Vector Potentials

in Three-Dimensional Nonsmooth Domains

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Abstract: This paper presents several results concerning the vector potential which can be associated with a divergence-free function in a bounded three-dimensional domain. Different types of boundary conditions are given, for which the existence, uniqueness and regularity of the potential are studied. This is applied firstly to the finite element discretization of these potentials and secondly to a new formulation of incompressible viscous flow problems.

Résumé : On présente dans cet article un certain nombre de résultats concernant le potentiel vecteur associé à une fonction à divergence nulle dans un ouvert borné de dimension trois. En particulier, plusieurs types de conditions aux limites sont proposés, pour lesquels on discute l'existence, l'unicité et la régularité du potentiel vecteur. On analyse la convergence d'une discrétisation par éléments finis de ces potentiels et on indique une application concernant l'approximation de fluides visqueux incompressibles.

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1. Introduction.

A potential associated with a *divergence-free* function $\boldsymbol{u}$ is a function $\boldsymbol{\psi}$ such that:

$$\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}.$$

In two-dimensional domains, the divergence-free property of a vector field is commonly handled by introducing a scalar potential called stream-function. This leads to a number of boundary value problems and numerical techniques where the stream-function is the new unknown and where either a direct or a mixed variational formulation is used. For instance, in the case of an incompressible viscous flow governed by the Navier-Stokes equations, the stream-function is the solution of a fourth-order problem which can be discretized by Hermite finite elements or spectral methods. It is also possible to introduce the vorticity as a new unknown; this leads to the so-called "$\psi - \omega$" equivalent formulation of the Navier-Stokes equations, which can be discretized either by usual Lagrange finite elements or by spectral techniques.

However, in the three-dimensional case, the situation is much more complicated: a potential can also be used for handling the divergence-free property, however the conditions that must be enforced in order to ensure its existence and uniqueness are not so simple. Three main difficulties appear:

(i) In contrast to the two-dimensional case, the potential is no longer a scalar function, but a vector function with three components, so it is called *vector potential*. As the curl of a gradient is always zero, a global condition must be added to eliminate such gradients; it is called the "gauge condition" and it usually concerns the divergence of the potential. More precisely, the condition that we propose imposes that the vector potential is itself divergence-free. But such a condition is generally not sufficient for uniqueness: for instance, the gradient of any harmonic scalar function is curl-free and divergence-free.

(ii) Adequate boundary conditions must be enforced on the vector potential: they concern either its normal or its tangential component (but not both). Consequently, the functional spaces in which the vector potential is sought for, are no longer standard Sobolev spaces and their imbeddings in these Sobolev spaces depend on the boundary conditions and on the regularity of the domain.

(iii) The geometry of the domain is more complex than in the two-dimensional case, even when it is bounded, since the simple-connexity is no longer linked to the number of connected components of the boundary. So, two different parameters are necessary to characterize the geometry of the domain, related to the number of components of the boundary and to the homotopy group of the domain. They are closely related to the dimensions of the kernels of the problems that we consider, which are spaces of curl- and divergence-free vector fields with null boundary conditions.

The main purpose of this paper is to exhibit the appropriate gauge and boundary conditions on the vector potential associated with a given divergence-free function, in order to ensure its existence and uniqueness. Most of these results are known in somewhat simpler geometries, see Bernardi [5], Bendali, Dominguez and Gallic [3][4] and Dubois [12] for instance; however we intend to prove them in the general case of any domain, possibly multiply-connected, with a Lipschitz-continuous boundary. They are very useful whenever solving problems involving the "**curl curl**" operator. As interesting applications, we study the approximation of a divergence-free function by divergence-free finite element functions constructed as the curl of vector potentials. We also write a variational formulation of the Stokes problem where the only unknown is the vector potential.

An outline of the paper is as follows.

In Section 2, we recall the definition of the functional spaces which are involved in the definition of the vector potentials and we give their main properties: traces, compactness, imbeddings.

Section 3 is devoted to the construction of vector potentials: depending on some boundary conditions on a given function $\boldsymbol{u}$, we prove the existence and uniqueness of an associated vector potential also satisfying some gauge and boundary conditions.

In Section 4, we apply our results along two directions: first, we present a finite element discretization of vector potentials and prove its convergence. Next we construct a new formulation of the Stokes system in terms of a vector potential for the velocity. In both applications, the advantage is that the divergence-free condition is exactly satisfied.

2. Basic properties of the functional spaces.

We give a precise definition of the type of geometry in which we are working, and we introduce the spaces of functions that are used in this paper. Next, we present the basic properties of these spaces, firstly of compactness and secondly of regularity.

2.a. Geometry and notation

Let us recall the definition of a Lipschitz-continuous domain (we refer for instance to Adams [1] and Nečas [24]). For the sake of conciseness, we just write Lipschitz domain.

Notation 2.1. A bounded domain Ω in $\mathbb{R}^3$ is said to be Lipschitz if, for any point $\boldsymbol{x}$ on the boundary $\partial\Omega$, there exist a system of orthogonal coordinates (y_1, y_2, y_3), a cube $U_{\boldsymbol{x}}$ containing $\boldsymbol{x}$, $U_{\boldsymbol{x}} = \prod_{i=1}^{3}]-a_i, a_i[$, and a Lipschitz-continuous mapping $\Phi_{\boldsymbol{x}}$ defined from $]-a_1, a_1[\times]-a_2, a_2[$ into $]-\frac{1}{2}a_3, \frac{1}{2}a_3[$ such that

$$\begin{aligned} \Omega \cap U_{\boldsymbol{x}} &= \{(y_1, y_2, y_3) \in U_{\boldsymbol{x}};\ y_3 > \Phi_{\boldsymbol{x}}(y_1, y_2)\}, \\ \partial\Omega \cap U_{\boldsymbol{x}} &= \{(y_1, y_2, y_3) \in U_{\boldsymbol{x}};\ y_3 = \Phi_{\boldsymbol{x}}(y_1, y_2)\}. \end{aligned} \tag{2.1}$$

The domain Ω is said to be of class $\mathscr{C}^{m,1}$, for an integer $m \geq 1$, if the mappings $\Phi_{\boldsymbol{x}}$ can be chosen m–times differentiable with Lipschitz-continuous partial derivatives of order m.

The Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{R}$, are well-defined on a Lipschitz domain Ω: for $s \geq 0$, $H^s(\Omega)$ is the space of restrictions to Ω of the elements in $H^s(\mathbb{R}^3)$ and $H^{-s}(\Omega)$ is the dual space of $H^s_0(\Omega)$, the closure of $\mathscr{D}(\Omega)$ in $H^s(\Omega)$, where, as usual, $\mathscr{D}(\Omega)$ is the space of all indefinitely differentiable functions with a compact support in Ω. On the boundary of Ω, the spaces $H^s(\partial\Omega)$ can be defined by local charts for $-1 < s < 1$ since Lipschitz mappings preserve such regularities. But, more generally, it is possible to define $H^s(\partial\Omega)$ for any $s > 0$ by the space of traces of $H^{s+\frac{1}{2}}(\Omega)$ and both definitions coincide for $0 < s < 1$. When Ω has a polyhedral boundary, its is possible to give a characterization of $H^s(\partial\Omega)$ by compatibility conditions between the restrictions to the faces of $\partial\Omega$. Similarly, if the domain Ω is of class $\mathscr{C}^{m,1}$, the spaces $H^s(\partial\Omega)$, $-(m+1) < s < m+1$, can be defined by local charts. However, we essentially need the spaces $H^s(\partial\Omega)$ for $s = \pm\frac{1}{2}$.

In all that follows, unless specified, Ω denotes a Lipschitz domain (bounded and connected) in $\mathbb{R}^3$. Then a unit exterior normal vector to the boundary can be defined almost everywhere on $\partial\Omega$; it is denoted by $\boldsymbol{n}$. The generic point in Ω (or $\mathbb{R}^3$) is $\boldsymbol{x} = (x_1, x_2, x_3)$.

Using the derivation in the distribution sense, we can define the operators **curl** and div on $L^2(\Omega)^3$. Indeed, let $< \cdot, \cdot >$ denote the duality pairing between $\mathscr{D}(\Omega)$ and its dual space $\mathscr{D}'(\Omega)$. For any function $\boldsymbol{v} = (v_1, v_2, v_3)$ in $L^2(\Omega)^3$, we have

$$\begin{aligned} &\forall \boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3) \in \mathscr{D}(\Omega)^3, \\ &\quad < \mathbf{curl}\ \boldsymbol{v}, \boldsymbol{\varphi} > = \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\ \boldsymbol{\varphi}\, d\boldsymbol{x} \\ &\qquad = \int_\Omega \Big(v_1\big(\frac{\partial\varphi_3}{\partial x_2} - \frac{\partial\varphi_2}{\partial x_3}\big) + v_2\big(\frac{\partial\varphi_1}{\partial x_3} - \frac{\partial\varphi_3}{\partial x_1}\big) + v_3\big(\frac{\partial\varphi_2}{\partial x_1} - \frac{\partial\varphi_1}{\partial x_2}\big)\Big)\, d\boldsymbol{x}, \end{aligned}$$

$$\forall \varphi \in \mathscr{D}(\Omega), \quad < \mathrm{div}\, \boldsymbol{v}, \varphi > = -\int_\Omega \boldsymbol{v} \cdot \mathbf{grad}\, \varphi \, d\boldsymbol{x} = -\int_\Omega (v_1 \frac{\partial \varphi}{\partial x_1} + v_2 \frac{\partial \varphi}{\partial x_2} + v_3 \frac{\partial \varphi}{\partial x_3})\, d\boldsymbol{x}.$$

This leads to the following definitions.

