(\Lambda)}^{N,\tau_1,\tau_0}$ the norm of the interpolate space of index $\frac{\tau_1-\tau}{\tau_1-\tau_0}$ between $\mathbb{P}_N(\Lambda)$ provided with the norm $\|\cdot\|_{H^{\tau_1}(\Lambda)}$ and this same space provided with the norm $\|\cdot\|_{H^{\tau_0}(\Lambda)}$.

The next corollary is an easy consequence of Theorem 4.2, combined with the reiteration theorem [23, Chap. 1, Thm 6.1].

Corollary 4.4. *For any real numbers τ_0, τ_1 and τ such that $\tau_0 < \tau < \tau_1$, there exists a positive constant c such that, for any nonnegative integer N and for any polynomial φ_N in $\mathbb{P}_N(\Lambda)$, the following inequalities hold*

$$\|\varphi_N\|_{H^\tau(\Lambda)} \leq \|\varphi_N\|_{H^\tau(\Lambda)}^{N,\tau_1,\tau_0} \leq c\,\|\varphi_N\|_{H^\tau(\Lambda)}. \tag{4.5}$$

We present now the analogue of Theorem 4.2 for the spaces $\mathbb{P}_N^{m,0}(\Lambda)$ of polynomials in $\mathbb{P}_N(\Lambda)$ which vanish at ± 1 together with their derivatives up to the order $m-1$.

Notation 4.5. For any positive real number τ and any integer $k > \tau$, we denote by $\|\cdot\|_{H^\tau(\Lambda)}^{N,(m),k,0}$ the norm of the interpolate space of index $1-\frac{\tau}{k}$ between $\mathbb{P}_N^{m,0}(\Lambda)$ provided with the norm $\|\cdot\|_{H^k(\Lambda)}$ and this same space provided with the norm $\|\cdot\|_{L^2(\Lambda)}$.

We expect that the cancellation condition at the ends of the interval Λ may have some influence on the norm of the interpolate, since for instance when $k = m$, the space $\mathbb{P}_N^{m,0}(\Lambda)$ is imbedded in $H_0^m(\Lambda)$, which in turn coincides with the weighted space $V^m(\Lambda)$ by virtue of Theorem I.3.6. The spaces $V^\tau(\Lambda)$ forming an interpolation scale, cf [32], their norms will naturally appear for interpolate polynomial spaces. For the reader's convenience, let us recall that for any $\tau \geq 0$

$$V^\tau(\Lambda) = \{\varphi \in H^\tau(\Lambda);\ |d^\ell\varphi|^2(1-\zeta^2)^{\ell-\tau} \in L^1(\Lambda),\ \ell = 0, \dots, [\tau]\}.$$

In limit cases, the norms of $H^\tau(\Lambda)$ and $V^\tau(\Lambda)$ are not equivalent to each other, thus the norm $V^\tau(\Lambda)$ appears there:

Theorem 4.6. *Let m be a positive integer. For any positive real number τ and for any integer $k > \tau$, there exists a positive constant c such that, for any nonnegative integer N and for any polynomial φ_N in $\mathbb{P}_N^{m,0}(\Lambda)$, the following inequalities hold*
(i) when $\tau \notin \{\frac{1}{2}, \dots, m-\frac{1}{2}\}$,

$$\|\varphi_N\|_{H^\tau(\Lambda)} \leq \|\varphi_N\|_{H^\tau(\Lambda)}^{N,(m),k,0} \leq c\,\|\varphi_N\|_{H^\tau(\Lambda)}, \tag{4.6}$$

(ii) when $\tau \in \{\frac{1}{2}, \dots, m-\frac{1}{2}\}$,

$$\|\varphi_N\|_{V^\tau(\Lambda)} \leq \|\varphi_N\|_{H^\tau(\Lambda)}^{N,(m),k,0} \leq c\,\|\varphi_N\|_{V^\tau(\Lambda)}. \tag{4.7}$$

Proof. We use similar arguments as previously for Theorem 4.2, relying now on Corollary 3.17 with $\alpha = 0$, $\beta = 2(k-\tau)-1$ and $s = k$.

As before, Notation 4.5 can be extended to any triple $\tau_0 < \tau < \tau_1$, and the reiteration theorem allows to extend the validity of (4.6) to any such triple when none of τ_0, τ, τ_1 belongs to $\{\frac{1}{2}, \dots, m - \frac{1}{2}\}$.

To conclude this subsection, let us mention that Theorems 4.2 and 4.6 are specially interesting for applications in numerical analysis when $k = 1$, $\tau = \frac{1}{2}$, and $m = 1$. Concerning Theorem 4.2, we obtain that any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies

$$\|\varphi_N\|_{H^{\frac{1}{2}}(\Lambda)} \leq \|\varphi_N\|^{N,1,0}_{H^{\frac{1}{2}}(\Lambda)} \leq c\, \|\varphi_N\|_{H^{\frac{1}{2}}(\Lambda)}, \tag{4.8}$$

with a constant c independent of N. We also see that this situation enters the limit case of Theorem 4.6: More precisely the following inequality is satisfied by any polynomial φ_N in $\mathbb{P}^{1,0}_N(\Lambda)$,

$$\|\varphi_N\|_{H^{\frac{1}{2}}_{00}(\Lambda)} \leq \|\varphi_N\|^{N,(1),1,0}_{H^{\frac{1}{2}}(\Lambda)} \leq c\, \|\varphi_N\|_{H^{\frac{1}{2}}_{00}(\Lambda)}, \tag{4.9}$$

with a constant c independent of N. This result is clearly what should be expected, since $\mathbb{P}^{1,0}_N(\Lambda)$ is a subspace of $H^1_0(\Lambda)$ and there holds

$$[H^1_0(\Lambda), L^2(\Lambda)]_{\frac{1}{2}} = H^{\frac{1}{2}}_{00}(\Lambda).$$

The equivalences (4.8) and (4.9) can be used to find discrete analogues of the characterization of $H^{1/2}$ and $H^{1/2}_{00}$ norms by Fourier coefficients on eigenvector bases of one-dimensional Neumann and Dirichlet problems, respectively.

4.2. Extension to weighted Sobolev spaces.

We now state the weighted analogues of the interpolation results between spaces of polynomials.

Notation 4.7. Let α be a real number > -1. For any nonnegative real number τ and for any integer $k > \tau$, let us denote by $\|\cdot\|^{N,k,0}_{H^\tau_\alpha(\Lambda)}$ the norm of the interpolate space of index $1 - \frac{\tau}{k}$ between $\mathbb{P}_N(\Lambda)$ provided with the norm $\|\cdot\|_{H^k_\alpha(\Lambda)}$ and this same space provided with the norm $\|\cdot\|_{L^2_\alpha(\Lambda)}$.

Theorem 4.8. *Let α be a real number > -1. For any nonnegative real number τ such that $\tau \neq \frac{\alpha+1}{2}$ if $\alpha < 0$, and for any integer $k > \tau$, there exists a positive constant c such that, for any nonnegative integer N and for any polynomial φ_N in $\mathbb{P}_N(\Lambda)$, the following inequalities hold*

$$\|\varphi_N\|_{H^\tau_\alpha(\Lambda)} \leq \|\varphi_N\|^{N,k,0}_{H^\tau_\alpha(\Lambda)} \leq c\, \|\varphi_N\|_{H^\tau_\alpha(\Lambda)}. \tag{4.10}$$

The proof of this theorem follows the same lines as Theorem 4.2, using Corollary 3.18 with $\beta = 2(k-\tau) - 1$.

Notation 4.7 can be extended to triples $\tau_0 < \tau < \tau_1$ of nonnegetive numbers and a general statement as Corollary 4.4 proved by reiteration, provided $\tau_0, \tau, \tau_1 \neq \frac{\alpha+1}{2}$ if $\alpha < 0$.

We end this section by extending Theorem 4.6 in weighted spaces.

Notation 4.9. Let α be a real number > -1. For any nonnegative real number τ and for any integer $k > \tau$, let us denote by $\|\cdot\|^{N,(m),k,0}_{H^\tau_\alpha(\Lambda)}$ the norm of the interpolate space of index $1 - \frac{\tau}{k}$ between $\mathbb{P}^{m,0}_N(\Lambda)$ provided with the norm $\|\cdot\|_{H^k_\alpha(\Lambda)}$ and this same space provided with the norm $\|\cdot\|_{L^2_\alpha(\Lambda)}$.

Theorem 4.10. *Let m be a positive integer. Let α be a real number > -1. For any positive real number τ and for any integer $k > \tau$, there exists a positive constant c such that, for any nonnegative integer N and for any polynomial φ_N in $\mathbb{P}^{m,0}_N(\Lambda)$, the following inequalities hold*
(i) when $\tau - \frac{\alpha}{2} \notin \{\frac{1}{2}, \dots, m - \frac{1}{2}\}$,

$$\|\varphi_N\|_{H^\tau_\alpha(\Lambda)} \leq \|\varphi_N\|^{N,(m),k,0}_{H^\tau_\alpha(\Lambda)} \leq c\,\|\varphi_N\|_{H^\tau_\alpha(\Lambda)}, \tag{4.11}$$

(ii) when $\tau - \frac{\alpha}{2} \in \{\frac{1}{2}, \dots, m - \frac{1}{2}\}$,

$$\|\varphi_N\|_{V^\tau_\alpha(\Lambda)} \leq \|\varphi_N\|^{N,(m),k,0}_{H^\tau_\alpha(\Lambda)} \leq c\,\|\varphi_N\|_{V^\tau_\alpha(\Lambda)}. \tag{4.12}$$

5. Lifting of traces into the cube.

The arguments here are very similar to those used in the previous sections for the lifting onto a square. Indeed, as explained in Corollary I.6.11, the compatibility conditions in this case are reduced to conditions on the edges. So we have rather briefly recall the main steps of the proof and only state the general lifting theorems. See Figure I.7.1 for the notation of the cube faces.

5.1. Construction of liftings into the cube.

With the same notation as for Corollary I.6.11, let $\mathbb{P}^{(m)}_N(\partial\Xi)$ be the subspace of polynomials $G = (G^1, \dots, G^6)$ in $\prod_{j=1}^6 \mathbb{P}_N(\Omega_j)^m$, with each $G^j = (g^j_0, \dots, g^j_{m-1})$, satisfying the conditions, for $1 \leq \ell \leq 12$,

$$\partial^n_{\tau_{\ell-}} g^{j-(\ell)}_k = \partial^k_{\tau_{\ell+}} g^{j+(\ell)}_n \quad \text{on } \Gamma_\ell, \qquad 0 \leq k, n \leq m-1. \tag{5.1}$$

STEP 1. Lifting from a plane to an infinite slab
Let us now consider the mapping

$$F_k^*(\varphi)(x, y, z) = \frac{z^k}{k!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi(v)\chi(w)\varphi(x + zv, y + zw)\, dv\, dw. \qquad (5.2)$$

If the function χ satisfies the same assumptions as in Proposition 2.2, the operator F^* defined by

$$F^*(g_0, \cdots, g_{m-1}) = \sum_{k=0}^{m-1} F_k^*(g_k), \qquad (5.3)$$

is continuous from $\prod_{k=0}^{m-1} H^{s-k-\frac{1}{2}}(\mathbb{R}^2)$ into $H^s(\mathbb{R}^2 \times \mathcal{I})$ and satisfies

$$\forall x \in \mathbb{R}, \quad \partial_z^k\big(F^*(g_0, \cdots, g_{m-1})\big)(x, y, 0) = g_k(x, y), \quad 0 \le k \le m - 1. \qquad (5.4)$$

Moreover, it maps $\mathbb{P}_N(\mathbb{R}^2)^m$ into $\mathcal{P}_{2N+m-1}(\mathbb{R}^2 \times \mathcal{I})$, more precisely into its intersection with the space of polynomials with degree smaller than N with respect to x and y and smaller than $2N + m - 1$ with respect to z.

