CONTINUITY PROPERTIES OF THE
INF-SUP CONSTANT FOR THE DIVERGENCE

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ABSTRACT. The inf-sup constant for the divergence, or LBB constant, is explicitly known for only few domains. For other domains, upper and lower estimates are known. If more precise values are required, one can try to compute a numerical approximation. This involves, in general, approximation of the domain and then the computation of a discrete LBB constant that can be obtained from the numerical solution of an eigenvalue problem for the Stokes system. This eigenvalue problem does not fall into a class for which standard results about numerical approximations can be applied. Indeed, many reasonable finite element methods do not yield a convergent approximation. In this article, we show that under fairly weak conditions on the approximation of the domain, the LBB constant is an upper semi-continuous shape functional, and we give more restrictive sufficient conditions for its continuity with respect to the domain. For numerical approximations based on variational formulations of the Stokes eigenvalue problem, we also show upper semi-continuity under weak approximation properties, and we give stronger conditions that are sufficient for convergence of the discrete LBB constant towards the continuous LBB constant. Numerical examples show that our conditions are, while not quite optimal, not very far from necessary.

1. INTRODUCTION

We define the usual inf-sup constant for the divergence or Ladyzhenskaya-Babuška-Brezzi (LBB) constant $\beta(\Omega)$ as

$$(1.1) \qquad \beta(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{\langle \operatorname{div} \boldsymbol{v}, q \rangle_\Omega}{|\boldsymbol{v}|_{1,\Omega} \, \|q\|_{0,\Omega}} .$$

The constant thus depends on a domain (connected open set) $\Omega \subset \mathbb{R}^d$ and on two function spaces, the space ("pressure space") $L^2_\circ(\Omega)$ of square integrable functions with mean value zero, with norm $\|\cdot\|_{0,\Omega}$ and scalar product $\langle \cdot, \cdot \rangle_\Omega$, and the "velocity space", which is the vector-valued version of the standard Sobolev space $H^1_0(\Omega)$, closure of the space of smooth functions of compact support in Ω with respect to the H^1 seminorm, which for vector-valued functions is defined by

$$|\boldsymbol{v}|_{1,\Omega} = \|\operatorname{\mathbf{grad}} \boldsymbol{v}\|_{0,\Omega} = \Big(\sum_{k=1}^d \sum_{j=1}^d \|\partial_{x_j} v_k\|_{0,\Omega}^2 \Big)^{1/2}.$$

The present article is devoted to the study of the variation of the LBB constant with respect to changes of the domain and also of the two function spaces.

The inf-sup condition $\beta(\Omega) > 0$ plays an important role in the existence and stability of solutions of incompressible fluid models such as the Stokes and Navier–Stokes equations, see for example [18, Chap. I, Theorem 4.1]. The value of the LBB constant influences the convergence rate of iterative algorithms such as the Uzawa algorithm, see [14]. The inf-sup condition has been known for a long time to be true for bounded domains satisfying a uniform cone condition, that is, Lipschitz domains [6] and has more recently been shown for the larger class of John domains defined by a "twisted cone" condition [1]. It is, however, not satisfied for domains having an exterior cusp [15, 28].

If one defines an analogous "discrete LBB constant" β_n by

$$(1.2) \qquad \beta_n = \inf_{q \in M_n} \sup_{\boldsymbol{v} \in X_n} \frac{\langle \operatorname{div} \boldsymbol{v}, q \rangle_\Omega}{|\boldsymbol{v}|_{1,\Omega} \, \|q\|_{0,\Omega}} \, ,$$

where

$$M_n \subset L^2_\circ(\Omega) \quad \text{and} \quad X_n \subset H^1_0(\Omega)^d$$

are subspaces, then it is known that the discrete LBB condition

$$(1.3) \qquad \forall \, n, \quad \beta_n \geq \beta_\star > 0$$

plays an equally important role for the stability of Galerkin approximation methods for the Stokes system defined by a sequence of finite-dimensional subspaces $(X_n, M_n)_n$ [2, 8], [18, Chap. II, Theorem 1.1], as well as for the convergence of solution methods for the corresponding linear systems. There exists a large body of finite element literature proving discrete LBB conditions (1.3) for many pairs of velocity/pressure spaces (X_n, M_n), often referred to as *inf-sup stable* elements.

Much less is known about the question whether β_n is an approximation of $\beta(\Omega)$, namely

$$(1.4) \qquad \lim_{n \to \infty} \beta_n = \beta(\Omega) \, .$$

It turns out that there is a large class of finite element methods for which a uniform lower bound (1.3) has been shown, but the convergence (1.4) has not been established, and sometimes is even false. One of the main results of this work is to give conditions on the approximation properties of the spaces X_n and M_n so that the convergence (1.4) is true, see Theorem 5.1.

Under much weaker approximation properties, concerning only the pressure spaces M_n, we show upper semi-continuity (see Theorem 2.1), which implies in particular that the uniform lower bound $\beta_\star$ in the discrete LBB condition (1.3) can never be better than the (continuous) LBB constant $\beta(\Omega)$:

$$(1.5) \qquad \beta_\star \leq \beta(\Omega) \, .$$

The latter result has been shown under more restrictive hypotheses on the regularity of the domain in [9, Theorem 1.2].

Regarding the approximation of the domain Ω, we prove two general results. For inner approximations, we show upper semi-continuity of $\beta(\Omega)$ under the weak condition that the difference of the domains tends to zero in measure, see Theorem 2.2. We describe several examples where one does not have convergence of $\beta(\Omega)$ in this situation.

The second main result of this paper is that under convergence with respect to Lipschitz deformations of a bounded Lipschitz domain Ω, one does have continuity of the constant

$\beta(\Omega)$, see Theorem 4.4. As a corollary, we find that the LBB constant converges if a smooth plane domain is approximated by polygonal interpolation. This has an important application in finite element discretizations, where the first step often consists in replacing a curved domain by a union of triangles or polyhedra. Our result shows that this replacement essentially does not deteriorate the stability of the continuous problem. A second application is in the numerical approximation of the inf-sup constant itself, where one might want to approximate both the function spaces and the domain.

The outline of this article is as follows:

• We first prove in Section 2 upper semi-continuity results of the inf-sup constant both with respect to the function spaces and with respect to interior approximations of the domain.

• Section 3 is devoted to several examples exhibiting various types of converging domains, resulting in convergence or non-convergence of the inf-sup constant.

• Then in Section 4 we prove the continuity with respect to Lipschitz deformations of the domain and, as a corollary, with respect to polygonal approximations of smooth plane domains.

• Finally, in Section 5 we prove that under certain conditions, the discrete inf-sup constants converge to the continuous inf-sup constant. We give several numerical examples exhibiting convergence or non-convergence.

2. UPPER SEMI-CONTINUITY

We first prove a general result about the upper semi-continuity of the inf-sup constants β_n defined in (1.2) as n tends to infinity. We then show that this result can be used in two ways: First as a statement on the behavior of the sequence of the inf-sup constants $\beta(\Omega_n)$ for subdomains $\Omega_n \subset \Omega$ converging to Ω, and second as a statement of the relation between the discrete LBB condition (1.3) and the LBB constant $\beta(\Omega)$.

2.1. **Upper semi-continuity with respect to the function spaces.** We now state and prove a key result.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $M_n \subset L^2_\circ(\Omega)$ and $X_n \subset H^1_0(\Omega)^d$, $n \in \mathbb{N}$, be a sequence of closed subspaces. Let β_n be the inf-sup constant defined by these spaces according to (1.2). We assume that the sequence M_n is asymptotically dense in the sense that for every $q \in L^2_\circ(\Omega)$ there exists a sequence $(q_n)_{n\in\mathbb{N}}$ with $q_n \in M_n$ converging to q in $L^2(\Omega)$. Then*

$$\text{(2.1)} \qquad \limsup_{n\to\infty} \beta_n \le \beta(\Omega).$$

Proof. For $q \in L^2_\circ(\Omega)$ define

$$\text{(2.2)} \qquad J(q) = \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{\left\langle \operatorname{div} \boldsymbol{v}, q \right\rangle_\Omega}{|\boldsymbol{v}|_{1,\Omega}} \quad \text{and} \quad J_n(q) = \sup_{\boldsymbol{v} \in X_n} \frac{\left\langle \operatorname{div} \boldsymbol{v}, q \right\rangle_\Omega}{|\boldsymbol{v}|_{1,\Omega}}.$$

Owing to the inequality

$$\| \operatorname{div} \boldsymbol{v} \|_{0,\Omega} \le |\boldsymbol{v}|_{1,\Omega},$$

the functionals J and J_n satisfy for all $q \in L^2_\circ(\Omega)$

$$|J(q)| \leq \|q\|_{0,\Omega}, \qquad |J_n(q)| \leq \|q\|_{0,\Omega}.$$

Moreover, the functional J is continuous in $L^2_\circ(\Omega)$, as can be seen by introducing the function $\boldsymbol{w}(q) \in H^1_0(\Omega)^d$ defined as the solution of the Dirichlet problem $\Delta \boldsymbol{w}(q) = \mathbf{grad}\, q$, or in variational form

$$(2.3) \qquad \forall \boldsymbol{v} \in H^1_0(\Omega)^d, \quad \big\langle \mathbf{grad}\, \boldsymbol{w}(q), \mathbf{grad}\, \boldsymbol{v} \big\rangle_\Omega = \big\langle \operatorname{div} \boldsymbol{v}, q \big\rangle_\Omega.$$

It is easy to see that the supremum in $J(q)$ is attained by the function $\boldsymbol{v} = \boldsymbol{w}(q)$. The mapping $q \mapsto \boldsymbol{w}(q)$ being continuous from $L^2_\circ(\Omega)$ to $H^1_0(\Omega)^d$, the continuity of the functional J follows from the formula

$$J(q) = \frac{\big\langle \operatorname{div} \boldsymbol{w}(q), q \big\rangle_\Omega}{|\boldsymbol{w}(q)|_{1,\Omega}} = |\boldsymbol{w}(q)|_{1,\Omega}.$$

We assume now that, possibly after passing to a subsequence, the sequence $(\beta_n)_n$ converges to some β_∞. Let a nontrivial function $q \in L^2_\circ(\Omega)$ be given and choose $q_n \in M_n$ such that $q_n \to q$ in $L^2(\Omega)$. We have

$$\frac{J(q)}{\|q\|_{0,\Omega}} = \lim_{n\to\infty} \frac{J(q_n)}{\|q_n\|_{0,\Omega}}$$

and

$$J(q_n) \geq J_n(q_n) \geq \beta_n \|q_n\|_{0,\Omega},$$

which together imply

$$\frac{J(q)}{\|q\|_{0,\Omega}} \geq \beta_\infty.$$

From the definition

$$\beta(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \frac{J(q)}{\|q\|_{0,\Omega}}$$

we therefore get the desired inequality $\beta(\Omega) \geq \beta_\infty$. $\qquad\qquad \square$