Definition 2.2. The space $H(\mathbf{curl}, \Omega)$ is defined by

$$H(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3;\ \mathbf{curl}\ \boldsymbol{v} \in L^2(\Omega)^3\}, \tag{2.2}$$

and is provided with the norm

$$\|\boldsymbol{v}\|_{H(\mathbf{curl},\Omega)} = (\|\boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\mathbf{curl}\ \boldsymbol{v}\|^2_{L^2(\Omega)^3})^{\frac{1}{2}}.$$

The space $H(\mathrm{div}, \Omega)$ is defined by

$$H(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3;\ \mathrm{div}\ \boldsymbol{v} \in L^2(\Omega)\}, \tag{2.3}$$

and is provided with the norm

$$\|\boldsymbol{v}\|_{H(\mathrm{div},\Omega)} = (\|\boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\mathrm{div}\ \boldsymbol{v}\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Finally, we set

$$X(\Omega) = H(\mathbf{curl}, \Omega) \cap H(\mathrm{div}, \Omega). \tag{2.4}$$

It is provided with the norm

$$\|\boldsymbol{v}\|_{X(\Omega)} = (\|\boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\mathbf{curl}\ \boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\mathrm{div}\ \boldsymbol{v}\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

These definitions will also be used with Ω replaced by $\mathbb{R}^3$.

Let us firstly recall a basic result which is proven in Girault & Raviart [19, Chapter I, Thms 2.4 & 2.10] or Temam [30, Chapter 1, Thm 1.1] for instance.

Proposition 2.3. *The space $\mathscr{D}(\overline{\Omega})^3$ of the restrictions to $\overline{\Omega}$ of functions of $\mathscr{D}(\mathbb{R}^3)^3$ is dense both in $H(\mathbf{curl}, \Omega)$ and in $H(\mathrm{div}, \Omega)$. It is also dense in $X(\Omega)$.*

As proven in [19, Chapter I, §2], these properties of density allow for defining tangential or normal traces for the functions of these spaces. More precisely, any function $\boldsymbol{v}$ in $H(\mathbf{curl}, \Omega)$ has a tangential trace $\boldsymbol{v} \times \boldsymbol{n}$ in $H^{-\frac{1}{2}}(\partial\Omega)^3$, defined by

$$\forall \boldsymbol{\varphi} \in H^1(\Omega)^3, \quad < \boldsymbol{v} \times \boldsymbol{n}\,,\, \boldsymbol{\varphi} >_{\partial\Omega} = \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\ \boldsymbol{\varphi}\, d\boldsymbol{x} - \int_\Omega \mathbf{curl}\ \boldsymbol{v} \cdot \boldsymbol{\varphi}\, d\boldsymbol{x}, \tag{2.5}$$

where the symbol $< \cdot, \cdot >_{\partial\Omega}$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. Any function $\boldsymbol{v}$ in $H(\mathrm{div}, \Omega)$ has a normal trace $\boldsymbol{v} \cdot \boldsymbol{n}$ in $H^{-\frac{1}{2}}(\partial\Omega)$, defined by

$$\forall \varphi \in H^1(\Omega), \quad < \boldsymbol{v} \cdot \boldsymbol{n}\,,\, \varphi >_{\partial\Omega} = \int_\Omega \boldsymbol{v} \cdot \mathbf{grad}\ \varphi\, d\boldsymbol{x} + \int_\Omega (\mathrm{div}\ \boldsymbol{v})\varphi\, d\boldsymbol{x}. \tag{2.6}$$

We can define the "homogeneous" spaces:

$$H_0(\mathbf{curl},\Omega)=\{\boldsymbol{v}\in H(\mathbf{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n}=\mathbf{0}\text{ on }\partial\Omega\},$$
$$H_0(\mathrm{div},\Omega)=\{\boldsymbol{v}\in H(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n}=0\text{ on }\partial\Omega\}.$$

It is proven in [30] and in [19] that $\mathscr{D}(\Omega)^3$ is dense in $H_0(\mathbf{curl},\Omega)$ and in $H_0(\mathrm{div},\Omega)$.

Definition 2.4. Let $X_N(\Omega)$, $X_T(\Omega)$ and $X_0(\Omega)$ be the following subspaces of $X(\Omega)$

$$\begin{aligned}X_N(\Omega)&=\{\boldsymbol{v}\in X(\Omega);\ \boldsymbol{v}\times\boldsymbol{n}=\mathbf{0}\text{ on }\partial\Omega\},\\ X_T(\Omega)&=\{\boldsymbol{v}\in X(\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n}=0\text{ on }\partial\Omega\}\end{aligned}\tag{2.7}$$

and

$$X_0(\Omega)=X_N(\Omega)\cap X_T(\Omega).\tag{2.8}$$

The aim of this section is to recall or prove some results related to the regularity of the spaces defined above. More precisely, two questions will be discussed: are they imbedded in $H^1(\Omega)^3$? Is the imbedding into $L^2(\Omega)^3$ compact? As will appear, the answer strongly depends on the boundary conditions. The following result is standard (*cf.* [19, Chapter I, Lemma 2.5] for instance).

Theorem 2.5. *The space $X_0(\Omega)$ coincides with $H_0^1(\Omega)^3$.*

Proof: Since the imbedding of $H_0^1(\Omega)^3$ into $X_0(\Omega)$ is obvious, we study the inverse imbedding. Let $\boldsymbol{v}$ be any function in $X_0(\Omega)$. We define the extension

$$\overline{\boldsymbol{v}}=\begin{cases}\boldsymbol{v} & \text{in }\Omega,\\ \mathbf{0} & \text{in }\mathbb{R}^3\setminus\overline{\Omega}.\end{cases}$$

Since $\boldsymbol{v}$ is in $X_N(\Omega)$, it is easy to check from (2.5) that $\mathbf{curl}\,\overline{\boldsymbol{v}}$ belongs to $L^2(\mathbb{R}^3)^3$. Similarly, the fact that $\boldsymbol{v}$ is in $X_T(\Omega)$ implies that $\mathrm{div}\,\overline{\boldsymbol{v}}$ belongs to $L^2(\mathbb{R}^3)$. Next, the function $\overline{\boldsymbol{v}}$ has a compact support, so that its Fourier transform $\hat{\boldsymbol{v}}=(\hat{v}_1,\hat{v}_2,\hat{v}_3)$ is analytic, and the previous properties are written equivalently (here, ξ_1, ξ_2 and ξ_3 stand for the dual variables of x_1, x_2 and x_3):

$$(\xi_2\hat{v}_3-\xi_3\hat{v}_2,\ \xi_3\hat{v}_1-\xi_1\hat{v}_3,\ \xi_1\hat{v}_2-\xi_2\hat{v}_1)\in L^2(\mathbb{R}^3)^3\quad\text{and}\quad\xi_1\hat{v}_1+\xi_2\hat{v}_2+\xi_3\hat{v}_3\in L^2(\mathbb{R}^3).$$

It is then easy to check that, for $1\le i,j\le 3$,

$$\|\xi_i\hat{v}_j\|_{L^2(\mathbb{R}^3)}\le(\|\mathbf{curl}\,\overline{\boldsymbol{v}}\|_{L^2(\mathbb{R}^3)^3}+\|\mathrm{div}\,\overline{\boldsymbol{v}}\|_{L^2(\mathbb{R}^3)}).$$

Hence, $\mathbf{grad}\,\overline{\boldsymbol{v}}$ belongs to $L^2(\mathbb{R}^3)^{3\times 3}$, and we obtain the theorem.

Remark 2.6. By integrating by parts and using a density argument, the following identity is readily checked for any function $\boldsymbol{v}$ in $H_0^1(\Omega)^3$:

$$|\boldsymbol{v}|^2_{H^1(\Omega)^3}=\|\mathbf{curl}\,\boldsymbol{v}\|^2_{L^2(\Omega)^3}+\|\mathrm{div}\,\boldsymbol{v}\|^2_{L^2(\Omega)}.\tag{2.9}$$

Consequently, the isomorphism of Theorem 2.5 is in fact an isometry. Formula (2.9) is a particular case of a more general formula which holds for smooth functions on smooth domains without any boundary conditions on the functions (see Grisvard [22, Thm 3.1.1.2]) and which will be used later on (see Lemma 2.11).

2.b. Compactness properties

As a by-product, Theorem 2.5 implies that the space $X(\Omega)$ is contained in $H^1_{\text{loc}}(\Omega)^3$ (even when Ω is unbounded). However, it is not so clear to decide whether or not the spaces $X_N(\Omega)$, $X_T(\Omega)$ or $X(\Omega)$ are imbedded in $H^1(\Omega)^3$. Let us begin with a result of non-compactness, suggested to us by Murat [23], according to an idea of Tartar.

Proposition 2.7. *The imbedding of $X(\Omega)$ into $L^2(\Omega)^3$ is not compact.*

Proof: Let $(g_k)_k$ be a sequence which tends to 0 weakly but not strongly in $H^{\frac{1}{2}}(\partial\Omega)$. For any k, we consider the unique solution χ_k of the problem

$$\begin{cases} \Delta\chi_k = 0 & \text{in } \Omega, \\ \chi_k = g_k & \text{on } \partial\Omega. \end{cases}$$

Then, the sequence $(\chi_k)_k$ is bounded in $H^1(\Omega)$, and it tends to 0 weakly but not strongly in $H^1(\Omega)$. Finally, for any k, the function $\boldsymbol{v}_k = \mathbf{grad}\ \chi_k$ satisfies

$$\mathbf{curl}\ \boldsymbol{v}_k = \mathbf{0} \quad \text{and} \quad \operatorname{div}\ \boldsymbol{v}_k = 0 \quad \text{in } \Omega.$$

The sequence $(\boldsymbol{v}_k)_k$ tends to $\mathbf{0}$ in $L^2(\Omega)^3$ weakly but not strongly (indeed, if it converged strongly, due to the compact imbedding of $H^1(\Omega)$ into $L^2(\Omega)$, the sequence $(\chi_k)_k$ would converge strongly to 0 in $H^1(\Omega)$, which is forbidden by the choice of $(g_k)_k$).

What can be said in the intermediate cases? We only state a result which is proven by Weber [32].

Theorem 2.8. *The imbeddings of $X_N(\Omega)$ and $X_T(\Omega)$ into $L^2(\Omega)^3$ are compact.*

So homogeneous normal or tangential boundary conditions are sufficient to enforce compactness.

