ASYMPTOTICS FOR 2D WHISPERING GALLERY MODES IN OPTICAL MICRO-DISKS WITH RADIALLY VARYING INDEX

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ABSTRACT. Whispering gallery modes [WGM] are resonant modes displaying special features: They concentrate along the boundary of the optical cavity at high angular frequencies and they are associated with (complex) scattering resonances very close to the real axis. As a classical simplification of the full Maxwell system, we consider two-dimensional Helmholtz equations governing transverse electric [TE] or magnetic [TM] modes. Even in this 2D framework, very few results provide asymptotic expansion of WGM resonances at high angular frequency. In this work, using multiscale expansions, we design a unified procedure to construct asymptotic quasi-resonances and associate quasi-modes that have the WGM structure in disk cavities with a radially varying optical index. We show using the black-box scattering approach that quasi-resonances are asymptotically close to true resonances. More specifically, using a Schrödinger analogy we highlight three typical behaviors in such optical micro-disks, leading to three distinct asymptotic expansions for the quasi-resonances and quasi-modes.

CONTENTS

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1. INTRODUCTION

1.1. **Helmholtz equations for optical micro-disks.** The motivation of our work is the study of light-wave propagation in optical micro-resonators. These optical devices, with micrometric size, came to be important components in the photonic toolbox. They are basically composed of a dielectric cavity coupled to waveguides or fibers for light input and output [10]. When resonance conditions are met, it is possible to confine light-waves in the cavity and to access a wide range of optical phenomena. The study of such scattering resonances is the subject of this paper. More specifically, we focus our study on *"Whispering Gallery Modes"* that are modes essentially localized inside the cavity and concentrated in a boundary layer, and that are associated with scattering resonances close to the real axis.

If a complete, 3-dimensional, modeling would require to solve the full Maxwell system, in many situations, the solution of 2-dimensional scalar harmonic equations brings insight in resonance phenomena and constitutes a reference configuration in the literature in optics. Moreover, the 2-dimensional model can be obtained as an approximation of the 3-dimensional ones by using an approach referred in the optical literature as the effective index method [26].

In the 2-dimensional model, the optical cavity is represented by a bounded plane domain, that we denote by Ω. This cavity is associated with an optical index $n > 1$ which is a regular function of the position in the closure $\overline{\Omega}$ of Ω. Outside $\overline{\Omega}$, the index n is equal to1. There are two relevant scalar Helmholtz equations associated with such a configuration, corresponding to *Transverse Electric* (TE) modes or *Transverse Magnetic* (TM) modes: For such a bi-dimensional optical cavity Ω with index n, the resonance pairs (k, u) are obtained by solving the following problem where $p = 1$ for TM modes and $p = -1$ for TE modes:

$$\begin{cases} \text{Find } k \in \mathbb{C},\ u\big|_{\Omega} \in \mathrm{H}^2(\Omega),\ u\big|_{\mathbb{R}^2\setminus\Omega} \in \mathrm{H}^2_{\mathsf{loc}}(\mathbb{R}^2 \setminus \Omega) & \text{s.t.} \\ -\operatorname{div}(n^{p-1}\,\nabla u) - k^2 n^{p+1}\, u = 0 & \text{in } \Omega \text{ and } \mathbb{R}^2 \setminus \overline{\Omega} \\ [u] = 0 & \text{across } \partial\Omega \\ [n^{p-1}\,\partial_\nu u] = 0 & \text{across } \partial\Omega \end{cases} \tag{1.1a}$$

with a radiation condition at infinity. This radiation condition imposes that the solution u to problem (1.1a), outside any disk $D(0, R_\Omega)$ which contains Ω, has an expansion in terms of Hankel functions of the first kind $\mathsf{H}^{(1)}_m$ in the following form in polar coordinates:

$$u(x, y) = \sum_{m\in\mathbb{Z}} C_m\, \mathrm{e}^{\mathrm{i}m\theta}\, \mathsf{H}^{(1)}_m(kr) \qquad \forall\theta \in [0, 2\pi],\ r > R_\Omega. \tag{1.1b}$$

Owing to Rellich theorem (see [16] for instance) the radiation condition (1.1b) implies that the imaginary part of k is negative and the modes, exponentially increasing at infinity. Such wave-numbers k are the poles of the extension of the resolvent of underlying Helmholtz operators when coming from the upper half complex plane.

We note that, in the case when n is constant inside Ω, problem (1.1a)–(1.1b) appears also as a modeling of scattering by a transparent obstacle (see MOIOLA and SPENCE [17, Remark 2.1] for a discussion of the models in acoustics and electromagnetics). In contrast with impenetrable obstacles (see SJÖSTRAND and ZWORSKI [25]), transparent obstacles or dielectric cavities may have resonances super-algebraically close to the real axis due to almost total internal reflection on a convex boundary (see POPOV and VODEV [23], and also GALKOWSKI[8]). These resonances correspond to the whispering gallery modes we are interested in.

This work is motivated by the following observation found in the literature in optics: Whispering Gallery Mode (WGM) resonators are in most cases formed of dielectric materials with constant optical index n but this constitutes a potential limit in their performance and in the range of their applications. Resonators with spatially varying optical index, that fall under the category of "graded index" structures [9], offer new opportunities to improve and enlarge the field of applications of these devices and start to be investigated in optics. Among the graded index structures investigated in the literature in optics, we can quote a modified form of the "Maxwell's fish eye", that can be implemented using dielectric material, where the optical index varies with the radial position in the micro-disk resonator as [4, 19]

$$n(r) = \alpha \left(1 + \frac{r^2}{R^2}\right)^{-1}$$

where $\alpha > 2$ and R is the disk radius. In [28] the authors consider a micro-cavity made of a quadratic-index glass doped with dye molecules and the refractive index is written as $n(r) = \alpha - \frac{1}{2}\beta r^2$, where $\alpha, \beta > 0$. In [31], an analysis of hollow cylindrical whispering gallery mode resonator is carried out where the refractive index of the cladding varies according to $n(r) = \beta/r$, $\beta > R$.

Motivated by the above mentioned examples, we found interesting to investigate the case when Ω is a disk (with radius denoted by R), and the optical index is a radial function of the position: In polar coordinates (r, θ) centered at the disk center, $n = n(r)$. Taking advantage of the invariance by rotation of equations (1.1), it is easily proved that any solution u associated with a $p \in \{\pm 1\}$ and a resonance $k \in \mathbb{C}$ can be expanded as a Fourier sum

$$u(x, y) = \sum_{m \in \mathbb{Z}} w_m(r)\, \mathrm{e}^{\mathrm{i}m\theta}$$

and that each term $u_m(x, y) := w_m(r)\, \mathrm{e}^{\mathrm{i}m\theta}$ is a solution of problem (1.1) associated with the same p and the same k. Hence it is sufficient to solve, for any $m \in \mathbb{Z}$, problem (1.1) with u of the form $w_m(r)\, \mathrm{e}^{\mathrm{i}m\theta}$. Here $m \in \mathbb{Z}$ is referred as the *polar mode index*. The radial problem satisfied by $w : r \mapsto w(r)$ when $u = w(r)\, \mathrm{e}^{\mathrm{i}m\theta}$ is plugged into problem (1.1) is the following radial problem (1.2a)–(1.2b) $\equiv$ (1.2) depending on m:

$$\begin{cases} \text{Find } k \in \mathbb{C}, \quad w\big|_{(0,R)} \in \mathrm{H}^2((0,R), r\,\mathrm{d}r), \quad w\big|_{(R,\infty)} \in \mathrm{H}^2_{\mathsf{loc}}([R,\infty), r\,\mathrm{d}r) \ \text{ s.t.} \\ -\frac{1}{r}\partial_r(n^{p-1} r \partial_r w) + n^{p-1}\left(\frac{m^2}{r^2} - k^2 n^2\right) w = 0 & \text{in } (0,R) \text{ and } (R,+\infty) \\ [w] = 0 & \text{for } r = R \\ [n^{p-1} w'] = 0 & \text{for } r = R \\ w(0) = 0 & \text{if } m \neq 0 \end{cases} \tag{1.2a}$$

with the following outgoing wave condition at infinity deduced from (1.1b):

$$w(r) = C\, \mathsf{H}^{(1)}_m(kr) \qquad \text{when } r > R, \qquad \text{with a constant } C \neq 0. \tag{1.2b}$$

1.2. **Circular cavity with constant optical index.** As a fundamental illustrative example, let us consider a circular cavity with *constant* optical index $n \equiv n_0 > 1$ in $\overline{\Omega}$. Though apparently simple, this case is indeed already very rich. The use of partly analytic formulas provides a lot of information on the resonance set and the associated modes. Let us sketch this now. For both TM and TE modes, solutions w of (1.2) have the form

$$w(r) = \begin{cases} \mathsf{J}_m(n_0 kr) & \text{if } r \leq R \\ \dfrac{\mathsf{J}_m(n_0 kR)}{\mathsf{H}^{(1)}_m(kR)} \mathsf{H}^{(1)}_m(kr) & \text{if } r > R. \end{cases} \tag{1.3}$$

Here J_m refers to the order m Bessel function of the first kind and $\mathsf{H}^{(1)}_m$ refers to the order m Hankel function of the first kind. In both TE and TM cases, the resonance k is obtained as a solution to the following non-linear equation, termed *modal equation* (recall that $p = 1$ for TM modes and $p = -1$ for TE modes)

$$n_0^p \, \mathsf{J}'_m(n_0 Rk) \, \mathsf{H}^{(1)}_m(Rk) - \mathsf{J}_m(n_0 Rk) \, \mathsf{H}^{(1)\prime}_m(Rk) = 0. \tag{1.4}$$

For each value of the polar mode index m, the modal equation (1.4) has infinitely many solutions $k \in \mathbb{C}$. We denote by $\mathscr{R}_p[n_0, R](m)$ this set. Because $\mathsf{J}_{-m}(\rho) = (-1)^m \mathsf{J}_m(\rho)$ and the same for $\mathsf{H}^{(1)}_m$, the two integers $\pm m$ provide the same resonance values, which reflects a degeneracy of resonances for disk-cavities. Thus, we can restrict the discussion to nonnegative integer values of m.

In Figure 1, we show the complex roots of equation (1.4) when $n_0 = 1.5$, $R = 1$, and $0 \leq m \leq 60$, the values of m being distinguished by a color scale. This figure displays clearly the general features of the set of resonances. For each m the set of resonances $\mathscr{R}_p[n_0, R](m)$ can be split into two parts, see [3],

- an infinite part $\mathscr{R}_{p,\,\mathsf{inner}}[n_0, R](m)$ made of *inner resonances* for which the modes are essentially supported inside the disk Ω
- a finite part $\mathscr{R}_{p,\,\mathsf{outer}}[n_0, R](m)$ made of *outer resonances* for which the modes are essentially supported outside the disk Ω.

The sets of inner and outer resonances are given by

$$\mathscr{R}_{p,\,\mathsf{inner}}[n_0, R] = \bigcup_{m \in \mathbb{N}} \mathscr{R}_{p,\,\mathsf{inner}}[n_0, R](m) \quad \text{and} \quad \mathscr{R}_{p,\,\mathsf{outer}}[n_0, R] = \bigcup_{m \in \mathbb{N}} \mathscr{R}_{p,\,\mathsf{outer}}[n_0, R](m)$$

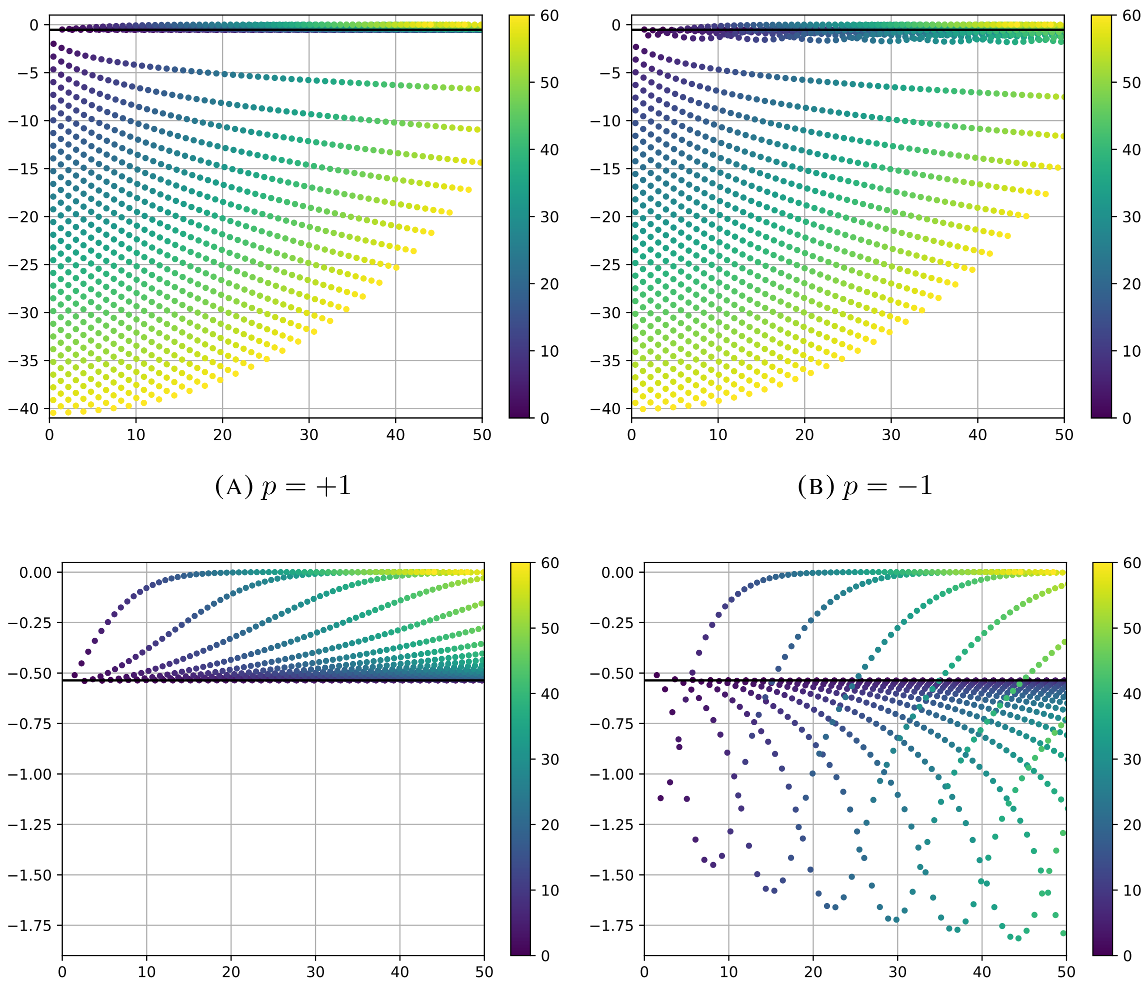
respectively. It appears that for TM modes ($p = 1$) there exists a negative threshold τ such that the outer resonances satisfy $\operatorname{Im} k < \tau$, and the inner ones, $\operatorname{Im} k \geq \tau$. We can clearly see on Fig 1 some organization in sub-families, not indexed by m (i.e., m varies along these families). Observation of the associated modes shows that these families depend on another parameter, j, which can be called a *radial mode index*: This is the number of sign changes (or nodal points) of the real part of an associated mode. For inner resonances, the sign changes occur inside the disk. For outer resonances, there is no such interpretation in term of sign changes but they can be linked to the resonances of the exterior Dirichlet problem: One can see in [2, Eq. (49)] that when $m \to +\infty$, outer resonances tend to zeros of the Hankel function $k \mapsto \mathsf{H}^{(1)}_m(Rk)$.

Inner resonances will be denoted by $k_{p;j}(m)$, with $p = 1$ and $p = -1$ according to the TM and TE cases respectively, and with m and j the polar and radial mode indices, respectively. Then there exist two distinct asymptotics for $k_{p;j}(m)$ according to $j \to \infty$ or $m \to \infty$.

On the one hand, direct calculations yield, [18, Section 3.3.1], when $p = \pm 1$ and $j \to +\infty$

$$k_{p;j}(m) \sim \frac{j\pi}{Rn_0} + \frac{(2m + 2 - p)\pi}{4Rn_0} + \frac{\mathrm{i}}{2n_0} \ln \left(\frac{n_0 - 1}{n_0 + 1} \right).$$

Thus, as $j \to \infty$, the imaginary part of $k_{p;j}(m)$ tends to the negative value $\frac{1}{2n_0} \ln(\frac{n_0 - 1}{n_0 + 1})$. For the example displayed in Figure 1, this value is -0.53648. This same value can also be found in the physical literature, see [6, Eq. (13)] for example.



(A) $p = +1$ (B) $p = -1$

(C) $p = +1$ (D) $p = -1$

FIGURE 1. Roots of (1.4) when $p \in \{\pm 1\}$, $n_0 = 1.5$, $R = 1$, and $0 \leq m \leq 60$. The first row gives a global view, whereas the second row provides a zoom on inner resonances.

On the other hand, using asymptotic expansions of Bessel's functions involved in the modal equation (1.4), one obtains that resonances $k_{p;j}(m)$ for a given radial index j satisfy the following asymptotic expansion when $m \to +\infty$:

$$k_{p;j}(m) = \frac{m}{Rn_0}\left[1 + \frac{\mathsf{a}_j}{2}\left(\frac{2}{m}\right)^{\frac{2}{3}} - \frac{n_0^p}{2(n_0^2-1)^{\frac{1}{2}}}\left(\frac{2}{m}\right) + \frac{3\mathsf{a}_j^2}{40}\left(\frac{2}{m}\right)^{\frac{4}{3}} - \mathsf{a}_j\frac{n_0^p(3n_0^2 - 2n_0^{2p})}{12(n_0^2-1)^{\frac{3}{2}}}\left(\frac{2}{m}\right)^{\frac{5}{3}} + \mathscr{O}\left(m^{-2}\right)\right]. \quad (1.5)$$

For details, we refer to [15] where computations were carried out for a sphere and therefore spherical Bessel's functions appears in this latter case in the modal equation instead of cylindrical Bessel's functions. In the asymptotic expansion (1.5), $0 < \mathsf{a}_0 < \mathsf{a}_1 < \mathsf{a}_2 < \cdots$ are the successive roots of the flipped Airy function $\mathsf{A} : z \in \mathbb{C} \mapsto \mathsf{Ai}(-z)$ where Ai denotes the Airy function. It is important to note that the terms in the asymptotic expansions are *real*: Hence the imaginary part of $k_{p;j}(m)$ is contained in the remainder. This part of the resonance set correspond to typical whispering gallery modes.

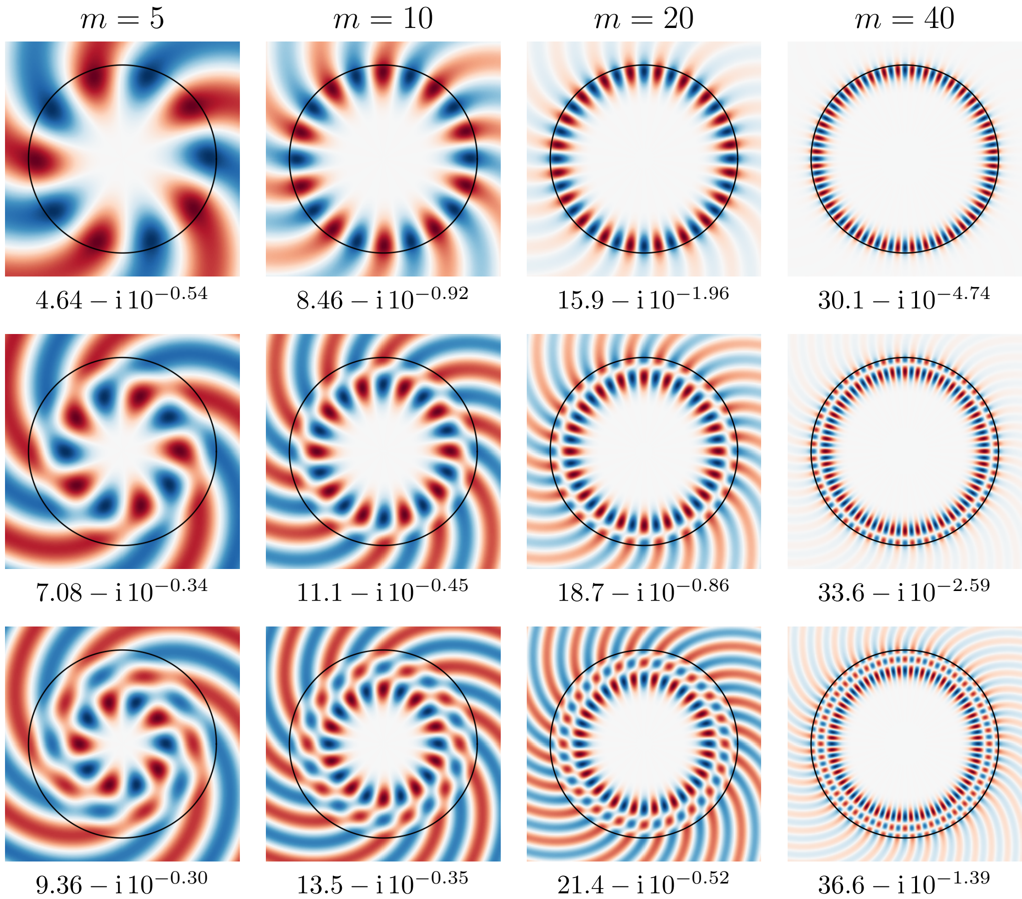


FIGURE 2. Plots of real parts of TM modes u in a circular cavity of radius $R = 1$ and index $n = 1.5$ (computed by solving the modal equation (1.4) using complex integration [22]). Below each plot, we give values of computed resonances $k \in \mathbb{C}$. Each row corresponds to a distinct value of j, from 0 to 2.

We observe the following whispering gallery mode features when j is chosen and m gets large, see Fig. 2:

(1) The analytic resonances obtained by solving the modal equation (1.4) have a *negative imaginary part* which tends to zero rapidly (exponentially) when $m \to \infty$.
(2) The analytic modes $u = w_m(r)\,\mathrm{e}^{\mathrm{i}m\theta}$ with w_m given by (1.3) and k solution of the modal equation (1.4) concentrate around the interface between the disk and the exterior medium.

1.3. **Radially varying index: Main results.** The proof of the formula (1.5) given in [15] relies on the modal equation (1.4) and makes use of asymptotic formulas for Bessel functions [20, 1]. Such an approach is specific to disks with constant optical index, and the number of terms in the expansion is limited by the asymptotics as $m \to \infty$ of Bessel functions available in the literature.

In this paper we develop a more versatile approach, based on multiscale expansions and semiclassical analysis. The idea is to consider $h = \frac{1}{m}$ as small parameter and to take advantage of the factor m^2 in front of $\frac{1}{r^2}$ in the first equation of (1.2a) to transform this equation into a semiclassical 1-dimensional Schrödinger operator with a singular potential V. Generically, V will have a potential well at $r = R$. We perform an asymptotic construction of quasi-resonances and quasi-modes in the vicinity of $r = R$, in such a way that we can rely on general arguments to deduce the existence of true resonances close to quasi-resonances modulo $\mathscr{O}(m^{-\infty})$, i.e., more rapidly than any polynomial in $\frac{1}{m}$. Unless explicitly mentioned, we suppose the following.