2.2. **Inner approximations of the domain.** A first corollary of Theorem 2.1 is obtained by defining the subspaces X_n and M_n via the natural inclusion of the spaces $H^1_0(\Omega_n)^d$ and $L^2_\circ(\Omega_n)$, respectively, where Ω_n is a subdomain of Ω.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Assume that the sequence of domains $(\Omega_n)_{n\in\mathbb{N}}$ converges to the limiting domain Ω in the following sense*

$$(2.4) \qquad \forall n \in \mathbb{N},\ \Omega_n \subset \Omega \quad \text{and} \quad \operatorname{meas}(\Omega \setminus \Omega_n) \xrightarrow[n\to\infty]{} 0,$$

where meas *denotes the Lebesgue measure. Then there holds*

$$(2.5) \qquad \limsup_{n\to\infty} \beta(\Omega_n) \leq \beta(\Omega).$$

Proof. For a function q defined on Ω_n, let $\tilde{q}$ be its extension by 0 to Ω. This mapping $\mathcal{Z}_n : q \mapsto \tilde{q}$ defines the natural inclusions of $L^2(\Omega_n)$ into $L^2(\Omega)$ and of $H_0^1(\Omega_n)$ into $H_0^1(\Omega)$. We define

$$M_n = \mathcal{Z}_n L_\circ^2(\Omega_n) \subset L_\circ^2(\Omega) \quad \text{and} \quad X_n = \mathcal{Z}_n H_0^1(\Omega_n)^d \subset H_0^1(\Omega)^d$$

and check that β_n as defined in (1.2) then coincides with $\beta(\Omega_n)$. We have to verify the hypothesis on the approximation property of M_n: Choose $q \in L_\circ^2(\Omega)$, let us set on Ω_n:

$$q_n = q\big|_{\Omega_n} - \frac{1}{\text{meas}(\Omega_n)} \int_{\Omega_n} q(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

and let $\tilde{q}_n$ be the extension by 0 of q_n to Ω. Then it is easy to see that

$$q_n \in L_\circ^2(\Omega_n), \quad \tilde{q}_n \in M_n \subset L_\circ^2(\Omega), \quad \text{and} \quad \tilde{q}_n \xrightarrow[n\to\infty]{} q \quad \text{in} \quad L^2(\Omega).$$

Theorem 2.1 can now be applied and gives the inequality (2.5). $\qquad\square$

We discuss in Section 3 several examples of domains Ω_n tending to a domain Ω and observe whether or not $\beta(\Omega_n)$ converges to $\beta(\Omega)$.

2.3. **Discrete and continuous LBB conditions.** In a conforming finite element discretization of the Stokes or Navier–Stokes equations, the trial space of the pressure variable is a finite-dimensional subspace M_n of $L_\circ^2(\Omega)$, and the trial space for the velocity variable is a finite-dimensional subspace X_n of $H_0^1(\Omega)^d$. The index n may be representative for the sum of the dimensions of these spaces. The discrete inf-sup constant is defined as in (1.2), and the uniform positivity of β_n, as expressed by the discrete LBB condition (1.3) plays an important role in the analysis of the method.

Proving uniform lower bounds for β_n for various pairs of finite element spaces (X_n, M_n) is an important subject of many papers in numerical fluid dynamics. A standard procedure in such proofs is the construction of a Fortin operator, see [19]. This is a projection operator

$$\Pi_n : H_0^1(\Omega)^d \to X_n$$

satisfying for each $q \in M_n$

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d, \quad \big\langle \operatorname{div} \Pi_n \boldsymbol{v}, q \big\rangle_\Omega = \big\langle \operatorname{div} \boldsymbol{v}, q \big\rangle_\Omega.$$

The Fortin operator provides a lower bound for the discrete inf-sup constant via the inequality

$$\beta(\Omega)\|q\|_{0,\Omega} \leq J(q) = \sup_{\boldsymbol{v} \in H_0^1(\Omega)^d} \frac{\big\langle \operatorname{div} \boldsymbol{v}, q \big\rangle_\Omega}{|\boldsymbol{v}|_{1,\Omega}} = \sup_{\boldsymbol{v} \in H_0^1(\Omega)^d} \frac{\big\langle \operatorname{div} \Pi_n \boldsymbol{v}, q \big\rangle_\Omega}{|\Pi_n \boldsymbol{v}|_{1,\Omega}} \frac{|\Pi_n \boldsymbol{v}|_{1,\Omega}}{|\boldsymbol{v}|_{1,\Omega}}$$

$$\leq \sup_{\boldsymbol{v} \in X_n} \frac{\big\langle \operatorname{div} \boldsymbol{v}, q \big\rangle_\Omega}{|\boldsymbol{v}|_{1,\Omega}} \|\Pi_n\| = J_n(q)\|\Pi_n\|$$

valid for all $q \in M_n$. Here $\|\Pi_n\|$ is the operator norm of Π_n in $H_0^1(\Omega)^d$ associated with the norm $|\cdot|_{1,\Omega}$. Dividing by $\|q\|_{0,\Omega}$ and taking the infimum over $q \in M_n$, one finds

$$\beta(\Omega) \leq \beta_n \|\Pi_n\|.$$

Uniformly bounding $\|\Pi_n\|$ is then the main work in the proof of the discrete LBB condition.

The resulting lower bound $\beta_n \geq \frac{\beta(\Omega)}{\|\Pi_n\|}$ lies always below $\beta(\Omega)$, because the norm of the projection operator Π_n is always at least 1.

The available theoretical estimates of the discrete LBB conditions may be very pessimistic compared to what is really observed in the numerical algorithms, but the fact that the uniform discrete LBB bound is always below the inf-sup constant of the domain is a corollary to the upper semi-continuity shown in Theorem 2.1.

Theorem 2.3. *Suppose that the sequence of spaces $(X_n, M_n)_{n \in \mathbb{N}}$ satisfies the discrete LBB condition*

$$\inf_{n \in \mathbb{N}} \beta_n = \beta_\star > 0$$

and that the sequence M_n is asymptotically dense in $L^2_\circ(\Omega)$. Then

$$\beta_\star \leq \beta(\Omega) .$$

Proof. This is an immediate consequence of Theorem 2.1. □

3. EXAMPLES OF DEPENDENCE WITH RESPECT TO THE DOMAIN

We present several examples of sequences of domains Ω_n that converge to a limiting domain Ω in various senses. All examples satisfy the assumption of Theorem 2.2, after a possible rescaling to achieve $\Omega_n \subset \Omega$. Some of them satisfy a stronger assumption (namely, the transformations tend to the identity in Lipschitz norm) and convergence occurs. As was already mentioned in [30] for conformal mappings, the absence of such a condition results in an absence of convergence.

3.1. **Cusp domains tending to a Lipschitz domain.** For each integer $n \geq 1$, let Ω_n be the set of points (x, y) such that

$$0 < x < 1, \quad -x^{\frac{1}{n}+1} < y < x^{\frac{1}{n}+1}.$$

For any n, the domain Ω_n has a cusp at the origin. By similar arguments as in [28], we derive that $\beta(\Omega_n) = 0$. In the limit $n \to \infty$, we obtain the isosceles right triangle

$$\Omega = \{(x, y) \in \mathbb{R}^2, \quad 0 < x < 1, \quad -x < y < x\}.$$

So Ω is a Lipschitz domain and its inf-sup constant is positive. We note that for any $n \geq 1$

$$\Omega_n \subset \Omega,$$

and the distance of $\partial\Omega_n$ to $\partial\Omega$ is smaller than $\sup_{0 \leq x \leq 1} x(1 - x^{\frac{1}{n}})$. This implies that the measure of $\Omega \setminus \Omega_n$ tends to zero when n tends to infinity. So we are in the framework of Theorem 2.1. We have the upper semi-continuity without the continuity:

$$\beta(\Omega_n) = 0 < \beta(\Omega).$$

3.2. **A self-similar excrescence.** Another example where the convergence does not hold is obtained if we add to a disk Ω a small equilateral triangle whose surface tends to 0. Let us say that Ω_n is the union of the unit disk centered at the origin and the equilateral triangle with side length $1/n$ and top vertex at $(0, 1 + \frac{1}{2n})$, see Figure 1. The inf-sup constants of Ω_n do not converge to that of the disk as $n \to \infty$ because the presence of the angle $\frac{\pi}{3}$ on the boundary of Ω_n implies by virtue of [12, Theorem 3.3] that we have the upper bound for all $n \geq 1$

$$\beta(\Omega_n) \leq \sqrt{\frac{1}{2} - \frac{\sin \omega}{2\omega}} \quad \text{with} \quad \omega = \frac{\pi}{3}.$$

This upper bound is smaller than $\beta(\Omega)$:

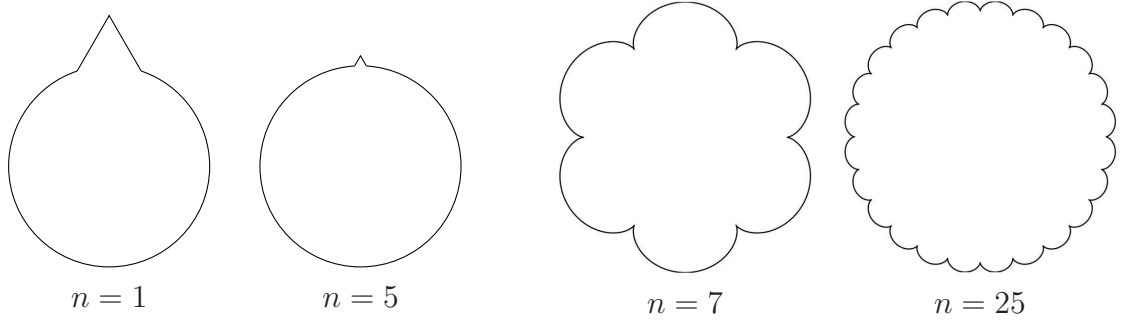$$\beta(\Omega_n) \leq \sqrt{\frac{1}{2} - \frac{3\sqrt{3}}{4\pi}} < \sqrt{\frac{1}{2}} = \beta(\Omega).$$

$$n = 1 \qquad n = 5 \qquad n = 7 \qquad n = 25$$

FIGURE 1. Disks with hats from § 3.2 (left)   Epitrochoids from § 3.4 (right)

3.3. **Regular polygons tending to a disk.** Let Ω be the unit disk in $\mathbb{R}^2$ and let Ω_n be a regular convex polygon with n edges which is inscribed in Ω. The Horgan–Payne angle (cf. [27]) of Ω_n with respect to the center of the ball is $\frac{\pi}{2} - \frac{\pi}{n}$. Thus, it follows from [23] and [13, Theorem 5.1] that

$$\sin(\frac{\pi}{4} - \frac{\pi}{2n}) \leq \beta(\Omega_n) \leq \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}.$$

Since the inf-sup constant of the disk Ω is $\frac{1}{\sqrt{2}}$, we find

$$0 \leq \beta(\Omega) - \beta(\Omega_n) \leq \sin\frac{\pi}{4} - \sin\left(\frac{\pi}{4} - \frac{\pi}{2n}\right) \leq \frac{\pi}{2n}.$$

So we have (at least) a convergence of order 1. This example pertains typically to the framework of polygonal approximation of a regular domain. We prove generally the convergence of the inf-sup constant in this case, see Theorem 4.6 further on.