2.c. Regularity properties

The end of this section deals with the imbeddings of both spaces $X_N(\Omega)$ and $X_T(\Omega)$ into $H^1(\Omega)^3$, when the domain Ω is either of class $\mathscr{C}^{1,1}$ (Theorems 2.9 and 2.12) or convex (Theorem 2.17). In the case of smooth domains, the results are due to Friedrichs [16] (see also Gobert [20]). For the space $X_T(\Omega)$, the proof can be found in the book of Duvaut & Lions [13, Chap. 7, Th. 6.1], in [19, Chapter I, §3.5] and in the paper of Foias & Temam [15]. However, we prefer to give a complete proof, in order to extend the result to the case of $X_N(\Omega)$ and also to make more precise assumptions on the regularity of the domain.

Theorem 2.9. *Assume that the domain Ω is of class $\mathscr{C}^{1,1}$. Then, the space $X_T(\Omega)$ is continuously imbedded in $H^1(\Omega)^3$.*

The proof of this theorem involves two lemmas. The first one, a density result, is proven in [13, Chap. 7, Lemme 6.1].

Lemma 2.10. *Assume that the domain Ω is of class $\mathscr{C}^{1,1}$. Then, the space $H^1(\Omega)^3 \cap X_T(\Omega)$ is dense in the space $X_T(\Omega)$.*

Proof: Let $\boldsymbol{v}$ be any function in $X_T(\Omega)$. Applying Proposition 2.3 yields that there is a sequence $(\boldsymbol{v}_k)_k$ of $\mathscr{D}(\overline{\Omega})^3$ which converges to $\boldsymbol{v}$ in $X(\Omega)$. Next, for each k, we consider the unique solution χ_k in $H^1(\Omega)$ with zero mean value, of the problem:

$$\forall \varphi \in H^1(\Omega), \quad \int_\Omega \mathbf{grad}\ \chi_k \cdot \mathbf{grad}\ \varphi \, d\boldsymbol{x} = \int_\Omega \boldsymbol{v}_k \cdot \mathbf{grad}\ \varphi \, d\boldsymbol{x}.$$

Equivalently, it can be noted that χ_k solves the Neumann problem:

$$\begin{cases} \Delta \chi_k = \operatorname{div} \boldsymbol{v}_k & \text{in } \Omega, \\ \partial_n \chi_k = \boldsymbol{v}_k \cdot \boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

Due to the regularity assumption on the domain Ω, for each k, the function χ_k belongs to $H^2(\Omega)$, so that the function $\boldsymbol{v}_k - \mathbf{grad}\ \chi_k$ is in $H^1(\Omega)^3$. Finally, due to the convergence of $(\boldsymbol{v}_k)_k$ in $X(\Omega)$, it is easy to check that the sequence $(\chi_k)_k$ converges in $H^1(\Omega)$ towards the solution χ of the problem:

$$\begin{cases} \Delta \chi = \operatorname{div} \boldsymbol{v} & \text{in } \Omega, \\ \partial_n \chi = \boldsymbol{v} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

This solution also belongs to $H^2(\Omega)$. Hence, the sequence $(\boldsymbol{v}_k - \mathbf{grad}\ \chi_k + \mathbf{grad}\ \chi)_k$ is in $H^1(\Omega)^3 \cap X_T(\Omega)$ and converges to $\boldsymbol{v}$ in $X_T(\Omega)$, which proves the lemma.

The second lemma can be found in [22, Thm 3.1.1.2]. It involves the curvature tensor of the boundary which will be denoted by $\mathscr{B}$: assuming that the boundary is of class $\mathscr{C}^{1,1}$ and using Notation 2.1, we see that the function $\Phi_{\boldsymbol{x}}$ has a.e. second derivatives and that $\mathscr{B}$ coincides with the matrix $(\frac{\partial^2 \Phi_{\boldsymbol{x}}}{\partial y_i \partial y_j})_{1 \le i,j \le 2}$ of these derivatives for appropriate coordinates. Let $\operatorname{Tr} \mathscr{B}$ denote the trace of this operator.

Lemma 2.11. *Assume that the domain Ω is of class $\mathscr{C}^{1,1}$.*
(i) Any function $\boldsymbol{v}$ in $H^1(\Omega)^3 \cap X_N(\Omega)$ satisfies

$$|\boldsymbol{v}|^2_{H^1(\Omega)^3} = \|\mathbf{curl}\ \boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\operatorname{div} \boldsymbol{v}\|^2_{L^2(\Omega)} - \int_{\partial\Omega} (\operatorname{Tr} \mathscr{B})(\boldsymbol{v} \cdot \boldsymbol{n})^2 \, d\boldsymbol{\tau}. \tag{2.10}$$

(ii) Any function $\boldsymbol{v}$ in $H^1(\Omega)^3 \cap X_T(\Omega)$ satisfies

$$|\boldsymbol{v}|^2_{H^1(\Omega)^3} = \|\mathbf{curl}\ \boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\operatorname{div} \boldsymbol{v}\|^2_{L^2(\Omega)} - \int_{\partial\Omega} \mathscr{B}(\boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{v} \times \boldsymbol{n}) \, d\boldsymbol{\tau}. \tag{2.11}$$

By applying this lemma to any function $\boldsymbol{v}$ in $H^1(\Omega)^3 \cap X_N(\Omega)$ or $H^1(\Omega)^3 \cap X_T(\Omega)$ and by noting that (*cf.* [22, Thm 1.5.1.10] for instance)

$$|\int_{\partial\Omega} \mathscr{B}(\boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{v} \times \boldsymbol{n}) \, d\boldsymbol{\tau}| \le c \int_{\partial\Omega} |\boldsymbol{v}|^2 \, d\boldsymbol{\tau} \le \frac{1}{2} |\boldsymbol{v}|^2_{H^1(\Omega)^3} + c' \, \|\boldsymbol{v}\|^2_{L^2(\Omega)^3},$$

with a similar inequality for $\int_{\partial\Omega}(\mathrm{Tr}\,\mathscr{B})(\boldsymbol{v}\cdot\boldsymbol{n})^2\,d\boldsymbol{\tau}$, we obtain in both cases the inequality

$$|\boldsymbol{v}|^2_{H^1(\Omega)^3} \le c\,(\|\boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\mathbf{curl}\ \boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\mathrm{div}\ \boldsymbol{v}\|^2_{L^2(\Omega)}). \tag{2.12}$$

We are now in a position to prove the theorem.

Proof of Theorem 2.9: Let $\boldsymbol{v}$ be any function in $X_T(\Omega)$. Due to Lemma 2.10, there exists a sequence $(\boldsymbol{v}_k)_k$ of $H^1(\Omega)^3 \cap X_T(\Omega)$ which converges to $\boldsymbol{v}$ in $X(\Omega)$. Applying the estimate (2.12) to $\boldsymbol{v}_k$ for each k, we see that the sequence $(\boldsymbol{v}_k)_k$ is bounded in $H^1(\Omega)^3$. Hence, it admits a subsequence which converges weakly in $H^1(\Omega)^3$. Of course, the limit of this subsequence is nothing else but $\boldsymbol{v}$. The continuity of the imbedding follows from (2.12).

Now, we state the corresponding theorem for the space $X_N(\Omega)$. Indeed, its proof requires the result of Theorem 2.9.

Theorem 2.12. *Assume that the domain Ω is of class $\mathscr{C}^{1,1}$. Then, the space $X_N(\Omega)$ is continuously imbedded in $H^1(\Omega)^3$.*

Due to (2.12), this theorem can be derived by exactly the same arguments as for Theorem 2.9, once the following lemma is proven.

Lemma 2.13. *Assume that the domain Ω is of class $\mathscr{C}^{1,1}$. Then the space $H^1(\Omega)^3 \cap X_N(\Omega)$ is dense in the space $X_N(\Omega)$.*

Proof: Let $\boldsymbol{v}$ be any function in $X_N(\Omega)$. Applying Proposition 2.3 yields that there is a sequence $(\boldsymbol{v}_k)_k$ of $\mathscr{D}(\overline{\Omega})^3$ which converges to $\boldsymbol{v}$ in $X(\Omega)$. Let us introduce the space

$$V_T(\Omega) = \{\boldsymbol{w} \in X_T(\Omega);\ \mathrm{div}\ \boldsymbol{w} = 0 \text{ in } \Omega\}. \tag{2.13}$$

Owing to Theorem 2.9, the space $V_T(\Omega)$ is contained in $H^1(\Omega)^3$. Next, for each k, we solve the problem: *find $\boldsymbol{\zeta}_k$ in $V_T(\Omega)$ such that*

$$\forall \boldsymbol{\varphi} \in V_T(\Omega), \quad \int_\Omega \boldsymbol{\zeta}_k \cdot \boldsymbol{\varphi}\,d\boldsymbol{x} + \int_\Omega \mathbf{curl}\ \boldsymbol{\zeta}_k \cdot \mathbf{curl}\ \boldsymbol{\varphi}\,d\boldsymbol{x} = \int_\Omega \boldsymbol{v}_k \cdot \mathbf{curl}\ \boldsymbol{\varphi}\,d\boldsymbol{x}. \tag{2.14}$$

It has a unique solution $\boldsymbol{\zeta}_k$ which belongs to $H^1(\Omega)^3$. To interpret this problem, it is convenient to remove the constraint on the test functions. For this, let $\widetilde{\boldsymbol{\varphi}}$ be any function in $X_T(\Omega)$, and let χ be a solution in $H^2(\Omega)$ of the Neumann problem

$$\begin{cases} \Delta\chi = \mathrm{div}\ \widetilde{\boldsymbol{\varphi}} & \text{in } \Omega, \\ \partial_n\chi = \widetilde{\boldsymbol{\varphi}}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that the function $\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\ \chi$ belongs to $V_T(\Omega)$ and that

$$\int_\Omega \boldsymbol{\zeta}_k \cdot (\mathbf{grad}\ \chi)\,d\boldsymbol{x} = -\int_\Omega (\mathrm{div}\ \boldsymbol{\zeta}_k)\chi\,d\boldsymbol{x} + \int_{\partial\Omega} (\boldsymbol{\zeta}_k\cdot\boldsymbol{n})\chi\,d\boldsymbol{\tau} = 0.$$