Assumption 1.1. The radial function $n : r \mapsto n(r)$ satisfies the following properties

(1) $n(r) = 1$ if $r > R$;
(2) The function $r \mapsto n(r)$ belongs to $\mathscr{C}^\infty([0, R])$ and $n(r) > 1$ for all $r \leq R$.

This assumption motivates the following notations.

Notation 1.2. Let $n(R)$, $n'(R)$, $n''(R)$ denote the limit values of $n(r)$ and its derivatives as $r \nearrow R$; We set

$$n_0 = n(R), \quad n_1 = n'(R), \quad n_2 = n''(R). \tag{1.6}$$

Let $\breve{\kappa}$ be the effective adimensional curvature defined by

$$\breve{\kappa} := R\left(\frac{1}{R} + \frac{n_1}{n_0}\right). \tag{1.7}$$

In this paper we prove expansions of resonances as $m \to \infty$ in three distinct cases discriminated by the sign of $\breve{\kappa}$. Such expansions are "modulo $\mathscr{O}(m^{-\infty})$" in a sense defined below.

Notation 1.3. Let $(a_m)_{m\in\mathbb{N}}$ be a sequence of numbers:

$$a_m = \mathscr{O}(m^{-\infty}) \quad \text{means that } \forall N \in \mathbb{N},\ \exists C_N \text{ such that } |a_m| \leq C_N\, m^{-N}, \quad \forall m \in \mathbb{N}.$$

Theorem 1.A. *Assume that the radial function n satisfies Assumption 1.1 and*

$$\breve{\kappa} > 0. \tag{1.8}$$

Choose $p \in \{\pm 1\}$ and denote by $\mathscr{R}_p[n, R]$ the resonance set solution to problem (1.1). Then for any $j \in \mathbb{N}$, there exists a smooth real function $\mathrm{K}_{p;j} \in \mathscr{C}^\infty([0, 1]) : t \mapsto \mathrm{K}_{p;j}(t)$ defining distinct sequences

$$\underline{k}_{p;j}(m) = m\, \mathrm{K}_{p;j}\left(m^{-\frac{1}{3}}\right), \quad \forall m \geq 1 \tag{1.9}$$

that are close modulo $\mathscr{O}(m^{-\infty})$ to the resonance set $\mathscr{R}_p[n, R]$, i.e. for each m, there exists $k_m \in \mathscr{R}_p[n, R]$ such that $\underline{k}_{p;j}(m) - k_m = \mathscr{O}(m^{-\infty})$. Let $\mathrm{K}^\ell_{p;j}$ be the coefficients of the Taylor expansion of $\mathrm{K}_{p;j}$ at $t = 0$. We have, with numbers a_j being the successive roots of the flipped Airy function,

$$\mathrm{K}^0_{p;j} = \frac{1}{Rn_0}, \quad \mathrm{K}^1_{p;j} = 0, \quad \mathrm{K}^2_{p;j} = \frac{1}{Rn_0}\,\frac{\mathsf{a}_j}{2}(2\breve{\kappa})^{\frac{2}{3}}, \tag{1.10}$$

All coefficients $\mathrm{K}^\ell_{p;j}$ are calculable, being the solution of an explicit algorithm involving matrix products and matrix inversions in finite dimensions.

We refer to Section 4.1 for more details. As a consequence of Theorem 1.A, for each chosen p and j, there exists a sequence of true resonances $m \mapsto k_{p;j}(m) \in \mathscr{R}_p[n, R]$ such that

$$k_{p;j}(m) = m\left[\sum_{\ell=0}^{N-1} \mathrm{K}^\ell_{p;j}\left(\frac{1}{m}\right)^{\frac{\ell}{3}} + \mathscr{O}\left(\frac{1}{m}\right)^{\frac{N}{3}}\right] \quad \forall N \geq 1. \tag{1.11}$$

This clearly generalizes (1.5).

When $\breve{\kappa}$ is zero, the powers of $m^{-\frac{1}{3}}$ are replaced by powers of $m^{-\frac{1}{2}}$.

Theorem 1.B. *Assume that the radial function n satisfies Assumption 1.1 and that*

$$\breve{\kappa} = 0, \quad \textit{and} \quad \breve{\mu} := 2 - \frac{R^2 n_2}{n_0} > 0. \tag{1.12}$$

Then for any $j \in \mathbb{N}$, there exists a smooth real function $\mathrm{K}_{p;j} \in \mathscr{C}^\infty([0,1]) : t \mapsto \mathrm{K}_{p;j}(t)$ defining distinct sequences

$$\underline{k}_{p;j}(m) = m\,\mathrm{K}_{p;j}\left(m^{-\frac{1}{2}}\right), \quad \forall m \geq 1 \tag{1.13}$$

that are close modulo $\mathscr{O}(m^{-\infty})$ to the resonance set $\mathscr{R}_p[n,R]$. The first coefficients of the Taylor expansion of $\mathrm{K}_{p;j}$ at $t=0$ are

$$\mathrm{K}^0_{p;j} = \frac{1}{Rn_0}, \quad \mathrm{K}^1_{p;j} = 0, \quad \mathrm{K}^2_{p;j} = \frac{1}{Rn_0}\,\frac{(4j+3)\sqrt{\breve{\mu}}}{2}. \tag{1.14}$$

We refer to Section 5.1 for more details. Note that, in contrast to the case $\breve{\kappa} > 0$, the coefficients $\mathrm{K}^\ell_{p;j}$ are not calculable (except the first four of them) in the sense that their determination needs the inversion of infinite dimensional matrices.

Unlike the two previous cases for which the quasi-modes are localized near the interface $r = R$, in the third case the quasi-modes are localized near an internal circle $r = R_0$ with some $R_0 < R$.

Theorem 1.C. *Assume that the radial function n satisfies Assumption 1.1 and that $\breve{\kappa} < 0$. Let $R_0 \in (0,R)$ such that $1 + \frac{R_0 n'(R_0)}{n(R_0)} = 0$ and assume further that*

$$\breve{\mu} := 2 - \frac{R_0^2 n''(R_0)}{n(R_0)} > 0. \tag{1.15}$$

Then a similar statement as in Theorem 1.B holds. The first coefficients of the Taylor expansion of $\mathrm{K}_{p;j}$ at $t=0$ are now, instead of (1.14),

$$\mathrm{K}^0_{p;j} = \frac{1}{R_0 n(R_0)}, \quad \mathrm{K}^1_{p;j} = 0, \quad \mathrm{K}^2_{p;j} = \frac{1}{R_0 n(R_0)}\,\frac{(2j+3)\sqrt{\breve{\mu}}}{2}. \tag{1.16}$$

We refer to Section 6.1 for more details. Note that in this case the coefficients $\mathrm{K}^\ell_{p;j}$ are all calculable in the sense introduced in Theorem 1.A.

Remark 1.4. In all cases covered by Theorems 1.A–1.C, the quasi-resonances $\underline{k}_{p;j}(m)$ are real. We will see in the proofs that the associated quasi-modes $\underline{u}_{p;j}(m)$ are localized in the radial variable (close to the interface $r = R$ in the fisrt two cases and close to the internal circle $r = R_0$ in the third one). The couples $(\underline{k}_{p;j}(m), \underline{u}_{p;j}(m))$ are in fact quasi-pairs for a transmission problem in a larger bounded domain D containing Ω. Such a problem can be viewed as self-adjoint. This explains why that the $\underline{k}_{p;j}(m)$ are real. They can be bridged with the true *complex resonances* solution of problem (1.1) through general results by TANG and ZWORSKI [29], and STEFANOV [27].

1.4. **Organization of the paper.** In section 2, we make precise the notion of quasi-pairs, quasi-resonances, and quasi-modes. In section 3, by means of the Schrödinger analogy, we classify the three main types of localized resonance modes that can be observed in circular dielectric cavities. In sections 4–6, we construct quasi-pairs associated with localized resonances in these three cases. Finally, in section 7, we show that the quasi-resonances just constructed are asymptotically close to true resonances of the cavity, hence ending the proof of Theorems 1.A–1.C.

The set of non-negative integers is denoted by $\mathbb{N}$ and the set of positive integer by $\mathbb{N}^*$. We denote by $\mathrm{L}^2(\Omega)$ the space of square-integrable functions on the open set Ω, and by $\mathrm{H}^\ell(\Omega)$

the Sobolev space of functions in $\mathrm{L}^2(\Omega)$ such that their derivatives up to order ℓ belong to $\mathrm{L}^2(\Omega)$. Finally, $\mathscr{S}(I)$ denotes the space of Schwartz functions on the unbounded interval I.

2. FAMILIES OF RESONANCE QUASI-PAIRS

Inspired by quasi-pair constructions used to investigate ground states in semiclassical analysis of Schrödinger operators (see SIMON [24] for instance), we are going to construct families of resonance quasi-pairs for problem (1.1). Here appears a fundamental difference: The quasi-pair construction in semiclassical analysis consists in building approximate eigenpairs (λ_h, u_h) that solve $A_h u_h = \lambda_h u_h$ with increasingly small error as $h \to 0$ where A_h is for instance the operator $-h^2\Delta + V$. In our case, we do not have any given semiclassical parameter h. However, the term $\frac{m^2}{r^2}$ in the first equation of (1.2a) may play the role of a confining potential in a semiclassical framework if we set

$$h = \frac{1}{|m|}\,.$$

This means that an internal frequency parameter m can be viewed as a driving parameter for an asymptotic study. This leads to the next definition for quasi-pairs, adapted to our problem.

Definition 2.1. Choose $p \in \{\pm 1\}$. A *family of resonance quasi-pairs* $\mathfrak{F}_p$ for problem (1.1) is formed by a sequence $\mathfrak{K}_p = (\underline{k}(m))_{m\geq 1}$ of real numbers called quasi-resonances and a sequence $\mathfrak{U}_p = (\underline{u}(m))_{m\geq 1}$ of complex valued functions called quasi-modes, where for each $m \geq 1$, the couple $(\underline{k}(m), \underline{u}(m))$ is a quasi-pair for problem (1.1) with an error in $\mathscr{O}\left(m^{-\infty}\right)$ when $m \to \infty$. More precisely, we mean that

(1) For any $m \geq 1$, the functions $\underline{u}(m)$ belong to the domain of the operator and are normalized,

$$\underline{u}(m) \in \mathrm{H}^2_p(\mathbb{R}^2, \Omega) \quad \text{and} \quad \|\underline{u}(m)\|_{\mathrm{L}^2(\mathbb{R}^2)} = 1$$

where

$$\mathrm{H}^2_p(\mathbb{R}^2,\Omega) = \Big\{u \in \mathrm{L}^2(\mathbb{R}^2) \;\Big|\; u\big|_{\Omega} \in \mathrm{H}^2(\Omega),\; u\big|_{\mathbb{R}^2\setminus\overline{\Omega}} \in \mathrm{H}^2(\mathbb{R}^2\setminus\overline{\Omega}), \quad [u]_{\partial\Omega} = 0, \;\text{ and }\; [n^{p-1}\,\partial_\nu u]_{\partial\Omega} = 0\Big\}. \tag{2.1}$$

(2) We have the following quasi-pair estimate as $m \to +\infty$,

$$\left\| -\operatorname{div}\left(n^{p-1}\,\nabla \underline{u}(m)\right) - \underline{k}(m)^2\, n^{p+1}\,\underline{u}(m)\right\|_{\mathrm{L}^2(\mathbb{R}^2)} = \mathscr{O}\left(m^{-\infty}\right). \tag{2.2}$$

(3) Uniform localization: There exists a function $\mathfrak{X} \in \mathscr{C}^\infty_0(\mathbb{R}^2)$, $0 \leq \mathfrak{X} \leq 1$, such that

$\|\mathfrak{X}\underline{u}(m)\|_{\mathrm{L}^2(\mathbb{R}^2)} \geq \frac{1}{2}$ and (2.2) holds with $\mathfrak{X}\underline{u}(m)$ replacing $\underline{u}(m)$.

(4) Regularity with respect to m: There exist a positive real number β and a smooth function $\mathrm{K} \in \mathscr{C}^\infty([0,1]) : t \mapsto \mathrm{K}(t)$ such that

$$\frac{\underline{k}(m)}{m} = \mathrm{K}(m^{-\beta}) \quad \forall m \geq 1. \tag{2.3}$$

If the cut-off function $\mathfrak{X}$ in item (3) can be taken as any function that is $\equiv 1$ in a neighborhood of $\partial\Omega$ for m large enough, we say that the family $\mathfrak{F}_p$ is a family of *whispering gallery type*.

Remark 2.2. By Taylor expansion of the function K at $t = 0$, we obtain that a consequence of (2.3) is the existence of coefficients K_ℓ, $\ell \in \mathbb{N}$, and constants C_N such that

$$\forall N \geq 1, \qquad \left| \frac{\underline{k}(m)}{m} - \sum_{\ell=0}^{N-1} \mathrm{K}_\ell\, m^{-\ell\beta} \right| \leq C_N\, m^{-N\beta}. \tag{2.4}$$

Note that the asymptotics (1.5) satisfies such an estimate with $\beta = \frac{1}{3}$ and $N = 6$.

Remark 2.3. The estimate (2.2) implies a bound from below for the resolvent of the underlying operator, compare with [17, §6]: At quasi-resonances, we have a *blow up of the resolvent*.

3. Classification of the three typical behaviors by a Schrödinger analogy

In our way to prove Theorems 1.A–1.C, we transform the family of problems (1.2a) when m spans $\mathbb{N}^*$ into a family of 1-dimensional Schrödinger operators depending on the semi-classical parameter $h = \frac{1}{m}$.

Namely, choosing a polar mode index $m \in \mathbb{N}^*$ and coming back to the ODE contained in problem (1.2a) divided by n^{p+1}, we obtain the equation:

$$-\frac{1}{rn^{p+1}}\partial_r(n^{p-1}r\partial_r w) + \frac{m^2}{r^2n^2}w - k^2\, w = 0 \tag{3.1}$$

As a start, we write a quasi-resonance as (compare with (1.5))

$$\underline{k}(m)^2 = m^2\Lambda$$

where the number Λ depends on h and has to be found. Multiplying (3.1) by $h^2 = 1/m^2$, we find that (3.1) takes the form of a one dimensional semiclassical Schrödinger modal equation

$$-h^2\, \mathscr{H}w + Ww = \Lambda w, \tag{3.2}$$

where $\mathscr{H}$ is the second order differential operator

$$\mathscr{H} = \frac{1}{rn^{p+1}}\partial_r(n^{p-1}r\partial_r) \tag{3.3}$$

and W is the potential

$$W(r) = \left(\frac{1}{r\,n(r)}\right)^2. \tag{3.4}$$

The operator $-h^2\,\mathscr{H} + W$ is self-adjoint on $\mathrm{L}^2(\mathbb{R}_+, n^{p+1}r\,\mathrm{d}r)$. We note that

$$\lim_{r\nearrow R} W(r) = \left(\frac{1}{R\,n_0}\right)^2 \quad \text{and} \quad \lim_{r\searrow R} W(r) = \left(\frac{1}{R}\right)^2. \tag{3.5}$$

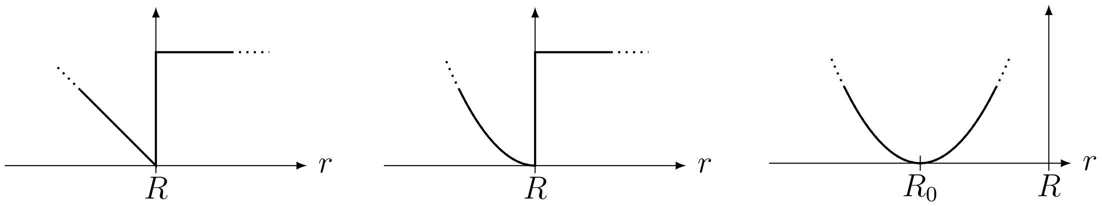
Since $n_0 > 1$, we have a potential barrier at $r = R$. The first and second derivatives of W on $(0, R]$ are given by

$$W'(r) = -2\left(\frac{1}{r\,n(r)}\right)^2\left[\frac{1}{r} + \frac{n'(r)}{n(r)}\right], \tag{3.6a}$$

$$W''(r) = 2\left(\frac{1}{r\,n(r)}\right)^2\left[3\left(\frac{1}{r} + \frac{n'(r)}{n(r)}\right)^2 - \frac{2n'(r)}{rn(r)} - \frac{n''(r)}{n(r)}\right]. \tag{3.6b}$$

The local minima (potential wells) of W cause the existence of resonances near these energy levels and their asymptotic structure as $h \to 0$ is determined by the Taylor expansion of W at its local minima. Let us recall that $\breve{\kappa} = R\big(\frac{1}{R} + \frac{n'(R)}{n(R)}\big)$. The sign of $\breve{\kappa}$ (if it is positive, zero, or negative) discriminates three typical behaviors in which case we will be able to construct families of resonance quasi-pairs (see Theorems 1.A, 1.B, 1.C):

(A) $\breve{\kappa} > 0$. Then W is decreasing on a left neighborhood of R and has a local minimum at R. In a two-sided neighborhood of R, W is tangent to a *half-triangular potential well*, see Fig. 3 (A).



(A) $\breve\kappa > 0$ (B) $\breve\kappa = 0$ (C) $\breve\kappa < 0$

FIGURE 3. The three typical local behaviors of the potential W: half-triangular potential well ($\breve\kappa > 0$), half-quadratic potential well ($\breve\kappa = 0$) and quadratic potential well ($\breve\kappa < 0$).

(B) $\breve\kappa = 0$. In this case, we assume that $W''(R) > 0$, which is ensured by the condition

$$\frac{2}{R^2} - \frac{n''(R)}{n(R)} > 0 \quad \text{with} \quad \frac{1}{R} + \frac{n'(R)}{n(R)} = 0. \tag{3.7}$$

Then W has a local minimum at R. In a two-sided neighborhood of R, W is tangent to a *half-quadratic potential well*, see Fig. 3 (B).

(C) $\breve\kappa < 0$. Then W has no local minimum at R. But, since $\lim_{r\to 0^+} W(r) = +\infty$, it has at least one local interior minimum R_0 over $(0, R)$. Now we assume that $W''(R_0) > 0$, which is ensured by the condition

$$\frac{2}{R_0^2} - \frac{n''(R_0)}{n(R_0)} > 0 \quad \text{with} \quad \frac{1}{R_0} + \frac{n'(R_0)}{n(R_0)} = 0. \tag{3.8}$$

Then W has a local non-degenerate minimum at R_0 where it is tangent to a *quadratic potential well*, see Fig. 3 (C).

4. CASE (A) HALF-TRIANGULAR POTENTIAL WELL

The case $\breve\kappa > 0$ is in a certain sense the most canonical one, since it includes constant optical indices $n \equiv n_0$ inside Ω. In this section, after stating the result, we perform the details of construction of families of resonance quasi-pairs.

4.1. Statements.

Recall that Assumption 1.1 is supposed to hold and Notation 1.2 is in use. We give now, in the case when $\breve\kappa$ is positive, the complete description of the quasi-pairs that we construct in the rest of this section. This statement has to be combined with Theorem 7.D to imply Theorem 1.A.

Theorem 4.A. *Choose $p \in \{\pm 1\}$. If $\breve\kappa > 0$, there exists for each natural integer j, a family of resonance quasi-pairs $\mathfrak{F}_{p;j} = (\mathfrak{K}_{p;j}, \mathfrak{U}_{p;j})$ of whispering gallery type (cf Definition 2.1) for which the sequence of numbers $\mathfrak{K}_{p;j} = (\underline{k}_{p;j}(m))_{m\geq 1}$ and the sequence of functions $\mathfrak{U}_{p;j} = (\underline{u}_{p;j}(m))_{m\geq 1}$ have the following properties:*

(i) The regularity property (2.3)–(2.4) with respect to m holds with $\beta = \frac{1}{3}$: There exist coefficients $\mathrm{K}^{\ell}_{p;j}$ for any $\ell \in \mathbb{N}$, and constants C_N such that

$$\forall N \geq 1, \qquad \left| \frac{\underline{k}_{p;j}(m)}{m} - \sum_{\ell=0}^{N-1} \mathrm{K}^{\ell}_{p;j}\, m^{-\frac{\ell}{3}} \right| \leq C_N\, m^{-N/3}. \tag{4.1}$$

The coefficients $\mathrm{K}^0_{p;j}$ (degree 0) are all equal to $\frac{1}{Rn_0}$, the coefficients of degree 1 are zero, and the coefficients of degree 2 are all distinct with j, see (4.5).

(ii) The functions $\underline{u}_{p;j}(m)$ forming the sequence $\mathfrak{U}_{p;j}$ have the form

$$\underline{u}_{p;j}(m;x,y) = \underline{w}_{p;j}(m;r)\,\mathrm{e}^{\mathrm{i}m\theta} \tag{4.2}$$

where the radial functions $\underline{w}_{p;j}(m)$ have a boundary layer structure around $r = R$ with different scaled variables σ as $r < R$ and ρ as $r > R$:

$$\sigma = m^{\frac{2}{3}}\left(\frac{r}{R}-1\right) \quad \textit{if} \quad r < R \quad \textit{and} \quad \rho = m\left(\frac{r}{R}-1\right) \quad \textit{if} \quad r > R. \tag{4.3}$$

This means that there exist smooth functions $\Phi_{p;j} \in \mathscr{C}^\infty([0,1],\mathscr{S}(\mathbb{R}_-)) : (t,\sigma) \mapsto \Phi_{p;j}(t,\sigma)$ and $\Psi_{p;j} \in \mathscr{C}^\infty([0,1],\mathscr{S}(\mathbb{R}_+)) : (t,\rho) \mapsto \Psi_{p;j}(t,\rho)$ such that

$$\underline{w}_{p;j}(m;r) = \mathfrak{X}(r)\Big(\mathbb{1}_{r<R}(r)\,\Phi_{p;j}(m^{-\frac{1}{3}},\sigma) + \mathbb{1}_{r>R}(r)\,\Psi_{p;j}(m^{-\frac{1}{3}},\rho)\Big) \tag{4.4}$$

where $\mathfrak{X} \in \mathscr{C}_0^\infty(\mathbb{R}_+)$, $\mathfrak{X} \equiv 1$ in a neighborhood of R.

The first terms of the expansions of the quasi-pairs $(\underline{k}_{p;j}(m), \underline{w}_{p;j}(m))$ in powers of $m^{\frac{1}{3}}$ are given below.

4.1.1. *Resonances.* The asymptotics of $\underline{k}_{p;j}(m)$ starts as

$$\underline{k}_{p;j}(m) = \frac{m}{Rn_0}\left[1 + \frac{\mathsf{a}_j}{2}\left(\frac{2\breve{\kappa}}{m}\right)^{\frac{2}{3}} - \frac{n_0^p}{2\sqrt{n_0^2-1}}\left(\frac{2\breve{\kappa}}{m}\right)\right.$$
$$\left. + \mathrm{k}^4_{p;j}\left(\frac{2\breve{\kappa}}{m}\right)^{\frac{4}{3}} + \mathrm{k}^5_{p;j}\left(\frac{2\breve{\kappa}}{m}\right)^{\frac{5}{3}} + \mathscr{O}\left(m^{-2}\right)\right] \tag{4.5}$$

where, as before, the a_j are the successive roots of the flipped Airy function and the coefficients $\mathrm{k}^4_{p;j}$ and $\mathrm{k}^5_{p;j}$ are given by

$$\mathrm{k}^4_{p;j} = \frac{\mathsf{a}_j^2}{15}\left(\frac{17}{8} - \frac{3}{\breve{\kappa}} + \frac{1}{\breve{\kappa}^2}\left(2 - \frac{R^2 n_2}{n_0}\right)\right),$$

$$\mathrm{k}^5_{p;j} = -\frac{\mathsf{a}_j\,n_0^p}{12\sqrt{n_0^2-1}}\left(\frac{3n_0^2 - 2n_0^{2p}}{n_0^2-1} + 2 - \frac{6}{\breve{\kappa}} + \frac{2}{\breve{\kappa}^2}\left(2 - \frac{R^2 n_2}{n_0}\right)\right).$$

Remark 4.1. Note that the second term of (4.5) separates the families $\mathfrak{F}_{p;j}$, while the third term distinguishes the TM ($p = 1$) and TE ($p = -1$) modes.