3.4. **Conformal mappings and epitrochoids.** The situation where plane domains are transformed by conformal mappings has been investigated by Zsuppán [30]: Corollary 3.9 *loc.cit.* can be stated as follows

Theorem 3.1 (Zsuppán). *Let Ω and $\widetilde{\Omega}$ be two simply connected plane domains with piecewise smooth boundaries. Let g denote the bijective conformal mapping of Ω onto $\widetilde{\Omega}$. If $|g' - 1| \le \varepsilon < 1$ in the closure of Ω, then we have*

$$\left| \beta(\Omega) - \beta(\widetilde{\Omega}) \right| \le \frac{\sqrt{2}\,\varepsilon}{1 - \varepsilon}.$$

We will generalize this statement to Lipschitz diffeomorphisms in Theorem 4.4 below. Still in [30], an explicit example of a family of conformal mappings is investigated. The domain Ω is chosen as the unit disk and the mapping g depends on two parameters: an integer $n \ge 2$ and a real number $c > 0$:

$$g_{n;c}(z) = z - \frac{c}{n} z^n, \quad z \in \mathbb{C}.$$

For any $c \le 1$, the transformation $g_{n;c}$ is bijective on the unit disk Ω. Then Ω_n is defined as the image of Ω under $g_{n;c}$. This is an epitrochoid, see examples with $c = 1$ in Figure 1. From [30]

$$\beta(\Omega_n)^2 = \begin{cases} \frac{1}{2} - \frac{c}{4}(1 + \frac{1}{n}), & \text{for } n \text{ odd}, \\ \frac{1}{2} - \frac{c}{4}\sqrt{1 + \frac{2}{n}}, & \text{for } n \text{ even}. \end{cases}$$

As $n \to \infty$, Ω_n tends to Ω, because $g_{n;c}(z)$ tends to z uniformly in z for $|z| \le 1$. Nevertheless, the assumption of Theorem 3.1 is not satisfied, and

$$\beta(\Omega_n) \to \beta_\infty = \sqrt{\tfrac{1}{2} - \tfrac{c}{4}} < \sqrt{\tfrac{1}{2}} = \beta(\Omega).$$

In contrast, if we make c depend on n via the law $c = n^{-\alpha}$ for some positive α, these Ω_n enter the framework of Theorem 3.1 and the convergence of $\beta(\Omega_n)$ to $\beta(\Omega)$ occurs.

4. CONTINUITY WITH RESPECT TO THE DOMAIN

4.1. **The continuity theorem.** We recall that d is the space dimension. We denote by $\mathcal{I}_d$ the identity matrix in $\mathbb{R}^d$ and by $\mathcal{E}_d$ the space of endomorphisms of $\mathbb{R}^d$ equipped with the norm subordinated to the Euclidean norm on $\mathbb{R}^d$. In this section $c(d)$ refers to various constants *depending only on* d (and not on the domains or functions under consideration).

Definition 4.1. Let Ω and $\widetilde{\Omega}$ be two bounded Lipschitz domains in $\mathbb{R}^d$, and let ε be a positive number. We say that Ω and $\widetilde{\Omega}$ are ε-*close in Lipschitz norm* if there exists a diffeomorphism $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_d)$ from $\widetilde{\Omega}$ onto Ω that satisfies

$$(4.1) \qquad \begin{cases} \mathcal{F} \in W^{1,\infty}(\widetilde{\Omega})^d \quad \text{and} \quad \|D\mathcal{F} - \mathcal{I}_d\|_{L^\infty(\widetilde{\Omega};\mathcal{E}_d)} \le \varepsilon, \\ \mathcal{F}^{-1} \in W^{1,\infty}(\Omega)^d \quad \text{and} \quad \|D\mathcal{F}^{-1} - \mathcal{I}_d\|_{L^\infty(\Omega;\mathcal{E}_d)} \le \varepsilon. \end{cases}$$

Remark 4.2. Condition (4.1) is slightly redundant in the following sense: By an expansion of $D\mathcal{F}^{-1}$ as a Neumann series of $D\mathcal{F} - \mathcal{I}_d$ we find that if the first line of (4.1) is satisfied with some value $\varepsilon_0 < 1$ of ε, then the second line is satisfied for $\varepsilon = \varepsilon_0/(1 - \varepsilon_0)$.

If condition (4.1) holds, the Jacobian determinant $\mathcal{J}_{\mathcal{F}}$ of $\mathcal{F}$ satisfies the estimate

$$(4.2) \qquad \|1 - \mathcal{J}_{\mathcal{F}}\|_{L^{\infty}(\widetilde{\Omega})} \leq c(d)\varepsilon.$$

Thus, the change of variables $\mathcal{F}$ in integrals and partial derivatives yields:

Lemma 4.3. *We assume that the diffeomorphism $\mathcal{F}$ satisfies property* (4.1). *Then there holds*
(i) For $q \in L^2(\Omega)$, let $\tilde{q}$ denote $q \circ \mathcal{F}$. Then $\tilde{q}$ belongs to $L^2(\widetilde{\Omega})$ and

$$(4.3) \qquad \frac{1}{1 + c(d)\,\varepsilon}\,\|q\|_{0,\Omega} \leq \|\tilde{q}\|_{0,\widetilde{\Omega}} \leq \big(1 + c(d)\,\varepsilon\big)\|q\|_{0,\Omega}.$$

(ii) For $v \in H_0^1(\Omega)$, let $\tilde{v}$ denote $v \circ \mathcal{F}$. Then $\tilde{v}$ belongs to $H_0^1(\widetilde{\Omega})$ and satisfies the estimates

$$(4.4) \qquad \forall j = 1, \ldots, d, \qquad \|\partial_{\tilde{x}_j}\tilde{v} - (\partial_{x_j}v) \circ \mathcal{F}\|_{0,\widetilde{\Omega}} \leq c(d)\,\varepsilon\,|v|_{1,\Omega},$$

$$(4.5) \qquad \frac{1}{1 + c(d)\varepsilon}|v|_{1,\Omega} \leq |\tilde{v}|_{1,\widetilde{\Omega}} \leq (1 + c(d)\varepsilon)|v|_{1,\Omega}.$$

Thanks to Lemma 4.3, we are in a position to compare $\beta(\widetilde{\Omega})$ with $\beta(\Omega)$. Recall that both $\beta(\widetilde{\Omega})$ and $\beta(\Omega)$ are positive since $\widetilde{\Omega}$ and Ω are bounded and Lipschitz.

Theorem 4.4. *There exists a constant $c(d)$ depending only on the dimension d such that, if Ω and $\widetilde{\Omega}$ are ε-close in Lipschitz norm, the following estimate holds*

$$(4.6) \qquad \big|\beta(\Omega) - \beta(\widetilde{\Omega})\big| \leq c(d)\,\varepsilon.$$

Proof. Since the property (4.1) defining the ε-closeness is symmetric, it suffices to prove one inequality

$$(4.7) \qquad \beta(\widetilde{\Omega}) - c(d)\varepsilon \leq \beta(\Omega)$$

to prove estimate (4.6).

Thus, let q be any function in $L_\circ^2(\Omega)$. Setting $\tilde{q} = q \circ \mathcal{F}$, since the mean value of $\tilde{q}$ is not necessarily zero, we consider instead

$$\tilde{q}_0 = \tilde{q}\,\mathcal{J}_{\mathcal{F}}$$

that belongs to $L_\circ^2(\widetilde{\Omega})$. By (4.2), it satisfies

$$(4.8) \qquad \|\tilde{q} - \tilde{q}_0\|_{0,\widetilde{\Omega}} \leq c(d)\,\varepsilon\,\|\tilde{q}\|_{0,\widetilde{\Omega}}.$$

Combining this with estimates (4.3), we find

$$(4.9) \qquad \frac{1}{1 + c(d)\,\varepsilon}\,\|q\|_{0,\Omega} \leq \|\tilde{q}_0\|_{0,\widetilde{\Omega}} \leq \big(1 + c(d)\,\varepsilon\big)\|q\|_{0,\Omega}.$$

Now it is well known (see for example [18, Chap. I, Lemma 4.1]) that the inf-sup condition on $\widetilde{\Omega}$ yields the existence of a vector function $\tilde{\boldsymbol{v}}$ in $H_0^1(\widetilde{\Omega})^d$ such that

$$(4.10) \qquad \operatorname{div}\tilde{\boldsymbol{v}} = \tilde{q}_0 \quad \text{and} \quad \beta(\widetilde{\Omega})|\tilde{\boldsymbol{v}}|_{1,\widetilde{\Omega}} \leq \|\tilde{q}_0\|_{0,\widetilde{\Omega}}.$$

We now set:

$$\boldsymbol{v} = \tilde{\boldsymbol{v}} \circ \mathcal{F}^{-1}.$$

The idea is to bound the quantity $\int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x}) q(\boldsymbol{x}) \, d\boldsymbol{x}$ from below as follows. Using (4.4) we immediately obtain

$$(4.11) \qquad \| \operatorname{div} \tilde{\boldsymbol{v}} - (\operatorname{div} \boldsymbol{v}) \circ \mathcal{F} \|_{0,\widetilde{\Omega}} \leq c(d)\, \varepsilon \, |\boldsymbol{v}|_{1,\Omega},$$

which allows to write

$$(4.12) \qquad \int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x}) q(\boldsymbol{x}) \, d\boldsymbol{x} = \int_{\widetilde{\Omega}} (\operatorname{div} \tilde{\boldsymbol{v}})(\tilde{\boldsymbol{x}}) \tilde{q}_0(\tilde{\boldsymbol{x}}) \, d\tilde{\boldsymbol{x}} + \mathcal{E}$$

where the error term $\mathcal{E}$ is given by

$$\mathcal{E} = \int_{\widetilde{\Omega}} \big((\operatorname{div} \boldsymbol{v}) \circ \mathcal{F} - \operatorname{div} \tilde{\boldsymbol{v}} \big)(\tilde{\boldsymbol{x}}) \, \tilde{q}_0(\tilde{\boldsymbol{x}}) \, d\tilde{\boldsymbol{x}}.$$

We bound this term using (4.11) first,

$$|\mathcal{E}| \leq c(d)\, \varepsilon \, |\boldsymbol{v}|_{1,\Omega} \| \tilde{q}_0 \|_{0,\widetilde{\Omega}}$$

and next (4.9)

$$(4.13) \qquad |\mathcal{E}| \leq c(d)\, \varepsilon \, |\boldsymbol{v}|_{1,\Omega} \| q \|_{0,\Omega}.$$

By substituting (4.13) and the choice (4.10) of $\tilde{\boldsymbol{v}}$ into (4.12), we obtain successively

$$\int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x}) q(\boldsymbol{x}) \, d\boldsymbol{x} \geq \int_{\widetilde{\Omega}} \tilde{q}_0(\tilde{\boldsymbol{x}})^2 \, d\tilde{\boldsymbol{x}} - c(d)\, \varepsilon \, |\boldsymbol{v}|_{1,\Omega} \| q \|_{0,\Omega}$$

$$\geq \beta(\widetilde{\Omega}) |\tilde{\boldsymbol{v}}|_{1,\widetilde{\Omega}} \| \tilde{q}_0 \|_{0,\widetilde{\Omega}} - c(d)\, \varepsilon \, |\boldsymbol{v}|_{1,\Omega} \| q \|_{0,\Omega}$$

$$\geq \big(\beta(\widetilde{\Omega}) - c(d)\, \varepsilon \big) |\boldsymbol{v}|_{1,\Omega} \| q \|_{0,\Omega},$$

where we have used (4.5) and (4.9) for the last line (recall that $c(d)$ denotes a generic constant that may change between lines). This proves that $\beta(\Omega)$ is larger than $\beta(\widetilde{\Omega}) - c(d)\,\varepsilon$, which is our aim. □

4.2. Polygonal approximation of plane curved domains.
An important application of Theorem 4.4 is the finite element approximation of curved domains. Let Ω be a two-dimensional curved polygon with a Lipschitz-continuous and piecewise $\mathcal{C}^2$ boundary. This means that the boundary of Ω is a finite union of $\mathcal{C}^2$-arcs Γ_j that touch at corners $\boldsymbol{c}_k$ and determine opening angles distinct from 0 and 2π, thus excluding outward and inward cusps.