Therefore, $\boldsymbol{\varphi}$ can be replaced by $\widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi$ in (2.14), and thus $\boldsymbol{\zeta}_k$ is also a solution in $V_T(\Omega)$ of the problem

$$\forall \widetilde{\boldsymbol{\varphi}} \in X_T(\Omega), \quad \int_\Omega \boldsymbol{\zeta}_k \cdot \widetilde{\boldsymbol{\varphi}}\, d\boldsymbol{x} + \int_\Omega \mathbf{curl}\,\boldsymbol{\zeta}_k \cdot \mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\, d\boldsymbol{x} = \int_\Omega \boldsymbol{v}_k \cdot \mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\, d\boldsymbol{x},$$

or equivalently of

$$\begin{cases} \boldsymbol{\zeta}_k - \Delta \boldsymbol{\zeta}_k = \mathbf{curl}\,\boldsymbol{v}_k & \text{in } \Omega, \\ \operatorname{div}\,\boldsymbol{\zeta}_k = 0 & \text{in } \Omega, \\ \boldsymbol{\zeta}_k \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{\zeta}_k \times \boldsymbol{n} = \boldsymbol{v}_k \times \boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

This boundary value problem is an elliptic system according to the definition of Agmon, Douglis & Nirenberg [2], so that the function $\boldsymbol{\zeta}_k$ belongs to $H^2(\Omega)^3$. Similarly, the solution $\boldsymbol{\zeta}$ of the problem

$$\begin{cases} \boldsymbol{\zeta} - \Delta \boldsymbol{\zeta} = \mathbf{curl}\,\boldsymbol{v} & \text{in } \Omega, \\ \operatorname{div}\,\boldsymbol{\zeta} = 0 & \text{in } \Omega, \\ \boldsymbol{\zeta} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{\zeta} \times \boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

belongs to $H^2(\Omega)^3$. Finally, the sequence $(\boldsymbol{v}_k - \mathbf{curl}\,\boldsymbol{\zeta}_k + \mathbf{curl}\,\boldsymbol{\zeta})_k$ belongs to the space $H^1(\Omega)^3 \cap X_N(\Omega)$ and converges to $\boldsymbol{v}$, which ends the proof.

Remark 2.14. Note that the results of Theorems 2.9 and 2.12 can be extended to the case where the boundary conditions $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ or $\boldsymbol{v} \times \boldsymbol{n} = \mathbf{0}$ are replaced by inhomogeneous ones. More precisely, when the domain Ω is of class $\mathscr{C}^{1,1}$, the space of functions $\boldsymbol{v}$ in $X(\Omega)$ such that $\boldsymbol{v} \times \boldsymbol{n}$ belongs to $H^{\frac{1}{2}}(\partial\Omega)^3$ (resp. $\boldsymbol{v} \cdot \boldsymbol{n}$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$) is imbedded in $H^1(\Omega)^3$.

Next, applying the previous remark to the derivatives of the functions, we derive the following corollary exactly in the same way (see also [15]).

Corollary 2.15. *Assume that the domain Ω is of class $\mathscr{C}^{m,1}$ for an integer $m \geq 1$. Then, the spaces of functions*

$$\{\boldsymbol{v} \in L^2(\Omega)^3;\ \mathbf{curl}\,\boldsymbol{v} \in H^{m-1}(\Omega)^3,\ \operatorname{div}\,\boldsymbol{v} \in H^{m-1}(\Omega) \text{ and } \boldsymbol{v} \times \boldsymbol{n} \in H^{m-\frac{1}{2}}(\partial\Omega)^3\}$$

and

$$\{\boldsymbol{v} \in L^2(\Omega)^3;\ \mathbf{curl}\,\boldsymbol{v} \in H^{m-1}(\Omega)^3,\ \operatorname{div}\,\boldsymbol{v} \in H^{m-1}(\Omega) \text{ and } \boldsymbol{v} \cdot \boldsymbol{n} \in H^{m-\frac{1}{2}}(\partial\Omega)\}$$

are both continuously imbedded in $H^m(\Omega)^3$.

However, the imbeddings of $X_N(\Omega)$ and $X_T(\Omega)$ in $H^1(\Omega)^3$ are no longer valid in general for Lipschitz domains, as will be proven by the following counter-example. Assume that the domain Ω has a "re-entrant edge", *i.e.*, for some local set of cylindrical coordinates (r, θ, z) and for a real number ω with $\pi < \omega < 2\pi$, the two faces

$$0 < r < 1,\ \ \theta = 0,\ \ -1 < z < 1 \quad \text{and} \quad 0 < r < 1,\ \ \theta = \omega,\ \ -1 < z < 1$$

are contained in $\partial\Omega$ while the set

$$0 < r < 1,\ \ 0 < \theta < \omega,\ \ -1 < z < 1$$

is contained in Ω. Now, let χ be an indefinitely differentiable function of one variable, with compact support in $[-R, R]$, which is equal to 1 in a neighbourhood of 0. By choosing R small enough and setting

$$\xi_N(r, \theta, z) = \chi(r)\chi(z) r^{\pi/\omega} \sin(\pi\theta/\omega) \quad \text{and} \quad \xi_T(r, \theta, z) = \chi(r)\chi(z) r^{\pi/\omega} \cos(\pi\theta/\omega),$$

it is easy to check that neither the function ξ_N nor the function ξ_T belong to $H^2(\Omega)$, but that $\Delta\xi_N$ and $\Delta\xi_T$ belong to $L^2(\Omega)$. Taking $\boldsymbol{v}_N = \mathbf{grad}\ \xi_N$ and $\boldsymbol{v}_T = \mathbf{grad}\ \xi_T$, we see that $\boldsymbol{v}_N$ belongs to $X_N(\Omega)$, that $\boldsymbol{v}_T$ belongs to $X_T(\Omega)$ but that none of them belongs to $H^1(\Omega)^3$.

Remark 2.16. However, a regularity result, which is due to Costabel [7], holds for Lipschitz domains: the space $X_N(\Omega)$ (resp. $X_T(\Omega)$) is imbedded in $H^{\frac{1}{2}}(\Omega)^3$ and, more generally, the space of functions $\boldsymbol{v}$ in $X(\Omega)$ such that $\boldsymbol{v} \times \boldsymbol{n}$ belongs to $L^2(\partial\Omega)^3$ (resp. $\boldsymbol{v} \cdot \boldsymbol{n}$ belongs to $L^2(\partial\Omega)$) is imbedded in $H^{\frac{1}{2}}(\Omega)^3$.

We conclude with a result in the case of a convex domain.

Theorem 2.17. *Assume that the domain Ω is convex. Then, the spaces $X_N(\Omega)$ and $X_T(\Omega)$ are both continuously imbedded in $H^1(\Omega)^3$.*

This result has already been established by Nedelec [26] for the space $X_T(\Omega)$ (the proof is also written in [19, Chapter I, Thm 3.9]) and by Saranen [28] for the space $X_N(\Omega)$. It makes use of the following "approximation" of convex domains which can be found for instance in [22, Lemma 3.2.1.1].

Lemma 2.18. *For any convex domain Ω, there exists an increasing sequence of convex open sets $(\Omega_k)_k$ such that each Ω_k is of class $\mathscr{C}^{1,1}$ and that*

$$\forall k, \quad \overline{\Omega}_k \subset \Omega \quad \textit{and} \quad \Omega = \textstyle\bigcup_k \Omega_k.$$

Proof of Theorem 2.17: Let us begin with the case of $X_T(\Omega)$. With any function $\boldsymbol{v}$ in $X_T(\Omega)$ and for each k, we associate the solution χ_k in $H^1(\Omega_k)$ of the problem

$$\forall \varphi \in H^1(\Omega_k), \quad \int_{\Omega_k} \chi_k \varphi\, d\boldsymbol{x} + \int_{\Omega_k} \mathbf{grad}\ \chi_k \cdot \mathbf{grad}\ \varphi\, d\boldsymbol{x} = <\boldsymbol{v} \cdot \boldsymbol{n}, \varphi>_{\partial\Omega_k}. \tag{2.15}$$

We note that χ_k is the solution of the Neumann problem

$$\begin{cases} \chi_k - \Delta\chi_k = 0 & \text{in } \Omega_k, \\ \partial_n \chi_k = \boldsymbol{v} \cdot \boldsymbol{n} & \text{on } \partial\Omega_k, \end{cases}$$

and that it satisfies the estimate

$$\|\chi_k\|_{H^1(\Omega_k)} \le \|\boldsymbol{v}\|_{H(\mathrm{div},\Omega)}. \tag{2.16}$$

We denote by 1_{Ω_k} the characteristic function of Ω_k. Setting: $\boldsymbol{w}_k = \boldsymbol{v} - \mathbf{grad}\ \chi_k$, we are going to prove that there exist:
1) a subsequence of $(1_{\Omega_k}\, \boldsymbol{w}_k)_k$ which converges weakly in $L^2(\Omega)^3$ towards a function $\boldsymbol{w}$ of $H^1(\Omega)^3$,
2) a subsequence of $(1_{\Omega_k}\, \mathbf{grad}\ \chi_k)_k$ which converges weakly to $\mathbf{0}$ in $L^2(\Omega)^3$.
This of course implies that $\boldsymbol{v}$ coincides with $\boldsymbol{w}$, hence belongs to $H^1(\Omega)^3$.