STEP 2. Lifting from a square face to a pyramid
Let $\mathcal{P}$ denotes the pyramid with vertices $(-1, -1, 0)$, $(1, -1, 0)$, $(1, 1, 0)$, $(1, -1, 0)$ and $(0, 0, \sqrt{3})$. It can be noted that its square face coincides with $\Theta \times \{0\}$ and that the angles of the four other faces with this one is equal to $\frac{\pi}{3}$. So, when the support of the function χ is contained in $[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$, the previous operator F^* lifts traces defined on $\Theta \times \{0\}$ into $\mathcal{P}$.

STEP 3. Lifting on one face of the cube
We use the one-to-one mapping

$$(x, y, z) = \mathscr{F}^*(X, Y, Z) = \Big((1 - \frac{1 + Z}{2\sqrt{3}})X\ ,\ (1 - \frac{1 + Z}{2\sqrt{3}})Y\ ,\ \frac{1 + Z}{2}\Big). \qquad (5.5)$$

which sends the cube $\Xi =]-1, 1[^3$ onto the trapezoid $\mathcal{P}^* = \{(x, y, z) \in \mathcal{P}; z < 1\}$. By this argument, denoting by Ω_2 the face of Ξ which is contained in the plane $Z = -1$, we construct a lifting operator from $\mathbb{P}_N(\Omega_2)$ into the subspace of polynomials in $\mathbb{P}_{2N+3m-2}(\Xi)$ made of polynomials vanishing together with its normal derivatives up to the order $m - 1$ on the opposite face to Ω_2. Thus, applying the degree reduction operator $r_{2N,3m-2}^m$ now with respect to the Z-variable produces a lifting operator from $\mathbb{P}_N(\Omega_2)$ into the subspace of $\mathbb{P}_{2N}(\Xi)$. Moreover, this operator is continuous in appropriate norms.

STEP 4. Lifting of half-flat and flat traces on one face of the cube
Still relying on the tensorization properties of the spaces $H^s(\Xi)$, we can construct by the same arguments as in Section 3.2
• a lifting of polynomials of $\mathbb{P}_N(\Omega_2)$ vanishing together with their Y-derivatives up

to the order $m-1$ on the edges contained in the planes $\textsc{x} = \pm 1$ (half-flat traces) into a polynomial in $\mathbb{P}_{2N}(\Xi)$ vanishing together with its normal derivatives up to the order $m-1$ on the opposite face to Ω_2 and on the two faces contained in the planes $\textsc{x} = \pm 1$,

- a lifting of polynomials of $\mathbb{P}_N(\Omega_2)$ vanishing together with their normal derivatives up to the order $m-1$ on the four edges of Ω_2 (flat traces) into a polynomial in $\mathbb{P}_{2N}(\Xi)$ vanishing together with its normal derivatives up to the order $m-1$ on the five other faces of Ξ but Ω_2.

Step 5. The general polynomial lifting onto the cube
The idea is the following:

- By using the operator exhibited in Step 3, we lift the traces on the faces contained in the planes $\textsc{z} = \pm 1$ into $\mathbb{P}_{2N}(\Xi)$.
- We now lift the half-flat traces contained in the planes $\textsc{y} = \pm 1$ by using the first operator introduced in Step 4 into $\mathbb{P}_{3N}(\Xi)$ (indeed, the previous arguments can be extended to the case of traces with degree $\leq N$ with respect to $\textsc{x}$ and $\leq 2N$ with respect to $\textsc{z}$).
- Finally we lift the flat traces contained in the planes $\textsc{x} = \pm 1$ by using the second operator of Step 4 into $\mathbb{P}_{5N}(\Xi)$.

Combining all this leads to the next statements.

Theorem 5.1. *Let m and M be two positive integers with $m \leq M$. For each integer $N \geq 2M$, there exists a linear operator $\mathcal{R}^{\Xi}_{N,m}$:*
(i) which acts from $\mathbb{P}_N^{(m)}(\partial\Xi)$ into $\mathbb{P}_{5N}(\Xi)$;
(ii) such that

$$\begin{aligned} \forall G = (G^1, \dots, G^6) \in \mathbb{P}_N^{(m)}(\partial\Xi),\ \text{with } G^j = (g_0^j, \dots, g_{m-1}^j), \\ \partial_{n_j}^k \mathcal{R}^{\Xi}_{N,m}(G)\big|_{\Omega_j} = g_k^j, \quad 0 \leq k \leq m-1, \end{aligned} \tag{5.6}$$

(iii) which satisfies the following continuity property for any real number s such that $m - \frac{1}{2} < s \leq M$ and $s \notin \{m, m+1, \dots, 2m-1\}$:

$$\begin{aligned} \forall G = (G^1, \dots, G^6) \in \mathbb{P}_N^{(m)}(\partial\Xi),\ \text{with } G^j = (g_0^j, \dots, g_{m-1}^j), \\ \|\mathcal{R}^{\Xi}_{N,m}(G)\|_{H^s(\Xi)} \leq c \sum_{j=1}^{6} \sum_{k=0}^{m-1} \|g_k^j\|_{H^{s-k-\frac{1}{2}}(\Omega_j)}. \end{aligned} \tag{5.7}$$

The limit cases require augmented norms, according to Corollary I.6.11. We just give the statement for $s = m$.

Theorem 5.2. *When s is equal to m, the operator $\mathcal{R}^{\Xi}_{N,m}$ introduced in Theorem*

5.1 *satisfies the following continuity property*

$$\forall G = (G^1, \dots, G^6) \in \mathbb{P}_N^{(m)}(\partial\Xi),\ \textit{with}\ G^j = (g_0^j, \dots, g_{m-1}^j),$$
$$\begin{aligned}\|\mathcal{R}_{N,m}^{\Xi}(G)\|_{H^m(\Xi)} \le c \Big\{ &\sum_{j=1}^{6} \sum_{k=0}^{m-1} \big\| g_k^j \|_{H^{m-k-\frac{1}{2}}(\Omega_j)} \\ &+ \sum_{\ell=1}^{12} \sum_{k=0}^{m-1} \Big(\int_{\Gamma_\ell} \int_0^1 |(\partial_{\tau_{\ell_-}}^{m-1-k} g_k^{j_-(\ell)})(\boldsymbol{x} - t\,\boldsymbol{\tau}_{\ell_-}) \\ &\qquad - (\partial_{\tau_{\ell_+}}^k g_{m-1-k}^{j_+(\ell)})(\boldsymbol{x} - t\,\boldsymbol{\tau}_{\ell_+})\Big|^2 \frac{dt}{t}\, d\boldsymbol{x} \Big)^{\frac{1}{2}} \Big\}.\end{aligned} \tag{5.8}$$

The operator $\mathcal{R}_{N,m}^{\Xi}$ has the disadvantage of lifting polynomial traces with degree $\le N$ into $\mathbb{P}_{5N}(\Xi)$, which is a little deceitful in view of the results in dimension $d = 2$. So we now propose a solution to this problem, but only in the case $m = 1$ of a single trace. This requires a further lemma.

We recall from [11, §4] for instance that the nodes of the Gauss–Lobatto formula which is exact on $\mathbb{P}_{2N-1}(\Lambda)$, are the zeros of the polynomial $(1-\zeta^2)\, L_N'(\zeta)$. Let i_N denote the Lagrange interpolation operator at these nodes with values in $\mathbb{P}_N(\Lambda)$.

Lemma 5.3. *The operator i_N maps $\mathscr{C}^0(\overline{\Lambda})$ into $\mathbb{P}_N(\Lambda)$, satisfies*

$$\forall \varphi \in \mathscr{C}^0(\overline{\Lambda}), \quad i_N\varphi(\pm 1) = \varphi(\pm 1), \tag{5.9}$$

Moreover, the following uniform continuity property holds: There exists a positive constant c such that, for each positive integer k,

$$\forall \varphi_N \in \mathbb{P}_{kN}(\Lambda), \quad \|i_N(\varphi_N)\|_{H^s(\Lambda)} \le c(1+k)^{1-s}\, \|\varphi_N\|_{H^s(\Lambda)}, \quad 0 \le s \le 1. \tag{5.10}$$

Proof. We recall from [11, form. (13.27) and (13.28)] the following properties

$$\forall \varphi_N \in H^1(\Lambda), \quad \|i_N(\varphi_N)\|_{H^1(\Lambda)} \le c\, \|\varphi_N\|_{H^1(\Lambda)},$$

and

$$\forall \varphi_N \in \mathbb{P}_{kN}(\Lambda), \quad \|i_N(\varphi_N)\|_{L^2(\Lambda)} \le c(1+k)\, \|\varphi_N\|_{L^2(\Lambda)}.$$

Thus, applying the principal theorem of interpolation, see [23, Chap. 1, Th. 5.1], leads to, with the notation of Section 4,

$$\forall \varphi_N \in \mathbb{P}_{kN}(\Lambda), \quad \|i_N(\varphi_N)\|_{H^s(\Lambda)} \le c(1+k)^{1-s}\, \|\varphi_N\|_{H^s(\Lambda)}^{kN,1,0}, \quad 0 \le s \le 1.$$

So, the desired result follows from Corollary 4.4.

We go back to the previous Steps 1 to 5 in the case $m = 1$. Steps 1, 2, 4 and 5 remain unchanged but, in Step 3, we use the operator i_N instead of the degree reduction operator $r^1_{2N,1}$. The next statements are then easily derived from Lemma 5.3 applied with $k = 3$.

Theorem 5.4. *For each integer $N \geq 2$, there exists a linear operator $\mathcal{R}^{\Xi *}_{N,1}$:*
(i) which acts from $\mathbb{P}^{(1)}_N(\partial\Xi)$ into $\mathbb{P}_N(\Xi)$;
(ii) such that

$$\forall G = (g^1, \dots, g^6) \in \mathbb{P}^{(1)}_N(\partial\Xi), \quad \mathcal{R}^{\Xi *}_{N,m}(G)\big|_{\Omega_j} = g^j, \tag{5.11}$$

(iii) which satisfies the following continuity properties:
- For any real number s such that $\frac{1}{2} < s < 1$:

$$\forall G = (g^1, \dots, g^6) \in \mathbb{P}^{(1)}_N(\partial\Xi),, \qquad \|\mathcal{R}^{\Xi *}_{N,1}(G)\|_{H^s(\Xi)} \leq c \sum_{j=1}^{6} \|g^j\|_{H^{s-\frac{1}{2}}(\Omega_j)}. \tag{5.12}$$

- For $s = 1$:

$$\begin{aligned} &\forall G = (g^1, \dots, g^6) \in \mathbb{P}^{(1)}_N(\partial\Xi), \\ &\quad \|\mathcal{R}^{\Xi *}_{N,1}(G)\|_{H^1(\Xi)} \leq c \Big\{ \sum_{j=1}^{6} \|g^j\|_{H^{\frac{1}{2}}(\Omega_j)} \\ &\qquad + \sum_{\ell=1}^{12} \Big(\int_{\Gamma_\ell} \int_0^1 \big| g^{j-(\ell)}(\boldsymbol{x} - t\,\boldsymbol{\tau}_{\ell-}) - g^{j+(\ell)}(\boldsymbol{x} - t\,\boldsymbol{\tau}_{\ell_+}) \big|^2 \frac{dt}{t}\, d\boldsymbol{x} \Big)^{\frac{1}{2}} \Big\}. \end{aligned} \tag{5.13}$$

5.2. Continuity of liftings in weighted Sobolev spaces.

It is also possible to construct polynomial lifting operators which are continuous between weighted Sobolev spaces, like in the trace Theorem I.7.8.