4.1.2. *Modes.* The asymptotic expansions of the radial part of the quasi-modes $\underline{w}_{p;j}(m)$ in (4.2) starts as

$$\underline{w}_{p;j}(m;r) = \mathfrak{X}(r)\left(\mathrm{w}^0_{p;j}(m;r) + \left(\frac{1}{m}\right)^{\frac{1}{3}}\mathrm{w}^1_{p;j}(m;r)\right) + \mathscr{O}\left(m^{-\frac{2}{3}}\right), \tag{4.6}$$

where, using the scaled variables $\sigma = m^{\frac{2}{3}}(\frac{r}{R}-1)$ and $\rho = m(\frac{r}{R}-1)$

$$\mathrm{w}^0_{p;j}(m;r) = \begin{cases} \mathsf{A}\left(\mathsf{a}_j + (2\breve{\kappa})^{\frac{1}{3}}\sigma\right) & \text{if } r < R, \\ 0 & \text{if } r \geq R, \end{cases} \tag{4.7}$$

and

$$\mathrm{w}^1_{p;j}(m;r) = \frac{-n_0^p\,(2\breve{\kappa})^{\frac{1}{3}}}{\sqrt{n_0^2-1}}\begin{cases} \mathsf{A}'\big(\mathsf{a}_j + (2\breve{\kappa})^{\frac{1}{3}}\sigma\big) & \text{if } r < R, \\ \mathsf{A}'(\mathsf{a}_j)\exp\big(-\frac{\sqrt{n_0^2-1}}{n_0}\,\rho\big) & \text{if } r \geq R\,. \end{cases} \tag{4.8}$$

4.1.3. *Special case of a constant optical index.* In the constant index case $n \equiv n_0 > 1$, we have $\breve{\kappa} = 1$ and we find the following 8-term expansion of the resonances for the circular cavity. For an improved readability, we distinguish the TM and TE case and we denote $\underline{k}_{p;j}$ by $\underline{k}_j^{\mathsf{TM}}$ if $p = 1$ and by $\underline{k}_j^{\mathsf{TE}}$ if $p = -1$. We have

$$
\begin{aligned}
\underline{k}_j^{\mathsf{TM}}(m) = \frac{m}{Rn_0}\Bigg[1 &+ \frac{\mathsf{a}_j}{2}\left(\frac{2}{m}\right)^{\frac{2}{3}} - \frac{n_0}{2(n_0^2-1)^{\frac12}}\left(\frac{2}{m}\right) + \frac{3\,\mathsf{a}_j^2}{40}\left(\frac{2}{m}\right)^{\frac43} \\
&- \frac{\mathsf{a}_j\,n_0^3}{12(n_0^2-1)^{\frac32}}\left(\frac{2}{m}\right)^{\frac53} + \frac{10-\mathsf{a}_j^3}{2800}\left(\frac{2}{m}\right)^{2} + \frac{\mathsf{a}_j^2\,n_0^2\,(n_0^2-4)}{80(n_0^2-1)^{\frac52}}\left(\frac{2}{m}\right)^{\frac73} \\
&- \frac{\mathsf{a}_j}{144}\left(\frac{1}{175} + \frac{479\,\mathsf{a}_j^3}{7000} + \frac{2n_0^6}{(n_0^2-1)^3}\right)\left(\frac{2}{m}\right)^{\frac83} + \mathscr{O}\left(m^{-3}\right)\Bigg]
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
\underline{k}_j^{\mathsf{TE}}(m) = \frac{m}{Rn_0}\Bigg[1 &+ \frac{\mathsf{a}_j}{2}\left(\frac{2}{m}\right)^{\frac{2}{3}} - \frac{1}{2n_0(n_0^2-1)^{\frac12}}\left(\frac{2}{m}\right) + \frac{3\,\mathsf{a}_j^2}{40}\left(\frac{2}{m}\right)^{\frac43} \\
&- \frac{\mathsf{a}_j\,(3n_0^2-2)}{12n_0^3(n_0^2-1)^{\frac32}}\left(\frac{2}{m}\right)^{\frac53} + \frac{1}{8}\left(\frac{1}{35} - \frac{\mathsf{a}_j^3}{350} + \frac{1}{n_0^4(n_0^2-1)^2}\right)\left(\frac{2}{m}\right)^{2} \\
&+ \frac{\mathsf{a}_j^2\,(3n_0^8+12n_0^6-12n_0^4-8n_0^2+8)}{80n_0^5(n_0^2-1)^{\frac52}}\left(\frac{2}{m}\right)^{\frac73} \\
&- \frac{\mathsf{a}_j}{144}\left(\frac{1}{175} + \frac{479\,\mathsf{a}_j^3}{7000} + \frac{18n_0^8-45n_0^6+12n_0^4+45n_0^2-28}{n_0^6(n_0^2-1)^3}\right)\left(\frac{2}{m}\right)^{\frac83} \\
&+ \mathscr{O}\left(m^{-3}\right)\Bigg]
\end{aligned}
\tag{4.10}
$$

4.2. **Proof: General concepts.** As explained in Sect. 3, the problem under consideration has the form (3.2) of the semi-classical Schrödinger equation $-h^2\mathscr{H}w + Ww = \Lambda w$, where $\mathscr{H}$ is a modified Laplacian, W is a potential, discontinuous at the interface $r = R$, and $h = \frac{1}{m}$ is the semiclassical parameter. Recall that in both cases (A) and (B), the potential W has a local minimum at R, with the distinctive feature that for $r < R$, the shape of W is triangular in case (A), and quadratic in case (B). The rationale of the quasi-resonance construction is to localize equation around the well bottom $r = R$ and to scale variables appropriately so that equation (3.2) can be solved by a multiscale power expansion. In this section, we describe the general concepts of the proof, common to the two cases (A) and (B).

4.2.1. *Localization around the interface.* The localization starts with the introduction of the dimensionless variable $\xi = \frac{r}{R} - 1 \in (-1, +\infty)$ for which the disk boundary is translated to the origin. Accordingly, we denote by $\tilde{n}$ the optical index function in this new variable, viz

$$\tilde{n} : \xi \mapsto n\big(R(1+\xi)\big)$$

and, for all $q \in \mathbb{N}$, we set $\tilde{n}_q = \tilde{n}^{(q)}(0)$, the q-th derivative of $\tilde{n}$ at 0. Referring to Notation 1.2, we have $\tilde{n}_q = R^q n_q$ and (cf (1.7) and (1.12))

$$\breve{\kappa} = 1 + \frac{\tilde{n}_1}{\tilde{n}_0} \quad \text{and} \quad \breve{\mu} = 2 - \frac{\tilde{n}_2}{\tilde{n}_0}. \tag{4.11}$$

Since $\tilde{n}_0 = n_0$, we will most often use the notation n_0.

The minimum of W at $r = R$ is its left limit $W_0 := \lim_{r\nearrow R} W(r) = (Rn_0)^{-2}$. Using the change of variables $r \mapsto \xi$, we set

$$\mathscr{L}(\xi, \partial_\xi) = R^2 n_0^2 \mathscr{H}(r, \partial_r), \quad V(\xi) = R^2 n_0^2 (W(r) - W_0), \quad \text{and} \quad \widetilde{\Lambda} = R^2 n_0^2 (\Lambda - W_0),$$

so that equation (3.2) is transformed into

$$-h^2 \mathscr{L} v + V v = \widetilde{\Lambda} v, \tag{4.12a}$$

with the new unknown function $v(\xi) = w(R(1+\xi))$. We have

$$\mathscr{L}(\xi, \partial_\xi) = \frac{n_0^2}{\tilde{n}(\xi)^2}\, \partial_\xi^2 + n_0^2 \left(\frac{1}{(1+\xi)\, \tilde{n}(\xi)^2} + (p-1) \frac{\tilde{n}'(\xi)}{\tilde{n}(\xi)^3} \right) \partial_\xi$$

$$V(\xi) = \left(\frac{n_0}{(1+\xi)\, \tilde{n}(\xi)} \right)^2 - 1.$$

Note that the potential V has 0 as local minimum at $\xi = 0$ (well bottom).

The unknown function v satisfies furthermore the following jump condition at $\xi = 0$ deduced from (1.2a)

$$[v]_{\{0\}} = 0 \quad \text{and} \quad \left[\tilde{n}^{p-1}\, \partial_\xi v \right]_{\{0\}} = 0. \tag{4.12b}$$

and the decay conditions in Schwarz spaces when $\xi \to \pm\infty$:

$$v^- := v\big|_{\mathbb{R}_-} \in \mathscr{S}(\mathbb{R}_-) \quad \text{and} \quad v^+ := v\big|_{\mathbb{R}_+} \in \mathscr{S}(\mathbb{R}_+). \tag{4.12c}$$

Our concern now is to construct quasi-resonances and quasi-modes localized around the well bottom $\xi = 0$, solutions to (4.12a)–(4.12c) in an asymptotic sense.

4.2.2. *Principal part of the Schrödinger modal equation.* The structure of quasi-pairs is determined by the *principal part* of problem (4.12a)–(4.12c) defined as:

$$-\, h^2\, \underline{\mathscr{L}}_0 v + \underline{V}_0 v = \underline{\Lambda} v \tag{4.13a}$$

where

(1) The operator $\underline{\mathscr{L}}_0 = (\underline{\mathscr{L}}_0^-, \underline{\mathscr{L}}_0^+) = (\partial_\xi^2, n_0^2 \partial_\xi^2)$ is the principal part of $\mathscr{L}$ frozen at $\xi = 0$ on the left and on the right,

(2) The associated jump conditions are

$$v^-(0) = v^+(0) \qquad \text{and} \qquad n_0^{p-1}\, \partial_\xi v^-(0) = \partial_\xi v^+(0). \tag{4.13b}$$

and the decay condition is the same as above

$$v^- \in \mathscr{S}(\mathbb{R}_-) \quad \text{and} \quad v^+ \in \mathscr{S}(\mathbb{R}_+). \tag{4.13c}$$

(3) The potential $\underline{V}_0 = (\underline{V}_0^-, \underline{V}_0^+)$ is the first nonzero term in the left and right Taylor expansions of V at $\xi = 0$. In any of the cases (A) and (B), $\underline{V}_0^+ = n_0^2 - 1 > 0$, whereas

$$\underline{V}_0^-(\xi) = \begin{cases} -2\breve{\kappa}\, \xi & \text{in case (A)} \\ \breve{\mu}\, \xi^2 & \text{in case (B).} \end{cases}$$

In order to cover both cases (A) and (B) in a unified way, we will assume more generally that

$$\underline{V}_0^-(\xi) = \gamma |\xi|^{\varkappa}, \quad \xi < 0, \quad \text{with} \quad \gamma > 0,\ \varkappa > 0 \tag{4.13d}$$

The system (4.13a)–(4.13c) can be solved by a formal series expansion according to the following procedure:

(i) Scale the variable ξ differently on the left and on the right of the origin, introducing

$$\sigma = \xi/h^{\alpha} \text{ for } \xi < 0 \quad \text{and} \quad \rho = \xi/h^{\alpha'} \text{ for } \xi > 0 \tag{4.14}$$

with α and $\alpha' > 0$ chosen in order to homogenize the operators $-h^2\underline{\mathscr{L}}_0^- + \underline{V}_0^-$ and $-h^2\underline{\mathscr{L}}_0^+ + \underline{V}_0^+$. We find that $-h^2\underline{\mathscr{L}}_0^- + \underline{V}_0^-$ becomes $-h^{2-2\alpha}\partial_\sigma^2 + \gamma h^{\varkappa\alpha}|\sigma|^{\varkappa}$, and $-h^2\underline{\mathscr{L}}_0^+ + \underline{V}_0^+$ becomes $-h^{2-2\alpha'}\partial_\rho^2 + n_0^2 - 1$, implying to choose

$$\alpha = \frac{2}{2+\varkappa} \quad \text{and} \quad \alpha' = 1. \tag{4.15}$$

(ii) Expand the new functions

$$\varphi(\sigma) := v(\xi) \text{ for } \xi < 0 \quad \text{and} \quad \psi(\rho) := v(\xi) \text{ for } \xi > 0, \tag{4.16}$$

and $\underline{\Lambda}$ in series of type $\sum_{q\in\mathbb{N}} a_q h^{q\beta}$ for some suitable $\beta > 0$. The jump condition (4.12b) being transformed into the following matching condition at $\sigma = \rho = 0$

$$\varphi(0) = \psi(0) \quad \text{and} \quad n_0^{p-1}\, h^{-\alpha}\partial_\sigma\varphi(0) = h^{-\alpha'}\partial_\rho\psi(0), \tag{4.17}$$

we find that α, α', and $\alpha' - \alpha$ should be integer multiples of β.

4.2.3. *Back to the full Schrödinger modal equation.* Now, we take advantage of the choices made in (i)–(ii) to treat the system (4.12a)–(4.12c) in its general form. Hence we know that $\alpha' = 1$ and leave α in equations for further determination. By the change of variables (4.14)–(4.15) and the change of functions (4.16), the equation (4.12a) is transformed into the following two equations set on each side of the interface $\sigma = \rho = 0$

$$\begin{cases} h^{2-2\alpha}(-\mathscr{L}_h^-\varphi + V_h^-\varphi) &= \widetilde{\Lambda}\varphi, \qquad \sigma \in (-\infty, 0) \\ -\mathscr{L}_h^+\psi + V_h^+\psi &= \widetilde{\Lambda}\psi, \qquad \rho \in (0, +\infty) \end{cases} \tag{4.18a}$$

with the matching condition

$$\varphi(0) = \psi(0) \quad \text{and} \quad n_0^{p-1}\, h^{1-\alpha}\partial_\sigma\varphi(0) = \partial_\rho\psi(0), \tag{4.18b}$$

the decay condition

$$\varphi \in \mathscr{S}(\mathbb{R}_-) \quad \text{and} \quad \psi \in \mathscr{S}(\mathbb{R}_+), \tag{4.18c}$$

and where the operators $\mathscr{L}_h^-$ and $\mathscr{L}_h^+$ are defined by

$$\begin{aligned} \mathscr{L}_h^- &= \frac{n_0^2}{\tilde{n}(h^\alpha\sigma)^2}\partial_\sigma^2 + h^\alpha n_0^2 \left(\frac{1}{(1+h^\alpha\sigma)\,\tilde{n}(h^\alpha\sigma)^2} + (p-1)\frac{\tilde{n}'(h^\alpha\sigma)}{\tilde{n}(h^\alpha\sigma)^3} \right)\partial_\sigma \\ \mathscr{L}_h^+ &= n_0^2\left(\partial_\rho^2 + \frac{h}{1+h\rho}\partial_\rho\right) \end{aligned} \tag{4.19}$$

and the potentials V_h^- and V_h^+ are given by

$$\begin{aligned} V_h^-(\sigma) &= h^{2\alpha-2}\left(\left(\frac{n_0}{(1+h^\alpha\sigma)\,\tilde{n}(h^\alpha\sigma)}\right)^2 - 1\right), \\ V_h^+(\rho) &= \frac{n_0^2}{(1+h\rho)^2} - 1. \end{aligned} \tag{4.20}$$

4.2.4. *Formal series of operators.* The next step is to associate a formal series of operators to the system (4.18a)–(4.18c), using a Taylor expansion at $\sigma = \rho = 0$ of their coefficients: For any smooth coefficient f, this association reads

$$f(h^\alpha\sigma) \sim \sum_{\ell\in\mathbb{N}} h^{\alpha\ell}\,\frac{f^{(\ell)}(0)}{\ell!}\,\sigma^\ell \quad\text{and}\quad f(h\rho) \sim \sum_{\ell\in\mathbb{N}} h^{\ell}\,\frac{f^{(\ell)}(0)}{\ell!}\,\rho^\ell.$$

This defines a formal series of operators in terms of powers of h^β

$$-\mathscr{L}_h^\pm + V_h^\pm \sim \sum_{q\in\mathbb{N}} h^{q\beta}\,\mathsf{A}_q^\pm \tag{4.21}$$

inviting to look for φ, ψ, and $\widetilde{\Lambda}$ in the form of the formal series

$$\varphi(\sigma) = \sum_{q\in\mathbb{N}} h^{q\beta}\,\varphi_q(\sigma), \quad \psi(\rho) = \sum_{q\in\mathbb{N}} h^{q\beta}\,\psi_q(\rho), \quad\text{and}\quad \widetilde{\Lambda} = \sum_{q\in\mathbb{N}} h^{q\beta}\,\widetilde{\Lambda}_q. \tag{4.22}$$

Note that in the general framework (4.13d) we have

$$\mathsf{A}_0^- = -\partial_\sigma^2 + \gamma|\sigma|^\varkappa \quad\text{and}\quad \mathsf{A}_0^+ = -n_0^2\partial_\rho^2 + n_0^2 - 1. \tag{4.23}$$

Finally, since we want to construct quasi-modes with a whispering gallery structure, we give priority in (4.18a) to the equation in $\mathbb{R}_-$, which means that we look for an expansion of $\widetilde{\Lambda}$ starting as $h^{2-2\alpha}$. This motivates the introduction of

$$\lambda = h^{2\alpha-2}\widetilde{\Lambda} \sim \sum_{q\in\mathbb{N}} h^{q\beta}\lambda_q\,, \tag{4.24}$$

so that Equations (4.18a) read

$$\begin{cases} -\mathscr{L}_h^-\varphi + V_h^-\varphi &= \lambda\varphi, & \sigma\in(-\infty,0) \\ -\mathscr{L}_h^+\psi + V_h^+\psi &= h^{2-2\alpha}\lambda\psi, & \rho\in(0,+\infty) \end{cases} \tag{4.25}$$

still coupled with the matching condition (4.18b) and the decay condition (4.18c).

4.3. **Proof: Specifics in case (A).** In case (A), $\breve{\kappa}$ is positive and the above general framework applies with the quantities

$$\varkappa = 1, \quad \gamma = 2\breve{\kappa}, \quad \alpha = \tfrac{2}{3}, \quad \alpha' = 1, \quad \beta = \tfrac{1}{3}.$$

This case is very close to the "toy model" considered in [5, Sec. III]. From expressions (4.19)–(4.21), we find that the first terms of the operator series $\mathsf{A}_q^\pm$ are as follows

$$\begin{cases} \mathsf{A}_0^- = -\partial_\sigma^2 + 2\breve{\kappa}|\sigma|, \quad \mathsf{A}_1^- = 0, \quad \mathsf{A}_2^- = 2\frac{\tilde{n}_1}{n_0}\sigma\,\partial_\sigma^2 - \Big(1 + (p-1)\frac{\tilde{n}_1}{n_0}\Big)\,\partial_\sigma + c_2^-\sigma^2, & (4.26) \\ \mathsf{A}_0^+ = -n_0^2\partial_\rho^2 + n_0^2 - 1, \quad \mathsf{A}_1^+ = 0, \quad \mathsf{A}_2^+ = 0, \quad \mathsf{A}_3^+ = -n_0^2(\partial_\rho + 2\rho), & (4.27) \end{cases}$$

where $c_2^- = 3 + 4\frac{\tilde{n}_1}{n_0} + 3\frac{\tilde{n}_1^2}{n_0^2} - \frac{\tilde{n}_2}{n_0}$. For a comprehensive description of the general terms $\mathsf{A}_q^\pm$ we need the introduction of polynomial spaces.

Notation 4.2. For $q\in\mathbb{N}$, let $\mathbb{P}^q$ denote the space of polynomials in one variable with degree $\leq q$ and $\mathbb{P}_*^q$ the subspace of $\mathbb{P}^q$ formed by polynomials P such that $P(0) = 0$.

A Taylor expansion at $\sigma = \rho = 0$ of the coefficients of $\mathscr{L}_h^\pm$ and of $V_h^\pm$ allows to prove that

Lemma 4.B. *For any integer $q \geq 1$, there holds*

$$\begin{aligned} &\mathsf{A}_q^- = A_q^-(\sigma)\,\partial_\sigma^2 + B_q^-(\sigma)\,\partial_\sigma + C_q^-(\sigma) \quad &&\textit{with}\ \ A_q^- \in \mathbb{P}^{[\frac{q}{2}]},\ \ B_q^- \in \mathbb{P}^{[\frac{q}{2}]-1},\ \ C_q^- \in \mathbb{P}^{[\frac{q}{2}]+1} \\ &\mathsf{A}_q^+ = B_q^+(\rho)\,\partial_\rho + C_q^+(\rho) &&\textit{with}\ \ B_q^+ \in \mathbb{P}^{[\frac{q}{3}]-1},\ \ C_q^+ \in \mathbb{P}^{[\frac{q}{3}]}. \end{aligned}$$

By the identifications (4.21)–(4.22), the system (4.25) with jump conditions (4.18b) is associated with the formal series system of equations

$$\begin{cases} \left(\sum_{\ell\in\mathbb{N}} h^{\frac{\ell}{3}}\,\mathsf{A}^-_\ell\right)\left(\sum_{\ell\in\mathbb{N}} h^{\frac{\ell}{3}}\,\varphi_\ell\right) &= \left(\sum_{\ell\in\mathbb{N}} h^{\frac{\ell}{3}}\,\lambda_\ell\right)\left(\sum_{\ell\in\mathbb{N}} h^{\frac{\ell}{3}}\,\varphi_\ell\right) & \text{in } \mathbb{R}_- \\ \left(\sum_{\ell\in\mathbb{N}} h^{\frac{\ell}{3}}\,\mathsf{A}^+_\ell\right)\left(\sum_{\ell\in\mathbb{N}} h^{\frac{\ell}{3}}\,\psi_\ell\right) &= h^{\frac{2}{3}}\left(\sum_{\ell\in\mathbb{N}} h^{\frac{\ell}{3}}\,\lambda_\ell\right)\left(\sum_{\ell\in\mathbb{N}} h^{\frac{\ell}{3}}\,\psi_\ell\right) & \text{in } \mathbb{R}_+ \\ \left(\sum_{\ell\in\mathbb{N}} h^{\frac{\ell}{3}}\,\varphi_\ell\right) &= \left(\sum_{\ell\in\mathbb{N}} h^{\frac{\ell}{3}}\,\psi_\ell\right) & \text{at } \{0\} \\ n_0^{p-1}\,h^{\frac{1}{3}}\left(\sum_{\ell\in\mathbb{N}} h^{\frac{\ell}{3}}\,\varphi'_\ell\right) &= \left(\sum_{\ell\in\mathbb{N}} h^{\frac{\ell}{3}}\,\psi'_\ell\right) & \text{at } \{0\} \end{cases} \tag{4.28}$$

This system is equivalent to an infinite collection of systems obtained by equating the series coefficients: Namely, for q spanning $\mathbb{N}$,

$$\begin{cases} \sum_{\ell=0}^{q} \mathsf{A}^-_\ell\,\varphi_{q-\ell} &= \sum_{\ell=0}^{q} \lambda_\ell\,\varphi_{q-\ell} & \text{in } \mathbb{R}_- \\ \sum_{\ell=0}^{q} \mathsf{A}^+_\ell\,\psi_{q-\ell} &= \sum_{\ell=2}^{q} \lambda_{\ell-2}\,\psi_{q-\ell} & \text{in } \mathbb{R}_+ \\ \varphi_q(0) &= \psi_q(0) \\ \psi'_q(0) &= n_0^{p-1}\,\varphi'_{q-1}(0) \end{cases} \tag{4.29}$$

where we agree that the right hand side of the second line is 0 when $q = 0$ or $q = 1$, and, likewise, the right hand side of the fourth line is 0 when $q = 0$.