1) Let us first consider $(\boldsymbol{w}_k)_k$. It can be checked that for each k, the function $\boldsymbol{w}_k$ belongs to $X_T(\Omega_k)$. Thus, using Theorem 2.9, we derive that it belongs to $H^1(\Omega_k)^3$. Moreover, from Lemma 2.11 (since Ω_k is convex, the curvature tensor of its boundary is nonnegative), we obtain

$$\|\boldsymbol{w}_k\|_{H^1(\Omega_k)^3} \le \|\boldsymbol{w}_k\|_{X(\Omega_k)}$$

and computing the right-hand side with the help of (2.16) leads to

$$\|\boldsymbol{w}_k\|_{H^1(\Omega_k)^3} \le \|\boldsymbol{v}\|_{X(\Omega)}.$$

It follows from this estimate that the sequences $(1_{\Omega_k}\, \boldsymbol{w}_k)_k$ and $(1_{\Omega_k}\, \mathbf{grad}\ \boldsymbol{w}_k)_k$ are bounded respectively in $L^2(\Omega)^3$ and in $L^2(\Omega)^{3\times3}$. Hence, there exists a subsequence $(\boldsymbol{w}_{k'})_{k'}$ such that $(1_{\Omega_{k'}}\, \boldsymbol{w}_{k'})_{k'}$ converges to a function $\boldsymbol{w}$ weakly in $L^2(\Omega)^3$ and $(1_{\Omega_{k'}}\, \mathbf{grad}\ \boldsymbol{w}_{k'})_{k'}$ converges to $\boldsymbol{z}$ weakly in $L^2(\Omega)^{3\times3}$. Next, for any function $\boldsymbol{\mu}$ in $\mathscr{D}(\Omega)^{3\times3}$, since there exists an integer ℓ such that the support of $\boldsymbol{\mu}$ is contained in Ω_ℓ, we have

$$\begin{aligned}\int_\Omega \boldsymbol{z}\otimes\boldsymbol{\mu}\,d\boldsymbol{x} &= \int_{\Omega_\ell} \boldsymbol{z}\otimes\boldsymbol{\mu}\,d\boldsymbol{x} = \lim_{k'\to+\infty}\int_{\Omega_\ell} \mathbf{grad}\ \boldsymbol{w}_{k'}\otimes\boldsymbol{\mu}\,d\boldsymbol{x}\\ &= -\lim_{k'\to+\infty}\int_{\Omega_\ell} \boldsymbol{w}_{k'}\cdot \mathrm{div}\ \boldsymbol{\mu}\,d\boldsymbol{x} = -\int_{\Omega_\ell} \boldsymbol{w}\cdot \mathrm{div}\ \boldsymbol{\mu}\,d\boldsymbol{x} = -<\boldsymbol{w}, \mathrm{div}\ \boldsymbol{\mu}>,\end{aligned}$$

so that $\boldsymbol{z}$ coincides with $\mathbf{grad}\ \boldsymbol{w}$. Hence, the function $\boldsymbol{w}$ belongs to $H^1(\Omega)^3$. Moreover, the lower semi-continuity of the norm for the weak topology implies that

$$\|\boldsymbol{w}\|_{H^1(\Omega)^3} \le \|\boldsymbol{v}\|_{X(\Omega)}.$$

2) Due to (2.16), the sequences $(1_{\Omega_k}\, \chi_k)_k$ and $(1_{\Omega_k}\, \mathbf{grad}\ \chi_k)_k$ are bounded respectively in $L^2(\Omega)$ and in $L^2(\Omega)^3$. Hence, there exists a subsequence $(1_{\Omega_{k'}}\, \chi_{k'})_{k'}$ which converges to χ weakly in $L^2(\Omega)$ and such that the subsequence $(1_{\Omega_{k'}}\, \mathbf{grad}\ \chi_{k'})_{k'}$ converges to $\boldsymbol{\zeta}$ weakly in $L^2(\Omega)^3$. Exactly as previously, it can be checked that $\boldsymbol{\zeta}$ coincides with $\mathbf{grad}\ \chi$. Moreover, χ is a solution of the problem

$$\forall\varphi\in H^1(\Omega),\qquad \int_\Omega \chi\varphi\,d\boldsymbol{x} + \int_\Omega \mathbf{grad}\ \chi\cdot\mathbf{grad}\ \varphi\,d\boldsymbol{x} = <\boldsymbol{v}\cdot\boldsymbol{n}\,,\,\varphi>_{\partial\Omega} = 0,$$

which means that it is 0.

Hence, we have derived the result for $X_T(\Omega)$. The proof for $X_N(\Omega)$ is very similar.

Indeed, with each function $\boldsymbol{v}$ of $X_N(\Omega)$ and for each k, we associate the solution $\boldsymbol{\zeta}_k$ in $V_T(\Omega_k)$ (see (2.13)) of the problem

$$\forall \boldsymbol{\varphi} \in V_T(\Omega_k), \quad \int_{\Omega_k} \boldsymbol{\zeta}_k \cdot \boldsymbol{\varphi} \, d\boldsymbol{x} + \int_{\Omega_k} \mathbf{curl}\ \boldsymbol{\zeta}_k \cdot \mathbf{curl}\ \boldsymbol{\varphi} \, d\boldsymbol{x} = <\boldsymbol{v} \times \boldsymbol{n}\,,\, \boldsymbol{\varphi}>_{\partial\Omega_k}.$$

We set: $\boldsymbol{w}_k = \boldsymbol{v} - \mathbf{curl}\ \boldsymbol{\zeta}_k$. Exactly as previously, we check that there exist a subsequence $(\boldsymbol{w}_{k'})_{k'}$ which converges to a function $\boldsymbol{w}$ of $H^1(\Omega)^3$ and, moreover, a subsequence $(\boldsymbol{\zeta}_{k'})_{k'}$ which converges to the solution $\boldsymbol{\zeta}$ of the following system

$$\begin{cases} \boldsymbol{\zeta} - \Delta\boldsymbol{\zeta} = \mathbf{0} & \text{in } \Omega, \\ \text{div}\ \boldsymbol{\zeta} = 0 & \text{in } \Omega, \\ \boldsymbol{\zeta} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\ \boldsymbol{\zeta} \times \boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

This solution is $\mathbf{0}$, so that the result is proven.

Remark 2.19. We refer to Dauge [9] for the regularity properties of the Neumann problem leading to the following extension of Theorem 2.17: if Ω is a convex polyhedron, there exists a real number $p_\Omega > 2$ such that for all p, $2 \leq p < p_\Omega$, any function $\boldsymbol{v}$ in $X_N(\Omega)$ or $X_T(\Omega)$ with

$$\mathbf{curl}\ \boldsymbol{v} \in L^p(\Omega)^3, \quad \text{div}\ \boldsymbol{v} \in L^p(\Omega),$$

belongs to $W^{1,p}(\Omega)^3$ and satisfies

$$\|\boldsymbol{v}\|_{W^{1,p}(\Omega)^3} \leq c\,\big(\|\boldsymbol{v}\|_{L^p(\Omega)^3} + \|\mathbf{curl}\ \boldsymbol{v}\|_{L^p(\Omega)^3} + \|\text{div}\ \boldsymbol{v}\|_{L^p(\Omega)}\big). \tag{2.17}$$

The upper limit p_Ω only depends on the geometry of the domain Ω, more precisely on the apertures along the edges and the solid angles at the vertices.

3. Vector potentials.

In order to exhibit the results concerning the vector potentials, we need some more precise notation about the geometry of the domain, and we recall a basic result about the vector potential without boundary conditions. Next, we prove successively the existence and uniqueness of tangential and normal vector potentials. In each case, an important tool is the characterization of the kernel. We also give some applications and extensions. Finally, we present other types of boundary conditions.

3.a. Geometry

In order to study the vector potentials, we have to describe with more precision the geometry of the domain. We first need the following definition which extends that of a Lipschitz domain to the case where the domain is not necessarily on one side of its boundary (see Dauge [8, Chapter 1] for similar definitions).

Definition 3.1. A bounded domain Ω in $\mathbb{R}^3$ is called pseudo–Lipschitz if for any point $\boldsymbol{x}$ on the boundary $\partial\Omega$ there exist an integer $r(\boldsymbol{x})$ equal to 1 or 2 and a strictly positive real number ρ_0 such that for all real numbers ρ with $0 < \rho < \rho_0$, the intersection of Ω with the ball with center $\boldsymbol{x}$ and radius ρ, has $r(\boldsymbol{x})$ connected components, each one being Lipschitz.

Remark 3.2. When $r(\boldsymbol{x})$ is equal to 1, Ω lies on only one side of its boundary in the neighbourhood of $\boldsymbol{x}$. If $r(\boldsymbol{x})$ is equal to 2, then Ω lies on both sides of its boundary in the neighbourhood of $\boldsymbol{x}$. This allows to introduce cuts in multiply-connected Lipschitz domains. It is important to note that pseudo–Lipschitz domains have the cone property (*cf.* [1, §4.7]).

As in section 2, we assume that Ω is a Lipschitz domain in $\mathbb{R}^3$.

(i) We do not assume that its boundary $\partial\Omega$ is connected and we denote by Γ_i, $0 \le i \le I$, the connected components of $\partial\Omega$, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$. We also fix a smooth open set $\mathcal{O}$ with a connected boundary (a ball, for instance), such that $\overline{\Omega}$ is contained in $\mathcal{O}$, and we denote by Ω_i, $0 \le i \le I$, the connected component of $\mathcal{O} \setminus \overline{\Omega}$ with boundary Γ_i ($\Gamma_0 \cup \partial\mathcal{O}$ for $i = 0$).

(ii) We do not assume that Ω is simply-connected. Observe that each component Γ_i, $0 \le i \le I$, is an orientable manifold of dimension two and hence is homeomorphic to a torus with p_i holes (we refer to Gramain [21] for these properties). We set $J = \sum_{i=0}^{I} p_i$ and we make the following assumption that permits to "cut" adequately Ω in order to reduce it to a simply-connected region.

Hypothesis 3.3. There exist J connected open surfaces Σ_j, $1 \le j \le J$, called "cuts", contained in Ω, such that:
(i) each surface Σ_j is an open part of a smooth manifold $\mathscr{M}_j$,
(ii) the boundary of Σ_j is contained in $\partial\Omega$ for $1 \le j \le J$,
(iii) the intersection $\overline{\Sigma}_i \cap \overline{\Sigma}_j$ is empty for $i \ne j$,
(iv) the open set

$$\Omega^\circ = \Omega \setminus \bigcup_{j=1}^{J} \Sigma_j$$

is pseudo–Lipschitz and simply-connected.
Any set of such surfaces is called an "admissible set of cuts". This assumption is satisfied in all examples of geometry we have in mind, see the example of a ring ($J = 1$) with $I = 3$ cubic holes in Figure 1.

Note that on $\partial\Omega^\circ$, $r(\boldsymbol{x})$ is equal to 2 if and only if $\boldsymbol{x}$ belongs to a cut $\overline{\Sigma}_j$.