Theorem 5.5. *Let α, β and γ be real numbers > -1. We assume without restriction that $\alpha \leq \beta \leq \gamma$. Let m and M be two positive integers with $m \leq M$. For each integer $N \geq 2M$, there exists a linear operator $\mathcal{R}^{\alpha\beta\gamma}_{N,m}$*
(i) which acts from $\mathbb{P}^{(m)}_N(\partial\Xi)$ into $\mathbb{P}_{5N}(\Xi)$;
(ii) such that, for $j = 1, \dots, 6$,

$$\begin{aligned} &\forall G = (G^1, \dots, G^6) \in \mathbb{P}^{(m)}_N(\partial\Xi), \text{ with } G^j = (g^j_0, \dots, g^j_{m-1}), \\ &\qquad \partial^k_{n_j} \mathcal{R}^{\alpha\beta\gamma}_{N,m}(G)\big|_{\Omega_j} = g^j_k, \quad 0 \leq k \leq m-1, \end{aligned} \tag{5.14}$$

(iii) which satisfies the following continuity property for any real number $s \le M$ with $s > m - \frac{1}{2} + \frac{\gamma}{2}$, and such that neither $s - \frac{\alpha+\beta}{2}$, nor $s - \frac{\alpha+\gamma}{2}$, nor $s - \frac{\beta+\gamma}{2}$ belongs to $\{m, m+1, \dots, 2m-1\}$,

$$
\begin{aligned}
&\forall G = (G^1, \dots, G^6) \in \mathbb{P}_N^{(m)}(\partial\Xi), \text{ with } G^j = (g_0^j, \dots, g_{m-1}^j), \\
&\|\mathcal{R}_{N,m}^{\alpha\beta\gamma}(G)\|_{H^s_{\alpha\beta\gamma}(\Xi)} \\
&\qquad \le c \sum_{k=0}^{m-1} \Big(\|g_k^1\|_{H_{\alpha\beta}^{s-k-\frac{1+\gamma}{2}}(\Omega_1)} + \|g_k^2\|_{H_{\alpha\gamma}^{s-k-\frac{1+\beta}{2}}(\Omega_2)} + \|g_k^3\|_{H_{\beta\gamma}^{s-k-\frac{1+\alpha}{2}}(\Omega_3)} \\
&\qquad\qquad + \|g_k^4\|_{H_{\alpha\beta}^{s-k-\frac{1+\gamma}{2}}(\Omega_4)} + \|g_k^5\|_{H_{\alpha\gamma}^{s-k-\frac{1+\beta}{2}}(\Omega_5)} + \|g_k^6\|_{H_{\beta\gamma}^{s-k-\frac{1+\alpha}{2}}(\Omega_6)} \Big).
\end{aligned}
\tag{5.15}
$$

In the case $m = 1$ of a single trace, it is also possible to construct another operator which is similar to $\mathcal{R}_{N,1}^{\alpha\beta\gamma}$ but now takes its values in $\mathbb{P}_N(\Xi)$. Its construction relies on the weighted analogue of Lemma 5.3, more precisely on some properties of the Lagrange interpolation operator i_N^β at the nodes of the Gauss–Lobatto formula for the weighted measure $(1-\zeta^2)^\beta\, d\zeta$. We refer to [11, form. (21.6)] for the exact definition of this operator.

Lemma 5.6. *For any real number β, $-1 < \beta < 1$, the operator i_N^β maps $\mathscr{C}^0(\overline{\Lambda})$ into $\mathbb{P}_N(\Lambda)$, satisfies*

$$\forall \varphi \in \mathscr{C}^0(\overline{\Lambda}), \quad i_N^\beta \varphi(\pm 1) = \varphi(\pm 1), \tag{5.16}$$

Moreover, the following uniform continuity property holds: There exists a positive constant c such that, for each positive integer k,

$$\forall \varphi_N \in \mathbb{P}_{kN}(\Lambda), \quad \|i_N^\beta(\varphi_N)\|_{H^s_\beta(\Lambda)} \le c(1+k)\, \|\varphi_N\|_{H^s_\beta(\Lambda)}, \quad 0 \le s \le 1. \tag{5.17}$$

Proof. It is performed in three steps, according to the values of s.
1) For $s = 1$, we derive from [11, form. (21.11) & (21.12)] and a Hardy inequality that (note that this requires that β is < 1)

$$\forall \varphi_N \in \mathbb{P}_{kN}^0(\Lambda), \quad \|i_N^\beta(\varphi_N)\|_{H^1_\beta(\Lambda)} \le c(1+k)\, \|\varphi_N\|_{H^1_\beta(\Lambda)}.$$

Then, for any polynomial φ_N in $\mathbb{P}_{kN}(\Lambda)$, we use the expansion

$$\varphi_N = \varphi_N^0 + \varphi, \qquad \text{with} \quad \varphi = \varphi_N(-1)\, \frac{1-\zeta}{2} + \varphi_N(-1)\, \frac{1+\zeta}{2},$$

and derive from the identities $\varphi_N^0(\pm 1) = 0$ and $i_N(\varphi) = \varphi$ that

$$\|i_N^\beta(\varphi_N)\|_{H^1_\beta(\Lambda)} \le c\,(1+k)\, \big(\|\varphi_N\|_{H^1_\beta(\Lambda)} + \|\varphi\|_{H^1_\beta(\Lambda)}\big) + \|\varphi\|_{H^1_\beta(\Lambda)}.$$

It follows from Theorem I.7.2 that

$$\|\varphi\|_{H^1_\beta(\Lambda)} \le c\,\big(|\varphi_N(-1)| + |\varphi_N(1)|\big) \le c'\,\|\varphi_N\|_{L^\infty(\Lambda)} \le c''\,\|\varphi_N\|_{H^1_\beta(\Lambda)}.$$

All this gives the estimate

$$\forall \varphi_N \in \mathbb{P}_{kN}(\Lambda), \quad \|i_N^\beta(\varphi_N)\|_{H^1_\beta(\Lambda)} \le c(1+k)\,\|\varphi_N\|_{H^1_\beta(\Lambda)}. \tag{5.18}$$

2) For $s=0$, a closer look at the proof of [11, Thm 21.1] gives

$$\forall \varphi \in H^1_{\beta,0}(\Lambda), \quad \|i_N^\beta(\varphi)\|_{L^2_\beta(\Lambda)} \le c\,\Big(\|\varphi\|_{L^2_\beta(\Lambda)} + N^{-1}\,\|\varphi\,(1-\zeta^2)^{-\frac{1}{2}}\|_{L^2_\beta(\Lambda)} + N^{-1}\,\|\varphi'(1-\zeta^2)^{-\frac{1}{2}}\|_{L^2_\beta(\Lambda)}\Big).$$

Thus, combining a Hardy inequality and a standard inverse inequality yields

$$\forall \varphi_N \in \mathbb{P}^0_{kN}(\Lambda), \quad \|i_N^\beta(\varphi_N)\|_{L^2_\beta(\Lambda)} \le c(1+k)\,\|\varphi_N\|_{L^2_\beta(\Lambda)}.$$

We introduce the polynomials $\chi_{N-1} = J_N^{\beta\prime}/J_N^{\beta'}(1)$ and its analogue χ_N, where the J_n^β are the Jacobi polynomials orthogonal in $L^2_\beta(\Lambda)$ and, for any polynomial φ_N in $\mathbb{P}_{kN}(\Lambda)$, we write the expansion

$$\varphi_N = \tilde{\varphi}_N^0 + \tilde{\varphi}, \qquad \text{with} \quad \tilde{\varphi} = \varphi_N(-1)\,\frac{(-1)^{N-1}(\chi_{N-1}-\chi_N)}{2} + \varphi_N(1)\,\frac{\chi_{N-1}+\chi_N}{2},$$

and as previously we derive

$$\|i_N^\beta(\varphi_N)\|_{L^2_\beta(\Lambda)} \le c\,(1+k)\,\big(\|\varphi_N\|_{L^2_\beta(\Lambda)} + \|\tilde{\varphi}\|_{L^2_\beta(\Lambda)}\big) + \|\tilde{\varphi}\|_{L^2_\beta(\Lambda)}.$$

Since both $\|\chi_{N-1}\|_{L^2_\beta(\Lambda)}$ and $\|\chi_N\|_{L^2_\beta(\Lambda)}$ are smaller than $c\,N^{-1-\beta}$, we have

$$\|\tilde{\varphi}\|_{L^2_\beta(\Lambda)} \le c\,N^{-1-\beta}\,|\varphi_N(\pm 1)|$$

Writing the expansion of φ_N in the basis $\{J_0^\beta, \dots, J_N^\beta\}$ and using standard properties of the Jacobi polynomials J_n^β (see [11, form. (19.1) and (19.9)]) thus give

$$\|\tilde{\varphi}\|_{L^2_\beta(\Lambda)} \le c\,\|\varphi_N\|_{L^2_\beta(\Lambda)}.$$

Combining all this leads to

$$\forall \varphi_N \in \mathbb{P}_{kN}(\Lambda), \quad \|i_N^\beta(\varphi_N)\|_{L^2_\beta(\Lambda)} \le c(1+k)\,\|\varphi_N\|_{L^2_\beta(\Lambda)}. \tag{5.19}$$

3) Finally applying the principal theorem of interpolation (see [23, Chap. 1, Th. 5.1]) between (5.19) and (5.20) and Theorem 4.8 leads to the desired result for $0 \le s \le 1$.