Definition 4.5. Let Ω be a two-dimensional curved Lipschitz polygon with a piecewise $\mathcal{C}^2$-boundary. Let h be a positive number. A *polygonal h-approximation* of Ω denotes a polygon Ω_h with straight sides such that

   (i) Its set of corners contains the set $\{\boldsymbol{c}_k\}$ of corners of Ω,
   (ii) Its corners belong to the boundary $\partial\Omega$ of Ω,
   (iii) The length of each side is less than h.

We now state the main result of this section.

Theorem 4.6. *Let Ω be a two-dimensional curved Lipschitz polygon with a piecewise $\mathcal{C}^2$-boundary. There exists a constant $c(\Omega)$ such that for all h-approximation Ω_h of Ω*

$$(4.14) \qquad \big| \beta(\Omega) - \beta(\Omega_h) \big| \leq c(\Omega)\, h.$$

Proof. In view of Theorem 4.4, it would suffice to prove that Ω and Ω_h are ε-close for an $\varepsilon = c(\Omega)h$. In fact, we are going to construct a finite number of intermediate domains Ω_h^k, $k = 1, \ldots, K - 1$, such that

- The number of these domains is independent of h,
- Setting $\Omega_h^0 = \Omega$ and $\Omega_h^K = \Omega_h$, each pair of consecutive domains $(\Omega_h^{k-1}, \Omega_h^k)$ are ε-close for an $\varepsilon = c(\Omega)h$, for $k = 1, \ldots, K$.

Taking Definition 4.1 and Remark 4.2 into account, this amounts to prove for each k that there exists a Lipschitz mapping $\mathcal{F}_h^k$ in $W^{1,\infty}(\Omega_h^{k-1})^2$ which is a diffeomorphism from Ω_h^{k-1} onto Ω_h^k and satisfies

$$(4.15) \qquad \mathcal{F}_h^k = \mathrm{Id} + \mathcal{G}_h^k, \quad \text{with} \quad \|D\mathcal{G}_h^k\|_{L^\infty(\Omega_h^{k-1};\mathcal{E}_2)} \leq C(\Omega)h,$$

where the constant $C(\Omega)$ depends only on Ω. Here Id denotes the identity mapping $\boldsymbol{x} \mapsto \boldsymbol{x}$ in $\mathbb{R}^2$. To construct this mapping, we proceed in several steps.

STEP 1: PARTITION OF THE DOMAIN. After the possible adjunction of extra points inside the original sides of Ω, that we will still denote by $\boldsymbol{c}_k$, we may assume that each new smaller side Γ_k is the graph of a $\mathcal{C}^2$ function in some coordinate system. Thus we can cover the boundary of the domain Ω by the closure of open sets $\mathcal{U}^1, \ldots, \mathcal{U}^K$ so that for each k, $1 \leq k \leq K$, after a possible rigid motion $\mathcal{M}_k$,

- the set $\overline{\mathcal{U}}^k$ is a rectangle $[a_k, b_k] \times [0, r_k]$,
- the local parts of the boundaries $\partial\Omega \cap \mathcal{U}^k$ and $\partial\Omega_h \cap \mathcal{U}^k$ are the graphs of a $\mathcal{C}^2$ map φ^k and a Lipschitz map φ_h^k, respectively, defined on $[a_k, b_k]$ with values in $[0, r_k]$
- $(a_k, \varphi(a_k)) = (a_k, \varphi_h^k(a_k)) = \boldsymbol{c}_{k-1}$ and $(b_k, \varphi(b_k)) = (b_k, \varphi_h^k(b_k)) = \boldsymbol{c}_k$.
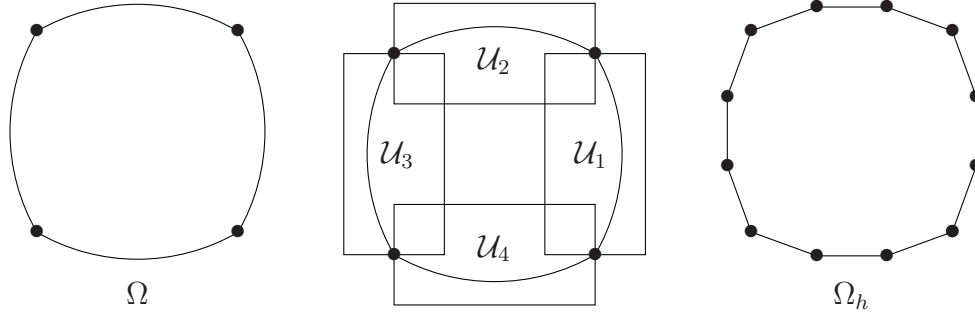

FIGURE 2. A curved square, its map neighborhoods, and its polygonal approximation

Then we introduce intermediate domains $\Omega_h^0 = \Omega, \Omega_h^1, \ldots, \Omega_h^K = \Omega_h$ so that

$$\Omega_h^{k-1} \cap \mathcal{U}^k = \Omega \cap \mathcal{U}^k, \qquad \Omega_h^k \cap \mathcal{U}^k = \Omega_h \cap \mathcal{U}^k, \qquad \Omega_h^{k-1} \cap \complement\mathcal{U}^k = \Omega_h^k \cap \complement\mathcal{U}^k.$$

Here $\complement\mathcal{U}$ stands for $\mathbb{R}^d \setminus \overline{\mathcal{U}}$. It follows that Ω_h^k is a local polygonal h-approximation of Ω_h^{k-1} subordinate to the neighborhood $\mathcal{U}^k$ (in the sense that Ω_h^{k-1} and Ω_h^k coincide outside $\mathcal{U}^k$ and that the properties (i) to (iii) of Definition 4.5 hold for the parts $\partial\Omega \cap \mathcal{U}^k$). Indeed, we intend to construct a diffeomorphism $\mathcal{F}_h^k$ from Ω_h^{k-1} onto Ω_h^k which is equal to the identity outside of $\mathcal{U}^k$.

STEP 2: CONSTRUCTION OF THE DIFFEOMORPHISM AT STEP k. From now on, we drop the exponent k and restrict the discussion to a local polygonal h-approximation in a rectangle $\mathcal{U} = (a, b) \times (0, r)$ such that

$$\overline{\Omega} \cap \mathcal{U} = \left\{ (x_1, x_2) \in \mathbb{R}^2, \quad a < x_1 < b, \ 0 < x_2 \leq \varphi(x_1) \right\},$$
$$\overline{\Omega}_h \cap \mathcal{U} = \left\{ (x_1, x_2) \in \mathbb{R}^2, \quad a < x_1 < b, \ 0 < x_2 \leq \varphi_h(x_1) \right\}.$$

Without restriction, we assume that φ is bounded from below by a constant $\eta_0 > 0$

(4.16) $$\exists \eta_0 > 0, \quad \forall x_1 \in [a, b], \quad \varphi(x_1) \geq \eta_0.$$

Since, in this situation, Ω and Ω_h coincide outside $\mathcal{U}$, then $\varphi(a) = \varphi_h(a)$ and $\varphi(b) = \varphi_h(b)$. We define the diffeomorphism $\mathcal{F}_h$ by

$$\mathcal{F}_h \boldsymbol{x} = \begin{cases} \boldsymbol{x} & \text{if} \ \ \boldsymbol{x} \in \Omega \cap \complement \mathcal{U} \\ (x_1, \frac{\varphi_h(x_1)}{\varphi(x_1)} x_2) & \text{if} \ \ \boldsymbol{x} \in \Omega \cap \mathcal{U} . \end{cases}$$

So the mapping $\mathcal{F}_h$ has a continuous extension to $\overline{\Omega}$ since it is the identity on $\partial \mathcal{U} \cap \Omega$ (i.e., on the three segments $\{x_1 \in [a, b], \ x_2 = 0\}$, $\{x_1 = a, \ x_2 \in [0, \varphi(a)]\}$, and $\{x_1 = b, \ x_2 \in [0, \varphi(b)]\}$). We write

$$\mathcal{F}_h = \mathrm{Id} + \mathcal{G}_h \quad \text{with} \quad \mathcal{G}_h \boldsymbol{x} = \begin{cases} 0 & \text{if} \ \ \boldsymbol{x} \in \Omega \cap \complement \mathcal{U} \\ (0, \frac{\varphi_h(x_1) - \varphi(x_1)}{\varphi(x_1)} x_2) & \text{if} \ \ \boldsymbol{x} \in \Omega \cap \mathcal{U} . \end{cases}$$

Therefore, we have the bound for $D\mathcal{G}_h$, with a constant $c(\Omega)$ depending only on Ω:

$$\| D\mathcal{G}_h \|_{L^\infty(\Omega; \mathcal{E}_2)} \leq c(\Omega) \left\| \frac{\varphi_h - \varphi}{\varphi} \right\|_{W^{1,\infty}(a,b)}.$$

Since φ' is bounded from above and φ satisfies (4.16), we obtain, for another constant $c(\Omega)$,

$$\| D\mathcal{G}_h \|_{L^\infty(\Omega; \mathcal{E}_2)} \leq c(\Omega) \| \varphi_h - \varphi \|_{W^{1,\infty}(a,b)}.$$

By definition of the polygonal h-approximation, the interval $[a, b]$ is the union of smaller intervals $[a', b']$ of length less than h and such that

$$\varphi_h(a') - \varphi(a') = 0, \qquad \varphi_h(b') - \varphi(b') = 0, \qquad \text{and} \qquad \forall x_1 \in [a', b'], \ \varphi_h''(x_1) = 0.$$

Thus, φ_h is a piecewise affine interpolate of φ (see Definition 4.5). Using that φ is of class $\mathcal{C}^2$, one obtains immediately

$$\| \varphi_h - \varphi \|_{W^{1,\infty}(a,b)} \leq \| \varphi'' \|_{L^\infty(a,b)} (h + h^2).$$

Then, reintroducing the exponent k, we find, as a direct consequence of Theorem 4.4:

(4.17) $$\left| \beta(\Omega_h^{k-1}) - \beta(\Omega_h^k) \right| \leq c(\Omega) \, h, \quad k = 1, \ldots, K.$$

STEP 3: CONCLUSION. Finally, the bound for $\left| \beta(\Omega) - \beta(\Omega_h) \right|$ follows from the triangle inequality

$$\left| \beta(\Omega) - \beta(\Omega_h) \right| \leq \sum_{k=1}^{K} \left| \beta(\Omega_h^{k-1}) - \beta(\Omega_h^k) \right|$$

and estimates (4.17), since the number K does not depend on h. $\square$

Remark 4.7. Along the same lines as for the two-dimensional case, one can prove convergence of the inf-sup constant for polyhedral approximations of certain classes of three-dimensional domains.