4.3.1. *Initialization stage.* For $q = 0$, the system (4.29) reads

$$\begin{cases} \mathsf{A}^-_0\,\varphi_0 &= \lambda_0\varphi_0 & \text{in } \mathbb{R}_- \\ \mathsf{A}^+_0\,\psi_0 &= 0 & \text{in } \mathbb{R}_+ \\ \varphi_0(0) &= \psi_0(0) \\ \psi'_0(0) &= 0 \end{cases} \tag{4.30}$$

for which we look for solutions $\varphi_0 \in \mathscr{S}(\mathbb{R}_-)$ and $\psi_0 \in \mathscr{S}(\mathbb{R}_+)$.

Since the equation $\mathsf{A}^+_0\,\psi_0 = 0$ with the Neumann condition at 0 has no non-zero solution in $\mathscr{S}(\mathbb{R}_+)$, it is natural to take $\psi_0 = 0$ in (4.30). Then we are left with the following Airy eigen-problem on $\mathbb{R}_-$ for φ_0

$$-\varphi''_0(\sigma) - 2\breve{\kappa}\,\sigma\varphi_0(\sigma) = \lambda_0\varphi_0(\sigma) \ \text{ for } \sigma \in (-\infty, 0), \qquad \text{and} \qquad \varphi_0(0) = 0$$

whose decaying solutions can be expressed in terms of the mirror Airy function A. Recall that a_j for $j \in \mathbb{N}$, denote the successive roots of A. We obtain immediately:

Lemma 4.C. *Let $j \in \mathbb{N}$. The couple of functions (φ_0, ψ_0) and the number λ_0 defined by*

$$\varphi_0(\sigma) = \mathsf{A}\big(\mathsf{a}_j + (2\breve{\kappa})^{\frac{1}{3}}\sigma\big), \quad \psi_0(\rho) = 0, \quad \textit{and} \quad \lambda_0 = \mathsf{a}_j(2\breve{\kappa})^{\frac{2}{3}}$$

solve (4.30) in $\mathscr{S}(\mathbb{R}_-) \times \mathscr{S}(\mathbb{R}_+)$.

Remark 4.3. The quasi-mode construction requires a cut-off at infinity at some stage. Such a cut-off will be harmless to the satisfied equations if the functions (φ_0, ψ_0), and more generally (φ_q, ψ_q), are exponentially decreasing when $\sigma \to -\infty$ and $\rho \to +\infty$. It is easy to see that any such solution of (4.30) is proportional to one of the solutions given in Lemma 4.C.

4.3.2. *Sequence of nested problems and recurrence.* Reordering the terms in the system (4.29) of rank q, we can write it in the following form

$$(\mathscr{R}_q^{(\mathrm{A})})\quad \begin{cases} -\varphi_q''(\sigma) - (2\breve{\kappa}\sigma + \lambda_0)\varphi_q(\sigma) = \lambda_q\,\varphi_0(\sigma) + S_q^{\varphi}(\sigma) & \sigma \in \mathbb{R}_- \quad (4.31a)\\ -n_0^2\psi_q''(\rho) + \big(n_0^2 - 1\big)\,\psi_q(\rho) = S_q^{\psi}(\rho) & \rho \in \mathbb{R}_+ \quad (4.31b)\\ \varphi_q(0) = \psi_q(0) & (4.31c)\\ \psi_q'(0) = n_0^{p-1}\varphi_{q-1}'(0) & (4.31d) \end{cases}$$

with right hand terms S_q^{φ} and S_q^{ψ} defined as (recall that $\mathbf{A}_1^+ = 0$)

$$S_q^{\varphi} = -\mathbf{A}_q^-\,\varphi_0 + \sum_{\ell=1}^{q-1}(\lambda_\ell - \mathbf{A}_\ell^-)\,\varphi_{q-\ell} \quad\text{and}\quad S_q^{\psi} = \sum_{\ell=2}^{q}(\lambda_{\ell-2} - \mathbf{A}_\ell^+)\,\psi_{q-\ell}. \tag{4.32}$$

Proposition 4.D. *Choose $j \in \mathbb{N}$ and define $(\varphi_0, \psi_0, \lambda_0)$ according to Lemma 4.C. Then there exist, for any $q \geq 1$,*

- *a unique $\lambda_q \in \mathbb{R}$*
- *unique polynomials $P_q^{\varphi} \in \mathbb{P}_*^q$, $Q_q^{\varphi} \in \mathbb{P}^{q-1}$, and $P_q^{\psi} \in \mathbb{P}^{q-1}$*

such that setting

$$\begin{aligned} \varphi_q(\sigma) &= P_q^{\varphi}(\sigma)\mathsf{A}\big(\mathsf{a}_j + (2\breve{\kappa})^{\frac13}\sigma\big) + Q_q^{\varphi}(\sigma)\mathsf{A}'\big(\mathsf{a}_j + (2\breve{\kappa})^{\frac13}\sigma\big) \quad \forall\sigma \in \mathbb{R}_-\\ \psi_q(\rho) &= P_q^{\psi}(\rho)\exp\big(-\rho\sqrt{1-n_0^{-2}}\big) \quad \forall\rho \in \mathbb{R}_+ \end{aligned} \tag{4.33}$$

the collection $(\varphi_0, \dots, \varphi_q, \psi_0, \dots, \psi_q, \lambda_0, \dots, \lambda_q)$ solves the sequence of problems $(\mathscr{R}_\ell^{(\mathrm{A})})$ introduced in (4.31) for $\ell = 0, \dots, q$.

Proof. We proceed by induction on q. For $q = 0$, lemma 4.C provides λ_0, φ_0, and ψ_0 solutions to $(\mathscr{R}_0^{(\mathrm{A})})$ and we readily obtain the polynomials $P_0^{\varphi} = 1$, $Q_0^{\varphi} = 0$, and $P_0^{\psi} = 0$. Let $q \geq 1$ and suppose that $(\lambda_\ell)_{0\leq\ell\leq q-1}$, $(\varphi_\ell)_{0\leq\ell\leq q-1}$, and $(\psi_\ell)_{0\leq\ell\leq q-1}$ are solutions to problems $(\mathscr{R}_\ell^{(\mathrm{A})})$ for $\ell = 0, \dots, q-1$, and satisfy (4.33).

Using the expression (4.32) of S_q^{ψ} combined with Lemma 4.B, we deduce from the induction assumption that there exists a polynomial $E_q^{\psi} \in \mathbb{P}^{q-2}$ such that

$$S_q^{\psi}(\rho) = E_q^{\psi}(\rho)\exp\big(-\rho\sqrt{1-n_0^{-2}}\big).$$

From Lemma A.1 in Appendix, there exists a unique polynomial $\widetilde{P}_q^{\psi} \in \mathbb{P}_*^{q-1}$ such that the function $\widetilde{\psi}$ defined by $\widetilde{\psi}_q(\rho) = \widetilde{P}_q^{\psi}(\rho)\exp(-\rho\sqrt{1-n_0^{-2}})$ is solution to (4.31b). It follows that the sought function ψ_q is given by

$$\psi_q(\rho) = \big(a_0 + \widetilde{P}_q^{\psi}(\rho)\big)\exp\big(-\rho\sqrt{1-n_0^{-2}}\big)$$

where the constant a_0 is determined from Neumann condition (4.31d). This defines the polynomial P_q^{ψ} as $a_0 + \widetilde{P}_q^{\psi}$ and hence $P_q^{\psi} \in \mathbb{P}^{q-1}$ as desired.

Let us now consider equation (4.31a). Using Lemma 4.B combined with the relation $\mathsf{A}''(z) = -z\mathsf{A}(z)$, we deduce from the expression (4.32) of S_q^{φ} and the induction assumption that there exist polynomials $R_q^{\psi} \in \mathbb{P}^q$ and $T_q^{\varphi} \in \mathbb{P}^{q-1}$ such that

$$S_q^{\varphi}(\sigma) = R_q^{\varphi}(\sigma)\mathsf{A}\big(\mathsf{a}_j + (2\breve{\kappa})^{\frac13}\sigma\big) + T_q^{\varphi}(\sigma)\mathsf{A}'\big(\mathsf{a}_j + (2\breve{\kappa})^{\frac13}\sigma\big).$$

From Lemma A.2 there exist unique polynomials $P_q^{\varphi} \in \mathbb{P}_*^q$ and $\widetilde{Q}_q^{\varphi} \in \mathbb{P}^{q-1}$ such that the function given by

$$\widetilde{\varphi}_q(\sigma) = P_q^{\varphi}(\sigma)\mathsf{A}\big(\mathsf{a}_j + (2\breve{\kappa})^{\frac13}\sigma\big) + \widetilde{Q}_q^{\varphi}(\sigma)\mathsf{A}'\big(\mathsf{a}_j + (2\breve{\kappa})^{\frac13}\sigma\big)$$

is a solution to the ODE (compare it with (4.31a))

$$-\widetilde{\varphi}_q''(\sigma) - (2\breve{\kappa}\sigma + \lambda_0)\widetilde{\varphi}_q(\sigma) = S_q^{\varphi}(\sigma)\,.$$

If we define φ_q as follows:

$$\varphi_q(\sigma) = \frac{\lambda_q}{(2\breve{\kappa})^{\frac{2}{3}}}\mathsf{A}'\big(\mathsf{a}_j + (2\breve{\kappa})^{\frac{1}{3}}\sigma\big) + \widetilde{\varphi}_q(\sigma) \quad \text{with} \quad \lambda_q = \frac{(2\breve{\kappa})^{\frac{2}{3}}}{\mathsf{A}'(\mathsf{a}_j)}\left(\psi_q(0) - \widetilde{\varphi}_q(0)\right),$$

then φ_q solves (4.31a) and the continuity condition (4.31c). (Note that we have $\mathsf{A}'(\mathsf{a}_j) \neq 0$ for all $j \in \mathbb{N}$, see [21, Sect. 9.9(ii)].) Finally, we set $Q_q^{\varphi} = \lambda_q(2\breve{\kappa})^{-\frac{2}{3}} + \widetilde{Q}_q^{\varphi} \in \mathbb{P}^{q-1}$. □

Calculating for $q = 1$ in Proposition 4.D provides $P_1^{\varphi} = 0$ and explicit values for Q_1^{φ}, P_1^{ψ} and λ_1, from which we deduce

Lemma 4.E. *We have* $\lambda_1 = \dfrac{-n_0^p\, 2\breve{\kappa}}{\sqrt{n_0^2 - 1}}$ *and, for* $(\sigma, \rho) \in \mathbb{R}_- \times \mathbb{R}_+$,

$$\varphi_1(\sigma) = -\frac{n_0^p\,(2\breve{\kappa})^{\frac{1}{3}}}{\sqrt{n_0^2-1}}\mathsf{A}'(\mathsf{a}_j + (2\breve{\kappa})^{\frac{1}{3}}\sigma), \quad \psi_1(\rho) = -\frac{n_0^p\,(2\breve{\kappa})^{\frac{1}{3}}\,\mathsf{A}'(\mathsf{a}_j)}{\sqrt{n_0^2-1}}\exp\left(-\rho\sqrt{1-n_0^{-2}}\right).$$

Remark 4.4. From the proof of Proposition 4.D, we see that, for each chosen j and $q \geq 1$, the four operators

$$\begin{aligned}
\{\lambda_0, \dots, \lambda_{q-1}, P_0^{\psi}, \dots, P_{q-1}^{\psi}\} &\longmapsto E_q^{\psi}\\
\{E_q^{\psi}, P_{q-1}^{\varphi}, Q_{q-1}^{\varphi}\} &\longmapsto P_q^{\psi}\\
\{\lambda_0, \dots, \lambda_{q-1}, P_0^{\varphi}, \dots, P_{q-1}^{\varphi}, Q_0^{\varphi}, \dots, Q_{q-1}^{\varphi}\} &\longmapsto \{R_q^{\varphi}, T_q^{\varphi}\}\\
\{R_q^{\varphi}, T_q^{\varphi}, P_q^{\psi}\} &\longmapsto \{P_q^{\varphi}, Q_q^{\varphi}, \lambda_q\}
\end{aligned}$$

act between finite dimensional spaces and can be identified to matrices. They result into an algorithm that can be derived and implemented in a computer algebra system to obtain the expression of $\lambda_q, \varphi_q, \psi_q$ for $q \geq 2$, see [18, Annexe D]. The coefficients of the polynomials $P_q^{\varphi}, Q_q^{\varphi}, P_q^{\varphi}$ are *rational functions* of the quantities $(2\breve{\kappa})^{\frac{1}{3}}$, $\sqrt{n_0^2-1}$, a_j, $\mathsf{A}'(\mathsf{a}_j)$, n_0^p, and n_ℓ for all $\ell \in \{0, \dots, q\}$.

4.3.3. *Convergence.* Choose $p \in \{\pm 1\}$ and a natural integer j. In a last stage, we have to prove that the formal series

$$\sum_{q\in\mathbb{N}} \lambda_q h^{\frac{q}{3}}, \qquad \sum_{q\in\mathbb{N}} \varphi_q h^{\frac{q}{3}}, \qquad \text{and} \qquad \sum_{q\in\mathbb{N}} \psi_q h^{\frac{q}{3}}, \tag{4.34}$$

obtained from Proposition 4.D give rise to a family of resonance quasi-pairs in the sense of Definition 2.1. Note that, by construction, the functions φ_q and ψ_q are exponentially decreasing at infinity, thus belong to $\mathscr{S}(\mathbb{R}_-)$ and $\mathscr{S}(\mathbb{R}_+)$, respectively. Relying on Borel's theorem [14, Thm. 1.2.6] and its variant given in Lemma A.4 in Appendix, we obtain the existence of smooth functions having $(\lambda_q)_q$, $(\varphi_q)_q$ and $(\psi_q)_q$ as Taylor terms at 0. Combined with Lemma A.5, this yields the following results for the remainders of truncated series expansions of formal series (4.34).

Lemma 4.F. *Let* $(\lambda_q)_{q\in\mathbb{N}}$, $(\varphi_q)_{q\in\mathbb{N}}$ *and* $(\psi_q)_{q\in\mathbb{N}}$ *given by Proposition 4.D. There exist smooth functions* $\underline{\lambda} \in \mathscr{C}^\infty([0,1])$, $\Phi \in \mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_-))$ *and* $\Psi \in \mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_+))$ *such that for all* $(h, \sigma, \rho) \in [0,1] \times \mathbb{R}_- \times \mathbb{R}_+$ *and for all integer* $N \geq 0$, *we have the following finite expansions with remainders*

$$\underline{\lambda}(h^{\frac{1}{3}}) = \sum_{q=0}^{N-1} h^{\frac{q}{3}}\,\lambda_q + h^{\frac{N}{3}} R_N^{\lambda}(h^{\frac{1}{3}}), \qquad \textit{with} \quad R_N^{\lambda} \in \mathscr{C}^\infty([0,1]) \tag{4.35a}$$

$$\Phi(h^{\frac{1}{3}};\sigma) = \sum_{q=0}^{N-1} h^{\frac{q}{3}}\,\varphi_q(\sigma) + h^{\frac{N}{3}} R_N^{\varphi}(h^{\frac{1}{3}};\sigma), \quad \textit{with} \quad R_N^{\varphi} \in \mathscr{C}^\infty([0,1],\mathscr{S}(\mathbb{R}_-)) \tag{4.35b}$$

$$\Psi(h^{\frac{1}{3}};\rho) = \sum_{q=0}^{N-1} h^{\frac{q}{3}}\,\psi_q(\rho) + h^{\frac{N}{3}} R_N^{\psi}(h^{\frac{1}{3}};\rho) \quad \textit{with} \quad R_N^{\psi} \in \mathscr{C}^\infty([0,1],\mathscr{S}(\mathbb{R}_+)) \tag{4.35c}$$

Note that the remainders at rank $N = 0$ simply coincide with the original function.

Definition 4.5. Choose a real number $\delta \in (0, \frac{1}{2})$ and a smooth cut-off function χ, $0 \le \chi \le 1$, such that $\chi(\xi) = 1$ for $|\xi| \le \delta$ and $\chi(\xi) = 0$ for $|\xi| \ge 2\delta$. We define for any integer $m \ge 1$ with the notation $h = m^{-1}$, the quantities:

$$\underline{k}(m) = \frac{m}{Rn_0}\sqrt{1 + h^{\frac{2}{3}}\,\underline{\lambda}(h^{\frac{1}{3}})},$$

$$\underline{v}(m;\xi) = \chi(\xi)\begin{cases} \Phi(h^{\frac{1}{3}}; h^{-\frac{2}{3}}\xi), & \xi \le 0 \\ \Psi(h^{\frac{1}{3}}; h^{-1}\xi), & \xi > 0 \end{cases} \qquad \xi \in (-1,+\infty)$$

$$\underline{u}(m;r,\theta) = \underline{v}\left(m; \tfrac{r}{R} - 1\right) \mathrm{e}^{\mathrm{i}m\theta} \qquad (r,\theta) \in (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}.$$

We now show that the sequence $(\underline{k}(m), \underline{u}(m))_{m\ge1}$ is a family of “almost” quasi-pairs in the sense of the following lemma. A further correction will have to be made to transform this family into a true family of resonance quasi-pairs in the sense of Definition 2.1.

Lemma 4.G. *The sequence $(\underline{k}(m), \underline{u}(m))_{m\ge1}$ defined above has the following properties:*

(i) *For all m, the function $\underline{u}(m)$ is supported in an annulus around the interface $r = R$*

$$\mathrm{supp}(\underline{u}(m)) \subset B(0, R(1+2\delta)) \setminus B(0, R(1-2\delta)).$$

(ii) *For all m, the function $\underline{u}(m)$ is piece-wise smooth up to the interface $r = R$:*

$$\underline{u}(m)\big|_{\overline{\Omega}} \in \mathscr{C}^\infty\left(\overline{\Omega}\right) \quad \textit{and} \quad \underline{u}(m)\big|_{\mathbb{R}^2\setminus\Omega} \in \mathscr{C}^\infty\left(\mathbb{R}^2 \setminus \Omega\right).$$

(iii) *We have the following estimates for the jumps across the interface when $m \to +\infty$*

$$[\underline{u}(m)]_{\partial\Omega} = \mathscr{O}\left(m^{-\infty}\right) \quad \textit{and} \quad \left[n^{p-1}\,\partial_\nu \underline{u}(m)\right]_{\partial\Omega} = \mathscr{O}\left(m^{-\infty}\right).$$

(iv) *Defining the residuals*

$$\underline{\varepsilon}(m) := \mathrm{div}\left(n^{p-1}\nabla \underline{u}(m)\right) + \underline{k}^2(m)\, n^{p+1}\,\underline{u}(m) \tag{4.36}$$

we have the following estimates in Ω and $\mathbb{R}^2 \setminus \overline{\Omega}$ when $m \to +\infty$

$$\frac{\|\underline{\varepsilon}(m)\|_{\mathrm{L}^2(\Omega)} + \|\underline{\varepsilon}(m)\|_{\mathrm{L}^2(\mathbb{R}^2\setminus\overline{\Omega})}}{\|\underline{u}(m)\|_{\mathrm{L}^2(\mathbb{R}^2)}} = \mathscr{O}\left(m^{-\infty}\right).$$

Proof. (i) and (ii) are obvious consequences of the definition of $\underline{u}(m)$.

(iii) From Definition 4.5, we have, for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and with $h = \frac{1}{m}$

$$[\underline{u}(m)]_{\partial\Omega}(\theta) = [\underline{v}(m;\xi)]_{\{\xi=0\}}\,\mathrm{e}^{\mathrm{i}m\theta} = \left(\Psi(h^{\frac{1}{3}};0) - \Phi(h^{\frac{1}{3}};0)\right)\mathrm{e}^{\mathrm{i}m\theta}.$$

Let $N \ge 1$. From (4.35b)–(4.35c), we deduce that

$$\Psi(h^{\frac{1}{3}};0) - \Phi(h^{\frac{1}{3}};0) = \sum_{q=0}^{N-1}\left(\psi_q(0) - \varphi_q(0)\right)h^{\frac{q}{3}} + h^{\frac{N}{3}}\left(R_N^{\psi}(h^{\frac{1}{3}};0) - R_N^{\varphi}(h^{\frac{1}{3}};0)\right). \tag{4.37}$$

Since by construction $\psi_q(0) - \varphi_q(0) = 0$ for any $q \in \mathbb{N}$, cf (4.31), we deduce from (4.37) that $\Psi(h^{\frac{1}{3}};0) - \Phi(h^{\frac{1}{3}};0) = \mathscr{O}(h^{\frac{N}{3}})$. This statement is true for any $N \geq 1$, hence $[\underline{u}(m)]_{\partial\Omega} = \mathscr{O}(m^{-\infty})$.

We proceed in a similar way for the second jump condition. We have

$$\left[n^{p-1}\partial_\nu \underline{u}(m)\right]_{\partial\Omega}(\theta) = R^{-1}\left[n^{p-1}\partial_\xi v(m,\xi)\right]_{\{\xi=0\}} \mathrm{e}^{\mathrm{i}m\theta}$$

and

$$\begin{aligned}\left[n^{p-1}\partial_\xi v(m,\xi)\right]_{\{\xi=0\}} &= h^{-1}\left(\partial_\rho\Psi(h^{\frac{1}{3}};0) - n_0^{p-1}h^{\frac{1}{3}}\partial_\sigma\Phi(h^{\frac{1}{3}};0)\right)\\ &= \sum_{q=1}^{N-1}\left(\psi_q'(0) - n_0^{p-1}\varphi_{q-1}'(0)\right)h^{\frac{q}{3}-1}\\ &\quad + h^{\frac{N}{3}-1}\left(-n_0^{p-1}\varphi_{N-1}'(0) + \partial_\rho R_N^{\psi}(h^{\frac{1}{3}};0) - n_0^{p-1}h^{\frac{1}{3}}\partial_\sigma R_N^{\varphi}(h^{\frac{1}{3}};0)\right).\end{aligned}$$

From the jump relation (4.31d) $\psi_q'(0) - n_0^{p-1}\varphi_{q-1}'(0)$ for any $q \geq 1$, we deduce that the above quantity is a $\mathscr{O}(h^{\frac{N}{3}-1})$ for any $N \geq 1$, hence a $\mathscr{O}(m^{-\infty})$.