Theorem 5.7. *Let α, β and γ be real numbers such that $-1 < \alpha, \beta, \gamma < 1$. For each integer $N \ge 2$, there exists a linear operator $\mathcal{R}_{N,1}^{\alpha\beta\gamma *}$:*
(i) which acts from $\mathbb{P}_N^{(1)}(\partial\Xi)$ into $\mathbb{P}_N(\Xi)$;
(ii) such that, for $j = 1, \dots, 6$,

$$\forall G = (g^1, \dots, g^6) \in \mathbb{P}_N^{(1)}(\partial\Xi), \quad \mathcal{R}_{N,1}^{\alpha\beta\gamma *}(G)\big|_{\Omega_j} = g^j, \tag{5.20}$$

(iii) which satisfies the following continuity property for any real number s such that $\max\{\frac{1+\alpha}{2}, \frac{1+\beta}{2}, \frac{1+\gamma}{2}\} < s \le 1$ such that $s \notin \{1 + \frac{\alpha+\beta}{2}, 1 + \frac{\beta+\gamma}{2}, 1 + \frac{\gamma+\alpha}{2}\}$:

$$\begin{aligned}
&\forall G = (g^1, \dots, g^6) \in \mathbb{P}_N^{(1)}(\Xi),\\
&\quad \|\mathcal{R}_{N,1}^{\alpha\beta\gamma *}(G)\|_{H^s(\Xi)} \le c \Big(\|g^1\|_{H_{\alpha\beta}^{s-\frac{1+\gamma}{2}}(\Omega_1)} + \|g^2\|_{H_{\alpha\gamma}^{s-\frac{1+\beta}{2}}(\Omega_2)} + \|g^3\|_{H_{\beta\gamma}^{s-\frac{1+\alpha}{2}}(\Omega_3)}\\
&\qquad\qquad + \|g^4\|_{H_{\alpha\beta}^{s-\frac{1+\gamma}{2}}(\Omega_4)} + \|g^5\|_{H_{\alpha\gamma}^{s-\frac{1+\beta}{2}}(\Omega_5)} + \|g^6\|_{H_{\beta\gamma}^{s-\frac{1+\alpha}{2}}(\Omega_6)} \Big).
\end{aligned} \tag{5.21}$$

Chapter III

Polynomial inverse inequalities

One of the applications of the previous trace properties is to derive optimal inverse inequalities on the polynomial spaces in Sobolev norms of non integral orders. Indeed, in the case $p = 2$ of Hilbertian Sobolev norms for instance, deriving such inequalities relies on the interpolation of spaces of polynomials by the method of traces, see [23, Chap. 3], so on the use of the polynomial lifting operators built and analyzed in Chapter II. Inverse inequalities can be also be derived by different arguments in the non Hilbertian case. Once all these inequalities are established, we provide counter-examples that prove their optimality with respect to the degree of the polynomials.

For simplicity, we only work on the interval $\Lambda =]-1, 1[$ with a symmetric weight. We first describe the corresponding Sobolev spaces, together with the Jacobi polynomials which are orthogonal with respect to the weight. Next, we state and prove the results concerning the interpolation of spaces of polynomials. We then derive inverse inequalities: The three parameters s, p and α which are involved in the definition of the space $W^{s,p}_\alpha(\Lambda)$ being called order, exponent and weight of the space, respectively, we prove inverse inequalities between spaces first of different exponents and second of different orders. Next, we prove these inequalities for different orders and exponents, and finally for different orders and weights in the Hilbertian case of exponent $p = 2$. We conclude by checking the optimality of the previous estimates.

1. Weighted Sobolev spaces and orthogonal polynomials.

Let α be a fixed real number. On the interval Λ, we introduce the weight

$$\rho_\alpha(\zeta) = (1 - \zeta^2)^\alpha.$$

For any p, $1 \leq p < +\infty$, we first consider the space

$$L^p_\alpha(\Lambda) = \big\{\varphi : \Lambda \to \mathbb{R} \text{ measurable; } \|\varphi\|_{L^p_\alpha(\Lambda)} = \big(\int_{-1}^{1} |\varphi(\zeta)|^p \, \rho_\alpha(\zeta)\, d\zeta\big)^{\frac{1}{p}} < +\infty\big\}. \tag{1.1}$$

As already explained in Section I.a, the natural extension for $p = \infty$ is

$$L^\infty_\alpha(\Lambda) = L^\infty(\Lambda). \tag{1.2}$$

The corresponding Sobolev spaces are defined as follows. For any nonnegative integer m and for any p in $[1,+\infty[$, $W^{m,p}_\alpha(\Lambda)$ is defined by

$$W^{m,p}_\alpha(\Lambda)=\big\{\varphi\in\mathscr{D}'(\Lambda);\ \|\varphi\|_{W^{m,p}_\alpha(\Lambda)}=\big(\sum_{k=0}^{m}\|d^k\varphi\|^p_{L^p_\alpha(\Lambda)}\big)^{\frac{1}{p}}<+\infty\big\}. \tag{1.3}$$

To define the spaces of non integral order, we rely on the spaces $W^{s,p}_\alpha(\mathcal{I})$ defined in (I.a.13). For instance, with any function v on Λ, we associate the functions v_- and v_+ on $\mathcal{I}$ given by

$$v_-(x)=v(-1+\frac{3x}{2})\quad\text{and}\quad v_+(x)=v(1-\frac{3x}{2}),\qquad x\in\overline{\mathcal{I}}.$$

Next, for any positive real number s which is not an integer, we define the space $W^{s,p}_\alpha(\Lambda)$ as

$$W^{s,p}_\alpha(\Lambda)=\big\{\varphi\in L^p_\alpha(\Lambda);\ (v_-,v_+)\in W^{s,p}_\alpha(\mathcal{I})^2\big\}. \tag{1.4}$$

Note that there is no contradiction with definition (1.3) since this property holds when s is an integer. Intrinsic norms can be constructed on these spaces $W^{s,p}_\alpha(\Lambda)$ thanks to the definitions in Section I.7.1 and moreover all the properties stated in this same section for the spaces $W^{s,p}_\alpha(\mathcal{I})$ still hold for the $W^{s,p}_\alpha(\Lambda)$. It can also be noted that, in the case $p=\infty$, all spaces $W^{s,\infty}_\alpha(\Lambda)$ coincide with $W^{s,\infty}(\Lambda)$. As usual, in the case $p=2$, the Hilbertian Sobolev spaces $W^{s,2}_\alpha(\Lambda)$ are denoted by $H^s_\alpha(\Lambda)$.

The Jacobi polynomials $J^{\alpha,\beta}_n$ form an orthogonal family on Λ for the weighted measure $(1-\zeta)^\alpha(1+\zeta)^\beta\,d\zeta$, for any real numbers $\alpha>-1$ and $\beta>-1$. We only consider here the case $\alpha=\beta$ of polynomials which are orthogonal in $L^2_\alpha(\Lambda)$ (these polynomials are usually called ultraspherical polynomials but with a different α). Indeed the corresponding orthogonal basis plays an important role, first for proving certain inverse inequalities (cf [28]) and second for stating the optimality of certain estimates and providing counterexamples.

Notation 1.1. For any fixed real number $\alpha>-1$, we denote by $(J^\alpha_n)_n$ the family of Jacobi polynomials: The polynomials J^α_n, $n\geq 0$, are of degree n, orthogonal to each other for the measure $\rho_\alpha(\zeta)\,d\zeta$ on Λ and satisfy the condition

$$J^\alpha_n(1)=\frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}. \tag{1.5}$$

We also introduce the associated orthonormal polynomials $J^{\alpha*}_n=\frac{J^\alpha_n}{\|J^\alpha_n\|_{L^2_\alpha(\Lambda)}}$, $n\geq 0$.

Note (see [19] for instance) that we have the following equivalences for the $L^2_\alpha(\Lambda)$ and the $L^\infty(\Lambda)$ norms of the J^α_n: Combining the equation

$$\|J^\alpha_n\|^2_{L^2_\alpha(\Lambda)}=\frac{2^{2\alpha+1}\,\Gamma(n+\alpha+1)^2}{(2n+2\alpha+1)\,n!\,\Gamma(n+2\alpha+1)} \tag{1.6}$$

with Stirling's formula, we observe that $\|J_n^\alpha\|_{L^2_\alpha(\Lambda)}$ behaves like $n^{-\frac{1}{2}}$ when n tends to $+\infty$, independently of α. The behaviour of the $L^\infty(\Lambda)$ norm is [30, form. (7.32.2)]

$$\begin{cases} \|J_n^\alpha\|_{L^\infty(\Lambda)} \simeq n^{-\frac{1}{2}} & \text{if } -1 < \alpha < -\frac{1}{2} \\ \|J_n^\alpha\|_{L^\infty(\Lambda)} \simeq n^{\alpha} & \text{if } \alpha \geq -\frac{1}{2} \end{cases} \quad \text{when} \quad n \to +\infty. \tag{1.7}$$

The change of behaviour of $\|J_n^\alpha\|_{L^\infty(\Lambda)}$ for $\alpha = -\frac{1}{2}$ is linked to the fact that J_n^α reaches its maximal absolute value in ± 1 when α is $> -\frac{1}{2}$ and near 0 when α is $< -\frac{1}{2}$. This basic example indicates that the behaviour of weighted Sobolev norms of polynomials is not always simple.

A rather important property of the Jacobi polynomials is that they are the eigenfunctions of a Sturm–Liouville type operator, more precisely they satisfy the following differential equation [19]

$$\big(\rho_{\alpha+1}\, J_n^{\alpha\,\prime}\big)' + n(n+2\alpha+1)\,\rho_\alpha\, J_n^\alpha = 0. \tag{1.8}$$

A consequence is the result of the following lemma.

Lemma 1.2. *The following formula holds for any real number $\alpha > -1$ and for any positive integer n:*

$$J_n^{\alpha\,\prime} = \frac{n+2\alpha+1}{2} J_{n-1}^{\alpha+1}. \tag{1.9}$$

Proof. From equation (1.8), we derive first that $J_n^{\alpha\,\prime}$ is orthogonal to all polynomials in $\mathbb{P}_{n-2}(\Lambda)$ for the measure $\rho_{\alpha+1}(\zeta)\,d\zeta$, hence is equal to a constant times $J_{n-1}^{\alpha+1}$, second that

$$J_n^{\alpha\,\prime}(1) = \frac{n(n+2\alpha+1)}{2(\alpha+1)}\, J_n^\alpha(1).$$

By comparing this formula with (1.5), we derive the desired result.

2. Inverse inequalities with different exponents.

We first recall the basic inverse inequality which can be deduced from [28] (see also [29]) and holds for any $\alpha \geq -\frac{1}{2}$, for any p and q, $1 \leq q \leq p \leq +\infty$, and for any φ_N in $\mathbb{P}_N(\Lambda)$,

$$\|\varphi_N\|_{L^p_\alpha(\Lambda)} \leq c\, N^{2(\alpha+1)(\frac{1}{q}-\frac{1}{p})}\, \|\varphi_N\|_{L^q_\alpha(\Lambda)}. \tag{2.1}$$

Applying iteratively this inequality to the derivatives of the polynomial leads to the following result.

Proposition 2.1. *Let α be a real number $\geq -\frac{1}{2}$ and p and q be such that $1 \leq q \leq p \leq +\infty$. For any nonnegative integer m, any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi_N\|_{W^{m,p}_\alpha(\Lambda)} \leq c\, N^{2(\alpha+1)(\frac{1}{q}-\frac{1}{p})}\, \|\varphi_N\|_{W^{m,q}_\alpha(\Lambda)}. \tag{2.2}$$

Estimate (2.2) still holds with the integer m replaced with any nonnegative real number s. However we prefer to skip the corresponding statement since this estimate is proved later on in a more general case.

3. Inverse inequalities with different integral orders.

These inequalities are well-known in the case $p = 2$, since they rely on the orthogonality property of the J_n^α. Proving them is much more complex for other values of p.