(1) Suppose that $\Omega \subset \mathbb{R}^3$ has a $\mathcal{C}^2$ boundary $\partial\Omega$. Then one may consider a family of polyhedral approximations Ω_h defined by regular triangulations of $\partial\Omega$. This means that the boundary $\partial\Omega_h$ of Ω_h is defined by planar triangles with vertices on $\partial\Omega$ whose diameter is bounded from above by Ch and whose inner radius is bounded from below by ch, where c and C are constants independent of h. In this situation, we can prove an error estimate of order h for the approximation of $\beta(\Omega)$ by $\beta(\Omega_h)$ as in (4.14). The proof has two steps: (i) construction of a global diffeomorphism $\mathcal{F}_h$, (ii) error estimates. For (i), we use the regularity of the inner unit normal field $\boldsymbol{n}$ on the $\mathcal{C}^2$ surface $\mathscr{S} := \partial\Omega$ and note that for a sufficiently small positive ε_0, the mapping

$$\begin{aligned} \Psi : \mathscr{S} \times [0, 2\varepsilon_0] &\longrightarrow & \overline{\mathcal{U}} \\ (\boldsymbol{y}, \rho) &\longmapsto & \boldsymbol{x} = \boldsymbol{y} + (\varepsilon_0 - \rho)\boldsymbol{n}(\boldsymbol{y}) \end{aligned}$$

is a diffeomorphism onto a tubular neighborhood $\overline{\mathcal{U}}$ of $\mathscr{S}$ in $\mathbb{R}^3$, of width $2\varepsilon_0$, that sends $\mathscr{S} \times \{\varepsilon_0\}$ onto $\partial\Omega$. Then for h small enough, the boundary of Ω_h is represented by the graph $\rho = \varphi_h(\boldsymbol{y})$ of a function $\varphi_h : \mathscr{S} \to [\frac{1}{2}\varepsilon_0, \frac{3}{2}\varepsilon_0]$:

$$\mathscr{S} \ni \boldsymbol{y} \longmapsto \boldsymbol{x} = \boldsymbol{y} + (\varepsilon_0 - \varphi_h(\boldsymbol{y}))\boldsymbol{n}(\boldsymbol{y}) \in \partial\Omega_h.$$

The diffeomorphism $\mathcal{F}_h$ is then defined as

$$\mathcal{F}_h\boldsymbol{x} = \begin{cases} \boldsymbol{x} & \text{if} \quad \boldsymbol{x} \in \Omega \cap \complement\mathcal{U} \\ \boldsymbol{y} + \left(\varepsilon_0 - \frac{\rho}{\varepsilon_0}\varphi_h(\boldsymbol{y})\right)\boldsymbol{n}(\boldsymbol{y}) & \text{if} \quad \boldsymbol{x} \in \Omega \cap \mathcal{U} \quad \text{with} \quad (\boldsymbol{y}, \rho) = \Psi^{-1}(\boldsymbol{x}). \end{cases}$$

(ii) Estimates for $\mathcal{F}_h - \mathrm{Id}$ in the Lipschitz norm then follow from standard $W^{1,\infty}$ estimates for interpolation with two-dimensional finite elements (see [10, Theorem 16.1], for example).

(2) We can extend the proof above in the spirit of Theorem 4.6 if Ω is a piecewise $\mathcal{C}^2$ domain with *straight edges* that form the boundaries of its faces f_j which are polygonal subdomains of $\mathcal{C}^2$ surfaces $\mathscr{S}_j$. Just as we imposed in 2D that corners of Ω be corners of Ω_h, we impose now that the edges of Ω be edges of Ω_h.

5. Continuity with respect to the function spaces

Whereas the uniform discrete LBB condition (1.3) is regularly addressed in the literature about numerical approximation of the solutions of incompressible fluid models, the convergence of the discrete inf-sup constants to the exact inf-sup constant is rarely (if ever) examined. In this section, we intend to fill this gap and give conditions on the sequence of finite-dimensional function spaces $X_n \subset H_0^1(\Omega)^d$, $M_n \subset L_\circ^2(\Omega)$, $n \in \mathbb{N}$ that guarantee that the discrete inf-sup constants β_n converge to the inf-sup constant $\beta(\Omega)$ of the domain. The convergence proof uses arguments in the spirit of the proof of the discrete LBB condition in the paper [29], namely a combination of inverse estimates for the pressure spaces and approximation properties of the velocity spaces.

5.1. **The continuity theorem.** For a regularity index $s > 0$, we introduce two characteristic constants associated with the function spaces X_n and M_n. The first constant is defined by comparing the Sobolev norm of order s on M_n with the equivalent L^2 norm (recall that M_n is finite-dimensional)

$$(5.1) \qquad \eta_{n,s} = \sup_{q \in M_n} \frac{\|q\|_{s,\Omega}}{\|q\|_{0,\Omega}}.$$

The second constant is defined by the approximation property of the spaces X_n

$$(5.2) \qquad \varepsilon_{n,s} = \sup_{\boldsymbol{u} \in H^{1+s}(\Omega) \cap H^1_0(\Omega)} \inf_{\boldsymbol{v} \in X_n} \frac{|\boldsymbol{u} - \boldsymbol{v}|_{1,\Omega}}{\|\boldsymbol{u}\|_{1+s,\Omega}}.$$

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We assume that $X_n \subset H^1_0(\Omega)^d$, $M_n \subset L^2_\circ(\Omega)$, $n \in \mathbb{N}$, are finite-dimensional subspaces and that for some $0 < s < \frac{1}{2}$, M_n is contained in the Sobolev space $H^s(\Omega)$. Then there is a constant C_s depending on the domain Ω such that with the constants $\eta_{n,s}$ and $\varepsilon_{n,s}$ defined above, we have*

$$(5.3) \qquad \beta_n \geq \beta(\Omega) - C_s\, \eta_{n,s}\, \varepsilon_{n,s}\,.$$

In particular, if the M_n are asymptotically dense in $L^2_\circ(\Omega)$ and

$$(5.4) \qquad if \quad \lim_{n \to \infty} \eta_{n,s}\, \varepsilon_{n,s} = 0 \quad then \quad \lim_{n \to \infty} \beta_n = \beta(\Omega)\,.$$

Proof. Define the functionals J and J_n as in (2.2). Recall that for J we have the relation

$$J(q) = |\boldsymbol{w}(q)|_{1,\Omega}$$

where $\boldsymbol{w}(q) = \Delta^{-1}\, \mathbf{grad}\, q$ is the solution of the harmonic Dirichlet problem with $\mathbf{grad}\, q$ as right-hand side. Likewise, the sup in J_n is attained at $\boldsymbol{w}_n(q) \in X_n$, which is the Galerkin solution $\boldsymbol{w}_n(q) = \Delta_n^{-1}\, \mathbf{grad}\, q$ of the Dirichlet problem with the same right-hand side:

$$(5.5) \qquad \forall\, \boldsymbol{v} \in X_n, \quad \big\langle \mathbf{grad}\, \boldsymbol{w}_n(q), \mathbf{grad}\, \boldsymbol{v} \big\rangle_\Omega = \big\langle \operatorname{div} \boldsymbol{v}, q \big\rangle_\Omega.$$

Taking $\boldsymbol{v} = \boldsymbol{w}_n(q)$ it follows that

$$J_n(q) = |\boldsymbol{w}_n(q)|_{1,\Omega}\,.$$

Now for each n, we solve in M_n the finite-dimensional eigenvalue problem for the Schur complement of the discretized Stokes system

$$(5.6) \qquad \mathcal{S}_n q = \sigma q \quad with \quad \big\langle \mathcal{S}_n q, p \big\rangle_\Omega = -\big\langle \boldsymbol{w}_n(q), \mathbf{grad}\, p \big\rangle \;\; \forall\, p \in M_n\,.$$

Let σ_n be the smallest eigenvalue and q_n a corresponding normalized eigenfunction. Thus q_n is a minimizer in M_n of the Rayleigh quotient

$$\frac{\big\langle \mathcal{S}_n q, q \big\rangle_\Omega}{\|q\|^2_{0,\Omega}} = \frac{\big\langle \operatorname{div} \boldsymbol{w}_n(q), q \big\rangle_\Omega}{\|q\|^2_{0,\Omega}} = \frac{|\boldsymbol{w}_n(q)|^2_{1,\Omega}}{\|q\|^2_{0,\Omega}} = \Big(\frac{J_n(q)}{\|q\|_{0,\Omega}} \Big)^2.$$

We see that q_n realizes the inf-sup condition, and if $\|q_n\|_{0,\Omega} = 1$, we simply have

$$\beta_n = J_n(q_n)\,.$$

Incidentally, we also have shown that $\sigma_n = \beta_n^2$.

Now we can write

$$\beta(\Omega) \le J(q_n) = J_n(q_n) + \big(J(q_n) - J_n(q_n)\big)$$
$$= \beta_n + \big(|\boldsymbol{w}(q_n)|_{1,\Omega} - |\boldsymbol{w}_n(q_n)|_{1,\Omega}\big)$$
$$\le \beta_n + |\boldsymbol{w}(q_n) - \boldsymbol{w}_n(q_n)|_{1,\Omega}\,.$$

Thus we have to estimate $|\boldsymbol{w}(q_n) - \boldsymbol{w}_n(q_n)|_{1,\Omega}$, which is the Galerkin error in the solution of the Dirichlet problem with $\mathbf{grad}\, q_n$ as right-hand side. We have

$$|\boldsymbol{w}(q_n) - \boldsymbol{w}_n(q_n)|_{1,\Omega} = \inf_{\boldsymbol{v}_n \in X_n} |\boldsymbol{w}(q_n) - \boldsymbol{v}_n|_{1,\Omega} \le \varepsilon_{n,s}\|\boldsymbol{w}(q_n)\|_{1+s,\Omega}$$

if $\boldsymbol{w}(q_n) \in H^{1+s}(\Omega)^d$. Now we use the fact that on a Lipschitz domain we have H^{1+s} regularity for the solution of the Dirichlet problem of the Laplacian, because $s \in (0, \frac{1}{2})$, see [11]:

$$|\boldsymbol{w}(q_n)|_{1+s,\Omega} \le C_s \|\mathbf{grad}\, q_n\|_{-1+s,\Omega} \le C_s\|q_n\|_{s,\Omega}\,.$$