(iv) In order to prove the estimates on the residuals, it is enough to prove that the L^2 norm of the residual $\underline{\varepsilon}(m)$ on Ω and on $\mathbb{R}^2 \setminus \overline{\Omega}$ is $\mathscr{O}(m^{-\infty})$ and that

$$\|\underline{u}(m)\|_{\mathrm{L}^2(\mathbb{R}^2)} = \gamma\, m^{-\frac{1}{3}} + \mathscr{O}(m^{-\frac{2}{3}}) \tag{4.38}$$

for some positive constant γ. Given a parameter $t > 0$, we introduce the following weighted L^2 (semi) norm on any interval $I \subset \mathbb{R}$:

$$\|w\|^2_{\mathrm{L}^2[t](I)} = \int_{I\cap(-2\delta/t,\infty)} |w(\tau)|^2\, t(1+\tau t)\,\mathrm{d}\tau.$$

Let us first prove (4.38). We readily obtain from Definition 4.5, having set $L := 2\pi R$,

$$\|\underline{u}(m)\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = L\left(\left\|\chi(\cdot h^{\frac{2}{3}})\Phi(h^{\frac{1}{3}};\cdot)\right\|^2_{\mathrm{L}^2[h^{\frac{2}{3}}](\mathbb{R}_-)} + \left\|\chi(\cdot h)\Psi(h^{\frac{1}{3}};\cdot)\right\|^2_{\mathrm{L}^2[h](\mathbb{R}_+)}\right). \tag{4.39}$$

From (4.35b) and (4.35c) considered with $N = 1$, we have

$$\begin{aligned}\Phi(h^{\frac{1}{3}};\sigma) &= \mathsf{A}(\mathsf{a}_j + (2\breve{\kappa})^{\frac{1}{3}}\sigma) + h^{\frac{1}{3}}R_1^{\varphi}(h^{\frac{1}{3}};\sigma)\\ \Psi(h^{\frac{1}{3}};\rho) &= h^{\frac{1}{3}}R_1^{\psi}(h^{\frac{1}{3}};\rho)\end{aligned}$$

where $R_1^{\varphi} \in \mathscr{C}^\infty([0,1],\mathscr{S}(\mathbb{R}_-))$ and $R_1^{\psi} \in \mathscr{C}^\infty([0,1],\mathscr{S}(\mathbb{R}_+))$. We deduce that

$$\left|\,\|\underline{u}(m)\|_{\mathrm{L}^2(\mathbb{R}^2)} - \sqrt{L}\left\|\chi(\cdot h^{\frac{2}{3}})\mathsf{A}(\mathsf{a}_j + (2\breve{\kappa})^{\frac{1}{3}}\cdot)\right\|_{\mathrm{L}^2[h^{\frac{2}{3}}](\mathbb{R}_-)}\right| \leq C_1 h^{\frac{1}{3}}(h^{\frac{1}{3}} + h^{\frac{1}{2}}) \leq C_1' h^{\frac{2}{3}}$$

for some constants C_1 and C_1'. We now have to estimate the quantity

$$\left\|\chi(\cdot h^{\frac{2}{3}})\mathsf{A}(\mathsf{a}_j + (2\breve{\kappa})^{\frac{1}{3}}\cdot)\right\|^2_{\mathrm{L}^2[h^{\frac{2}{3}}](\mathbb{R}_-)} = \int_{\mathbb{R}_-}\left|\chi(\sigma h^{\frac{2}{3}})\mathsf{A}(\mathsf{a}_j + (2\breve{\kappa})^{\frac{1}{3}}\sigma)\right|^2 h^{\frac{2}{3}}(1+\sigma h^{\frac{2}{3}})\,\mathrm{d}\sigma\,.$$

We split the integral according to $I_1 - I_2 - I_3$ with the three positive integrals

$$\begin{aligned}I_1 &= h^{\frac{2}{3}}\int_{\mathbb{R}_-}\left|\mathsf{A}(\mathsf{a}_j + (2\breve{\kappa})^{\frac{1}{3}}\sigma)\right|^2\,\mathrm{d}\sigma\\ I_2 &= h^{\frac{2}{3}}\int_{\mathbb{R}_-}\left(1 - \chi(\sigma h^{\frac{2}{3}})^2\right)\left|\mathsf{A}(\mathsf{a}_j + (2\breve{\kappa})^{\frac{1}{3}}\sigma)\right|^2\,\mathrm{d}\sigma\\ I_3 &= h^{\frac{4}{3}}\int_{\mathbb{R}_-}\left|\chi(\sigma h^{\frac{2}{3}})\mathsf{A}(\mathsf{a}_j + (2\breve{\kappa})^{\frac{1}{3}}\sigma)\right|^2\,|\sigma|\,\mathrm{d}\sigma\,.\end{aligned}$$

Since a primitive function of A^2 is $x \mapsto \mathsf{A}'(x)^2 + x\mathsf{A}(x)^2$, we find that

$$I_1 = h^{\frac{2}{3}}\,(2\breve{\kappa})^{-\frac{1}{3}}\,\mathsf{A}'(\mathsf{a}_j)^2.$$

Moreover, since A is exponentially decreasing over $\mathbb{R}_-$, we find that $I_3 \le C_3 h^{\frac{4}{3}}$. Finally, Lemma A.6 in Appendix shows that $I_2 = \mathcal{O}(h^\infty)$.

Let us now show that the L^2 norm on Ω and on $\mathbb{R}^2\setminus\overline{\Omega}$ of the residual $\underline{\varepsilon}(m)$ defined in (4.36) is $\mathcal{O}(m^{-\infty})$. Revisiting all variable changes and problem reformulations we find

$$\big\|\underline{\varepsilon}(m)\big\|_{\mathrm{L}^2(\Omega)} = \sqrt{L}\,\Big\|h^{\frac{2}{3}}\Big(-\mathscr{L}_h^- + V_h^- - \underline{\lambda}(h^{\frac{1}{3}})\Big)\Big(\chi(\cdot h^{\frac{2}{3}})\Phi(h^{\frac{1}{3}};\cdot)\Big)\Big\|_{\mathrm{L}^2[h^{\frac{2}{3}}](\mathbb{R}_-)} \tag{4.40a}$$

$$\big\|\underline{\varepsilon}(m)\big\|_{\mathrm{L}^2(\mathbb{R}^2\setminus\overline{\Omega})} = \sqrt{L}\,\Big\|\Big(-\mathscr{L}_h^+ + V_h^+ - h^{\frac{2}{3}}\underline{\lambda}(h^{\frac{1}{3}})\Big)\Big(\chi(\cdot h)\Psi(h^{\frac{1}{3}};\cdot)\Big)\Big\|_{\mathrm{L}^2[h](\mathbb{R}_+)} \tag{4.40b}$$

Introducing the commutators $\Big[\mathscr{L}_h^-,\chi(\cdot h^{\frac{2}{3}})\Big]$ and $\big[\mathscr{L}_h^+,\chi(\cdot h)\big]$ of the differential operators $\mathscr{L}_h^\pm$ with scaled cut-off functions, we deduce from (4.40a)–(4.40b) the inequalities

$$\big\|\underline{\varepsilon}(m)\big\|_{\mathrm{L}^2(\Omega)} \le \sqrt{L}\,h^{\frac{2}{3}}(\mathcal{N}_\varphi + \mathcal{N}'_\varphi) \tag{4.41a}$$

$$\big\|\underline{\varepsilon}(m)\big\|_{\mathrm{L}^2(\mathbb{R}^2\setminus\overline{\Omega})} \le \sqrt{L}\,(\mathcal{N}_\psi + \mathcal{N}'_\psi) \tag{4.41b}$$

where

$$\mathcal{N}_\varphi = \Big\|\chi(h^{\frac{2}{3}}\cdot)\,\big(-\mathscr{L}_h^- + V_h^- - \underline{\lambda}\big)\,\Phi(h^{\frac{1}{3}};\cdot)\Big\|_{\mathrm{L}^2[h^{\frac{2}{3}}](\mathbb{R}_-)} \tag{4.42a}$$

$$\mathcal{N}'_\varphi = \Big\|\Big[\mathscr{L}_h^-,\chi(h^{\frac{2}{3}}\cdot)\Big]\,\Phi(h^{\frac{1}{3}};\cdot)\Big\|_{\mathrm{L}^2[h^{\frac{2}{3}}](\mathbb{R}_-)} \tag{4.42b}$$

$$\mathcal{N}_\psi = \Big\|\chi(h\cdot)\,\Big(-\mathscr{L}_h^+ + V_h^+ - h^{\frac{2}{3}}\underline{\lambda}\Big)\,\Psi(h^{\frac{1}{3}};\cdot)\Big\|_{\mathrm{L}^2[h](\mathbb{R}_+)} \tag{4.42c}$$

$$\mathcal{N}'_\psi = \Big\|\big[\mathscr{L}_h^+,\chi(h\cdot)\big]\,\Psi(h^{\frac{1}{3}};\cdot)\Big\|_{\mathrm{L}^2[h](\mathbb{R}_+)}. \tag{4.42d}$$

Both operators $\chi(h^{\frac{2}{3}}\cdot)\,\big(-\mathscr{L}_h^- + V_h^-\big)$ and $\chi(h\cdot)\,\big(-\mathscr{L}_h^+ + V_h^+\big)$ are differential operators in the form $a_2^\pm\partial^2 + a_1^\pm\partial + a_0^\pm$ with coefficients $a_i^\pm(h^{\frac{1}{3}},\cdot)$ belonging to $\mathscr{C}^\infty_{\mathsf{bounded}}([0,1]\times\mathbb{R}_\pm)$, see(4.19)–(4.20). Hence, the formal series (4.21) gives rise to the following sequences of finite expansions with remainders: For any $N\ge 1$,

$$-\mathscr{L}_h^\pm + V_h^\pm = \sum_{q=0}^{N-1} h^{\frac{q}{3}}\,\mathbf{A}_q^\pm + h^{\frac{N}{3}}\,\mathbf{R}_N^\pm(h^{\frac{1}{3}};\cdot) \tag{4.43}$$

where the remainders $\mathbf{R}_N^\pm$ are differential operators of order 2 such that $\chi(h^{\frac{2}{3}}\cdot)\mathbf{R}_N^-(h^{\frac{1}{3}};\cdot)$ and $\chi(h\cdot)\mathbf{R}_N^+(h^{\frac{1}{3}};\cdot)$ have coefficients belonging to $\mathscr{C}^\infty_{\mathsf{bounded}}([0,1]\times\mathbb{R}_\pm)$. It follows that for any given $N, N'\in\mathbb{N}$

$$\begin{aligned}
&\big(-\mathscr{L}_h^- + V_h^- - \underline{\lambda}\big)\,\Phi(h^{\frac{1}{3}};\cdot)\\
&= \left(\sum_{q=0}^{N-1} h^{\frac{q}{3}}\,(\mathbf{A}_q^- - \lambda_q) + h^{\frac{N}{3}}\Big(\mathbf{R}_N^-(h^{\frac{1}{3}};\cdot) - R_N^\lambda(h^{\frac{1}{3}})\Big)\right)\left(\sum_{q=0}^{N'-1} h^{\frac{q}{3}}\,\varphi_q + h^{\frac{N'}{3}}\,R^\varphi_{N'}(h^{\frac{1}{3}};\cdot)\right)\\
&= h^{\frac{N}{3}}\left(\sum_{q=0}^{N-1}(\mathbf{A}_q^- - \lambda_q)R^\varphi_{N-q}(h^{\frac{1}{3}};\cdot) + \Big(\mathbf{R}_N^-(h^{\frac{1}{3}};\cdot) - R_N^\lambda(h^{\frac{1}{3}})\Big)\,R^\varphi_0(h^{\frac{1}{3}};\cdot)\right)
\end{aligned} \tag{4.44}$$

where the second equality is obtained from the relation $\sum_{\ell=0}^{q}(\mathbf{A}_\ell^- - \lambda_\ell)\varphi_{q-\ell} = 0$ for all $q \in \mathbb{N}$ deduced from (4.29). In a similar way, we show that

$$\big(-\mathscr{L}_h^+ + V_h^+ - h^{\frac{2}{3}}\underline{\lambda}\big)\Psi(h^{\frac{1}{3}};\cdot) = h^{\frac{N}{3}}\left(\sum_{q=0}^{N-1}(\mathbf{A}_q^+ - \lambda_{q-2})R_{N-q}^{\psi}(h^{\frac{1}{3}};\cdot)\right.$$
$$\left. + \Big(\mathbf{R}_N^+(h^{\frac{1}{3}};\cdot) - R_{N-2}^{\lambda}(h^{\frac{1}{3}})\Big)R_0^{\psi}(h^{\frac{1}{3}};\cdot)\right) \quad (4.45)$$

where $\lambda_{-1} = \lambda_{-2} = 0$. We deduce from from (4.44) and (4.45) that

$$\mathscr{N}_\varphi + \mathscr{N}_\psi \le h^{\frac{N}{3}} \sum_{q=0}^{N} \left(\left\| F_q^{\varphi}(h^{\frac{1}{3}};\cdot)\right\|_{\mathrm{L}^2[h^{\frac{2}{3}}](\mathbb{R}_-)} + \left\| F_q^{\psi}(h^{\frac{1}{3}};\cdot)\right\|_{\mathrm{L}^2[h](\mathbb{R}_+)}\right)$$

with $F_q^{\varphi} \in \mathscr{C}^\infty([0,1],\mathscr{S}(\mathbb{R}_-))$ and $F_q^{\psi} \in \mathscr{C}^\infty([0,1],\mathscr{S}(\mathbb{R}_+))$, and finally that

$$\mathscr{N}_\varphi + \mathscr{N}_\psi \le C_N h^{\frac{N}{3}} \quad (4.46)$$

for some constant C_N independent of h.

Let us consider now the two commutators norms $\mathscr{N}'_\varphi$ and $\mathscr{N}'_\psi$. We observe that the coefficients of the operators $\big[\mathscr{L}_h^-, \chi(h^{\frac{2}{3}}\cdot)\big]$ and $\big[\mathscr{L}_h^+, \chi(h\cdot)\big]$ are zero in the regions defined by $-\delta h^{-\frac{2}{3}} \le \sigma \le 0$ and $0 \le \rho \le \delta h^{-1}$, respectively. This allows us to deduce that

$$\mathscr{N}'_\varphi + \mathscr{N}'_\psi \le \left(\left\| G^{\varphi}(h^{\frac{1}{3}};\cdot)\right\|_{\mathrm{L}^2[h^{\frac{2}{3}}](-\infty,-\delta h^{-\frac{2}{3}})} + \left\| G^{\psi}(h^{\frac{1}{3}};\cdot)\right\|_{\mathrm{L}^2[h](\delta h^{-1},+\infty)}\right)$$

with functions $G^{\varphi} \in \mathscr{C}^\infty([0,1],\mathscr{S}(\mathbb{R}_-))$ and $G^{\psi} \in \mathscr{C}^\infty([0,1],\mathscr{S}(\mathbb{R}_+))$. Lemma A.6 shows that $\mathscr{N}'_\varphi + \mathscr{N}'_\psi = \mathscr{O}(h^\infty)$. Combined with (4.46) true for all $N \in \mathbb{N}$, this complete the proof of part (iv) of the Lemma. ☐

4.3.4. *Proof of Theorem 4.A.* Choose $p \in \{\pm 1\}$ and $j \in \mathbb{N}$. In order to meet all the requirements listed in Definition 2.1, we modify the sequence of functions $(\underline{u}(m))_{m\ge 1}$ constructed in Definition 4.5, so that each such function satisfies the jump conditions in (2.1). To lift the jumps of $\underline{u}(m)$, we define the "radial" function

$$v^*(m;\xi) = \tfrac{1}{2}\chi(\xi)\begin{cases} -\,[\underline{v}(m)]_{\xi=0} - n_0^{1-p}\xi\,[n^{p-1}\,\partial_r\underline{v}(m)]_{\xi=0} & \xi \le 0 \\ [\underline{v}(m)]_{\xi=0} + \xi\,[n^{p-1}\,\partial_\nu\underline{v}(m)]_{\xi=0} & \xi > 0 \end{cases} \quad (4.47)$$

where χ can be taken as the same cut-off function used in Definition 4.5. We set

$$\underline{u}_{p;j}(m;r,\theta) := \left(\underline{v}\left(m;\tfrac{r}{R}-1\right) - v^*\left(m;\tfrac{r}{R}-1\right)\right)\,\mathrm{e}^{\mathrm{i}m\theta}$$

and $\underline{k}_{p;j}(m) := \underline{k}(m)$. Using (4.38), we can normalize the function $\underline{u}_{p;j}(m)$ in the L^2 norm. Relying on Lemmas 4.F and 4.G it is easy to check that the family $(\mathfrak{K}_{p;j}, \mathfrak{U}_{p;j})$ where $\mathfrak{K}_{p;j} = (\underline{k}_{p;j}(m))_{m\ge1}$ and $\mathfrak{U}_{p;j} = (\underline{u}_{p;j}(m))_{m\ge1}$ satisfies the four conditions of Definition 2.1.

5. Case (b) Half-quadratic potential well

Our concern is now the case when $\breve{\kappa} = 0$ and $\breve{\mu} > 0$. According to the same plan as before, we start with the complete description of the quasi-pairs that are constructed in the rest of the section. The corresponding statement has to be combined with Theorem 7.D to imply Theorem 1.B. As mentioned earlier, Case (a) and Case (b) share general concepts in the way the asymptotic expansion of quasi-pairs is obtained. Hence, we do not provide a comprehensive proof of Theorem 5.A but instead highlight the differences with the proof of Theorem 4.A.

5.1. Statements.

Theorem 5.A. *Choose $p \in \{\pm 1\}$. Let Assumptions 1.1 be verified and according to (3.7) and notations (4.11) assume that $\breve\kappa = 0$ and $\breve\mu > 0$. Then there exists for each $j \in \mathbb{N}$, a family of resonance quasi-pairs $\mathfrak{F}_{p;j} = (\mathfrak{K}_{p;j}, \mathfrak{U}_{p;j})$ of whispering gallery type (cf Definition 2.1) with $\mathfrak{K}_{p;j} = (\underline{k}_{p;j}(m))_{m\geq 1}$ and $\mathfrak{U}_{p;j} = (\underline{u}_{p;j}(m))_{m\geq 1}$.*

(i) The regularity properties (2.3)–(2.4) with respect to m holds with $\beta = \frac{1}{2}$: There exist coefficients $\mathrm{K}^{\ell}_{p;j}$ for any $\ell \in \mathbb{N}$, and constants C_N so that

$$\forall N \in \mathbb{N}, \qquad \left| \frac{\underline{k}_{p;j}(m)}{m} - \sum_{\ell=0}^{N-1} \mathrm{K}^{\ell}_{p;j}\, m^{-\ell/2} \right| \leq C_N\, m^{-N/2}. \tag{5.1}$$

The coefficients $\mathrm{K}^0_{p;j}$ are all equal to $(R\,n(R))^{-1}$, the coefficients of degree 1 are zero, and the coefficients of degree 2 are all distinct with j, see (5.3).

(ii) The functions $\underline{u}_{p;j}(m)$ still have the form (4.2) with radial functions $\underline{w}_{p;j}(m)$ that have a boundary layer structure around $r = R$ with the different scaled variables σ as $r < R$ and ρ as $r > R$:

$$\sigma = m^{\frac{1}{2}}\left(\tfrac{r}{R} - 1\right) \quad \text{if} \quad r < R \qquad \text{and} \qquad \rho = m\left(\tfrac{r}{R} - 1\right) \quad \text{if} \quad r > R.$$

There exist smooth functions $\Phi_{p;j} \in \mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_-)) : (t,\sigma) \mapsto \Phi_{p;j}(t,\sigma)$ and $\Psi_{p;j} \in \mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_+)) : (t,\rho) \mapsto \Psi_{p;j}(t,\rho)$ such that

$$\underline{w}_{p;j}(m; r) = \mathfrak{X}(r)\Big(\mathbb{1}_{r<R}(r)\, \Phi_{p;j}(m^{-\frac{1}{2}}, \sigma) + \mathbb{1}_{r>R}(r)\, \Psi_{p;j}(m^{-\frac{1}{2}}, \rho) \Big) \tag{5.2}$$

where $\mathfrak{X} \in \mathscr{C}^\infty_0(\mathbb{R}_+)$, $\mathfrak{X} \equiv 1$ in a neighborhood of R.

The first terms of the expansions of the quasi-resonances $\underline{k}_{p;j}(m)$ in powers of $m^{-1/2}$ as $m \to \infty$ are as follows:

$$\underline{k}_{p;j}(m) = \frac{m}{Rn_0}\left[1 + \sum_{\ell=1}^{3} \mathrm{k}^{\ell}_{p;j} \left(\frac{\sqrt{\breve\mu}}{m} \right)^{-\frac{\ell}{2}} + \mathscr{O}\left(m^{-2}\right) \right] \tag{5.3}$$

where $\mathrm{k}^1_{p;j} = 0$,

$$\mathrm{k}^2_{p;j} = \frac{4j+3}{2}, \quad \text{and} \quad \mathrm{k}^3_{p;j} = \left((\Psi^{\mathsf{GH}}_{2j+1})'(0) \right)^2 \left(\frac{-n_0^p}{\sqrt{n_0^2 - 1}} + \frac{4j+3}{9\,\breve\mu^{\frac{3}{2}}} \left(6 + \frac{R^3 n_3}{n_0} - 6\breve\mu \right) \right).$$

Here, $\Psi^{\mathsf{GH}}_{\ell}$ denotes the Gauss-Hermite function of order ℓ, see [1, 20].

The asymptotic expansions of the radial part of the quasi-modes $\underline{w}_{p;j}(m)$ in (5.2) starts as

$$\underline{w}_{p;j}(m; r) = \mathfrak{X}(r) \left(\mathrm{w}^0_{p;j}(m; r) + \left(\frac{1}{m} \right)^{\frac{1}{2}} \mathrm{w}^1_{p;j}(m; r) \right) + \mathscr{O}\left(m^{-1}\right), \tag{5.4}$$

where, using the scaled variables $\sigma = m^{\frac{1}{2}}(\frac{r}{R} - 1)$ and $\rho = m(\frac{r}{R} - 1)$

$$\mathrm{w}^0_{p;j}(m; r) = \begin{cases} \Psi^{\mathsf{GH}}_{2j+1}\left(\breve\mu^{\frac{1}{4}}\sigma\right) & \text{if} \ \ r < R, \\ 0 & \text{if} \ \ r \geq R, \end{cases} \tag{5.5}$$

and

$$\mathrm{w}^1_{p;j}(m; r) = \frac{-n_0^p\, \breve\mu^{\frac{1}{4}}\, (\Psi^{\mathsf{GH}}_{2j+1})'(0)}{\sqrt{n_0^2 - 1}} \begin{cases} \frac{1}{\Psi^{\mathsf{GH}}_0(0)} \Psi^{\mathsf{GH}}_0\left(\breve\mu^{\frac{1}{4}}\sigma\right) + \widetilde\varphi_1(\sigma) & \text{if} \ \ r < R, \\ \exp\left(-\frac{\sqrt{n_0^2 - 1}}{n_0}\, \rho \right) & \text{if} \ \ r \geq R, \end{cases} \tag{5.6}$$

where $\widetilde{\varphi}_1 \in \mathrm{H}^1_0(\mathbb{R}_-) \cap \mathscr{S}(\mathbb{R}_-)$ is the unique solution to a Dirichlet problem posed on $\mathbb{R}_-$, see (5.13) later on. In contrast with Case (A), the expression of $\mathrm{w}^1_{p;j}$, though determined, is not explicit. This is the reason why we gave only two terms in the asymptotic expansions (5.3).

Remark 5.1. The quasi-resonances are organized in an asymptotic lattice with constant step: The gap between two resonances with consecutive polar mode index m and $m+1$ and the same radial mode index j is found to be

$$\underline{k}_{p;j}(m+1) - \underline{k}_{p;j}(m) = \frac{1}{Rn_0} + \mathscr{O}\left(m^{-\frac{3}{2}}\right) \tag{5.7}$$

whereas when m is fixed and j is incremented by 1, the gap between two resonance is found to be

$$\underline{k}_{p;j+1}(m) - \underline{k}_{p;j}(m) = \frac{2\sqrt{\breve{\mu}}}{n_0} + \mathscr{O}\left(m^{-\frac{1}{2}}\right).$$

This property is very interesting for various applications in optics, e.g. for the design of frequency combs.