For any function f in $L^1_\alpha(\Lambda)$, we denote by $c_n^\alpha(f)$, $n \geq 0$, its coefficient in the basis $(J_n^{\alpha *})_n$:

$$c_n^\alpha(f) = \int_{-1}^{1} f(\zeta) J_n^{\alpha *}(\zeta)\, \rho_\alpha(\zeta)\, d\zeta.$$

In the same spirit as for Cesaro's means, we fix a function λ of class $\mathscr{C}^\infty$ from $\mathbb{R}_+$ into $[0,1]$, which is equal to 1 on $[0,1]$ and vanishes on $[2,+\infty)$. Then, for any nonnegative integer N, we set

$$\mathcal{L}_N^\alpha f = \sum_{n=0}^{N} \lambda\left(\tfrac{n}{N}\right)\, c_n^\alpha(f) J_n^{\alpha\,*}. \tag{3.1}$$

It is easily checked that the operator $\mathcal{L}_N^\alpha$ is linear continuous from $L^2_\alpha(\Lambda)$ into $\mathbb{P}_{2N}(\Lambda)$ and that it coincides with the identity on $\mathbb{P}_N(\Lambda)$. Furthermore, it satisfies a very important stability property, which is derived in [28] from its analogue for the Cesaro's means of appropriate order j.

Lemma 3.1. *Let α be a real number $\geq -\frac{1}{2}$ and p be such that $2 \leq p \leq \infty$. any function f in $L^p_\alpha(\Lambda)$ satisfies*

$$\|\mathcal{L}_N^\alpha f\|_{L^p_\alpha(\Lambda)} \leq c\, \|f\|_{L^p_\alpha(\Lambda)}. \tag{3.2}$$

This property is the basic argument for the proof of the two propositions that follow.

Proposition 3.2. *Let α be a real number $\geq -\frac{1}{2}$ and p be such that $2 \leq p \leq \infty$. Any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi_N'\|_{L^p_\alpha(\Lambda)} \leq c\, N^2 \|\varphi_N\|_{L^p_\alpha(\Lambda)}. \tag{3.3}$$

Proof. This inequality is already known in two particular cases: This is Markov's inequality when p is equal to $+\infty$ [31] and it is proven in [9, §V] when p is equal to 2. We derive the result for $2 \le p \le \infty$ by the following interpolation argument, using the interpolation result between the spaces $L^p_\alpha(\Lambda)$ quoted in Remark I.7.1: Let us introduce the mapping defined on $L^1_\alpha(\Lambda)$ which, with any function f, associates $(\mathcal{L}^\alpha_N f)'$; from [31] and [9], loc. cit., it is linear and continuous from $L^2_\alpha(\Lambda)$ into itself, with norm $\le c\,N^2$, and also from $L^\infty(\Lambda)$ into itself, with norm $\le c\,N^2$; hence, it is continuous from $L^p_\alpha(\Lambda)$ into itself, with norm $\le c\,N^2$. We obtain the desired result by taking $f = \varphi_N$ in $\mathbb{P}_N(\Lambda)$.

Proposition 3.3. *Let α be a real number $\ge -\frac{1}{2}$ and p be such that $1 \le p \le 2$. Any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi_N'\|_{L^p_\alpha(\Lambda)} \le c\,N^2 \|\varphi_N\|_{L^p_\alpha(\Lambda)}. \tag{3.4}$$

Proof. Here, we use a duality argument. Defining q as the conjugate number of p, i.e., by $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\|\varphi_N'\|_{L^p_\alpha(\Lambda)} = \sup_{v \in L^q_\alpha(\Lambda)} \frac{\int_{-1}^1 \varphi_N'(\zeta)\, v(\zeta)\, \rho_\alpha(\zeta)\, d\zeta}{\|v\|_{L^q_\alpha(\Lambda)}}. \tag{3.5}$$

Next, we compute

$$\int_{-1}^1 \varphi_N'(\zeta)\, v(\zeta)\, \rho_\alpha(\zeta)\, d\zeta = \sum_{n=0}^{N-1} c_n^\alpha(\varphi_N')\, c_n^\alpha(v) = \int_{-1}^1 \varphi_N'(\zeta)\, (\mathcal{L}^\alpha_N v)(\zeta)\, \rho_\alpha(\zeta)\, d\zeta.$$

There exists (see [9, form. (V.35)]) a polynomial ξ_N in $\mathbb{P}_N(\Lambda)$ which is equal to 1 in -1, to 0 in 1 and which satisfies

$$\|\xi_N\|_{L^2_\alpha(\Lambda)} \le c\,N^{-\alpha-1}. \tag{3.6}$$

We set

$$v_N(\zeta) = \mathcal{L}^\alpha_N v(\zeta) - (\mathcal{L}^\alpha_N v)(-1)\, \xi_N(\zeta) - (\mathcal{L}^\alpha_N v)(1)\, \xi_N(-\zeta)$$

and, since $v_N(-1) = v_N(1) = 0$, we may write

$$\int_{-1}^1 \varphi_N'(\zeta)\, v(\zeta)\, \rho_\alpha(\zeta)\, d\zeta = \int_{-1}^1 \varphi_N'(\zeta)\, v_N(\zeta)\, \rho_\alpha(\zeta)\, d\zeta + A_- + A_+,$$

with

$$A_\pm = (\mathcal{L}^\alpha_N v)(\pm 1) \int_{-1}^1 \varphi_N'(\zeta)\, \xi_N(\mp\zeta)\, \rho_\alpha(\zeta)\, d\zeta.$$

By integration by parts, this yields

$$\int_{-1}^{1} \varphi_N'(\zeta)\, v(\zeta)\, \rho_\alpha(\zeta)\, d\zeta = -\int_{-1}^{1} \varphi_N(\zeta)\, (v_N\, \rho_\alpha)'(\zeta)\, d\zeta + A_- + A_+.$$

Next, note that

$$\int_{-1}^{1} \varphi_N(\zeta)\, (v_N\, \rho_\alpha)'(\zeta)\, d\zeta \\ = \int_{-1}^{1} \varphi_N(\zeta)\, v_N'(\zeta)\, \rho_\alpha(\zeta)\, d\zeta - 2\alpha \int_{-1}^{1} \varphi_N(\zeta)\, v_N(\zeta)\, \zeta\, \rho_{\alpha-1}(\zeta)\, d\zeta,$$

whence

$$|\int_{-1}^{1} \varphi_N(\zeta)\, (v_N\, \rho_\alpha)'(\zeta)\, d\zeta| \le c\, \|\varphi_N\|_{L^p_\alpha(\Lambda)} \big(\|v_N'\|_{L^q_\alpha(\Lambda)} + \|v_N\|_{L^q_{\alpha-q}(\Lambda)}\big).$$

When q differs from $\alpha+1$, using Hardy's inequality quoted in Lemma I.3.2 leads to

$$|\int_{-1}^{1} \varphi_N(\zeta)\, (v_N\, \rho_\alpha)'(\zeta)\, d\zeta| \le c\, \|\varphi_N\|_{L^p_\alpha(\Lambda)} (\|v_N\|_{L^q_\alpha(\Lambda)} + \|v_N'\|_{L^q_\alpha(\Lambda)}).$$

Finally, we obtain

$$|\int_{-1}^{1} \varphi_N'(\zeta)\, v(\zeta)\, \rho_\alpha(\zeta)\, d\zeta| \le \|\varphi_N\|_{L^p_\alpha(\Lambda)} (\|v_N\|_{L^q_\alpha(\Lambda)} + \|v_N'\|_{L^q_\alpha(\Lambda)}) \\ + \big(|(\mathcal{L}_N^\alpha v)(-1)| + |(\mathcal{L}_N^\alpha v)(1)|\big)\, \|\varphi_N'\|_{L^2_\alpha(\Lambda)} \|\xi_N\|_{L^2_\alpha(\Lambda)}.$$

Using the inverse inequality of Proposition 3.2, together with (3.6), leads to

$$|\int_{-1}^{1} \varphi_N'(\zeta)\, v(\zeta)\, \rho_\alpha(\zeta)\, d\zeta| \le c\, N^2\, \|\varphi_N\|_{L^p_\alpha(\Lambda)} \|v_N\|_{L^q_\alpha(\Lambda)} \\ + c'\, N^2\, \big(|(\mathcal{L}_N^\alpha v)(-1)| + |(\mathcal{L}_N^\alpha v)(1)|\big)\, \|\varphi_N\|_{L^2_\alpha(\Lambda)}\, N^{-\alpha-1}.$$

By using (2.1) for the pair (p, q) equal successively to (∞, q) and to $(2, p)$, we have

$$|(\mathcal{L}_N^\alpha v)(\pm 1)| \le c\, N^{\frac{2(\alpha+1)}{q}}\, \|\mathcal{L}_N^\alpha v\|_{L^q_\alpha(\Lambda)}, \\ \|\varphi_N\|_{L^2_\alpha(\Lambda)} \le c\, N^{2(\alpha+1)(\frac{1}{p}-\frac{1}{2})}\, \|\varphi_N\|_{L^p_\alpha(\Lambda)},$$

whence

$$|\int_{-1}^{1} \varphi_N'(\zeta)\, v(\zeta)\, \rho_\alpha(\zeta)\, d\zeta| \le c\, N^2\, \|\varphi_N\|_{L^p_\alpha(\Lambda)} \big(\|v_N\|_{L^q_\alpha(\Lambda)} + \|\mathcal{L}_N^\alpha v\|_{L^q_\alpha(\Lambda)}\big).$$

On the other hand, we estimate $\|v_N\|_{L^q_\alpha(\Lambda)}$ by using once more (3.6) and (2.1):

$$\begin{aligned}\|v_N\|_{L^q_\alpha(\Lambda)} &\leq \|\mathcal{L}^\alpha_N v\|_{L^q_\alpha(\Lambda)} + \big(|(\mathcal{L}^\alpha_N v)(-1)| + |(\mathcal{L}^\alpha_N v)(1)|\big)\, \|\xi_N\|_{L^q_\alpha(\Lambda)}\\ &\leq c\, \|\mathcal{L}^\alpha_N v\|_{L^q_\alpha(\Lambda)}.\end{aligned}$$

These last two estimates, combined with (3.5), give the desired inequality when q is not equal to $\alpha + 1$. In this last case, the result is obtained by an interpolation argument as in the proof of Proposition 3.2.

Applying Propositions 3.2 and 3.3 to the derivatives of polynomials gives the general result in integral order spaces.

Corollary 3.4. *Let α be a real number $\geq -\frac{1}{2}$ and p be such that $1 \leq p \leq \infty$. For any nonnegative integers ℓ and m, $\ell \leq m$, any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi_N\|_{W^{m,p}_\alpha(\Lambda)} \leq c\, N^{2(m-\ell)} \|\varphi_N\|_{W^{\ell,p}_\alpha(\Lambda)}. \tag{3.7}$$

We do not consider the case $-1 < \alpha < -\frac{1}{2}$, since we have no application for that.

4. Inverse inequalities with different orders.

The general inverse inequalities are derived in two steps.