With the definition of $\eta_{n,s}$ we can further estimate

$$\|q_n\|_{s,\Omega} \le \eta_{n,s}\|q_n\|_{0,\Omega} = \eta_{n,s}\,.$$

Altogether we have shown

$$\beta(\Omega) \le \beta_n + \varepsilon_{n,s}\, C_s\, \eta_{n,s}$$

as claimed in (5.3). Finally, if $\lim_{n\to\infty} \eta_{n,s}\varepsilon_{n,s} = 0$, then (5.3) implies

$$\liminf_{n\to\infty} \beta_n \ge \beta(\Omega),$$

which together with inequality (2.1) from Theorem 2.1 proves that $\lim_{n\to\infty} \beta_n = \beta(\Omega)$. $\qquad\square$

Remark 5.2. The main tools in the proof, namely introduction of Sobolev spaces of fractional order, the approximation property (5.2), the inverse estimate (5.1) and the H^{1+s} regularity for the Dirichlet problem, were in the end only used to prove the error estimate for the Galerkin approximation of the Dirichlet problem (5.5) with $\mathbf{grad}\, q_n$ as right-hand side, namely

(5.7) $$|\boldsymbol{w}(q_n) - \boldsymbol{w}_n(q_n)|_{1,\Omega} \le \delta_n\|q_n\|_{0,\Omega}\,,$$

where $\delta_n = \varepsilon_{n,s}\, C_s\, \eta_{n,s}$. The assumptions of Theorem 5.1 implied that $\lim_{n\to\infty} \delta_n = 0$, which allowed to complete the proof of the convergence of β_n. A weaker hypothesis that would still be sufficient for the convergence result would therefore be to assume directly that

for all $q_n \in M_n$, the estimate (5.7) holds with $\lim_{n\to\infty} \delta_n = 0\,.$

There is one situation where this hypothesis obviously holds, namely when $\boldsymbol{w}(q_n) \in X_n$ for all $q_n \in M_n$, that is, the Dirichlet problem

$$\Delta \boldsymbol{w} = \mathbf{grad}\, q_n$$

can be solved exactly in X_n, because then $\boldsymbol{w}(q_n) = \boldsymbol{w}_n(q_n)$. This can be made to happen in very particular cases, if the Laplacian with Dirichlet conditions can be inverted explicitly for right-hand sides of the form $\mathbf{grad}\, q_n$ where q_n runs through a basis of M_n. The space X_n can then simply be chosen as the space generated by these solutions of the Dirichlet

problem. Such a special situation is analyzed in [16], where the convergence $\beta_n \to \beta(\Omega)$ is then proved.

5.2. **Convergence of the first eigenfunction.**

For the approximation of the inf-sup constant $\beta(\Omega)$ in Theorem 5.1 we computed the discrete inf-sup constant β_n by solving an eigenvalue problem in M_n. In (5.6), this eigenvalue problem was described by the Schur complement of the discretized Stokes system. It is easy to see that the eigenvalue problem (5.6) is equivalent to the following discretized Stokes eigenvalue problem in variational form.
Find $(\boldsymbol{w}_n, q) \in X_n \times M_n$ and $\sigma \in \mathbb{C}$ such that $q \neq 0$ and

(5.8)
$$\forall \, \boldsymbol{v} \in X_n, \qquad \big\langle \mathbf{grad}\, \boldsymbol{w}_n, \mathbf{grad}\, \boldsymbol{v} \big\rangle_\Omega - \big\langle \operatorname{div} \boldsymbol{v}, q \big\rangle_\Omega = 0,$$
$$\forall \, p \in M_n, \qquad\qquad\qquad\qquad \big\langle \operatorname{div} \boldsymbol{w}_n, p \big\rangle_\Omega = \sigma \big\langle q, p \big\rangle_\Omega.$$

This is an eigenvalue problem in mixed form of the second type studied in [5, Section 4]. In the case of the Stokes system, however, the convergence analysis of [5] does not apply, because the basic assumption is not satisfied, namely the compactness of the solution operator $T_\Xi : g \mapsto p$ in $L^2_\circ(\Omega)$ of the Stokes system for $(\boldsymbol{u}, p) \in H^1_0(\Omega)^d \times L^2_\circ(\Omega)$

$$-\Delta \boldsymbol{u} + \mathbf{grad}\, p = 0$$
$$\operatorname{div} \boldsymbol{u} = g \, .$$

Indeed, T_Ξ is the inverse of the Schur complement operator $\mathcal{S} = \operatorname{div} \Delta^{-1} \mathbf{grad}$, and it has long been known from the analysis of the Cosserat eigenvalue problem that $\mathcal{S}$ has a non-trivial essential spectrum: Namely, if Ω has a smooth boundary, then as shown in [24], the essential spectrum is the set $\{\frac{1}{2}, 1\}$ with an eigenspace of infinite dimension at $\sigma = 1$ and an accumulation point of eigenvalues at $\sigma = \frac{1}{2}$. In [12], it is shown that if Ω is a polygonal plane domain with corner angles ω_j, then the essential spectrum is the union of $\{1\}$ and of the closed intervals

$$\left[\frac{1}{2} - \frac{\sin \omega_j}{2\omega_j}, \frac{1}{2} + \frac{\sin \omega_j}{2\omega_j} \right],$$

whereas for three-dimensional piecewise smooth domains, the essential spectrum contains at least such intervals associated with the opening angles of the edges. In many cases of non-smooth domains, for example a square in two dimensions, it is unknown whether $\sigma(\Omega)$ is the bottom of the essential spectrum of $\mathcal{S}$ or an isolated eigenvalue below the essential spectrum.

Due to the presence of the essential spectrum, the eigenvalue problem (5.8) does not fit into any known convergence analysis of eigenvalue approximation methods (and this is the main reason why we had to develop the analysis presented in this section). In particular, the discrete eigenfunctions q_n will, in general, not converge to an eigenfunction of the corresponding continuous eigenvalue problem. This is true even when our sufficient conditions for the convergence of the eigenvalues $\sigma_n \to \beta(\Omega)^2$ are satisfied. We can, however, prove convergence of q_n if the continuous eigenvalue $\sigma(\Omega) = \beta(\Omega)^2$ lies below the essential spectrum. Due to the inherent non-uniqueness of eigenfunctions, the convergence has to be formulated in a somewhat convoluted fashion.

Theorem 5.3. *Let the sequence of finite-dimensional subspaces X_n and M_n be such that the sufficient conditions of Theorem 5.1 for the validity of $\lim_{n \to \infty} \beta_n = \beta(\Omega)$ are satisfied. Assume further that $\sigma_0 = \beta(\Omega)^2$ is an isolated eigenvalue of the Schur complement operator $\mathcal{S} = \operatorname{div} \Delta^{-1} \mathbf{grad}$ and let $\mathcal{P}$ be the orthogonal projection operator in $L_\circ^2(\Omega)$ onto the eigenspace of $\mathcal{S}$ with eigenvalue σ_0. Define q_n to be a normalized eigenfunction of the operator $\mathcal{S}_n$ defined in (5.6) associated with its smallest eigenvalue $\sigma_n = \beta_n^2$. Then q_n converges to an eigenfunction of $\mathcal{S}$ in the following sense:*

$$(5.9) \qquad \lim_{n \to \infty} \| q_n - \mathcal{P} q_n \|_{0,\Omega} = 0 \,.$$

Proof. It follows from the selfadjointness of $\mathcal{S}$ that $\mathcal{P}$ and $\mathcal{S}$ commute. Since the smallest eigenvalue σ_0 is isolated, there exists $\delta > 0$ such that for all $q \in \ker \mathcal{P}$ there holds $\langle Sq, q \rangle_\Omega \geq (\sigma_0 + \delta) \| q \|_{0,\Omega}^2$.

We will now use the notation from the proof of Theorem 2.1. In particular, we have for any q

$$\langle \mathcal{S}q, q \rangle_\Omega = \langle \operatorname{div} \mathbf{w}(q), q \rangle_\Omega = | \mathbf{w}(q) |_{1,\Omega}^2 = J(q)^2 \,.$$

We can therefore estimate

$$\begin{aligned}
\sigma_0 &= \sigma_0 \big(\| \mathcal{P} q_n \|_{0,\Omega}^2 + \| q_n - \mathcal{P} q_n \|_{0,\Omega}^2 \big) \\
&\leq \sigma_0 \| \mathcal{P} q_n \|_{0,\Omega}^2 + (\sigma_0 + \delta) \| q_n - \mathcal{P} q_n \|_{0,\Omega}^2 \\
&\leq \langle \mathcal{S} \mathcal{P} q_n, q_n \rangle_\Omega + \langle \mathcal{S}(q_n - \mathcal{P} q_n), q_n \rangle_\Omega \\
&= \langle \mathcal{S} q_n, q_n \rangle_\Omega = J(q_n)^2 \,.
\end{aligned}$$

We also know from the proof of Theorem 5.1 that $J(q_n) \to \beta(\Omega)$ as $n \to \infty$. It follows that $\delta \| q_n - \mathcal{P} q_n \|_{0,\Omega}^2 \to 0$, hence (5.9). $\qquad\square$

Remark 5.4. Under the sufficient conditions of Theorem 5.1 for the convergence of $\beta_n \to \beta(\Omega)$, the discrete eigenfunctions q_n define a minimizing sequence both for the functional J and for the Rayleigh quotient of the operator $\mathcal{S}$. Therefore if there exists a convergent subsequence, it will converge to an eigenfunction of the Schur complement operator $\mathcal{S}$. If $\sigma_0 = \beta(\Omega)^2$ is below the essential spectrum of $\mathcal{S}$, hence of finite multiplicity, the previous theorem shows that every subsequence of q_n has a subsequence converging to an eigenfunction of $\mathcal{S}$ associated with σ_0. If, however, σ_0 is not an eigenvalue, then the sequence q_n has no subsequence that converges in $L^2(\Omega)$.

5.3. Consequences for the convergence of the discrete inf-sup constant in the p and h versions of the finite element method.

The sufficient conditions (5.4) for the convergence $\beta_n \to \beta(\Omega)$ given in Theorem 5.1 rely on the estimates of the quantities defined by (5.1) and (5.2). Such estimates are known to hold in many finite element methods, and we can therefore concretize the convergence theorem for these methods.

Let us assume first that the spaces X_n and M_n correspond to an h version finite element method. That is, they consist of piecewise polynomials of a fixed degree on meshes with meshsize tending to zero. We consider the case where velocity and pressure variables are discretized on two different families of meshes indexed by $n \in \mathbb{N}$. The inclusions $X_n \subset H_0^1(\Omega)^d$ and $M_n \subset L_\circ^2(\Omega)$ mean that X_n consists of globally continuous functions, whereas the elements of M_n may be discontinuous. For the mesh of the velocity space X_n, we define

$\overline{h}_{X_n}$ as the diameter of the largest element, and for the mesh of the pressure space M_n we define $\underline{h}_{M_n}$ as the smallest radius of the inscribed sphere of any element. Both $\overline{h}_{X_n}$ and $\underline{h}_{M_n}$ tend to zero as $n \to \infty$.