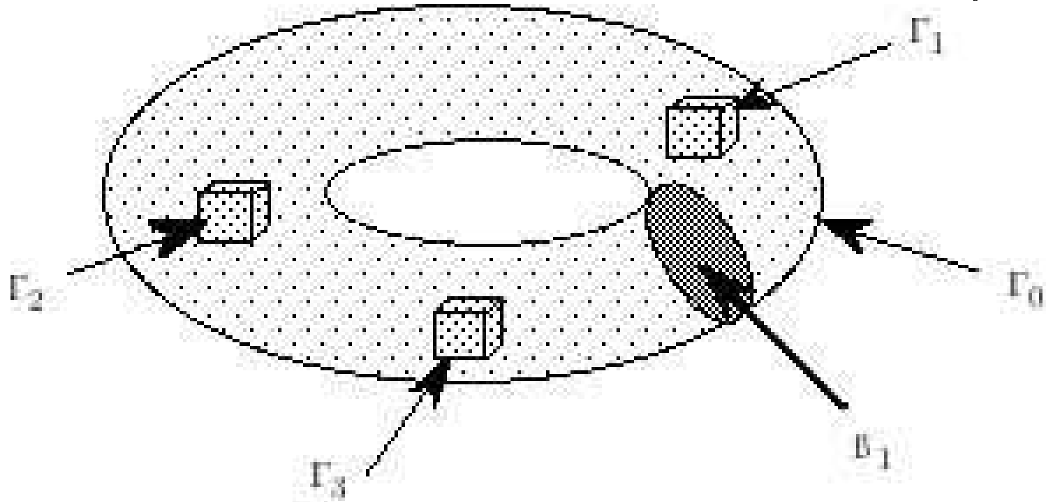

Figure 1

At the beginning of Section 2, we have recalled how to define Sobolev spaces $H^s(\partial\Omega)$ on the boundary of Ω, for $-1 < s < 1$. It is clear that such definitions can be extended to each connected component Γ_i of the boundary, since Γ_i is a manifold without boundary: thus $H^s(\Gamma_i)$ makes sense for $0 \le i \le I$. We also need Sobolev spaces on the cuts Σ_j.

Definition 3.4. For any real number s, the space $H^s(\Sigma_j)$ is the space of restrictions to Σ_j of the distributions belonging to $H^s(\mathscr{M}_j)$.

In the sequel we denote by $H^{\frac{1}{2}}(\Sigma_j)'$, the dual space of $H^{\frac{1}{2}}(\Sigma_j)$.

3.b. Vector potentials without boundary conditions

The results of this section mainly rely on the following basic lemma. Its proof is written in [19, Chapter I, Thm 3.4]; however it is worth recalling the argument since it is the corner stone of the forthcoming analysis.

Lemma 3.5. *For any function $\boldsymbol{u}$ in $H(\mathrm{div}, \Omega)$ which satisfies*

$$\begin{aligned} &\mathrm{div}\, \boldsymbol{u} = 0 \quad \textit{in } \Omega, \\ &< \boldsymbol{u} \cdot \boldsymbol{n}\, , 1 >_{\Gamma_i} = 0, \quad 0 \le i \le I, \end{aligned} \tag{3.1}$$

there exists a vector potential $\boldsymbol{\psi}_0$ in $H^1(\Omega)^3$ such that

$$\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}_0 \quad \textit{in } \Omega \quad \textit{and} \quad \mathrm{div}\, \boldsymbol{\psi}_0 = 0 \quad \textit{in } \Omega. \tag{3.2}$$

Conversely, for any function $\boldsymbol{\psi}_0$ in $H^1(\Omega)^3$, the function $\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}_0$ satisfies (3.1).

Remark 3.6. Note that the condition $< \boldsymbol{u} \cdot \boldsymbol{n}\, , 1 >_{\Gamma_i} = 0$ makes sense because, thanks to (2.6), the restriction of $\boldsymbol{u} \cdot \boldsymbol{n}$ to each Γ_i belongs to $H^{-\frac{1}{2}}(\Gamma_i)$.

Proof of Lemma 3.5: Let $\boldsymbol{u} = (u_1, u_2, u_3)$ be any function satisfying (3.1). Using the above notation, for $0 \le i \le I$, we consider the solution χ_i in $H^1(\Omega_i)$ of the following Neumann problem

$$\begin{cases} -\Delta \chi_0 = 0 \quad \text{in } \Omega_0, \\ \\ \partial_n \chi_0 = \boldsymbol{u}\,.\boldsymbol{n} \quad \text{on } \Gamma_0 \quad \text{and} \quad \partial_n \chi_0 = 0 \quad \text{on } \partial\mathcal{O}, \end{cases} \tag{3.3}$$

$$\begin{cases} -\Delta \chi_i = 0 \quad \text{in } \Omega_i, \\ \\ \partial_n \chi_i = \boldsymbol{u}\,.\boldsymbol{n} \quad \text{on } \Gamma_i, \end{cases} \qquad 1 \le i \le I, \tag{3.4}$$

(recall that $\boldsymbol{n}$ denotes the unit outward normal to Ω and $\mathcal{O}$). Then the function $\overline{\boldsymbol{u}}$ defined by

$$\overline{\boldsymbol{u}} = \begin{cases} \boldsymbol{u} & \text{in } \Omega, \\ \textbf{grad}\ \chi_i & \text{in } \Omega_i,\ 0 \le i \le I, \\ \boldsymbol{0} & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \end{cases} \tag{3.5}$$

belongs to $H(\text{div}, \mathbb{R}^3)$ and is divergence-free in $\mathbb{R}^3$. Taking its Fourier transform (and denoting it by $\hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$) leads to the equation

$$\xi_1 \hat{u}_1 + \xi_2 \hat{u}_2 + \xi_3 \hat{u}_3 = 0. \tag{3.6}$$

Next, observe that conditions (3.2) are satisfied by a function $\boldsymbol{\psi}_0 = (\psi_{01}, \psi_{02}, \psi_{03})$ if and only if

$$\hat{u}_1 = \xi_2 \hat{\psi}_{03} - \xi_3 \hat{\psi}_{02}, \quad \hat{u}_2 = \xi_3 \hat{\psi}_{01} - \xi_1 \hat{\psi}_{03}, \quad \hat{u}_3 = \xi_1 \hat{\psi}_{02} - \xi_2 \hat{\psi}_{01}, \tag{3.7}$$

and

$$\xi_1 \hat{\psi}_{01} + \xi_2 \hat{\psi}_{02} + \xi_3 \hat{\psi}_{03} = 0. \tag{3.8}$$

In $L^2(\mathbb{R}^3)$, system (3.7) (3.8) is equivalent to

$$\hat{\psi}_{01} = \frac{\xi_3 \hat{u}_2 - \xi_2 \hat{u}_3}{|\boldsymbol{\xi}|^2}, \quad \hat{\psi}_{02} = \frac{\xi_1 \hat{u}_3 - \xi_3 \hat{u}_1}{|\boldsymbol{\xi}|^2}, \quad \hat{\psi}_{03} = \frac{\xi_2 \hat{u}_1 - \xi_1 \hat{u}_2}{|\boldsymbol{\xi}|^2}, \tag{3.9}$$

where $|\boldsymbol{\xi}|^2$ stands for $\xi_1^2 + \xi_2^2 + \xi_3^2$. Let us define the function $\boldsymbol{\psi}_0$ by the equations (3.9). It suffices now to check that it belongs to $H^1_{\text{loc}}(\mathbb{R}^3)^3$. Its gradient is clearly in $L^2(\Omega)^{3\times 3}$, due to the inequalities $|\xi_j \hat{\psi}_{0k}| \le \sum_{l=1}^3 |\hat{u}_l|$. Moreover, we fix a function ω of class $\mathscr{C}^\infty$ which has a compact support in $\mathbb{R}^3$ and which is equal to 1 in a neighbourhood of $\boldsymbol{0}$ and we write

$$\hat{\boldsymbol{\psi}}_0(\boldsymbol{\xi}) = \omega(\boldsymbol{\xi}) \hat{\boldsymbol{\psi}}_0(\boldsymbol{\xi}) + \big(1 - \omega(\boldsymbol{\xi})\big) \hat{\boldsymbol{\psi}}_0(\boldsymbol{\xi}).$$

First, observe that the function $\omega \hat{\boldsymbol{\psi}}_0$ has a compact support, so that its inverse Fourier transform is analytic and the restriction of this last function to Ω belongs to $L^2(\Omega)^3$. Second, since $1 - \omega$ vanishes in a neighbourhood of $\boldsymbol{0}$, $(1 - \omega)\hat{\boldsymbol{\psi}}_0$ belongs to $L^2(\mathbb{R}^3)^3$ as well as its inverse Fourier transform. That ends the proof of (3.2).

Conversely, for any function ψ_0 in $H^1(\Omega)^3$, div ($\mathbf{curl}\ \psi_0$) is equal to 0. Moreover, for $0 \le i \le I$, let μ_i be a function of class $\mathscr{C}^\infty$ on $\overline{\Omega}$ which is equal to 1 in a neighbourhood of Γ_i and vanishes in a neighbourhood of Γ_k, $0 \le k \le I$, $k \ne i$. We have

$$< \boldsymbol{u} \cdot \boldsymbol{n}\,, 1 >_{\Gamma_i} = < \mathbf{curl}\ (\mu_i \psi_0) \cdot \boldsymbol{n}\,, 1 >_{\partial\Omega} = \int_\Omega \operatorname{div}\big(\mathbf{curl}\ (\mu_i \psi_0)\big)\, d\boldsymbol{x} = 0,$$

which is the desired condition.

As a first consequence, we give an improvement of the result quoted in Remark 2.16 in the case of a polyhedral domain.

Proposition 3.7. *If the domain Ω is a Lipschitz polyhedron, there exists a real number $s > \frac{1}{2}$ such that $X_T(\Omega)$ and $X_N(\Omega)$ are continuously imbedded in $H^s(\Omega)^3$.*

Proof: The arguments are those of [7], combined with the regularity results of [8]. Without loss of generality, it can be supposed that the domain Ω is simply-connected and has a connected boundary (in the general case, it is the finite union of open polyhedra Ω_k satisfying these properties; introducing a partition of unity by smooth functions χ_k with support in $\overline{\Omega}_k$ and proving the result for each $\chi_k \boldsymbol{u}$ yield the property in the whole domain).