Proposition 4.1. *Let α be a real number $\geq -\frac{1}{2}$ and p be such that $1 \leq p \leq \infty$. For any integer ℓ and nonnegative real number s, $s \geq \ell$, any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi_N\|_{W^{s,p}_\alpha(\Lambda)} \leq c\, N^{2(s-\ell)} \|\varphi_N\|_{W^{\ell,p}_\alpha(\Lambda)}. \tag{4.1}$$

Proof. In the case $1 < p < \infty$, we first derive from Corollary 3.4 the inequality (which is obvious when $[s]$ is equal to ℓ).

$$\|\varphi_N\|_{W^{[s],p}_\alpha(\Lambda)} \leq c\, N^{2([s]-\ell)} \|\varphi_N\|_{W^{\ell,p}_\alpha(\Lambda)}. \tag{4.2}$$

The space $\mathbb{P}_N(\Lambda)$ provided with the norm of $W^{[s],p}_\alpha(\Lambda)$ is imbedded
- in $W^{[s],p}_\alpha(\Lambda)$, and the norm of the embedding is 1,
- in $W^{[s]+1,p}_\alpha(\Lambda)$, and the norm of the imbedding is $\leq c\, N^2$ from Corollary 3.4.

We know from Theorem I.7.3 that $W^{s,p}_\alpha(\Lambda)$ is the interpolate space of index $[s] + 1 - s$ between $W^{[s]+1,p}_\alpha(\Lambda)$ and $W^{[s],p}_\alpha(\Lambda)$. So applying the principal theorem of interpolation [23, Chap. I, Thm 5.1] yields that the space $\mathbb{P}_N(\Lambda)$ provided with the norm of $W_\alpha^{[s],p}(\Lambda)$ is imbedded in $W_\alpha^{s,p}(\Lambda)$, with the norm of the imbedding $\leq c\,N^{2(s-[s])}$. Combined with (4.2), this yields the desired result. We refer to [7, §4.b] for the extension to the cases $p = 1$ and $p = \infty$.

Theorem 4.2. *Let α be a real number $\geq -\frac{1}{2}$ and p be such that $1 \leq p \leq \infty$. For any nonnegative real numbers t and s, $t \leq s$, any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi_N\|_{W_\alpha^{s,p}(\Lambda)} \leq c\,N^{2(s-t)}\|\varphi_N\|_{W_\alpha^{t,p}(\Lambda)}. \tag{4.3}$$

We establish this theorem only in the case $p = 2$ and refer to [7, Thm 4.b.1] for the proof in other cases which is much more technical (it requires a different but equivalent norm on the space $W_\alpha^{t,p}(\Lambda)$).

Proof in the case $p = 2$. We treat successively two cases.
1) When $[s]$ is $\geq t$, we write the inequality (4.2) for ℓ equal to $[s]$, which gives

$$\forall \varphi_N \in \mathbb{P}_N(\Lambda), \quad \|\varphi_N\|_{H_\alpha^s(\Lambda)} \leq cN^{2(s-[s])}\,\|\varphi_N\|_{H_\alpha^{[s]}(\Lambda)},$$

next for $\ell = 0$, which gives

$$\forall \varphi_N \in \mathbb{P}_N(\Lambda), \quad \|\varphi_N\|_{H_\alpha^s(\Lambda)} \leq cN^{2s}\,\|\varphi_N\|_{L_\alpha^2(\Lambda)}.$$

A standard interpolation argument gives

$$\forall \varphi_N \in \mathbb{P}_N(\Lambda), \quad \|\varphi_N\|_{H_\alpha^s(\Lambda)} \leq cN^{2(s-t)}\,\|\varphi_N\|_{H_\alpha^t(\Lambda)}^{N,\ell} \quad \text{with} \quad \ell = [s],$$

and the result follows from Theorem 2.2.
2) In the case $[s] \leq t \leq s$, we obtain the result from the inequalities (the first one is derived from the first part of the proof)

$$\forall \varphi_N \in \mathbb{P}_N(\Lambda), \quad \|\varphi_N\|_{H_\alpha^{[t]+1}(\Lambda)} \leq cN^{2([t]+1-t)}\,\|\varphi_N\|_{H_\alpha^t(\Lambda)},$$
$$\|\varphi_N\|_{H_\alpha^t(\Lambda)} \leq c\,\|\varphi_N\|_{H_\alpha^t(\Lambda)},$$

by noting from Theorem I.7.3 that $H_\alpha^s(\Lambda)$ is the interpolate of index $\frac{[t]+1-s}{[t]+1-t}$ between $H_\alpha^{[t]+1}(\Lambda)$ and $H_\alpha^t(\Lambda)$ and using once more the principal theorem of interpolation [23, Chap. 1, Thm 5.1].

5. Inverse inequalities with different orders and exponents.

Our problem is now to obtain more general inverse inequalities of the type

$$\forall \varphi_N \in \mathbb{P}_N(\Lambda), \quad \|\varphi_N\|_{W_\alpha^{s,p}(\Lambda)} \leq c\,N^{\gamma}\,\|\varphi_N\|_{W_\alpha^{t,q}(\Lambda)}, \tag{5.1}$$

for a convenient exponent γ when the space $W_\alpha^{s,p}(\Lambda)$ is contained in the space $W_\alpha^{t,q}(\Lambda)$, or in the case of critical exponents. In view of Theorem I.7.2, the following assumption is natural

$$\begin{cases} \quad t - \frac{1}{q} \le s - \frac{1}{p} \\ \text{and} \\ \quad t - \frac{\alpha}{q} - \frac{1}{q} \le s - \frac{\alpha}{p} - \frac{1}{p}. \end{cases} \tag{5.2}$$

We already know such inverse inequalities in certain particular cases: For a fixed $\alpha \ge -\frac{1}{2}$, if t and s denote two real numbers, $0 \le t \le s$, and p and q are such that $1 \le q \le p \le \infty$, we recall from (2.1) and (4.3) the next inequalities

$$\forall \varphi_N \in \mathbb{P}_N(\Lambda), \quad \|\varphi_N\|_{L^p_\alpha(\Lambda)} \le c\, N^{2(\alpha+1)(\frac{1}{q}-\frac{1}{p})}\, \|\varphi_N\|_{L^q_\alpha(\Lambda)},$$

and

$$\forall \varphi_N \in \mathbb{P}_N(\Lambda), \quad \|\varphi_N\|_{W^{s,p}_\alpha(\Lambda)} \le c\, N^{2(s-t)}\, \|\varphi_N\|_{W^{t,p}_\alpha(\Lambda)}.$$

We now prove that inverse inequalities (5.1) hold in the case when q is $\le p$ with the exponent γ equal to twice the difference between the "Sobolev characteristic exponents" of the two spaces, i.e. $2\Big((s - \frac{\alpha}{p} - \frac{1}{p}) - (t - \frac{\alpha}{q} - \frac{1}{q})\Big)$. We treat separately the cases $\alpha \ge 0$ and $\alpha < 0$, since in the latter case, we need a stronger assumption on t and s.

Theorem 5.1. *Let α be a real number ≥ 0 and p and q be such that $1 \le q \le p \le \infty$. For any nonnegative real numbers t and s, $t \le s$, any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi_N\|_{W^{s,p}_\alpha(\Lambda)} \le c\, N^{2(s-t)+2(\alpha+1)(\frac{1}{q}-\frac{1}{p})} \|\varphi_N\|_{W^{t,q}_\alpha(\Lambda)}. \tag{5.3}$$

Proof. Since (5.3) is already proven when $p = q$, we may assume that $q < p$. We first note that, as consequences of the assumptions, the inequalities in (5.2) are strictly satisfied. Thus it is possible to find two real numbers t' and s' such that:

$$t \le t', \quad s' \le s, \quad s' < t', \tag{5.4}$$

$$t' - \frac{\alpha}{q} - \frac{1}{q} = s' - \frac{\alpha}{p} - \frac{1}{p} \quad \text{with} \quad s' - \frac{\alpha}{p} - \frac{1}{p} \notin \mathbb{N}, \tag{5.5}$$

$$s' - \frac{1}{p} \le t' - \frac{1}{q}. \tag{5.6}$$

Indeed, (5.4) and (5.5) are compatible due to the inequality $t - \frac{\alpha}{q} - \frac{1}{q} < s - \frac{\alpha}{p} - \frac{1}{p}$ and (5.6) is a consequence of (5.5) because α is ≥ 0. Since p is $> q$, these three conditions on t' and s' infer that, by application of Theorem I.7.2, the following imbedding holds:

$$W_\alpha^{t',q}(\Lambda) \subset W_\alpha^{s',p}(\Lambda).$$

Then applying twice the inequality (4.3), we obtain for any φ_N in $\mathbb{P}_N(\Lambda)$,

$$\begin{aligned}\|\varphi_N\|_{W_\alpha^{s,p}(\Lambda)} &\le c\, N^{2(s-s')} \|\varphi_N\|_{W_\alpha^{s',p}(\Lambda)}\\ &\le c\, N^{2(s-s')} \|\varphi_N\|_{W_\alpha^{t',q}(\Lambda)}\\ &\le c\, N^{2(s-s')}\, N^{2(t'-t)} \|\varphi_N\|_{W_\alpha^{t,q}(\Lambda)}.\end{aligned}$$

Finally equation (5.5) yields that $s - s' + t' - t$ is equal to $(s-t) + (\alpha+1)(\frac{1}{q} - \frac{1}{p})$, whence the result.

Theorem 5.2. *Let α be a real number, $-\frac{1}{2} \le \alpha < 0$, and p and q be such that $1 \le q \le p \le \infty$. For any nonnegative real numbers t and s, $t \le [s]$, any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies the inverse estimate* (5.3).

Proof. The result follows by applying twice the basic inequalities (4.3) and once (2.1), i.e.

$$\begin{aligned}\|\varphi_N\|_{W_\alpha^{s,p}(\Lambda)} &\le c\, N^{2(s-[s])} \|\varphi_N\|_{W_\alpha^{[s],p}(\Lambda)}\\ &\le c\, N^{2(s-[s])}\, N^{2(\alpha+1)(\frac{1}{q}-\frac{1}{p})} \|\varphi_N\|_{W_\alpha^{[s],q}(\Lambda)}\\ &\le c\, N^{2(s-[s])}\, N^{2(\alpha+1)(\frac{1}{q}-\frac{1}{p})}\, N^{2([s]-t)} \|\varphi_N\|_{W_\alpha^{t,q}(\Lambda)}.\end{aligned}$$

When q is $> p$, we are only able to treat the non-weighted spaces ($\alpha = 0$). The formula giving the exponent of N in (5.3) is no more valid in this case. This can be seen in the fundamental situation above.

Lemma 5.3. *Any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi_N\|_{W^{1,1}(\Lambda)} \le c\, N\, \|\varphi_N\|_{L^\infty(\Lambda)}. \tag{5.7}$$

In Section 7, it is checked that this estimate is optimal. However, the formula of the exponent of N in (5.3) would have given 0. Note also that, although we are in a limit case, $W^{1,1}(\Lambda)$ is contained in $L^\infty(\Lambda)$.

Proof. Let φ_N be in $\mathbb{P}_N(\Lambda)$, and let $\zeta_1, \dots, \zeta_n$ denote the zeros of φ'_N which are in Λ. Then n is $\le N$. Setting $\zeta_0 = -1$ and $\zeta_{n+1} = 1$, we have

$$\begin{aligned}\int_\Lambda |\varphi'_N(\zeta)|\, d\zeta = \sum_{i=0}^n \Big| \int_{\zeta_i}^{\zeta_{i+1}} \varphi'_N(\zeta)\, d\zeta \Big| &= \sum_{i=0}^n |\varphi_N(\zeta_i) - \varphi_N(\zeta_{i+1})|\\ &\le 2(n+1)\, \|\varphi_N\|_{L^\infty(\Lambda)},\end{aligned}$$

which proves the lemma.