Under very mild conditions on local uniformity and shape regularity of the meshes one has standard finite element approximation properties and inverse estimates that imply the existence of a constant C independent of n such that, for some $s \in (0, \frac{1}{2})$,

$$(5.10) \qquad \forall \, \boldsymbol{u} \in (H^{1+s}(\Omega) \cap H_0^1(\Omega))^d, \quad \inf_{\boldsymbol{v} \in X_n} |\boldsymbol{u} - \boldsymbol{v}|_{1,\Omega} \leq C \, (\overline{h}_{X_n})^s \|\boldsymbol{u}\|_{1+s,\Omega}$$

and

$$(5.11) \qquad \forall \, q \in M_n, \quad \|q\|_{s,\Omega} \leq C \, (\underline{h}_{M_n})^{-s} \|q\|_{0,\Omega} \,.$$

The approximation estimate (5.10) can be deduced by Hilbert space interpolation from the $O(\overline{h}_{X_n})$ approximation property for $\boldsymbol{u} \in H^2(\Omega)^d$ that can be found in many textbooks on the mathematical theory of finite elements, see for example [7, 10]. The inverse estimate (5.11), which we need for discontinuous elements, is proved in [20] under very general hypotheses on the meshes that include highly anisotropic elements.

Thus if the estimates (5.10) and (5.11) are satisfied, we compare (5.10) with (5.2) and (5.11) with (5.1) and conclude $\varepsilon_{n,s} \leq C \overline{h}_{X_n}^s$ and $\eta_{n,s} \leq C \underline{h}_{M_n}^{-s}$. In this case, Theorem 5.1 has the following corollary.

Corollary 5.5. *In the h version of the finite element method,*

$$(5.12) \qquad \qquad if \quad \lim_{n \to \infty} \frac{\overline{h}_{X_n}}{\underline{h}_{M_n}} = 0 \quad then \quad \lim_{n \to \infty} \beta_n = \beta(\Omega) \,.$$

Remark 5.6. For many popular inf-sup stable elements, such as the Taylor-Hood pair of elements (see [18, Chap. II, Section 4.2], [22]), the velocity and pressure are discretized on the same mesh, in which case (5.12) is not satisfied and Corollary 5.5 gives no information on the convergence (or divergence) of β_n.

The asymptotic condition (5.12) may seem unusual, but it has been known for a long time that when the mesh size for the pressure is sufficiently larger than that for the velocity, then the corresponding pair of elements is inf-sup stable. An example, where the velocity mesh is a straightforward refinement of the pressure mesh, is given at the end of [18, Chap. II, Section 4.2] and in [25, Chap. 4, Section 3.2].

Second, we consider spectral methods or the p version of the finite element method. Here the meshes are fixed, and the spaces X_n and M_n consist of (piecewise) polynomials of degrees p_{X_n} and p_{M_n}, respectively, tending to infinity as $n \to \infty$. Then the typical approximation properties and inverse estimates are

$$(5.13) \qquad \forall \, \boldsymbol{u} \in (H^{1+s}(\Omega) \cap H_0^1(\Omega))^d, \quad \inf_{\boldsymbol{v} \in X_n} |\boldsymbol{u} - \boldsymbol{v}|_{1,\Omega} \leq C \, (p_{X_n})^{-s} \|\boldsymbol{u}\|_{1+s,\Omega}$$

and

$$(5.14) \qquad \forall \, q \in M_n, \quad \|q\|_{s,\Omega} \leq C \, (p_{M_n})^{2s} \|q\|_{0,\Omega} \,.$$

Again, the approximation property for continuous elements (5.13) is standard, see [3]. The inverse estimate (5.14) under very general assumptions on the meshes is proved in [17],

where one can even find corresponding estimates for the hp version of the finite element method.

Now if the estimates (5.13) and (5.14) are satisfied, we find $\varepsilon_{n,s} \leq Cp_{X_n}^{-s}$ and $\eta_{n,s} \leq Cp_{M_n}^{2s}$, and in this case we obtain the second corollary of Theorem 5.1.

Corollary 5.7. *In the p version of the finite element method and in spectral methods,*

$$(5.15) \qquad if \quad \lim_{n\to\infty} \frac{(p_{M_n})^2}{p_{X_n}} = 0 \quad then \quad \lim_{n\to\infty} \beta_n = \beta(\Omega)\,.$$

Remark 5.8. While the conditions (5.12) and (5.15) are sufficient for the convergence of the discrete inf-sup constants to the exact inf-sup constant of the domain, they might not be necessary. In Section 5.4 this question is studied via concrete examples of finite element approximations. It turns out that for the h version there are examples where (5.12) is not satisfied and the β_n converge to arbitrarily given small positive values. On the other hand, for the p version we only have numerical observations, strongly indicating that the condition (5.15) on the degrees is too restrictive: We only found situations where either the β_n converge to 0 (no discrete LBB condition) or else they converge to $\beta(\Omega)$. The optimal condition instead of (5.15) may be conjectured to be

$$\limsup_{n\to\infty} \frac{p_{M_n}}{p_{X_n}} < 1\,.$$

5.4. **Examples of finite element approximations.** We discuss two examples: the first one is devoted to the Scott-Vogelius triangular element in the h-version, and the second one to the p-version (or spectral discretization) on rectangular elements. The examples address the points made in Remark 5.8. The first example illustrates the situation of a finite element method that satisfies a discrete LBB condition, but the discrete LBB constants converge to a limit that may be arbitrarily small. The second example reports on numerical observations concerning the degree condition (5.15).

5.4.1. *Scott-Vogelius $\mathbb{P}_4$–$\mathbb{P}_3^{\mathrm{dc}}$ triangular element.* Let $\mathcal{T}$ be a triangulation of the Lipschitz polygonal domain Ω. We denote by $\overline{h}_{\mathcal{T}}$ the largest diameter of its elements. For any node $\boldsymbol{x}$ of $\mathcal{T}$, let $K_1, \ldots, K_J$ be the triangles of $\mathcal{T}$ that have $\boldsymbol{x}$ as vertex, ordered so that K_j and K_{j+1} have a common edge for all $j = 1, \ldots, J-1$. Let θ_j be the opening of K_j at its vertex $\boldsymbol{x}$. The regularity index at $\boldsymbol{x}$ is defined as

$$(5.16) \qquad \mathcal{R}(\boldsymbol{x}) = \max_{j=1,\ldots,J-1} |\theta_j + \theta_{j+1} - \pi|\,.$$

If $\mathcal{R}(\boldsymbol{x}) = 0$, $\boldsymbol{x}$ is said singular. Note that any interior singular point satisfies $J = 4$ and the edges to which $\boldsymbol{x}$ belongs are contained in two lines. We also denote

$$(5.17) \qquad \mathcal{R}(\mathcal{T}) = \min_{\boldsymbol{x} \text{ node of } \mathcal{T}} \mathcal{R}(\boldsymbol{x}).$$

Let us pick a quasiuniform family of triangulations $(\mathcal{T}_n)_n$ and choose the discrete $\mathbb{P}_4$–$\mathbb{P}_3^{\mathrm{dc}}$ spaces (continuous piecewise polynomials of (total) degree 4 for velocities, discontinuous

piecewise polynomials of degree 3 for pressures):

(5.18)
$$\begin{cases} X_n = \{\boldsymbol{v} \in H_0^1(\Omega)^2, \quad \boldsymbol{v}\big|_K \in \mathbb{P}_4(K)^2 \ \text{for all} \ K \in \mathcal{T}_n\}, \\ M_n = \{q \in L_\circ^2(\Omega), \quad q\big|_K \in \mathbb{P}_3(K) \ \text{for all} \ K \in \mathcal{T}_n\}. \end{cases}$$

The result of [26, Th.5.2] provides the following implication

(5.19) $\big(\exists \delta > 0 \text{ such that } \forall n, \ \mathcal{R}(\mathcal{T}_n) \geq \delta\big) \implies \big(\exists \beta_\star > 0 \text{ such that } \forall n, \ \beta_n \geq \beta_\star\big).$

However this implication does not guarantee the convergence of β_n to $\beta(\Omega)$. We now show that convergence to a value lower than $\beta(\Omega)$ may occur.

Proposition 5.9. *Let Ω be a Lipschitz polygonal domain. There exists $\beta_0 > 0$ such that for all $\beta_\infty \in (0, \beta_0]$ the following property holds: There exists a sequence of quasiuniform triangulations $(\mathcal{T}_n)_n$ such that $\overline{h}_{\mathcal{T}_n}$ tends to 0 and the inf-sup constant β_n associated with the spaces X_n and M_n given by (5.18) converges to β_∞ as n tends to ∞.*

Proof. We mesh Ω by a finite number of quadrilaterals Q that are images of the unit square $\widehat{Q} = (0, 1)^2$ under bi-affine diffeomorphisms and denote by $\mathcal{Q}_1$ this first mesh. For $n \geq 2$ we define the quadrangular mesh $\mathcal{Q}_n$ as the refinement of $\mathcal{Q}_1$ obtained by the images of the meshes associated with the grid $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}^2$ of $\widehat{Q}$. Each element Q of $\mathcal{Q}_n$ is itself bi-affinely diffeomorphic to $\widehat{Q}$. For a parameter $b \in [0, \frac{1}{2})$ and each element Q of $\mathcal{Q}_n$ we define the interior point $\boldsymbol{a}_Q(b)$ as the image of $(\frac{1}{2}, \frac{1}{2} + b)$. If $b = 0$, $\boldsymbol{a}_Q(b)$ is the center of Q. Then we associate with the four edges $\boldsymbol{e}$ of Q the four triangles having $\boldsymbol{e}$ as an edge and $\boldsymbol{a}_Q(b)$ as a vertex. Choosing $b = \frac{1}{4}$, for instance, we define in this way the triangular mesh $\mathcal{T}_n^0$ and we can check that the left-hand condition of (5.19) for the sequence $(\mathcal{T}_n^0)_n$ is satisfied. Then the right-hand condition of (5.19) defines $\beta_\star$ and we take $\beta_0 = \beta_\star$. Let us choose $\beta_\infty \in (0, \beta_0]$.

In a second step, we pick one element Q_n in each quadrangular mesh $\mathcal{Q}_n$. We consider another parameter $a \in (-\frac{1}{2}, \frac{1}{2})$ and the interior point $\boldsymbol{a}_{Q_n}(a)$. The triangular mesh $\mathcal{T}_n(a)$ is obtained by the method above with the interior point $\boldsymbol{a}_{Q_n}(a)$ for Q_n, and $\boldsymbol{a}_Q(\frac{1}{4})$ for the other quadrilaterals Q of $\mathcal{Q}_n$. Let $\beta_n(a)$ be the associated inf-sup constant. It is easy to see that it is a continuous function of a. Since $\beta_n(0) = 0$ (because the node $\boldsymbol{a}_{Q_n}(0)$ is singular) and $\beta_n(\frac{1}{4}) \geq \beta_0$, there exists $a_n \in (0, \frac{1}{4})$ such that $\beta_n(a_n) = \beta_\infty$. Choosing $\mathcal{T}_n = \mathcal{T}_n(a_n)$ proves the proposition.   □

We illustrate this proof by the example of Ω as the rectangle $(0, 4) \times (0, 1)$. In this case, we know [12, Section 5.1] that $\beta(\Omega)^2$ is an isolated eigenvalue of the Cosserat operator, and the most precise computations give a value of $\beta(\Omega) \simeq 0.218444$. In our computations with the Scott-Vogelius $\mathbb{P}_4$–$\mathbb{P}_3^{\mathrm{dc}}$ element, the quadrangular meshes consist of squares, specifically 4×1, 8×2, 12×3, and 24×6, see Figure 3 (left). The general "decentering" parameter b is chosen as 0.4 and the special parameter a is running from -0.49 to 0.49 by steps of 0.01. We have computed with the code FreeFem++ [21] the lowest eigenvalues σ_j, $j = 0, 1, \ldots$ of the discrete Stokes system (5.8). The first one σ_0 is always 0 and, as already mentioned in Section 5.2, the square root of σ_1 equals the inf-sup constant β_n of the discretization.