Let $\boldsymbol{u}$ belong to $X_T(\Omega)$. Applying Lemma 3.5 to the function $\mathbf{curl}\ \boldsymbol{u}$ yields the existence of a function $\boldsymbol{w}$ in $H^1(\Omega)^3$ such that: $\mathbf{curl}\ \boldsymbol{w} = \mathbf{curl}\ \boldsymbol{u}$, div $\boldsymbol{w} = 0$ in Ω. Thus, the function $\boldsymbol{u} - \boldsymbol{w}$ has a null **curl**, hence it is the gradient of a function χ in $H^1(\Omega)$. This function χ is a solution of the Neumann problem

$$\begin{cases} \Delta\chi = \operatorname{div}\boldsymbol{u} & \text{in } \Omega, \\ \partial_n \chi = -\boldsymbol{w} \cdot \boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

For each face F of $\partial\Omega$, $\boldsymbol{w} \cdot \boldsymbol{n}$ belongs to $H^{\frac{1}{2}}(F)$. As a consequence of [8, Corollary 23.5], χ belongs to $H^{\frac{3}{2}+\delta}(\Omega)$ for some $\delta > 0$. So, the desired result follows from the decomposition $\boldsymbol{u} = \boldsymbol{w} + \mathbf{grad}\ \chi$.

When $\boldsymbol{u}$ belongs to $X_N(\Omega)$, we denote by $\boldsymbol{z}$ the extension by zero of $\mathbf{curl}\ \boldsymbol{u}$ to $\mathcal{O}$, and we consider the function $\boldsymbol{w}$ in $H^1(\mathcal{O})^3$ given by Lemma 3.5 such that $\mathbf{curl}\ \boldsymbol{w} = \boldsymbol{z}$, div $\boldsymbol{w} = 0$ in $\mathcal{O}$. Next, since the function $\boldsymbol{w}$ is curl-free in $\mathcal{O} \setminus \overline{\Omega}$ which is connected and simply-connected, it is the gradient of a function μ in $H^2(\mathcal{O} \setminus \overline{\Omega})$. Then, it is readily checked that the function $\boldsymbol{u} - \boldsymbol{w}$ is the gradient of the solution χ of the Dirichlet problem

$$\begin{cases} \Delta\chi = \operatorname{div}\boldsymbol{u} & \text{in } \Omega, \\ \chi = -\mu|_{\partial\Omega} & \text{on } \partial\Omega. \end{cases}$$

Then, for each face F, $\mu|_F$ belongs to $H^{\frac{3}{2}}(F)$. As μ is in $H^2(\mathcal{O} \setminus \overline{\Omega})$, the traces of μ on the faces of Ω fit together along the edges, thus $\mu|_{\partial\Omega}$ belongs to $H^{1+\delta}(\partial\Omega)$ for some $\delta > 0$. We conclude like previously with [8, Corollary 18.15].

Remark 3.8. The imbedding of Proposition 3.7 is a special case of the following more general and sharper result. Let s_Ω be the optimal exponent of the Neumann problem, in the sense that the mapping: $g \mapsto \varphi$, where φ is the solution of the Neumann problem with null interior data and boundary data g, is continuous from $H^{s-\frac{1}{2}}(\partial\Omega)$ into $H^{s+1}(\Omega)$ for all s, $0 \le s < s_\Omega$, but not for $s = s_\Omega$. Similarly, let s_Ω^0 be the optimal exponent of the Dirichlet problem. Then, for any real number σ, $0 \le \sigma < \frac{1}{2}$, the subspace of functions $\boldsymbol{v}$ in $X_T(\Omega)$, resp. in $X_N(\Omega)$, such that

$$\mathbf{curl}\ \boldsymbol{v} \in H^\sigma(\Omega)^3, \quad \mathrm{div}\ \boldsymbol{v} \in H^\sigma(\Omega),$$

is continuously imbedded in $H^{\min\{s,\sigma+1\}}(\Omega)^3$ for all s, $0 \le s < s_\Omega$, resp. $0 \le s < s_\Omega^0$, the exponents s_Ω and s_Ω^0 being optimal limits.

3.c. Tangential vector potentials

The next theorem states the first important result: existence of a vector potential in $X_T(\Omega)$. Beforehand, we require the following preliminaries.

Notations 3.9. Let $\{\Sigma_j\,;\, 1 \le j \le J\}$ be an admissible set of cuts, let Ω° be defined as in Hypothesis 3.3 and let us fix a unit normal $\boldsymbol{n}$ on each Σ_j, $1 \le j \le J$.

(i) For any function q in $H^1(\Omega^\circ)$, let us denote by $[q]_j$ the jump of q through Σ_j (*i.e.* the differences of the traces of q) along $\boldsymbol{n}$.

(ii) For any function q in $H^1(\Omega^\circ)$, $\mathbf{grad}\ q$ is the gradient of q in the sense of distributions in $\mathscr{D}'(\Omega^\circ)$. It belongs to $L^2(\Omega^\circ)^3$ and therefore can be extended to $L^2(\Omega)^3$. In order to distinguish this extension from the gradient of q in $\mathscr{D}'(\Omega)$, we denote it by $\widetilde{\mathbf{grad}}\ q$.

(iii) We introduce the space

$$\Theta = \{r \in H^1(\Omega^\circ);\ [r]_j = \text{constant},\ 1 \le j \le J\}. \tag{3.10}$$

The first preliminary lemma is an extension of (2.6).

Lemma 3.10. *If $\boldsymbol{\psi}$ belongs to $H_0(\mathrm{div}, \Omega)$, the restriction of $\boldsymbol{\psi} \cdot \boldsymbol{n}$ to any Σ_j belongs to $H^{\frac{1}{2}}(\Sigma_j)'$ and the following Green's formula holds:*

$$\forall \chi \in H^1(\Omega^\circ), \quad \sum_{j=1}^{J} < \boldsymbol{\psi} \cdot \boldsymbol{n}\,,\, [\chi]_j >_{\Sigma_j} = \int_{\Omega^\circ} \boldsymbol{\psi} \cdot \mathbf{grad}\ \chi\, d\boldsymbol{x} + \int_{\Omega^\circ} (\mathrm{div}\ \boldsymbol{\psi})\, \chi\, d\boldsymbol{x}\,. \tag{3.11}$$

Proof: Let us fix an integer j with $1 \le j \le J$. For any μ in $H^{\frac{1}{2}}(\Sigma_j)$, we can find φ in $H^1(\Omega^\circ)$ such that $[\varphi]_k$ is equal to 0 for all $k \ne j$, $[\varphi]_j$ is equal to μ and φ satisfies the estimate

$$\|\varphi\|_{H^1(\Omega^\circ)} \le c\|\mu\|_{H^{\frac{1}{2}}(\Sigma_j)}\,. \tag{3.12}$$

Now, let $\boldsymbol{\psi}$ be any function in $\mathscr{D}(\Omega)^3$; then Green's formula gives

$$< \boldsymbol{\psi} \cdot \boldsymbol{n}\,,\, \mu >_{\Sigma_j} = \int_{\Omega^\circ} \boldsymbol{\psi} \cdot \mathbf{grad}\ \varphi\, d\boldsymbol{x} + \int_{\Omega^\circ} (\mathrm{div}\ \boldsymbol{\psi})\, \varphi\, d\boldsymbol{x}\,. \tag{3.13}$$

Therefore

$$|< \boldsymbol{\psi}\cdot\boldsymbol{n}\,,\,\mu>_{\Sigma_j}| \le c\|\boldsymbol{\psi}\|_{H(\mathrm{div},\Omega)}\,\|\mu\|_{H^{\frac12}(\Sigma_j)},$$

where c is the constant of (3.12). As a consequence, the restriction of $\boldsymbol{\psi}\cdot\boldsymbol{n}$ to Σ_j belongs to $H^{\frac12}(\Sigma_j)'$, and satisfies

$$\|\boldsymbol{\psi}\cdot\boldsymbol{n}\|_{H^{\frac12}(\Sigma_j)'} \le c\|\boldsymbol{\psi}\|_{H(\mathrm{div},\Omega)}\,.$$

Then the density of $\mathscr{D}(\Omega)^3$ in $H_0(\mathrm{div},\Omega)$ permits to extend the normal trace on Σ_j to functions of $H_0(\mathrm{div},\Omega)$ and the extension has the same properties as above. By using an adequate partition of unity, the Green's formula (3.11) follows easily from (3.13).

The second preliminary lemma gives a characterization of the space Θ.

Lemma 3.11. *Let r belong to $H^1(\Omega^\circ)$. Then r belongs to Θ if and only if*

$$\mathbf{curl}\;(\widetilde{\mathbf{grad}}\;r) = \mathbf{0} \quad \text{in}\;\Omega.$$

Proof: First observe that for all r in $H^1(\Omega^\circ)$ and all $\boldsymbol{\varphi}$ in $\mathscr{D}(\Omega)^3$, we always have

$$< \mathbf{curl}\;(\widetilde{\mathbf{grad}}\;r)\,,\,\boldsymbol{\varphi} >= \int_{\Omega^\circ} \mathbf{grad}\;r\cdot\mathbf{curl}\;\boldsymbol{\varphi}\,d\boldsymbol{x}\,.$$

Then, by applying (3.11), we deduce

$$\forall r\in H^1(\Omega^\circ)\,,\;\forall\boldsymbol{\varphi}\in\mathscr{D}(\Omega)^3\,,\quad < \mathbf{curl}\;(\widetilde{\mathbf{grad}}\;r)\,,\,\boldsymbol{\varphi} >= \sum_{j=1}^{J}\int_{\Sigma_j}[r]_j\,(\mathbf{curl}\;\boldsymbol{\varphi}\cdot\boldsymbol{n})\,d\boldsymbol{\tau}\,. \tag{3.14}$$

Besides that, it can be proved by the Stokes formula that

$$\forall\boldsymbol{\varphi}\in\mathscr{D}(\Omega)^3\,,\;\forall\mu\in L^2(\Sigma_j)\,,\quad \int_{\Sigma_j}\mu\,(\mathbf{curl}\;\boldsymbol{\varphi}\cdot\boldsymbol{n})\,d\boldsymbol{\tau} = <(\mathbf{grad}\;\times\boldsymbol{n})\,\mu\,,\,\boldsymbol{\varphi}>_{\Sigma_j}. \tag{3.15}$$

Now, let r belong to Θ; by applying (3.15) with $\mu = [r]_j$, we infer from (3.14) that $\mathbf{curl}\;(\widetilde{\mathbf{grad}}\;r)$ is equal to $\mathbf{0}$.