Theorem 5.4. *Let p and q be such that $1 \leq p \leq q \leq \infty$. For any nonnegative real numbers t and s, $t \leq [s]-1$, any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi_N\|_{W^{s,p}(\Lambda)} \leq c\, N^{2(s-t)+(\frac{1}{q}-\frac{1}{p})} \|\varphi_N\|_{W^{t,q}(\Lambda)}. \tag{5.8}$$

Proof. We begin with the case when $t=0$ and $s=1$. We use an interpolation argument between the estimate $W^{1,1}(\Lambda)$ — $L^\infty(\Lambda)$ and the estimate $W^{1,r}(\Lambda)$ — $L^r(\Lambda)$ for a convenient r. Indeed, setting

$$\theta = 1 + \frac{1}{q} - \frac{1}{p} \quad \text{and} \quad r = \theta q,$$

we note that θ and r are such that $0 \leq \theta \leq 1$ and $1 \leq r \leq \infty$. Moreover, we have

$$[L^1(\Lambda), L^r(\Lambda)]_\theta = L^p(\Lambda) \quad \text{and} \quad [L^\infty(\Lambda), L^r(\Lambda)]_\theta = L^q(\Lambda).$$

The application $f \mapsto (\mathcal{L}_N^0 f)'$ introduced in (3.1) is continuous from $L^\infty(\Lambda)$ into $L^1(\Lambda)$ with a norm $\leq c\,N$ (see Lemma 5.3) and is continuous from $L^r(\Lambda)$ into $L^r(\Lambda)$ with a norm $\leq c\,N^2$ (see Propositions 3.2 et 3.3). We deduce by interpolating that the same application is continuous from $L^q(\Lambda)$ into $L^p(\Lambda)$ with a norm $\leq c\,N^{1+\theta}$, i.e. $\leq c\,N^{2+\frac{1}{q}-\frac{1}{p}}$. This proves the theorem when $t=0$ and $s=1$, and this result obviously extends to the case where s is an integer and t is equal to $s-1$.
We treat the general case by applying twice the basic inequalities (4.3) and once the estimate (5.8) in the above case (for the exponents $[s]-1$ and $[s]$), which gives

$$\begin{aligned}
\|\varphi_N\|_{W_\alpha^{s,p}(\Lambda)} &\leq c\,N^{2(s-[s])} \|\varphi_N\|_{W_\alpha^{[s],p}(\Lambda)} \\
&\leq c\,N^{2(s-[s])}\, N^{2+(\frac{1}{q}-\frac{1}{p})} \|\varphi_N\|_{W_\alpha^{[s]-1,q}(\Lambda)} \\
&\leq c\,N^{2(s-[s])}\, N^{2+(\frac{1}{q}-\frac{1}{p})}\, N^{2([s]-1-t)} \|\varphi_N\|_{W_\alpha^{t,q}(\Lambda)}.
\end{aligned}$$

This concludes the proof of the theorem.

6. Inverse inequalities with different orders and weights.

For simplicity, we only consider the case of exponent $p=2$ and try to compare the norms of a polynomial in $L^2_\alpha(\Lambda)$ and $L^2_\beta(\Lambda)$, $\alpha \neq \beta$. The proof of the first statement relies on the formula, see [11, form. (19.11)]

$$(2n+2\alpha+1) J_n^\alpha = \frac{n+2\alpha+1}{n+\alpha+1}\, J_{n+1}^{\alpha\,\prime} - \frac{n+\alpha}{n+2\alpha}\, J_{n-1}^{\alpha\,\prime}, \tag{6.1}$$

Proposition 6.1. *Let α be a real number > -1 . Any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi_N\|_{L^2_\alpha(\Lambda)} \leq c\,N \|\varphi_N\|_{L^2_{\alpha+1}(\Lambda)}. \tag{6.2}$$

Proof. Writing the expansion $\varphi_N = \sum_{n=0}^N \varphi^n\, J_n^{\alpha+1}$, we have

$$\|\varphi_N\|_{L^2_{\alpha+1}(\Lambda)} = \Big(\sum_{n=0}^N (\varphi^n)^2\, \|J_n^{\alpha+1}\|^2_{L^2_{\alpha+1}(\Lambda)}\Big)^{\frac{1}{2}}.$$

On the other hand, we obtain by a Cauchy–Schwarz inequality

$$\begin{aligned}\|\varphi_N\|_{L^2_\alpha(\Lambda)} &\leq c \sum_{n=0}^N |\varphi^n|\, \|\, J_n^{\alpha+1}\|_{L^2_\alpha(\Lambda)} \\ &\leq \Big(\sum_{n=0}^N (\varphi^n)^2\, \|J_n^{\alpha+1}\|^2_{L^2_{\alpha+1}(\Lambda)}\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^N \frac{\|J_n^{\alpha+1}\|^2_{L^2_\alpha(\Lambda)}}{\|J_n^{\alpha+1}\|^2_{L^2_{\alpha+1}(\Lambda)}}\Big)^{\frac{1}{2}},\end{aligned}$$

whence

$$\|\varphi_N\|_{L^2_\alpha(\Lambda)} \leq \|\varphi_N\|_{L^2_{\alpha+1}(\Omega)} \Big(\sum_{n=0}^N \frac{\|J_n^{\alpha+1}\|^2_{L^2_\alpha(\Lambda)}}{\|J_n^{\alpha+1}\|^2_{L^2_{\alpha+1}(\Lambda)}}\Big)^{\frac{1}{2}}.$$

To evaluate this last quantity, we derive from formulas (1.9) and (6.1) that

$$J_n^{\alpha+1} = \frac{2}{n+2\alpha+2}\, J^{\alpha\,\prime}_{n+1} = \frac{2}{n+2\alpha+2} \sum_{0\leq 2m\leq n} \lambda_{n-2m}\, J^\alpha_{n-2m},$$

with

$$\lambda_{n-2m} = \frac{\Gamma(n+\alpha+2)\Gamma(n-2m+2\alpha+2)}{\Gamma(n-2m+\alpha+2)\Gamma(n+2\alpha+2)}\, \frac{(2(n-2m)+2\alpha+1)(n-2m+\alpha+1)}{n-2m+2\alpha+1}.$$

Recalling that $\|J_n^\alpha\|_{L^2_\alpha(\Lambda)}$ behaves like $n^{-\frac{1}{2}}$ thus gives that $\|J_n^{\alpha+1}\|_{L^2_\alpha(\Lambda)}$ is bounded independently of n, whence

$$\frac{\|J_n^{\alpha+1}\|^2_{L^2_\alpha(\Lambda)}}{\|J_n^{\alpha+1}\|^2_{L^2_{\alpha+1}(\Lambda)}} \leq c\,n.$$

Inserting this into the previous formula leads to the desired result.

The following result is now derived via an interpolation argument.

Corollary 6.2. *Let α and β be two real numbers, $\alpha > -1$ and $\beta > 0$, with $\alpha \leq \beta$. Any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi_N\|_{L^2_\alpha(\Lambda)} \leq c\, N^{\beta-\alpha} \|\varphi_N\|_{L^2_\beta(\Lambda)}. \tag{6.3}$$

Proof. When $\beta-\alpha$ is ≤ 1, we use the following interpolation argument: The identity operator is continuous from the space $\mathbb{P}_N(\Lambda)$ provided with the norm of $L^2_\beta(\Lambda)$ into $L^2_\beta(\Lambda)$ with norm 1 and into $L^2_{\beta-1}$ with norm $\leq c\,N$ as a consequence of Proposition 6.1, hence from this same space into $L^2_{\beta-\lambda}(\Lambda)$ with norm $\leq c\,N^\lambda$ for $0 \leq \lambda \leq 1$. Taking $\alpha = \beta - \lambda$ gives the desired result. When $\beta - \alpha$ is ≥ 1, we obtain by iterating inequality (6.2)

$$\|\varphi_N\|_{L^2_\alpha(\Lambda)} \leq c\, N^k \|\varphi_N\|_{L^2_{\alpha+k}(\Lambda)},$$

and combining this with the previous result leads to the desired estimate.

Of course, the previous inequality still holds with $\alpha > -2$ and $\beta > -1$, for polynomials φ_N vanishing in ± 1, with $\alpha > -3$ and $\beta > -2$, for polynomials φ_N vanishing in ± 1 together with their first derivative, and so on.

In a second step, we prove the inverse inequality related to Hardy's inequalities (I.3.6) and (I.3.7), see Lemma I.3.2. The next property plays a key role in the derivation of optimal approximation properties for the interpolation operator at the Gauss-Lobatto nodes, see [11, §21] for instance.

Proposition 6.3. *Let α be a real number > -1 . Any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi'_N\|_{L^2_{\alpha+1}(\Lambda)} \leq c\, N \|\varphi_N\|_{L^2_\alpha(\Lambda)}. \tag{6.4}$$

Proof. By writing the expansion $\varphi_N = \sum_{n=0}^N \varphi^n\, J_n^\alpha$, we have

$$\|\varphi_N\|_{L^2_\alpha(\Lambda)} = \Big(\sum_{n=0}^N (\varphi^n)^2\, \|J_n^\alpha\|^2_{L^2_\alpha(\Lambda)}\Big)^{\frac{1}{2}},$$

and, thanks to the differential equation (1.8),

$$\|\varphi'_N\|_{L^2_{\alpha+1}(\Lambda)} = \Big(\sum_{n=0}^N (\varphi^n)^2\, n(n+2\alpha+1)\, \|J_n^\alpha\|^2_{L^2_\alpha(\Lambda)}\Big)^{\frac{1}{2}}.$$

This gives the desired result.

It can be noted that this inequality is much better than (3.3), in the sense that the equivalence of norms hold with a constant times N instead of N^2. It is readily checked that the constant c in (6.4) is equal to $\big(2(\alpha+1)\big)^{\frac{1}{2}}$ and also, by taking φ_N equal to J_N^α, that the power 1 of N in (6.4) is optimal.

Another consequence of (6.4) is that, for all $\beta \geq \alpha+1$, and for any polynomial φ_N in $\mathbb{P}_N(\Lambda)$,

$$\|\varphi_N'\|_{L^2_\beta(\Lambda)} \leq c\, N \|\varphi_N\|_{L^2_\alpha(\Lambda)}, \tag{6.5}$$

and this inequality is also optimal, as can be proved by also taking φ_N equal to J_N^α and computing $\|J_N^{\alpha\,\prime}\|_{L^2_{\alpha+2}(\Lambda)}$ (this calculus is rather simple in the case $\alpha = 0$). We conclude with a more general result.