5.2. **Proof.** The general outline of the proof of Theorem 5.A is similar to the one of Theorem 4.A in Case (A). In particular, the general framework of section 4.2 applies with

$$\varkappa = 2, \quad \gamma = \breve{\mu}, \quad \alpha = \tfrac{1}{2}, \quad \alpha' = 1, \quad \beta = \tfrac{1}{2}.$$

We provide now the part of the proof of Theorem 5.A specific to Case (B).

From general expressions (4.19)–(4.21), we find that

$$\begin{cases} \mathsf{A}^-_0 = -\partial^2_\sigma + \breve{\mu}\sigma^2, \quad \mathsf{A}^-_1 = -2\partial^2_\sigma + (p-2)\partial_\sigma - \sigma^3\left(\dfrac{\tilde{n}_2}{n_0} + \dfrac{\tilde{n}_3}{3n_0}\right), & (5.8)\\ \mathsf{A}^+_0 = -n_0^2\partial^2_\rho + n_0^2 - 1, \quad \mathsf{A}^+_1 = 0, \quad \mathsf{A}^+_2 = -n_0^2(\partial_\rho + 2\rho). & (5.9)\end{cases}$$

The analog of Lemma 4.B describing the coefficients of the formal series of operators (4.21) in terms of powers of $h^\beta = h^{\frac{1}{2}}$ reads as follows.

Lemma 5.B. *For any integer $q \geq 1$,*

$$\mathsf{A}^-_q = A^-_q(\sigma)\,\partial^2_\sigma + B^-_q(\sigma)\,\partial_\sigma + C^-_q(\sigma) \quad \textit{with}\ \ A^-_q \in \mathbb{P}^q,\ \ B^-_q \in \mathbb{P}^{q-1},\ \ C^-_q \in \mathbb{P}^{q+1}$$

$$\mathsf{A}^+_q = B^+_q(\rho)\,\partial_\rho + C^+_q(\rho) \quad \textit{with}\ \ B^+_q \in \mathbb{P}^{[\frac{q}{2}]-1},\ \ C^+_q \in \mathbb{P}^{[\frac{q}{2}]}.$$

We proceed as in Section 4.3, associating to the system (4.25) a formal series system of equations like (4.28), in which the powers of h are modified according to the values of α, α', β, and $\varkappa$. As a matter of fact, equating the series coefficients, we obtain in Case (B) exactly the same infinite collection of systems (4.29) as in Case (A), but with the new expressions of operators $\mathsf{A}^\pm_q$. The coefficients of the formal series expansions (4.22) are obtained by solving (4.29) for q spanning $\mathbb{N}$.

5.2.1. *Initialization stage.* For $q=0$, the couple of functions (φ_0, ψ_0) and the number λ_0 are obtained by solving (4.30) with A^-_0 and A^+_0 given in (5.8)–(5.9). Since the equation $\mathsf{A}^+_0\,\psi_0 = 0$ with the Neumann condition at 0 has no non-zero solution in $\mathscr{S}(\mathbb{R}_+)$, it is natural to take $\psi_0 = 0$. Then, we are left with the following harmonic oscillator problem on $\mathbb{R}_-$

$$-\varphi_0''(\sigma) - \breve{\mu}\,\sigma^2\varphi_0(\sigma) = \lambda_0\varphi_0(\sigma) \ \text{ for } \ \sigma \in (-\infty, 0), \qquad \text{and} \qquad \varphi_0(0) = 0$$

whose bounded solutions are generated by the odd Gauss-Hermite functions $\left\{\Psi^{\mathsf{GH}}_{2j+1}\right\}_{j\in\mathbb{N}}$.

Lemma 5.C. *Let $j \in \mathbb{N}$. The couple of functions (φ_0, ψ_0) and the number λ_0 defined by*

$$\varphi_0(\sigma) = \Psi^{\mathsf{GH}}_{2j+1}\left(\breve{\mu}^{\frac{1}{4}}\sigma\right), \quad \psi_0(\rho) = 0, \quad \textit{and} \quad \lambda_0 = (4j+3)\sqrt{\breve{\mu}}$$

solve (4.30) for A^-_0 and A^+_0 given in (5.8)–(5.9).

5.2.2. *Sequence of nested problems and recurrence.* As in Case (A), reordering the terms in the system (4.29) taking into account Lemma 5.B, we obtain that the couple of functions (φ_q, ψ_q) and the number λ_q for $q \geq 1$ are solutions to

$$(\mathscr{R}_q^{(\mathrm{B})}) \quad \begin{cases} -\varphi_q''(\sigma) + (\breve{\mu}\sigma^2 - \lambda_0)\varphi_q(\sigma) = \lambda_q\, \varphi_0(\sigma) + S_q^{\varphi}(\sigma) & \sigma \in \mathbb{R}_- \quad (5.10a) \\ -n_0^2\psi_q''(\rho) + \left(n_0^2 - 1\right)\psi_q(\rho) = S_q^{\psi}(\rho) & \rho \in \mathbb{R}_+ \quad (5.10b) \\ \varphi_q(0) = \psi_q(0) & (5.10c) \\ \psi_q'(0) = n_0^{p-1}\varphi_{q-1}'(0) & (5.10d) \end{cases}$$

with right hand side terms S_q^{φ} and S_q^{ψ} defined as (recall that $\mathbf{A}_1^+ = 0$)

$$S_q^{\varphi} = -\mathbf{A}_q^-\, \varphi_0 + \sum_{\ell=1}^{q-1} (\lambda_\ell - \mathbf{A}_\ell^-)\, \varphi_{q-\ell} \quad \text{and} \quad S_q^{\psi} = \sum_{\ell=2}^{q} (\lambda_{\ell-2} - \mathbf{A}_\ell^+)\, \psi_{q-\ell}. \tag{5.11}$$

Notation 5.2. For a real number t let $\omega_t : \sigma \mapsto \exp\left(2^t|\sigma|\right)$. We denote by $\mathrm{L}^2(\mathbb{R}_-, \omega_t)$ and $\mathrm{H}^\ell(\mathbb{R}_-, \omega_t)$, the weighted Lebesgue and Sobolev spaces with measure $\omega_t(\sigma)\,\mathrm{d}\sigma$.

Proposition 5.D. *Choose $j \in \mathbb{N}$ and take $\varphi_0, \psi_0, \lambda_0$ as given in Lemma 5.C. For any $q \geq 1$, there exist*

- *a unique $\lambda_q \in \mathbb{R}$*
- *a unique real number c_q, a unique real sequence $m \in \mathbb{N} \mapsto b_q^i$ such that $b_q^j = 0$, and a unique polynomial $P_q^{\psi} \in \mathbb{P}^{q-1}$*

such that setting

$$\begin{aligned} &\varphi_q(\sigma) = c_q \Psi_0^{\mathsf{GH}}\big(\breve{\mu}^{\frac14}\sigma\big) + \widetilde{\varphi}_q(\sigma) \quad \textit{with} \quad \widetilde{\varphi}_q(\sigma) = \sum_{i\in\mathbb{N}} b_q^i \Psi_{2i+1}^{\mathsf{GH}}\big(\breve{\mu}^{\frac14}\sigma\big) \quad \forall\sigma \in \mathbb{R}_- \\ &\psi_q(\rho) = P_q^{\psi}(\rho)\exp\big(-\rho\sqrt{1-n_0^{-2}}\big) \quad \forall\rho \in \mathbb{R}_+ \end{aligned} \tag{5.12}$$

the collection $(\varphi_0, \dots, \varphi_q, \psi_0, \dots, \psi_q, \lambda_0, \dots, \lambda_q)$ solves the sequence of problems $(\mathscr{R}_\ell^{(\mathrm{B})})$ for $\ell = 0, \dots, q$. Moreover $\widetilde{\varphi}_q \in \mathrm{H}_0^1(\mathbb{R}_-) \cap \mathrm{H}^2(\mathbb{R}_-, \omega_{-q})$

Proof. The proof is quite similar to the one of Proposition 4.D and we will focus on the main differences. We proceed by induction on q. For $q = 0$, Lemma 5.C provides λ_0, φ_0, and ψ_0 solutions to $(\mathscr{R}_0^{(\mathrm{B})})$ and we readily obtain $c_0 = 0$, $\widetilde{\varphi}_0 = \Psi_{2j+1}^{\mathsf{GH}}(\breve{\mu}^{\frac14}\,\cdot) \in \mathrm{H}_0^1(\mathbb{R}_-)$, and $P_0 = 0$. Moreover, φ_0 belongs to $\mathrm{H}^2(\mathbb{R}_-, \omega_0)$ because $\Psi_{2j+1}^{\mathsf{GH}}$ is defined as the product of the $(2j+1)$-th order Hermite polynomial of degree $2j+1$ by $x \mapsto \exp(-\frac{x^2}{2})$.

Let $q \geq 1$ and suppose that $(\lambda_\ell)_{0\leq\ell\leq q-1}$, $(\varphi_\ell)_{0\leq\ell\leq q-1}$, and $(\psi_\ell)_{0\leq\ell\leq q-1}$ are solutions to problems $(\mathscr{R}_\ell^{(\mathrm{B})})$ for $\ell = 0, \dots, q-1$, and satisfy (5.12). Solving equation (5.10b) for ψ_q proceed in a way very similar to (4.31b) in the proof of Proposition 4.D to show that there exists $P_q^{\psi} \in \mathbb{P}^{q-1}$ such that

$$\psi_q(\rho) = P_q^{\psi}(\rho)\exp\big(-\rho\sqrt{1-n_0^{-2}}\big).$$

Let us now consider equation (5.10b) for φ_q. First of all, we obtain by induction that $\varphi_\ell \in \mathrm{H}^2(\mathbb{R}_-, \omega_{1-q})$ for all $\ell \in \{0, \dots, q-1\}$. Then, using Lemma 5.B and (5.11), it follows that $S_q^{\varphi} \in \mathrm{L}^2(\mathbb{R}_-, \omega_{1/2-q})$. Note that the value of the constant $q-1$ in the exponential weight is reduced by $\frac12$ to $\frac12 - q$ in order to absorb the polynomials behavior. To solve equation (5.10a) with the non-homogeneous boundary condition (5.10c) we introduce as new unknown $\widetilde{\varphi}_q = \varphi_q - c_q\Psi_0^{\mathsf{GH}}(\breve{\mu}^{\frac14}\,\cdot)$ where $c_q = \frac{\psi_q(0)}{\Psi_0^{\mathsf{GH}}(0)}$. It belongs to $\mathrm{H}^2(\mathbb{R}_-, \omega_{1-q})$ and the

Dirichlet problem (5.10a), (5.10c) becomes

$$\begin{cases} -\widetilde{\varphi}_q'' + (\breve{\mu}\sigma^2 - \lambda_0)\widetilde{\varphi}_q = \lambda_q \Psi^{\mathsf{GH}}_{2j+1}(\breve{\mu}^{\frac14}\cdot) + \widetilde{S}^{\varphi}_q \quad \forall \sigma \in (-\infty, 0) & (5.13a) \\ \widetilde{\varphi}_q(0) = 0 & (5.13b) \end{cases}$$

where $\widetilde{S}^{\varphi}_q = S^{\varphi}_q + 2(2j+1)\sqrt{\breve{\mu}}\, c_q \Psi^{\mathsf{GH}}_0(\breve{\mu}^{\frac14}\cdot)$. Problem (5.13) has a solution only when the left hand side function $\lambda_q \Psi^{\mathsf{GH}}_{2j+1}(\breve{\mu}^{\frac14}\cdot) + \widetilde{S}^{\varphi}_q$ is orthogonal to $\Psi^{\mathsf{GH}}_{2j+1}(\breve{\mu}^{\frac14}\cdot)$. It follows that we must have

$$\lambda_q = -2\breve{\mu}^{\frac14} \int_{-\infty}^{0} \Psi^{\mathsf{GH}}_{2j+1}(\breve{\mu}^{\frac14}\sigma)\, \widetilde{S}^{\varphi}_q(\sigma)\, \mathrm{d}\sigma. \tag{5.14}$$

Finally, from Lemma A.3, there exists a unique $\widetilde{\varphi}_q$ in $\mathrm{H}^1_0(\mathbb{R}_-) \cap \mathrm{H}^2(\mathbb{R}_-, \omega_{-q})$ and orthogonal to $\Psi^{\mathsf{GH}}_{2j+1}$ solution to (5.13). The formula giving $\widetilde{\varphi}_q$ is

$$\widetilde{\varphi}_q = \sum_{i=0,\ i\neq j}^{+\infty} \frac{1}{4(i-j)} \left(\widetilde{S}^{\varphi}_q,\, \varsigma_i \Psi^{\mathsf{GH}}_{2i+1}\right)_{\mathrm{L}^2(\mathbb{R}_-)} \varsigma_i \Psi^{\mathsf{GH}}_{2i+1} \tag{5.15}$$

where $\varsigma_i = \|\Psi^{\mathsf{GH}}_{2i+1}\|^{-1}_{\mathrm{L}^2(\mathbb{R}_-)}$. □

Remark 5.3. In contrast to Case (A), we cannot deduce from Proposition 5.D a finite algorithm to compute the terms of the sequence $(\varphi_q)_{q\in\mathbb{N}}$. The reason is that for $q \geq 1$, the sum of the series (5.15) cannot be computed explicitly. However, a few terms are explicit: We know $(\psi_0, \varphi_0, \lambda_0)$ so we can compute, first P^{ψ}_1, then c_1, and, after this, S^{φ}_1. With these latter quantities, we can deduce an explicit expression of $\widetilde{S}^{\varphi}_1$ as a finite sum of polynomials times Gauss-Hermite functions. Now, from the definition of the Gauss-Hermite functions [11, Eq. 1.3.8] and recurrence relations on Hermite polynomials [21, Sect. 18.9(i)], we deduce the following recurrence relations for $i \geq 0$ and $z \in \mathbb{R}$,

$$\partial_z \Psi^{\mathsf{GH}}_i(z) = \left(\tfrac{i}{2}\right)^{\frac12} \Psi^{\mathsf{GH}}_{i-1}(z) - \left(\tfrac{i+1}{2}\right)^{\frac12} \Psi^{\mathsf{GH}}_{i+1}(z), \tag{5.16a}$$

$$z \Psi^{\mathsf{GH}}_i(z) = \left(\tfrac{i}{2}\right)^{\frac12} \Psi^{\mathsf{GH}}_{i-1}(z) + \left(\tfrac{i+1}{2}\right)^{\frac12} \Psi^{\mathsf{GH}}_{i+1}(z). \tag{5.16b}$$

Hence we can rewrite $\widetilde{S}^{\varphi}_1$ as a finite sum of Gauss-Hermite functions and with this we can compute explicitly λ_1 given by (5.14). Nevertheless $\widetilde{\varphi}_1$ will be an infinite sum of Gauss-Hermite functions so, for $q \geq 2$, λ_q does not have a closed form.

Lemma 5.E. *For all $q \in \mathbb{N}$, we have $\varphi_q \in \mathscr{S}(\mathbb{R}_-)$ and $\psi_q \in \mathscr{S}(\mathbb{R}_+)$.*

Proof. From the expression (5.12) of ψ_q, it is obvious that it belongs to $\mathscr{S}(\mathbb{R}_+)$.

From Proposition 5.D we know that φ_q and its derivatives of order ≤ 2 are exponentially decaying as $\sigma \to \infty$. Concerning higher order derivatives $\varphi_q^{(i)}$, from the identity $\varphi_q'' = (\breve{\mu}\sigma^2 - \lambda_0)\varphi_q - \lambda_0\varphi_0 - S^{\varphi}_q$ deduced from (5.10a), from (5.11) and Lemma 5.B, we find that there exists families of polynomials $P^{\ell}_{q,i}, Q^{\ell}_{q,i}$ such that

$$\varphi_q^{(i)} = \sum_{\ell=0}^{q} \left(P^{\ell}_{q,i}\, \varphi_\ell + Q^{\ell}_{q,i}\, \varphi_\ell'\right). \tag{5.17}$$

Hence $\varphi_q^{(i)}$ is exponentially decaying too, and we have proved that φ_q belongs to $\mathscr{S}(\mathbb{R}_-)$. □

5.2.3. *Convergence.* The proof that the formal series

$$\sum_{q\in\mathbb{N}} \lambda_q h^{\frac{q}{2}}, \qquad \sum_{q\in\mathbb{N}} \varphi_q h^{\frac{q}{2}}, \qquad \text{and} \qquad \sum_{q\in\mathbb{N}} \psi_q h^{\frac{q}{2}}, \tag{5.18}$$

obtained from Proposition 5.D give rise to a family of resonance quasi-pairs in the sense of Definition 2.1 can be achieved exactly as in Section 4.3.3 for Case (A). Namely, Lemma 4.F and Definition 4.5 are respectively replaced by the following Lemma 5.F and Definition 5.4.

Lemma 5.F. *Let $(\lambda_q)_{q\in\mathbb{N}}$, $(\varphi_q)_{q\in\mathbb{N}}$ and $(\psi_q)_{q\in\mathbb{N}}$ given by Proposition 5.D. There exist smooth functions $\underline{\lambda} \in \mathscr{C}^\infty([0,1])$, $\Phi \in \mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_-))$ and $\Psi \in \mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_+))$ such that for all $(h,\sigma,\rho) \in [0,1] \times \mathbb{R}_- \times \mathbb{R}_+$ and for all integer $N \geq 0$, we have the following finite expansions with remainders*

$$\underline{\lambda}(h^{\frac12}) = \sum_{q=0}^{N-1} h^{\frac{q}{2}}\, \lambda_q + h^{\frac{N}{2}} R_N^\lambda(h^{\frac12}), \qquad \textit{with} \quad R_N^\lambda \in \mathscr{C}^\infty([0,1]) \tag{5.19a}$$

$$\Phi(h^{\frac12};\sigma) = \sum_{q=0}^{N-1} h^{\frac{q}{2}}\, \varphi_q(\sigma) + h^{\frac{N}{2}} R_N^\varphi(h^{\frac12};\sigma), \quad \textit{with} \quad R_N^\varphi \in \mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_-)) \tag{5.19b}$$

$$\Psi(h^{\frac12};\rho) = \sum_{q=0}^{N-1} h^{\frac{q}{2}}\, \psi_q(\rho) + h^{\frac{N}{2}} R_N^\psi(h^{\frac12};\rho) \quad \textit{with} \quad R_N^\psi \in \mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_+)) \tag{5.19c}$$

Definition 5.4. Choose a real number $\delta \in (0, \frac12)$ and a smooth cut-off function χ, $0 \leq \chi \leq 1$, such that $\chi(\xi) = 1$ for $|\xi| \leq \delta$ and $\chi(\xi) = 0$ for $|\xi| \geq 2\delta$. We define for any integer $m \geq 1$ with the notation $h = m^{-1}$, the quantities:

$$\underline{k}(m) = \frac{m}{Rn_0}\sqrt{1 + h\, \underline{\lambda}(h^{\frac12})},$$

$$\underline{v}(m;\xi) = \chi(\xi) \begin{cases} \Phi(h^{\frac12}; h^{-1}\xi), & \xi \leq 0 \\ \Psi(h^{\frac12}; h^{-1}\xi), & \xi > 0 \end{cases} \qquad \xi \in (-1, +\infty)$$

$$\underline{u}(m; r, \theta) = \underline{v}\left(m; \tfrac{r}{R} - 1\right) \mathrm{e}^{\mathrm{i}m\theta} \qquad (r,\theta) \in (0,+\infty) \times \mathbb{R}/2\pi\mathbb{Z}.$$

One can show that the sequence $(\underline{k}(m), \underline{u}(m))_{m\geq1}$ is a family of "almost" quasi-pairs in the sense of Lemma 4.G. The main difference with Case (A) in proving Lemma 4.G for the sequence $(\underline{k}(m), \underline{u}(m))_{m\geq1}$ introduced in Definition 5.4 is that we do not have anymore an explicit expression for φ_q but this does not prevent to obtain the same estimates as in Case (A). We refer to [18] for details.

5.2.4. *Proof of Theorem 5.A.* A further correction will have to be made to transform the sequence of functions $(\underline{u}(m))_{m\geq1}$ constructed in Definition 5.4 into a true family of resonance quasi-modes in the sense of Definition 2.1. We set

$$\underline{u}_{p;j}(m; r, \theta) := \left(\underline{v}\left(m; \tfrac{r}{R} - 1\right) - v^*\left(m; \tfrac{r}{R} - 1\right)\right) \mathrm{e}^{\mathrm{i}m\theta}$$

where v^* is defined as in (4.47) and $\underline{k}_{p;j}(m) := \underline{k}(m)$. Relying on Lemmas 5.F and the analogous of 4.G for Case (B), one can check that the family $(\mathfrak{K}_{p;j}, \mathfrak{U}_{p;j})$ where $\mathfrak{K}_{p;j} = (\underline{k}_{p;j}(m))_{m\geq1}$ and $\mathfrak{U}_{p;j} = (\underline{u}_{p;j}(m))_{m\geq1}$ satisfies the four conditions of Definition 2.1.

6. Case (C) Quadratic potential well

We are now under Assumption (3.8). We recall that Case (C) corresponds to a situation where $\breve{\kappa} < 0$ and the potential W has no local minimum at R but has at least one local inner minimum R_0 over $(0, R)$. This case falls into to the framework investigated by Helffer and

SJÖSTRAND [12]. Namely, the asymptotic expansions of quasi-resonances and quasi-modes are given in respectively Theorem 10.7 and Theorem 10.8 in [12]. Note however that the construction is not made explicit in [12]. Here, in contrast, we construct explicit families of resonance quasi-pairs $\mathfrak{F}_{p;j}$ localized around the circle $r = R_0$ inside the cavity Ω. Note that strictly speaking, these families of resonance quasi-pairs are not of whispering gallery type.

6.1. Statements.

Theorem 6.A. *Choose* $p \in \{\pm 1\}$. *Let Assumptions 1.1 be verified and assume* $\breve\kappa < 0$. *Let* $R_0 \in (0, R)$ *such that* $1 + \frac{R_0\, n'(R_0)}{n(R_0)} = 0$ *and* $\breve\mu := 2 - \frac{R_0^2\, n''(R_0)}{n(R_0)} > 0$, *cf* (1.15). *Then, for each* $j \in \mathbb{N}$, *there exists a family of resonance quasi-pairs* $\mathfrak{F}_{p;j} = (\mathfrak{K}_{p;j}, \mathfrak{U}_{p;j})$ *with* $\mathfrak{K}_{p;j} = (\underline{k}_{p;j}(m))_{m\geq 1}$ *and* $\mathfrak{U}_{p;j} = (\underline{u}_{p;j}(m))_{m\geq 1}$.

(i) The regularity property (2.3)–(2.4) *with respect to* m *holds with* $\beta = \frac{1}{2}$, *see* (5.1). *The coefficients* $\mathrm{K}^0_{p;j}$ *are all equal to* $(R_0\, n(R_0))^{-1}$, *the coefficients of degree* 1 *are zero, and the coefficients of degree* 2 *are all distinct with* j, *see* (6.2).