In Figure 3 (right) we observe a quasi-constant value for $\beta_n(a)$ when $|a|$ lies between 0.35 and 0.48. This value is a good approximation of the inf-sup constant $\beta(\Omega)$. Our computations approach the exact value from below, see Figure 3 (right). For $|a|$ less that 0.3, we
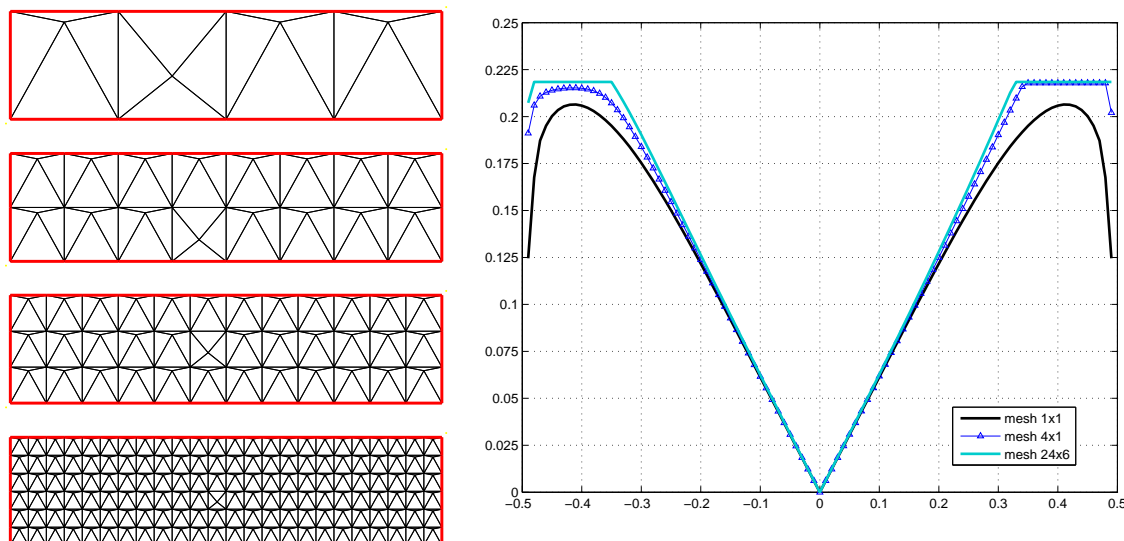
FIGURE 3. Meshes of $\Omega = (0, 4) \times (0, 1)$ for $a = -0.1$ and $b = 0.4$ (left). $\beta_n(a)$ for discrete spaces (5.18) versus $a \in (-0.5, 0.5)$ on a one-element mesh of the square and on two meshes of the rectangle (right)

observe a linear behavior of $\beta_n(a)$ as the central node $\boldsymbol{a}_{Q_n}(a)$ approaches the center (singular point), that is $\beta_n(a)$ behaves as a multiple of the regularity index (5.16) at $\boldsymbol{a}_{Q_n}(a)$. Interestingly, the same behavior with a very similar proportionality constant can already be observed on the one-element square mesh.
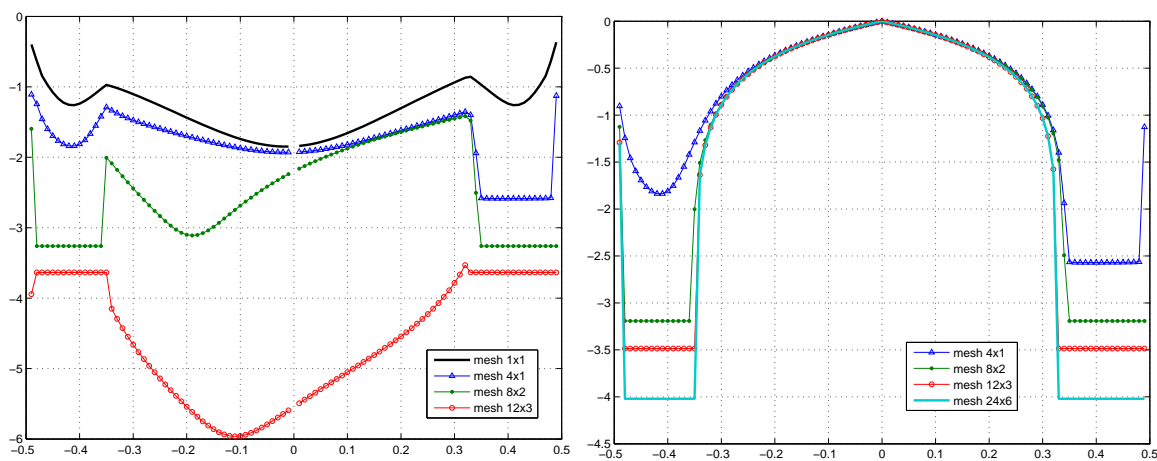

FIGURE 4. $\beta_n(a)$ for discrete spaces (5.18) versus $a \in (-0.5, 0.5)$ on four meshes of the rectangle: Difference with the finer mesh (left) and difference with $\beta(\Omega)$ (right) in log10 scale.

In Figure 4 we plot the (relative) difference between the constants β when the mesh grows finer (left) and the difference with $\beta(\Omega) \simeq 0.218444$ (right). More precisely, denoting by

β_n the inf-sup constant associated with the mesh $4n \times n$, $n = 1, 2, 3, 6$, and $\beta_\square$ the inf-sup constant of the one-element square mesh, we plot $a \mapsto \log_{10}\{(\beta_6(a) - \beta_n(a))/\beta_6(a)\}$, $n = \square, 1, 2, 3$ on the left part of Figure 4, and $a \mapsto \log_{10}(\beta(\Omega) - \beta_n(a))$, $n = 1, 2, 3, 6$, on the right part of Figure 4.

5.4.2. *p-version on rectangular elements.* For one spectral element in dimension 2, i.e. for the square $\Omega = (-1, 1)^2$ discretized by means of the space $\mathbb{Q}_p$ of polynomials with degree $\leq p$ with respect to each variable, we have the following results, quoted or proved in [4]: As discrete space for velocities let us take

$$X_n = H_0^1(\Omega)^2 \cap \mathbb{Q}_n^2, \quad n \geq 2\,.$$

Then, according to the choice of the pressure space M_n there holds

(1) If $M_n = \mathbb{Q}_{n-1}$, then $\beta_n = 0$.
(2) If $M_n = \mathbb{Q}_{n-d}$ with a chosen $d \geq 2$, then $\beta_n = \mathcal{O}(n^{-1})$ as $n \to \infty$.
(3) If $M_n = \mathbb{Q}_{n-\lambda n}$ with a chosen $\lambda \in (0, 1)$, then $\beta_n \geq \beta_\star > 0$.

In this paper, we have proved convergence of β_n to $\beta(\Omega)$ if $M_n = \mathbb{Q}_{\lambda_n \sqrt{n}}$ for any sequence of positive numbers λ_n such that $\lambda_n \to 0$ and $\lambda_n \sqrt{n} \to \infty$ as $n \to \infty$.

We have performed computations on rectangles of different aspect ratios with two meshes $\mathcal{T}$ with 1×1 or 2×2 isometric rectangular elements. Here we use the finite element library Mélina++. Now we set

(5.20)
$$\begin{cases} X_n = \{\boldsymbol{v} \in H_0^1(\Omega)^2, \quad \boldsymbol{v}\big|_K \in \mathbb{Q}_n(K)^2 \ \text{ for all } \ K \in \mathcal{T}\}, \\ M_n = \{q \in L_\circ^2(\Omega), \quad q\big|_K \in \mathbb{Q}_k(K) \ \text{ for all } \ K \in \mathcal{T}\}. \end{cases}$$

We observe

(1) If $k = n - 1$, then $\beta_n = 0$ on the mesh 1×1 and β_n behaves as in the next point on the mesh 2×2.
(2) If $k = n - d$ with $d = 2, 3$, then β_n may display a preasymptotic convergence to $\beta(\Omega)$, and eventually tends to 0 as $n \to \infty$.
(3) If $k = n/2$, then β_n converges to $\beta(\Omega)$.

In Figure 5, we present numerical results for the rectangle $\Omega = (0, 2) \times (0, 1)$ for which we know that $\beta(\Omega)^2$ is an isolated eigenvalue of the Cosserat operator, and our most precise computations yield the approximate value 0.387262 for $\beta(\Omega)$.

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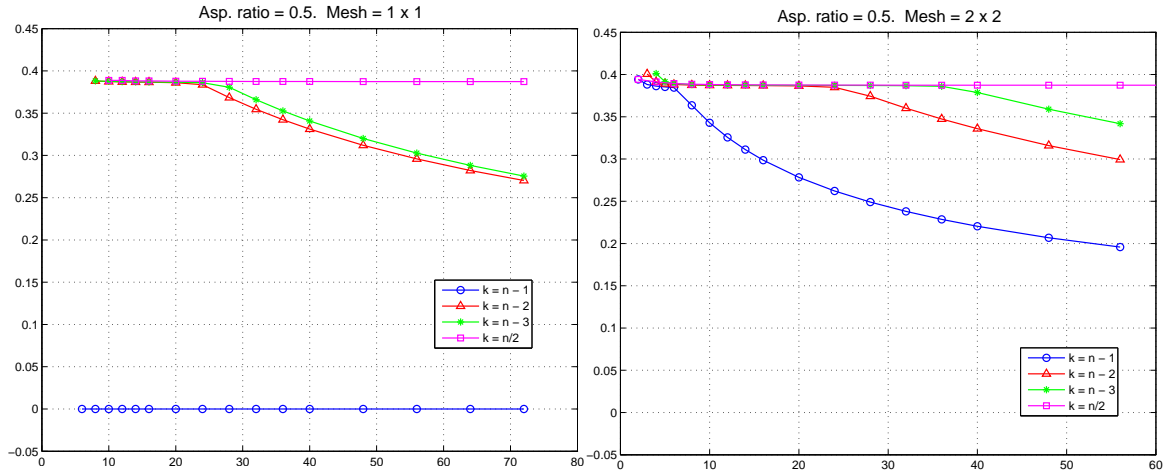
FIGURE 5. β_n for discrete spaces (5.20) versus n on two meshes of the rectangle $(0,2) \times (0,1)$.

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