Corollary 6.4. *Let α and β be two real numbers, $\alpha > -1$ and $\beta > 0$, and let k and m be two nonnegative integers with $k \geq m$. Any polynomial φ_N in $\mathbb{P}_N(\Lambda)$ satisfies*

$$\|\varphi_N\|_{H^k_\alpha(\Lambda)} \leq c\, N^{2(k-m)-\min\{\alpha-\beta, k-m\}}\, \|\varphi_N\|_{H^m_\beta(\Lambda)}. \tag{6.6}$$

Proof. By applying inequality (6.3) to the derivatives of φ_N, we derive the following inequality for any integer $s \geq 0$ and $\alpha \leq \beta$

$$\forall \varphi_N \in \mathbb{P}_N(\Lambda), \quad \|\varphi_N\|_{H^s_\alpha(\Lambda)} \leq c\, N^{\beta-\alpha} \|\varphi_N\|_{H^s_\beta(\Lambda)}, \tag{6.7}$$

and the same inequality when s is a nonnegative real number is then easily proved by an interpolation argument relying on Corollary 2.4. Inequality (6.7), combined with (4.3), yields (6.6) when α is $\leq \beta$, since $\alpha - \beta$ is $\leq k - m$ in this case.
The proof when α is $> \beta$ relies on the following arguments:
1) In the case $\beta < \alpha \leq \beta + 1$, we obtain by combining (6.3) and (6.4)

$$\|\varphi_N'\|_{L^2_\alpha(\Lambda)} \leq c\, N^{\beta+1-\alpha}\, \|\varphi_N'\|_{L^2_{\beta+1}(\Lambda)} \leq c'\, N^{2+\beta-\alpha}\, \|\varphi_N\|_{L^2_\beta(\Lambda)}.$$

2) Applying this result to the derivatives of φ_N up to the order m and using its analogue $k-m$ times iteratively gives (6.6) when $\alpha-\beta$ is $\leq k-m$. Then, using (6.5) allows for handling the case where $\alpha - \beta$ is $> k - m$.

To conclude, we observe that, thanks to an interpolation argument relying on Corollary 2.4, inequality (6.6) is still valid with k and m replaced with nonnegative real numbers s and t, with $s \geq t$,
(i) either when $\alpha - \beta$ is nonpositive,
(ii) or when $s - t$ is an integer,
(iii) or when s and t satisfy $[s] \geq [t] + 1$ and

$$\alpha - \beta \leq [s] - [t] - 1 \qquad \text{or} \qquad \alpha - \beta \geq [s] - [t] + 1,$$

(indeed, when this last condition is not satisfied, we can only prove the estimate with a slightly larger power of N).
So it is likely that estimate (6.6) holds for all s and t, $s \geq t$, however the interpolation theory does not seem sufficient to prove this result in all cases.

7. Optimality.

We now prove that certain basic inverse inequalities that we have just stated are unimprovable, namely estimate (2.1) for $(p,q)=(2,1)$ and for $(p,q)=(+\infty,2)$, estimate (4.3) for $(s,t)=(1,0)$ and $p=1,2,\infty$, and estimates (5.7) and (6.2)

To check this optimality, we determine the asymptotic behaviour of different norms of the Jacobi polynomials J_N^α and of their first and second derivatives. For the evaluation of the norms in $L^1_\alpha(\Lambda)$, we rely on the following result of Szegö [30, form. (7.34.1)]:

$$\|J_N^\alpha\|_{L^1_\beta(\Lambda)} \simeq \begin{cases} N^{-1/2} & \text{if } \alpha < 2\beta+\frac{3}{2} \\ N^{-1/2}\log N & \text{if } \alpha = 2\beta+\frac{3}{2} \\ N^{\alpha-2\beta-2} & \text{if } \alpha > 2\beta+\frac{3}{2} \end{cases} \qquad \text{when} \quad N \to +\infty. \tag{7.1}$$

For the norm of $L^\infty_\alpha(\Lambda)$, we rely on (1.7) while, for the norm of $L^2_\alpha(\Lambda)$, we use (1.6). All this can be combined with Lemma 1.2 and a further argument for estimating the first and second derivatives.

For simplicity, in the following tables, we omit the Λ in the norms. Table 7.1 gives asymptotic behaviours when $N \to +\infty$.

α	$-\frac{1}{2}$	$]-\frac{1}{2},\frac{1}{2}[$	$\frac{1}{2}$	$]\frac{1}{2},\frac{3}{2}[$	$\frac{3}{2}$	$]\frac{3}{2},\frac{5}{2}[$
$\Vert J_N^\alpha\Vert_{L^1_\alpha}$	$N^{-1/2}$	$N^{-1/2}$	$N^{-1/2}$	$N^{-1/2}$	$N^{-1/2}$	$N^{-1/2}$
$\Vert J_N^\alpha\Vert_{L^1}$	$N^{-1/2}$	$N^{-1/2}$	$N^{-1/2}$	$N^{-1/2}$	$N^{-1/2}\log N$	$N^{\alpha-2}$
$\Vert J_N^\alpha\Vert_{L^2_\alpha}$	$N^{-1/2}$	$N^{-1/2}$	$N^{-1/2}$	$N^{-1/2}$	$N^{-1/2}$	$N^{-1/2}$
$\Vert J_N^\alpha\Vert_{L^\infty}$	N^α	N^α	N^α	N^α	N^α	N^α
$\Vert J_N^{\alpha\prime}\Vert_{L^1_\alpha}$	$N^{1/2}\log N$	$N^{1/2}$	$N^{1/2}$	$N^{1/2}$	$N^{1/2}$	$N^{1/2}$
$\Vert J_N^{\alpha\prime}\Vert_{L^1}$	$N^{1/2}$	$N^{1/2}$	$N^{1/2}\log N$	N^α	N^α	N^α
$\Vert J_N^{\alpha\prime}\Vert_{L^2_\alpha}$	N	N	N	N	N	N
$\Vert J_N^{\alpha\prime}\Vert_{L^\infty}$	$N^{\alpha+2}$	$N^{\alpha+2}$	$N^{\alpha+2}$	$N^{\alpha+2}$	$N^{\alpha+2}$	$N^{\alpha+2}$
$\Vert J_N^{\alpha\prime\prime}\Vert_{L^1_\alpha}$	$N^{2-\alpha}$	$N^{2-\alpha}$	$N^{3/2}\log N$	$N^{3/2}$	$N^{3/2}$	$N^{3/2}$
$\Vert J_N^{\alpha\prime\prime}\Vert_{L^1}$	$N^{3/2}\log N$	$N^{\alpha+2}$	$N^{\alpha+2}$	$N^{\alpha+2}$	$N^{\alpha+2}$	$N^{\alpha+2}$
$\Vert J_N^{\alpha\prime\prime}\Vert_{L^2_\alpha}$	N^3	N^3	N^3	N^3	N^3	N^3
$\Vert J_N^{\alpha\prime\prime}\Vert_{L^\infty}$	$N^{\alpha+4}$	$N^{\alpha+4}$	$N^{\alpha+4}$	$N^{\alpha+4}$	$N^{\alpha+4}$	$N^{\alpha+4}$
$\Vert J_N^{\alpha\prime\prime\prime}\Vert_{L^1_\alpha}$	$N^{4-\alpha}$	$N^{4-\alpha}$	$N^{4-\alpha}$	$N^{4-\alpha}$	$N^{5/2}\log N$	$N^{5/2}$

Table 7.1

The results given in Tables 7.2, 7.3 and 7.4 are deduced from Table 7.1. They concern the asymptotic behaviour when $N \to +\infty$ of different ratios between couples of norms. For Table 7.2, the upper bound which appears in formula (2.1) is $N^{\alpha+1}$.

α	$-\frac{1}{2}$	$]-\frac{1}{2},\frac{1}{2}[$	$\frac{1}{2}$	$]\frac{1}{2},\frac{3}{2}[$	$\frac{3}{2}$	$]\frac{3}{2},\frac{5}{2}[$
$\frac{\\|J_N^{\alpha}{}''\\|_{L^2_\alpha}}{\\|J_N^{\alpha}{}''\\|_{L^1_\alpha}}$	$N^{\alpha+1}$	$N^{\alpha+1}$	$N^{\frac{3}{2}}\log^{-1}N$	$N^{\frac{3}{2}}$	$N^{\frac{3}{2}}$	$N^{\frac{3}{2}}$
$\frac{\\|J_N^{\alpha}{}''\\|_{L^\infty_\alpha}}{\\|J_N^{\alpha}{}''\\|_{L^2_\alpha}}$	$N^{\alpha+1}$	$N^{\alpha+1}$	$N^{\alpha+1}$	$N^{\alpha+1}$	$N^{\alpha+1}$	$N^{\alpha+1}$

Table 7.2

For Table 7.3, the upper bound which appears in formula (4.3) is N^2.

α	$-\frac{1}{2}$	$]-\frac{1}{2},\frac{1}{2}[$	$\frac{1}{2}$	$]\frac{1}{2},\frac{3}{2}[$	$\frac{3}{2}$	$]\frac{3}{2},\frac{5}{2}[$
$\frac{\\|J_N^{\alpha}{}''\\|_{W^{1,1}_\alpha}}{\\|J_N^{\alpha}{}''\\|_{L^1_\alpha}}$	N^2	N^2	$N^2\log^{-1}N$	$N^{\frac{5}{2}-\alpha}$	$N\log N$	N
$\frac{\\|J_N^{\alpha}{}'\\|_{W^{1,2}_\alpha}}{\\|J_N^{\alpha}{}'\\|_{L^2_\alpha}}$	N^2	N^2	N^2	N^2	N^2	N^2
$\frac{\\|J_N^{\alpha}\\|_{W^{1,\infty}}}{\\|J_N^{\alpha}\\|_{L^\infty}}$	N^2	N^2	N^2	N^2	N^2	N^2

Table 7.3

For Table 7.4, the upper bound which appears in formula (5.7) is N.

α	$-\frac{1}{2}$	$]-\frac{1}{2}, \frac{1}{2}[$	$\frac{1}{2}$	$]\frac{1}{2}, \frac{3}{2}[$	$\frac{3}{2}$	$]\frac{3}{2}, \frac{5}{2}[$
$\dfrac{\|J_N^\alpha\|_{W^{1,1}}}{\|J_N^\alpha\|_{L^\infty}}$	N	$N^{\frac{1}{2}-\alpha}$	$\log N$	1	1	1
$\dfrac{\|{J_N^\alpha}'\|_{W^{1,1}}}{\|{J_N^\alpha}'\|_{L^\infty}}$	$\log N$	1	1	1	1	1

Table 7.4

To conclude, we observe that, even if all previous inequalities are optimal, combining them does not necessarily lead to an optimal result. For instance, we derive from Proposition 2.1, Lemma 5.3 and formula (2.1) that, for all φ_N in $\mathbb{P}_N(\Lambda)$,

$$\|\varphi_N\|_{H^1(\Lambda)} \leq c\, N\, \|\varphi_N\|_{W^{1,1}(\Lambda)} \leq c'\, N^2\, \|\varphi_N\|_{L^\infty(\Lambda)} \leq c''\, N^3\, \|\varphi_N\|_{L^2(\Lambda)},$$

while the optimal inequality, stated in Proposition 3.2, reads

$$\|\varphi_N\|_{H^1(\Lambda)} \leq c\, N^2\, \|\varphi_N\|_{L^2(\Lambda)}.$$

In contrast, if φ_N vanishes in ± 1, we derive from Propositions 6.3 and 6.1 the optimal inequality

$$\|\varphi_N\|_{H^1(\Lambda)} \leq c\, N\, \|\varphi_N\|_{L^2_{-1}(\Lambda)} \leq c'\, N^2\, \|\varphi_N\|_{L^2(\Lambda)}.$$

This last inequality brings to light the importance of using weights when working with polynomials, which seems coherent with the differential equation (1.8).

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