(ii) The functions $\underline{u}_{p;j}(m)$ *still have the form* (4.2) *with radial functions* $\underline{w}_{p;j}(m)$ *that are smooth in the scaled variables* $\sigma = m^{\frac{1}{2}}(r/R_0 - 1)$. *There exists a smooth function* $\Phi_{p;j} \in \mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R})) : (t,\sigma) \mapsto \Phi_{p;j}(t,\sigma)$ *such that*

$$\underline{w}_{p;j}(m; r) = \mathfrak{X}(r)\, \Phi_{p;j}(m^{-\frac{1}{2}}, \sigma) \tag{6.1}$$

where $\mathfrak{X} \in \mathscr{C}_0^\infty(\mathbb{R}_+)$, $\mathfrak{X} \equiv 1$ *in a neighborhood of* R_0.

These families of resonance quasi-pairs are *not* of whispering gallery type: The quasi-modes are strictly localized inside the cavity. The first terms of the asymptotic expansion of $\underline{k}_{p;j}$ are:

$$\underline{k}_{p;j}(m) = \frac{m}{R_0 n(R_0)} \left[1 + \sum_{\ell=1}^{4} \mathrm{k}^\ell_{p;j} \left(\frac{\sqrt{\breve\mu}}{m}\right)^{\frac{\ell}{2}} + \mathscr{O}\left(m^{-\frac{5}{2}}\right)\right] \tag{6.2}$$

with $\mathrm{k}^1_{p;j} = 0$, $\mathrm{k}^2_{p;j} = \dfrac{2j+1}{2}$, $\mathrm{k}^3_{p;j} = 0$, and

$$\mathrm{k}^4_{p;j} = \frac{1}{64}\Bigg[13 - 16p + \frac{8p^2 - 16p - 5}{\breve\mu} - \frac{2\eta_3 - 3\eta_4}{3\breve\mu^2} - \frac{7\eta_3^2}{9\breve\mu^3} + (2j+1)^2\left(5 - \frac{35}{\breve\mu} + \frac{10\eta_3 + \eta_4}{\breve\mu^2} - \frac{5\eta_3^2}{3\breve\mu^3}\right)\Bigg].$$

where

$$\eta_3 = 6 + \frac{R_0^3\, n^{(3)}(R_0)}{n(R_0)} \quad \text{and} \quad \eta_4 = 24 - \frac{R_0^4\, n^{(4)}(R_0)}{n(R_0)}.$$

The asymptotic expansion of the quasi-modes starts with

$$\underline{u}_{p;j}(\pm m; x, y) = \mathfrak{X}(r)\Psi^{\mathsf{GH}}_j\left(\breve\mu^{\frac{1}{4}} m^{\frac{1}{2}} \left(\tfrac{r}{R_0} - 1\right)\right) \mathrm{e}^{\pm i m\theta} + \mathscr{O}\left(m^{-\frac{1}{2}}\right). \tag{6.3}$$

Remark 6.1. As in Case (B), the quasi-resonances are organized in an asymptotic lattice with constant step: The gap between two resonances with consecutive polar mode index m and $m+1$ and the same radial mode index j is found to be

$$\underline{k}_{p;j}(m+1) - \underline{k}_{p;j}(m) = \frac{1}{R_0 n(R_0)} + \mathscr{O}\left(m^{-2}\right),$$

whereas when m is fixed and j is incremented by 1, the gap between two resonance is found to be

$$\underline{k}_{p;j+1}(m) - \underline{k}_{p;j}(m) = \frac{\sqrt{\breve{\mu}}}{n(R_0)} + \mathcal{O}\left(m^{-\frac{1}{2}}\right).$$

6.2. **Proof.** The proof of Theorem 6.A can be seen as a simpler version of the proof of Theorem 5.A since the driving operator $\mathsf{A}_0^- = -\partial_\sigma^2 + \breve{\mu}\sigma^2$ of the asymptotic expansion is the same quadratic oscillator on both side of the potential well location R_0, *i.e.* $\mathsf{A}_0^+ = \mathsf{A}_0^-$. Therefore, we will not detail the entire proof of Theorem 6.A but we will focus on an interesting byproduct of our approach compared to the results of [12], viz a finite algorithm for computing the terms of the asymptotic expansion of the resonance quasi-pairs.

In the framework of the Schrödinger analogy introduced in Section 3, we start this time by introducing the dimensionless variable $\xi = \frac{r}{R_0} - 1$ (instead of $\xi = \frac{r}{R} - 1$ as in the two previous cases) and the unknown v such that $v(\xi) = w(R_0(1+\xi))$. This leads to the same equation (4.12a) where $\widetilde{\Lambda} = R_0^2\,\tilde{n}(0)^2\,(\Lambda - W_0)$ with $\tilde{n}(\xi) = n(R_0(1+\xi))$. Compared to the general framework introduced in Section 4.2 for Cases (A) and (B), the potential V is smooth at its local minimum at $\xi = 0$. As a consequence, it is not anymore necessary to introduce a different scaling on both side of $\xi = 0$. Moreover, it is still possible to take advantage of the framework of Section 4.2, but taking into account the fact the variable $\sigma = h^{-\alpha}\xi$ must be considered over $\mathbb{R}$ and not only over $\mathbb{R}_-$. This framework applies with the same relevant quantities as in Case (B) (the ones affecting $\mathscr{L}$ on $\mathbb{R}_-$). Denoting by φ the new unknown such that $\varphi(\sigma) = v(\xi)$, equation (4.12a) become $-\mathscr{L}_h\varphi + V_h\varphi = \lambda\varphi$, $\sigma \in \mathbb{R}$, where the operator $\mathscr{L}_h$ and the potentials V_h have the same expressions than $\mathscr{L}_h^-$ in (4.19) and V_h^- of (4.20) with $\tilde{n}(\xi) = n(R_0(1+\xi))$. The decay condition is $\varphi \in \mathscr{S}(\mathbb{R})$.

We define a formal series of operators in terms of powers of $h^{\frac{1}{2}}$, similarly to (4.21), as $-\mathscr{L}_h + V_h \sim \sum_{q\in\mathbb{N}} h^{\frac{q}{2}}\,\mathsf{A}_q$ and we look for a function φ and a scalar λ in the form of the formal series $\varphi = \sum_{q\in\mathbb{N}} h^{\frac{q}{2}}\,\varphi_q$ and $\lambda = \sum_{q\in\mathbb{N}} h^{\frac{q}{2}}\,\lambda_q$. One can show that the coefficients A_q, $q \in \mathbb{N}$, satisfy Lemma 5.B (the statement on A_q^-). Then, by the same arguments as in Cases (A) and (B) that can equally apply here, we obtain that (φ_0, λ_0) is solutions to the full harmonic oscillator equation (in opposition to the half harmonic oscillator of Case (B))

$$-\varphi_0''(\sigma) + \breve{\mu}\,\sigma^2\varphi_0(\sigma) = \lambda_0\varphi_0, \qquad \sigma \in \mathbb{R}, \qquad \varphi_0 \in \mathscr{S}(\mathbb{R}), \tag{6.4}$$

and that for $q \geq 1$, (φ_q, λ_q) are solutions to the sequence of problems

$$(\mathscr{R}_q^{(\mathrm{C})}) \quad \begin{cases} -\varphi_q''(\sigma) + (\breve{\mu}\sigma^2 - \lambda_0)\varphi_q(\sigma) = \lambda_q\,\varphi_0(\sigma) + S_q^\varphi(\sigma) & \sigma \in \mathbb{R} \quad (6.5a) \\ \varphi_q \in \mathscr{S}(\mathbb{R}) & (6.5b) \end{cases}$$

with the right hand side term S_q^φ defined as $S_q^\varphi = -\mathsf{A}_q\,\varphi_0 + \sum_{\ell=1}^{q-1}(\lambda_\ell - \mathsf{A}_\ell)\,\varphi_{q-\ell}$.

Solutions to the full harmonic oscillator equation (6.4) are

$$\varphi_0(\sigma) = \Psi_j^{\mathsf{GH}}\left(\breve{\mu}^{\frac{1}{4}}\sigma\right) \quad \text{and} \quad \lambda_0 = (2j+1)\sqrt{\breve{\mu}} \quad (j \in \mathbb{N}). \tag{6.6}$$

For $q \geq 1$, the features of the solution (φ_q, λ_q) to problem $(\mathscr{R}_q^{(\mathrm{C})})$ are detailed in the following proposition. Its proof below also provides an algorithm to compute φ_q and λ_q.

Proposition 6.B. *Let $j \in \mathbb{N}$ and let (φ_0, λ_0) given by (6.6). Then there exist, for any $q \geq 1$, a unique $\lambda_q \in \mathbb{R}$ and a unique $(b_q^i)_{i\in\{0,\dots,j+3q\}} \in \mathbb{R}^{j+3q+1}$ with $b_q^j = 0$ such that by setting*

$$\varphi_q(\sigma) = \sum_{i=0}^{j+3q} b_q^i\,\Psi_i^{\mathsf{GH}}\left(\breve{\mu}^{\frac{1}{4}}\sigma\right), \quad \forall\sigma \in \mathbb{R}, \tag{6.7}$$

the collection $(\varphi_0, \dots, \varphi_q, \lambda_0, \dots, \lambda_q)$ solves the sequence of problems $(\mathscr{R}_\ell^{(\mathrm{C})})_{\ell=0,\dots,q}$.

Proof. The relations (5.16) combined with Lemma 5.B show that there exists a $(j+3q+1)$-tuple $(d_q^i)_{i\in\{0,\dots,j+3q\}} \in \mathbb{R}^{j+3q+1}$ such that

$$S_q^{\varphi}(\sigma) = \sum_{i=0}^{j+3q} d_q^i \, \Psi_i^{\mathsf{GH}}\left(\breve{\mu}^{\frac{1}{4}}\sigma\right) \quad \forall \sigma \in \mathbb{R}. \tag{6.8}$$

Namely, on one hand, the relations (5.16) indicate that the ℓ-th derivative of Ψ_i^{GH} or the function obtained by multiplying Ψ_i^{GH} by z^ℓ can be expressed as a linear combination of Gauss-Hermite function up to order $i+\ell$. On the other hand, Lemma 5.B indicates that $\mathbf{A}_q^- \varphi_0$ and $(\lambda_\ell - \mathbf{A}_\ell^-)\varphi_{q-\ell}$, for $1 \le \ell \le q-1$ can respectively be expressed as a linear combination of Gauss-Hermite function up to order $j+q+2$ and $j+3q-2\ell+2$, these two numbers being bounded by $j+3q$.

Equation (6.5a) has a solution in $\mathrm{L}^2(\mathbb{R})$ if, and only if, $\lambda_q\,\varphi_0 + S_q^{\varphi}$ is orthogonal to $\Psi_j^{\mathsf{GH}}(\breve{\mu}^{\frac{1}{4}}\cdot)$; This implies that $\lambda_q = -d_q^j$. Moreover since the operator $-\partial_\sigma^2 + \breve{\mu}\,\sigma^2 - (2j+1)\sqrt{\breve{\mu}}$ is diagonalizable and inversible on $\operatorname{span}(\Psi_i^{\mathsf{GH}}(\breve{\mu}\,\cdot) \mid i \ge 0,\ i \ne j)$, we get $b_q^i = \frac{d_q^i}{2(i-j)}$ for $i \in \{0,\dots,j+3q\} \setminus \{j\}$ and $b_q^j = 0$. □

Remark 6.2. From the proof of Proposition 6.B we can deduce a finite algorithm for the computation of the terms in the asymptotic expansion of the resonance quasi-pairs because the expression of S_q^{φ} in (6.8) involves a finite sum and because the computation of the solution (λ_q, φ_q) is explicit form the coefficients of S_q^{φ}.

The proof that the formal series $\sum_{q\in\mathbb{N}} \lambda_q h^{\frac{q}{2}}$ and $\sum_{q\in\mathbb{N}} \varphi_q h^{\frac{q}{2}}$ obtained from Proposition 6.B give rise to a family of resonance quasi-pairs in the sense of Definition 2.1 can be achieved exactly as in Section 5.2.3 for Case (B). Note that in order to use Borel's Theorem and to obtain the required estimates, we have to show that $\varphi_q \in \mathscr{S}(\mathbb{R}) \cap \mathrm{H}^2(\mathbb{R}, \mathrm{e}^{|\sigma|}\,\mathrm{d}\sigma)$. This properties can be deduced directly from Equation (6.7).

Finally, we can conclude with the proof of Theorem 6.A in a way very similar to the one of Theorem 5.A as detailed in Section 5.2.4.

7. Proximity between quasi-resonances and true resonances

7.1. **Separation of quasi-resonances, quasi-orthogonality of quasi-modes.** For the three cases (A), (B), and (C), cf Theorems 4.A, 5.A, and 6.A, we have exhibited families of resonance quasi-pairs in the sense of Definition 2.1. Namely, for each $j \ge 0$ and $m \ge 1$, we have constructed a quasi-pair $(\underline{k}_{p;j}(m), \underline{u}_{p;j}(m))$ where $\underline{k}_{p;j}(m) \in \mathbb{R}_+$ is a quasi-resonance and $\underline{u}_{p;j}(m) \in \mathrm{H}_p^2(\mathbb{R}^2, \Omega)$ is a compactly supported quasi-mode. Actually, to each quasi-resonance $\underline{k}_{p;j}(m)$, we can associate two quasi-modes: $\underline{u}_{p;j}(m)$ and its conjugate. These quasi-modes are quasi-orthogonal with respect to j and m, as stated in the next lemma.

We consider the Hilbert space $\mathrm{L}^2(\mathbb{R}^2, n(x)^{p+1}\,\mathrm{d}x)$ and denote its scalar product by

$$\langle f, g\rangle = \int_{\mathbb{R}^2} \overline{f(x)}\, g(x)\, n(x)^{p+1}\,\mathrm{d}x \qquad \text{for } f, g \in \mathrm{L}^2(\mathbb{R}^2, n(x)^{p+1}\,\mathrm{d}x).$$

Lemma 7.A. *For all the three cases* (A), (B), *and* (C), *and for all* $i, j \ge 0$ *and* $m, m' \ge 1$, *we have* $\langle \overline{\underline{u}_{p;i}(m)}, \underline{u}_{p;j}(m')\rangle = 0$ *and*

$$\langle \underline{u}_{p;i}(m), \underline{u}_{p;j}(m')\rangle = \begin{cases} 1 & \text{if } m = m' \text{ and } i = j,\\ 0 & \text{if } m \ne m',\\ \mathscr{O}\left(m^{-\infty}\right) & \text{if } m = m' \text{ and } i \ne j. \end{cases}$$

For any $m \geq 1$ and $i, j \geq 0$, we have the separation property

$$\underline{k}_{p;i}(m)^2 - \underline{k}_{p;j}(m)^2 = \begin{cases} C_{ij}^{(\mathrm{A})} m^{\frac{4}{3}} + \mathscr{O}(m) & \text{in Case (A)}\,, \\ C_{ij}^{(\mathrm{X})} m + \mathscr{O}(m^{\frac{1}{2}}) & \text{in Cases (B),(C)}\,, \end{cases} \tag{7.1}$$

with $C_{ij}^{(\mathrm{X})} \neq 0$ if $i \neq j$.

Proof. The relation $\langle \underline{u}_{p;i}(m), \underline{u}_{p;j}(m') \rangle = 1$, for all $j \geq 0$ and $m \geq 1$, comes from the normalization of the quasi-mode in Definition 2.1. The relations $\langle \overline{\underline{u}_{p;i}(m)}, \underline{u}_{p;j}(m') \rangle = 0$, for all $i, j \geq 0$ and $m, m' \geq 1$, and $\langle \underline{u}_{p;i}(m), \underline{u}_{p;j}(m') \rangle = 0$, for all $i, j \geq 0$ and $m \neq m'$, $m, m' \geq 1$, are deduced from the identity $\int_0^{2\pi} \mathrm{e}^{\mathrm{i}q\theta}\, \mathrm{d}\theta = 0$ for all integer $q \neq 0$.

For the last estimate, we consider $i \neq j$, $i, j \geq 0$, and $m \geq 1$. By construction, there exists $R_q \in \mathrm{L}^2(\mathbb{R}^2)$, for $q \in \{i, j\}$, such that $\|R_q\|_{\mathrm{L}^2(\mathbb{R}^2)} = \mathscr{O}(m^{-\infty})$ and

$$\underline{k}_{p;q}(m)^2\, n^{p+1}\, \underline{u}_{p;q}(m) = -\operatorname{div}\left(n^{p-1}\, \nabla \underline{u}_{p;q}(m)\right) - R_q. \tag{7.2}$$

Using this identity, conjugated, for $q = i$, we deduce:

$$\underline{k}_{p;i}(m)^2 \int_{\mathbb{R}^2} \overline{\underline{u}_{p;i}(m)}\, \underline{u}_{p;j}(m)\, n^{p+1}\, \mathrm{d}x = \\ - \int_{\mathbb{R}^2} \operatorname{div}\left(n^{p-1}\, \nabla \overline{\underline{u}_{p;i}(m)}\right) \underline{u}_{p;j}(m)\, \mathrm{d}x - \int_{\mathbb{R}^2} \overline{R_i}\, \underline{u}_{p;j}(m)\, \mathrm{d}x.$$

Integrating by parts and using again (7.2), we get

$$\left(\underline{k}_{p;i}(m)^2 - \underline{k}_{p;j}(m)^2\right) \left\langle \underline{u}_{p;i}(m), \underline{u}_{p;j}(m) \right\rangle_{\mathrm{L}^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \left(\overline{\underline{u}_{p;i}(m)}\, R_j - \overline{R_i}\, \underline{u}_{p;j}(m) \right) \mathrm{d}x.$$

Taking the modulus and using Cauchy-Schwarz inequality, we obtain

$$\left|\underline{k}_{p;i}(m)^2 - \underline{k}_{p;j}(m)^2\right| \left| \left\langle \underline{u}_{p;i}(m), \underline{u}_{p;j}(m) \right\rangle_{\mathrm{L}^2(\mathbb{R}^2)} \right| \leq \|R_i\|_{\mathrm{L}^2(\mathbb{R}^2)} + \|R_j\|_{\mathrm{L}^2(\mathbb{R}^2)} \\ = \mathscr{O}\left(m^{-\infty}\right).$$

Then we use the separation property (7.1) (which is an obvious consequence of asymptotic formulas for $\underline{k}_{p;j}(m)$ in each case) and finally get the estimate $\langle \underline{u}_{p;i}(m), \underline{u}_{p;j}(m') \rangle = \mathscr{O}(m^{-\infty})$. ☐

7.2. **Spectral-like theorems for resonances.** We have constructed well separated quasi-pairs for the operator

$$P := -n^{-p-1} \operatorname{div}(n^{p-1}\, \nabla \cdot)$$

with domain $\mathrm{H}^2_p(\mathbb{R}^2, \Omega)$ on the Hilbert space $\mathrm{L}^2(\mathbb{R}^2, n(x)^{p+1}\, \mathrm{d}x)$. The operator P is self-adjoint and its spectrum $\Sigma(P)$ reduces to its essential spectrum, equal to $[0, +\infty)$. If we apply the spectral theorem [13, Theorem 5.9] to our quasi-resonances for the operator P, we get that for each quasi-resonance $\underline{k}_{p;j}(m)$ there exists an interval I of length $\mathscr{O}(m^{-\infty})$ such that the intersection $\Sigma(P) \cap I$ is non empty, which is useless, since we know already that $\Sigma(P) = [0, +\infty)$.

If the operator P has been defined as the Dirichlet realization of $-n^{-p-1} \operatorname{div}(n^{p-1}\, \nabla \cdot)$ on a bounded open set containing $\overline{\Omega}$, then its spectrum would have been discrete. In such case, the application of the spectral theorem would be more significant. Nevertheless, this procedure of cut-off would not inform us about resonances.

That is why we need to use a spectral-like theorem for resonances. Two statements are available in the literature: one from TANG and ZWORSKI [29], and another from STEFANOV [27]. Those theorems lie in the *black box scattering* framework. We are going to present main assumptions and results of these papers in a simplified way, convenient for our application.

In dimension 2, the main ingredients are

- A complex Hilbert space $\mathscr{H}$ with orthogonal decomposition (with positive ϱ_0)
$$\mathscr{H} = \mathscr{H}_{\varrho_0} \oplus \mathrm{L}^2(\mathbb{R}^2 \setminus B(0,\varrho_0))$$
- A family of unbounded selfadjoint operators $h \mapsto P(h)$ on $\mathscr{H}$ with domain independent of h, whose projection onto $\mathrm{L}^2(\mathbb{R}^2 \setminus B(0,\varrho_0))$ coincides with $\mathrm{H}^2(\mathbb{R}^2 \setminus B(0,\varrho_0))$.

We introduce the following assumptions
$$\mathbb{1}_{B(0,\varrho_0)}(P(h)-\mathrm{i})^{-1} \quad \text{compact} \quad \mathscr{H} \to \mathscr{H} \tag{H1}$$
and
$$\mathbb{1}_{\mathbb{R}^2\setminus B(0,\varrho_0)} P(h)u = -h^2 \Delta u\big|_{\mathbb{R}^2\setminus B(0,\varrho_0)}. \tag{H2}$$
Then we choose $\varrho \gg \varrho_0$ and periodize $P(h)$ outside $B(0,\varrho_0)$, obtaining an operator $P^\sharp(h)$ on the Hilbert space
$$\mathscr{H}^\sharp = \mathscr{H}_{\varrho_0} \oplus \mathrm{L}^2(M \setminus B(0,\varrho_0)) \quad \text{with} \quad M = (\mathbb{R}/\varrho\mathbb{Z})^2$$
Denoting by $N(P^\sharp(h), I)$ the number of eigenvalues in I, we write the third assumption as
$$N(P^\sharp(h),[-\lambda,\lambda]) = \mathscr{O}\left((\lambda/h^2)^{n^\sharp/2}\right) \quad \lambda\to\infty, \quad \text{for some} \quad n^\sharp \geq 2. \tag{H3}$$
Let us denote by $\mathfrak{X}(P(h))$ the set of poles of the resolvent $z \mapsto (P(h)-z)^{-1}$. In dimension 2, this set is a subset of the Riemann logarithmic surface, its elements satisfy $\arg z < 0$ with our convention for the definition of resonances.

Now we can state a simplified version of the main result of [29]:

Theorem 7.B ([29]). *Let $P(h)$ satisfy hypotheses (H1), (H2), and (H3). Assume that there exists for any $h \in (0,h_0]$ a quasi-pair $(E(h),u(h))$ with $E(h) \subset [E_0-h, E_0+h]$ for some real E_0, and with $u(h)$ normalized in $\mathscr{H}$ and compactly supported independently of h. The quasi-pairs are supposed to satisfy the residue estimate*
$$\|\,(P(h)-E(h))\,u(h)\|_{\mathscr{H}} = \mathscr{O}(h^\infty).$$
Then for any $h \in (0,h_0']$ with a positive h_0' small enough, there exists a resonance pole $z(h) \in \mathfrak{X}(P(h))$ such that
$$|E(h) - z(h)| = \mathscr{O}(h^\infty).$$

The result in [27] is more precise but requires one more hypothesis, according to which the number of resonance poles is not too large: For some positive integers N and N'
$$\operatorname{Card}\left\{z \in \mathfrak{X}(P(h)), \quad a_0 \leq |z| \leq b_0, \quad -\operatorname{Im} z < h^N\right\} \leq C_{a_0,b_0} h^{N'}. \tag{H4}$$

Our simplified version of the main result of [27] follows:

Theorem 7.C ([27]). *Let $\mathfrak{H}$ be a infinite subset of $(0,1]$ with accumulation point at 0. Let $P(h)$ satisfy hypotheses (H1), (H2), (H3), and (H4). Assume that, for any $h \in (0,h_0] \cap \mathfrak{H}$, there exists d quasi-pair $(E_\ell(h), u_\ell(h))$ with $E_1(h) = \ldots = E_d(h) \in [a_0,b_0]$, and with $u_\ell(h)$ compactly supported independently of h, and almost orthonormal: $|\langle u_i(h), u_\ell(h)\rangle_{\mathscr{H}} - \delta_{i\ell}| = \mathscr{O}(h^\infty)$. The quasi-pairs are supposed to satisfy the residue estimate*
$$\|\,(P(h)-E_\ell(h))\,u_\ell(h)\|_{\mathscr{H}} = \mathscr{O}(h^\infty), \quad \ell = 1,\ldots,d.$$
Then for any $h \in (0,h_0'] \cap \mathfrak{H}$ with a positive h_0' small enough, there exists d resonance poles $z_\ell(h) \in \mathfrak{X}(P(h))$ with repetition according to multiplicity, such that
$$|E_\ell(h) - z_\ell(h)| = \mathscr{O}(h^\infty), \quad \ell = 1,\ldots,d.$$

The distinction between the two latter theorems is the consideration of multiplicity in Theorem 7.C. The multiplicity of a resonance pole z_0 is understood as the rank of the operator

$$\frac{1}{2\mathrm{i}\pi}\int_{|z-z_0|=\varepsilon}(P(h)-z)^{-1}\,\mathrm{d}z$$

for $\varepsilon>0$ small enough to isolate the pole z_0 and $(P(h)-z)^{-1}$ is the meromophic extension of the resolvent [7, Definition 4.6].

7.3. **Application of the spectral-like theorems to disks with radially varying index.** We apply the above theorems to our situation. We set

$$P(h)=h^2P \quad \text{with} \quad P=-n^{-p-1}\operatorname{div}(n^{p-1}\nabla\cdot) \quad \text{on} \quad \mathscr{H}=\mathrm{L}^2(\mathbb{R}^2,n(x)^{p+1}\,\mathrm{d}x).$$

The subset $\mathfrak{H}$ is $\{h=\frac{1}{m},\ m\in\mathbb{N}^*\}$. Hypotheses (H1) and (H2) are easy to check. Concerning (H3), by using the max–min principle for eigenvalues [11, Theorem 11.12] and comparing the eigenvalues of $P^\sharp(h)$ with $(-h^2\Delta)^\sharp$ on a large torus $M=(\mathbb{R}/\varrho\mathbb{Z})^2$ for $\varrho\gg R$, we get that the counting function $N(P^\sharp(h),[-\lambda,\lambda])=\mathscr{O}(\lambda/h^2)$ for $\lambda\to\infty$, which yields $n^\sharp=2$.

Concerning (H4), we simply have to use the main theorem in [30] (with $\phi(t)=t^2$ and $a=1+\varepsilon$).

Then to bridge our families of resonance quasi-pairs with the formalism of [27], we set, for any chosen $p\in\{\pm1\}$ and any chosen $j\in\mathbb{N}$:

$$E_\ell(h)=h^2\,\underline{k}_{p;j}(\tfrac{1}{h})^2,\quad h\in\mathfrak{H},\quad \ell=1,2$$

with

$$u_1(h)=\underline{u}_{p;j}(\tfrac{1}{h})\quad\text{and}\quad u_2(h)=\overline{\underline{u}_{p;j}(\tfrac{1}{h})},\quad h\in\mathfrak{H}.$$

Then, as h tends to 0, the energy $E_\ell(h)$ converges to $1/(Rn(R))^2$ in Cases (A) and (B), and to $1/(R_0n(R_0))^2$ in Case (C). Applying Theorem 7.C and coming back to resonances by the formula

$$k_m=m\sqrt{z_1(\tfrac{1}{m})}\quad\text{and}\quad k'_m=m\sqrt{z_2(\tfrac{1}{m})},\quad m\ge1$$

we have proved:

Theorem 7.D. *For $p\in\{\pm1\}$, $j\in\mathbb{N}$, and m large enough, there exist two resonances k_m and k'_m (counted with multiplicity) such that, as $m\to+\infty$, we have*

$$\max\left(\left|\underline{k}_{p;j}(m)-k_m\right|,\left|\underline{k}_{p;j}(m)-k'_m\right|\right)=\mathscr{O}\left(m^{-\infty}\right).$$

Remark 7.1. *(i)* It is plausible that modes associated with the true resonances k_m and k'_m have m as polar mode index. The proof of this would require to apply a spectral theorem to the family of one dimensional resonance problems (1.2a)-(1.2b), which seemingly does not enter the general framework of [29] or [27]. Nevertheless, finite computations performed with perfectly matched layers displayed numerical modes complying with the structure of quasi-modes (see [18, Chapter 7] and a forthcoming paper of the authors).

(ii) Throughout the paper we have assumed that $p\in\{\pm1\}$, because of the physical motivation, but without any change, everything is true for $p\in\mathbb{R}$.

Appendix A. Technical lemmas

A.1. **Explicit solutions to some differential equations.**

Lemma A.1. *For all $\ell\in\mathbb{N}$, let denote by γ_ℓ the mapping $z\in\mathbb{R}\mapsto z^\ell\mathrm{e}^{-z}$ and for $d\ge1$ by $\mathscr{E}_d$ the set $\{\gamma_\ell;\ell=0,\dots,d\}$. The operator $-\partial_z^2+1$ is a bijection from the vector-space* $\operatorname{span}(\mathscr{E}_d\setminus\{\gamma_0\})$ *to the vector-space* $\operatorname{span}(\mathscr{E}_{d-1})$.

Proof. For all $\ell \geq 0$, we readily obtain $(-\partial_z^2+1)\gamma_\ell = \ell(\ell-1)\,\gamma_{\ell-2} - 2\ell\,\gamma_{\ell-1}$. Considering the vector basis $(\gamma_1,\dots,\gamma_d)$ and $(\gamma_0,\dots,\gamma_{d-1})$ of $\operatorname{span}(\mathscr{E}_d \setminus \{\gamma_0\})$ and $\operatorname{span}(\mathscr{E}_{d-1})$ respectively, the matrix of $-\partial_z^2+1$ from $\operatorname{span}(\mathscr{E}_d \setminus \{\gamma_0\})$ to $\operatorname{span}(\mathscr{E}_{d-1})$ is an upper triangular matrix with a determinant equal to $(-2)^d d! \neq 0$. □

Lemma A.2. *For all $\ell \in \mathbb{N}$, let denote by α_ℓ the mapping $z \in \mathbb{R} \mapsto z^\ell \mathsf{A}(z)$ and by β_ℓ the mapping $z \in \mathbb{R} \mapsto z^\ell \mathsf{A}'(z)$ where A is the mirror Airy's function. For all $d \in \mathbb{N}$, let $\mathscr{A}_d$ be the set $\{\alpha_\ell, \beta_\ell; \ell = 0,\dots,d\}$. The operator $-\partial_z^2 - z$ is a bijection from the vector-space $\operatorname{span}(\mathscr{A}_d \setminus \{\alpha_0\})$ to the vector-space $\operatorname{span}(\mathscr{A}_d \setminus \{\beta_d\})$.*

Proof. From the definition of Airy's function, we have $\mathsf{A}'' = -z\mathsf{A}$ and therefore, for $\ell \geq 0$,

$$(-\partial_z^2 - z)\alpha_\ell = -\ell(\ell-1)\,\alpha_{\ell-2} - 2\ell\,\beta_{\ell-1},$$
$$(-\partial_z^2 - z)\beta_\ell = -\ell(\ell-1)\,\beta_{\ell-2} + (2\ell+1)\,\alpha_\ell.$$

Considering the vector basis $(\beta_0,\alpha_1,\beta_1,\dots,\alpha_d,\beta_d)$ of $\operatorname{span}(\mathscr{A}_d \setminus \{\alpha_0\})$ and the vector basis $(\alpha_0,\beta_0,\alpha_1,\beta_1,\dots,\alpha_d)$ of $\operatorname{span}(\mathscr{A}_d \setminus \{\beta_d\})$, the matrix of $-\partial_z^2 - z$ considered from $\operatorname{span}(\mathscr{A}_d \setminus \{\alpha_0\})$ to $\operatorname{span}(\mathscr{A}_d \setminus \{\beta_d\})$ is an upper triangular matrix with a determinant equal to $(-1)^d(2d+1)! \neq 0$. □

A.2. **Half harmonic oscillator.** We recall that we denote by $\mathrm{L}^2(\mathbb{R}_-,\omega_x)$ and $\mathrm{H}^\ell(\mathbb{R}_-,\omega_x)$ the weighted Sobolev spaces with measure $\omega_x(\sigma)\,\mathrm{d}\sigma$ where $\omega_x : \sigma \mapsto \exp(2^x|\sigma|)$ for x real and that $\Psi^{\mathsf{GH}}_{2j+1}$ refers to the Gauss-Hermite function of order $2j+1$, see [1, 20].

Lemma A.3. *Let $\beta \in \mathbb{R}$, $\theta > 0$, and $j \in \mathbb{N}$. For any $S \in \mathrm{L}^2(\mathbb{R}_-,\omega_\beta) \cap \operatorname{span}(\Psi^{\mathsf{GH}}_{2j+1})^\perp$ there exists a unique solution to the problem: Find $w \in \mathrm{H}^2(\mathbb{R}_-) \cap \operatorname{span}(\Psi^{\mathsf{GH}}_{2j+1})^\perp$ such that*

$$\begin{cases} -w''(x) + (x^2 - 4j - 3)w(x) = S(x) & \forall x \in (-\infty,0) \\ w(0) = 0 \end{cases}. \tag{A.1}$$

Moreover, this solution belongs to $\mathrm{H}^1_0(\mathbb{R}_-,\omega_\beta) \cap \mathrm{H}^2(\mathbb{R}_-,\omega_{\beta-\theta})$.

Proof. Existence and unicity rely on the fact that the family $(\Psi^{\mathsf{GH}}_{2\ell+1})_{\ell\in\mathbb{N}}$ is a Hilbert basis of $\mathrm{L}^2(\mathbb{R}_-)$ and that the half harmonic oscillator operator is diagonalizable on $\operatorname{span}(\Psi^{\mathsf{GH}}_{2\ell+1} \mid \ell \in \mathbb{N})$. The solution to problem (A.1) can be written as

$$w = \sum_{\ell=0,\,\ell\neq j}^{+\infty} \frac{1}{4(\ell-j)}\,(S,\,\varsigma_\ell\Psi^{\mathsf{GH}}_{2\ell+1})_{\mathrm{L}^2(\mathbb{R}_-)}\,\varsigma_\ell\Psi^{\mathsf{GH}}_{2\ell+1} \tag{A.2}$$

where $\varsigma_\ell = \|\Psi^{\mathsf{GH}}_{2\ell+1}\|^{-1}_{\mathrm{L}^2(\mathbb{R}_-)}$ and we clearly have $w \in \mathrm{H}^2(\mathbb{R}_-) \cap \operatorname{span}(\Psi^{\mathsf{GH}}_{2j+1})^\perp$.

We set $J := \sqrt{4(j+1) + 2^{2\beta-2}}$ so that, for all $x \leq -J$, we have $V(x) := x^2 - 4j - 3 - 2^{2\beta-2} \geq 1$. Let $\phi \in \mathscr{C}^\infty(\mathbb{R})$ such that $0 \leq \phi \leq 1$, $\phi(x) = 0$ for all $x \leq 0$, and $\phi(x) = 1$ for all $x \geq 1$, let $b = (1+2^\beta)\max_{\mathbb{R}}|\phi'|$ and let $a > J + 2b$. We define a cut-off function $\chi_a \in \mathscr{C}^\infty_{\mathrm{comp}}(\mathbb{R}_-)$ by

$$\chi_a(x) = \phi(b^{-1}(x+a)) \cdot \phi(-b^{-1}(x+J)), \quad \forall x \in \mathbb{R}_-.$$

We also define $\chi(x) = \phi(-b^{-1}(x+J))$ for $x \in \mathbb{R}_-$. Note that, for all $a > J + 2b$, we have $|\chi_a'| \leq C$ where $C = (1+2^\beta)^{-1}$. Let also $\widehat{w} := w\,\omega_{\beta-1}$ and $\widehat{S} := S\,\omega_{\beta-1}$. Multiplying both sides of equation (A.1) by $\chi_a\,w\,\omega_\beta$ and integrating over $\mathbb{R}_-$, yields

$$\int_{-\infty}^0 \Big(w'\,(\chi_a\,w\,\omega_\beta)' + \chi_a\,\big(x^2 - 4j - 3\big)\,\widehat{w}^2\Big)\,\mathrm{d}x = \int_{-\infty}^0 \chi_a\,\widehat{S}\,\widehat{w}\,\mathrm{d}x.$$

Since $w'\,\omega_{\beta-1} = \widehat{w}' + 2^{\beta-1}\widehat{w}$ and $w'(x)(w(x)\,\omega_\beta)' = \widehat{w}'(x)^2 - 2^{2\beta-2}\widehat{w}^2(x)$, we deduce that

$$\int_{-\infty}^{0} \chi_a\,\widehat{w}'^2 + \chi_a\,V\,\widehat{w}^2\,\mathrm{d}x + \int_{-\infty}^{0} \chi_a'\widehat{w}\left(\widehat{w}' + 2^{\beta-1}\widehat{w}\right)\,\mathrm{d}x = \int_{-\infty}^{0} \chi_a\,\widehat{S}\,\widehat{w}\,\mathrm{d}x. \tag{A.3}$$

For the first term on the left hand side of (A.3), since $\chi_a\,V \geq \chi_a$, we have

$$\int_{-\infty}^{0} \chi_a\,\widehat{w}'^2 + \chi_a\,V\,\widehat{w}^2\,\mathrm{d}x \geq \int_{-\infty}^{0} \chi_a\left(\widehat{w}'^2 + \widehat{w}^2\right)\,\mathrm{d}x. \tag{A.4}$$

Then, for the second term on the left hand side of (A.3), since $\chi_a' \geq -C$, $1 \geq \chi_a$, and b is such that $C(1+2^\beta) = 1$, we have

$$\begin{aligned}\int_{-\infty}^{0} \chi_a'\widehat{w}\left(\widehat{w}' + 2^{\beta-1}\widehat{w}\right)\,\mathrm{d}x &\geq -C\int_{-\infty}^{0} \widehat{w}\,\widehat{w}' + 2^{\beta-1}\widehat{w}^2\,\mathrm{d}x \\ &\geq -\frac{C}{2}\int_{-\infty}^{0} \widehat{w}'^2 + (1+2^\beta)\widehat{w}^2\,\mathrm{d}x \\ &\geq -\frac{1}{2}\int_{-\infty}^{0} \chi_a\left(\widehat{w}'^2 + \widehat{w}^2\right)\,\mathrm{d}x. \end{aligned}\tag{A.5}$$

For the last term on the right hand side of (A.3), since $\chi_a^2 \leq \chi_a$, we have

$$\int_{-\infty}^{0} \chi_a\,\widehat{S}\,\widehat{w}\,\mathrm{d}x \leq \|\widehat{S}\|_{\mathrm{L}^2(\mathbb{R}_-)}\left(\int_{-\infty}^{0} \chi_a\widehat{w}^2\,\mathrm{d}x\right)^{\frac{1}{2}} \leq \|\widehat{S}\|_{\mathrm{L}^2(\mathbb{R}_-)}\,\mathcal{N}_w(a) \tag{A.6}$$

where $\mathcal{N}_w(a) = \sqrt{\int_{-\infty}^{0} \chi_a\left(\widehat{w}'^2 + \widehat{w}^2\right)\,\mathrm{d}x}$. Combining the estimates (A.4), (A.5), and (A.6) yields

$$\mathcal{N}_w(a)^2 \leq 2\|\widehat{S}\|_{\mathrm{L}^2(\mathbb{R}_-)}\,\mathcal{N}_w(a).$$

The function $a \in (J+2,+\infty) \mapsto \mathcal{N}_w(a)$ is not negative and not decreasing, so the function is either always zero or positive for a large enough but in any cases we have

$$\mathcal{N}_w(a) \leq 2\|\widehat{S}\|_{\mathrm{L}^2(\mathbb{R}_-)}.$$

By letting a tends towards $+\infty$, we obtain that

$$\int_{-\infty}^{0} \chi\left(\widehat{w}'^2 + \widehat{w}^2\right)\,\mathrm{d}x \leq 4\|\widehat{S}\|^2_{\mathrm{L}^2(\mathbb{R}_-)}$$

which implies that $\widehat{w}$ belongs to $\mathrm{H}^1_0(\mathbb{R}_-)$. It follows that w belongs to $\mathrm{L}^2(\mathbb{R}_-,\omega_\beta)$. From the relation $w'\,\omega_{\beta-1} = -2^{\beta-1}\widehat{w} - \widehat{w}'$, we deduce that $w \in \mathrm{H}^1_0(\mathbb{R}_-,\omega_\beta)$.

Finally, using the relation $w'' = (x^2 - 4j - 3)w - S$, we get

$$\int_{-\infty}^{0} w''^2\,\omega_{\beta-\theta}\,\mathrm{d}x \leq C\int_{-\infty}^{0} \left(w^2 + S^2\right)\,\omega_\beta\,\mathrm{d}x$$

where $C = \max_{x\in\mathbb{R}_-}(x^2 - 4j - 3)^2\,\omega^{-1}_{\beta+\widetilde{\theta}}(x) < +\infty$ with $\widetilde{\theta} = \ln(1-2^{-\theta})/\ln(2)$. This shows that $w \in \mathrm{H}^2(\mathbb{R}_-,\omega_{\beta-\theta})$. Note that the constant β is replaced by $\beta-\theta$ by the need to take into account the coefficient $x^2 - 4j - 3$. $\square$

A.3. **Borel's Theorem.** Our construction of quasi-modes requires to find a smooth function given its Taylor expansion. This can be achieved using a Borel's like theorem on the spaces of Schwartz functions $\mathscr{S}(\mathbb{R}_\pm)$. We denote by $p_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}_\pm} |x^\alpha \partial_x^\beta f(x)|$, $\alpha, \beta \in \mathbb{N}$, the usual family of semi-norms over $\mathscr{S}(\mathbb{R}_\pm)$.

Lemma A.4. *Let $(f_q)_{q\in\mathbb{N}}$ be a sequence of functions where $f_q \in \mathscr{S}(\mathbb{R}_\pm)$ for all $q \in \mathbb{N}$. There exists $f \in \mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_\pm))$ such that*

$$\partial_t^q f(0,x) = f_q(x), \quad \forall x \in \mathbb{R}_\pm.$$

Proof. This proof is inspired by the proof of [14, theorem 1.2.6] where smooth functions with compact support are replaced by Schwartz functions.

Let $g \in \mathscr{C}^\infty_{\text{comp}}(\mathbb{R})$ be a smooth cut-off function such that $g(t) = 1$ for all $t \in [-1, 1]$. For each $q \in \mathbb{N}$ we introduce the function

$$g_q : (t,x) \in \mathbb{R} \times \mathbb{R}_\pm \longmapsto g\left(\varepsilon_q^{-1} t\right) \frac{t^q}{q!} f_q(x)$$

for some positive number ε_q that will be specified later on. For all $d, \alpha, \beta \in \mathbb{N}$, we have

$$x^\alpha \partial_t^d \partial_x^\beta g_q(t,x) = \varepsilon_q^{q-d} G_q^{(d)}(\varepsilon_q^{-1} t)\, x^\alpha f_q^{(\beta)}(x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}_\pm, \tag{A.7}$$

where $G_q(s) = g(s)\, \frac{s^q}{q!}$. It follows that $\left|x^\alpha \partial_t^d \partial_x^\beta g_q(t,x)\right| \leq C_{q,d}^{\alpha,\beta}\, \varepsilon_q^{q-d}$ where

$$C_{q,d}^{\alpha,\beta} = \sup_{s\in\mathbb{R}} \left|G_q^{(d)}(s)\right|\; p_{\alpha,\beta}(f_q) < +\infty.$$

By choosing $\varepsilon_q = \min_{d+\alpha+\beta<q} (2^q\, C_{q,d}^{\alpha,\beta})^{-\frac{1}{q-d}}$, $q \geq 1$ and $\varepsilon_0 = 2\varepsilon_1$ we obtain that $\left|x^\alpha \partial_t^d \partial_x^\beta g_q(t,x)\right| \leq 2^{-q}$ for all $d, \alpha, \beta \in \mathbb{N}$ and for all $q > d + \alpha + \beta$. Therefore, the sum

$$f = \sum_{q\geq 0} g_q$$

is well defined because the series converge absolutely. Its successive derivatives are equal to the sum of the derivatives of g_g; As a consequence, $f \in \mathscr{C}^\infty([0,1] \times \mathbb{R}_\pm)$. Moreover, from the estimate

$$p_{\alpha,\beta}\left(\partial_t^d f(t,\cdot)\right) \leq \sum_{q=0}^{d+\alpha+\beta} C_{q,d}^{\alpha,\beta}\, \varepsilon_q^{q-d} + 2^{-d-\alpha-\beta}, \qquad \forall t \in [0,1],\ \forall d, \alpha, \beta \in \mathbb{N}$$

we obtain $f \in \mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_\pm))$. From (A.7), we deduce that for all $d \in \mathbb{N}$

$$\partial_t^d f(0,x) = \sum_{q=0}^{+\infty} \varepsilon_q^{q-d}\, G_q^{(d)}(0)\, f_q(x), \qquad \forall x \in \mathbb{R}_\pm.$$

Since g is constant equal to 1 around $t = 0$, we have $G_q^{(d)}(0) = \delta_{q,d}$ where $\delta_{q,d}$ is the Kronecker symbol. This implies that $\partial_t^d f(0,x) = f_d(x)$. □

By Taylor's formula with integral remainder we deduce immediately the following result.

Lemma A.5. *Let f be a function belonging to $\mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_\pm))$. For all integer $N \geq 1$ there exists $R_N \in \mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_\pm))$ such that*

$$f(t,x) = \sum_{q=0}^{N-1} \frac{\partial_t^q f(0,x)}{q!}\, t^q + t^N R_N(t,x), \quad \forall (t,x) \in [0,1] \times \mathbb{R}_\pm.$$

A.4. **Additional result.**

Lemma A.6. *Let* $F : (t, \sigma) \mapsto F(t, \sigma)$ *a function in* $\mathscr{C}^\infty([0,1], \mathscr{S}(\mathbb{R}_-))$. *Then*

$$\int_{-\infty}^{-\delta/t} |F(t,\tau)|^2 \,\mathrm{d}\tau = \mathscr{O}(t^\infty) \quad \textit{as} \quad t \to 0.$$

The same result holds with $\mathbb{R}_-$ *replaced by* $\mathbb{R}_+$ *and* $(-\infty, -\delta/t)$ *replaced by* $(\delta/t, \infty)$.

Proof. It suffices to notice that for any $N \geq 1$, there exists C_N such that

$$|\tau^N F(t,\tau)| \leq C_N, \quad \text{for all } \ (t,\tau) \in [0,1] \times \mathbb{R}_-.$$

Hence $\int_{-\infty}^{-\delta/t} |F(t,\tau)|^2 \,\mathrm{d}\tau \leq \left(\frac{t}{\delta}\right)^{2N-1}$, which proves the lemma. □

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