Conversely, let r belong to $H^1(\Omega^\circ)$ and satisfy $\mathbf{curl}\;(\widetilde{\mathbf{grad}}\;r) = \mathbf{0}$. It follows from (3.14) and (3.15) that

$$\forall\boldsymbol{\varphi}\in\mathscr{D}(\Omega)^3\,,\quad \sum_{j=1}^{J} <(\mathbf{grad}\;\times\boldsymbol{n})\,[r]_j\,,\,\boldsymbol{\varphi}>_{\Sigma_j} = 0\,.$$

Therefore, by choosing properly the support of $\boldsymbol{\varphi}$, we find that $(\mathbf{grad}\;\times\boldsymbol{n})\,([r]_j) = \mathbf{0}$, for $1\le j\le J$, and $[r]_j$ is constant.

Theorem 3.12. *A function $\boldsymbol{u}$ in $H(\mathrm{div}, \Omega)$ satisfies* (3.1)*:*

$$\mathrm{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega, \qquad < \boldsymbol{u}\cdot\boldsymbol{n}\,,1 >_{\Gamma_i} = 0, \quad 0 \le i \le I\,,$$

if and only if there exists a vector potential $\boldsymbol{\psi}$ in $X(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi} \quad \text{in } \Omega \quad \text{and} \quad \mathrm{div}\ \boldsymbol{\psi} = 0 \quad \text{in } \Omega,\\ &\boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega,\\ &< \boldsymbol{\psi}\cdot\boldsymbol{n}\,,1 >_{\Sigma_j} = 0, \quad 1 \le j \le J\,. \end{aligned} \tag{3.16}$$

This function $\boldsymbol{\psi}$ is unique.

Remark 3.13. The statement of Theorem 3.12 is independent of the particular choice of the admissible set of cuts $\{\Sigma_j\,;\, 1 \le j \le J\}$, in the following sense: if $\boldsymbol{\psi}$ is a vector potential satisfying (3.16) on an "admissible set of cuts", then it will satisfy (3.16) on any other "admissible set of cuts". This can be illustrated by the following simple situation. Let $\boldsymbol{\psi}$ be any function in $H(\mathrm{div}, \Omega)$ such that

$$\mathrm{div}\ \boldsymbol{\psi} = 0 \quad \text{in } \Omega \ \text{ and } \ \boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega\,.$$

Suppose for example that $\mathcal{U}$ is a Lipschitz open set contained in Ω. We have

$$< \boldsymbol{\psi}\cdot\boldsymbol{n}\,,1 >_{\partial\mathcal{U}} = \int_{\mathcal{U}} \mathrm{div}\ \boldsymbol{\psi}\ d\boldsymbol{x} = 0\,.$$

If $\mathcal{U}$ is chosen such that there exist two surfaces Σ and Σ' satisfying

$$\Sigma \cup \Sigma' \subset \partial\mathcal{U} \quad , \quad \partial\mathcal{U} \setminus \{\Sigma \cup \Sigma'\} \subset \partial\Omega \ \text{ and } \ \Sigma \cap \Sigma' = \emptyset\,,$$

then

$$< \boldsymbol{\psi}\cdot\boldsymbol{n}\,,1 >_{\Sigma} = - < \boldsymbol{\psi}\cdot\boldsymbol{n}\,,1 >_{\Sigma'}\,.$$

Thus if the flux of $\boldsymbol{\psi}$ through Σ is 0, then the flux of $\boldsymbol{\psi}$ through Σ' is also 0.

Clearly, the uniqueness of the function $\boldsymbol{\psi}$ will follow from the characterization of the kernel

$$K_T(\Omega) = \{\boldsymbol{w} \in X_T(\Omega);\ \mathbf{curl}\ \boldsymbol{w} = \mathbf{0} \text{ in } \Omega \text{ and } \mathrm{div}\ \boldsymbol{w} = 0 \text{ in } \Omega\}. \tag{3.17}$$

The following proposition was proven first in [15] in a different form and next by Dominguez [10] in smooth domains.

Proposition 3.14. *The dimension of the space $K_T(\Omega)$ is equal to J. It is spanned by the functions $\widetilde{\mathbf{grad}}\ q_j^T$, $1 \le j \le J$, where each q_j^T is the solution in $H^1(\Omega^\circ)$, unique up to an additive constant, of the problem*

$$\begin{cases} -\Delta q_j^T = 0 \quad \text{in } \Omega^\circ,\\ \partial_n q_j^T = 0 \quad \text{on } \partial\Omega,\\ [q_j^T]_k = \text{constant} \quad \text{and} \quad [\partial_n q_j^T]_k = 0, \quad 1 \le k \le J,\\ < \partial_n q_j^T\,,1 >_{\Sigma_k} = \delta_{jk}, \quad 1 \le k \le J. \end{cases} \tag{3.18}$$

Proof: Since Θ is a closed subspace of $H^1(\Omega^\circ)$, applying the Lax–Milgram lemma yields that, for $1 \le j \le J$, the problem: *find* q_j^T *in* Θ *such that*

$$\forall r \in \Theta, \qquad \int_{\Omega^\circ} \mathbf{grad}\ q_j^T \cdot \mathbf{grad}\ r\, d\boldsymbol{x} = [r]_j, \tag{3.19}$$

has a solution which is unique up to an additive constant. Using (3.19) with r in $\mathscr{D}(\Omega)$, we obtain:

$$< \mathrm{div}\ (\widetilde{\mathbf{grad}}\ q_j^T)\,,\, r >= -\int_{\Omega} \widetilde{\mathbf{grad}}\ q_j^T \cdot \mathbf{grad}\ r\, d\boldsymbol{x} = -\int_{\Omega^\circ} \mathbf{grad}\ q_j^T \cdot \mathbf{grad}\ r\, d\boldsymbol{x} = 0\,.$$

Hence $\widetilde{\mathbf{grad}}\ q_j^T$ belongs to $H(\mathrm{div},\Omega)$; moreover, on one hand $-\Delta q_j^T$ is equal to 0 in Ω° and on the other hand, Green's formula with $r \in H_0^1(\Omega)$ yields that the jump of $\partial_n q_j^T$ across any Σ_k is zero. Furthermore, using (3.19) with r in $H^1(\Omega)$ and applying again Green's formula, we obtain

$$0 = \int_{\Omega^\circ} \mathbf{grad}\ q_j^T \cdot \mathbf{grad}\ r\, d\boldsymbol{x} = < \partial_n q_j^T\,,\, r >_{\partial\Omega}.$$

As a consequence, $\partial_n q_j^T$ vanishes on $\partial\Omega$ and $\widetilde{\mathbf{grad}}\ q_j^T$ belongs to $H_0(\mathrm{div},\Omega)$. Thus, Lemma 3.10 implies that the restriction of $\partial_n q_j^T$ to Σ_k belongs to $H^{\frac{1}{2}}(\Sigma_k)'$. Finally, by fixing an integer k, choosing r in Θ such that $[r]_j = \delta_{jk}$ for any j and applying (3.11) with $\chi = r$ and (3.19), we easily derive the last equality in (3.18).
Conversely, it is easy to prove that every solution of (3.18) also solves (3.19). Thus (3.18) has a unique solution.

It stems from the above considerations and Lemma 3.11 that $\widetilde{\mathbf{grad}}\ q_j^T$ belongs to $K_T(\Omega)$. From the last property in (3.18), it is readily checked that the functions $\widetilde{\mathbf{grad}}\ q_j^T$, for $1 \le j \le J$, are linearly independent; thus it remains to show that they span $K_T(\Omega)$. Let $\boldsymbol{w}$ be any function in $K_T(\Omega)$ and consider the function

$$\boldsymbol{u} = \boldsymbol{w} - \sum_{j=1}^{J} < \boldsymbol{w}\cdot\boldsymbol{n}\,,\,1 >_{\Sigma_j} \widetilde{\mathbf{grad}}\ q_j^T\,.$$

It belongs to $K_T(\Omega)$ and satisfies $< \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1 >_{\Sigma_k} = 0$, for $1 \le k \le J$. Since Ω° is simply-connected and $\mathbf{curl}\ \boldsymbol{u}$ is equal to $\mathbf{0}$ in Ω°, there exists a function q in $H^1(\Omega^\circ)$ such that $\boldsymbol{u}$ coincides with $\widetilde{\mathbf{grad}}\ q$ and the fact that div $\boldsymbol{u} = 0$ implies that $\Delta q = 0$ in Ω°. In addition, since $\boldsymbol{u}$ belongs to $H_0(\mathrm{div},\Omega)$, it follows that $\partial_n q = 0$ on $\partial\Omega$ and for any j, the jump of $\partial_n q$ is zero almost everywhere across Σ_j. As $\mathbf{curl}\ \boldsymbol{u}$ is also equal to $\mathbf{0}$ in Ω, Lemma 3.11 implies that q belongs to Θ. Finally, since $< \partial_n q\,,\,1 >_{\Sigma_j} = 0$ for all j, the above properties show that q is the solution of the problem (3.19) with zero right-hand side. Hence q is a constant and $\boldsymbol{u}$ is zero. That ends the proof.

Remark 3.15. In [15], a slightly different basis is exhibited. However, since the q_j^T satisfying (3.18) are the solutions of the variational problem (3.19), they seem more appropriate for numerical computation.

Let us state an immediate consequence of Proposition 3.14.

Corollary 3.16. *On the space $X_T(\Omega)$, the seminorm*

$$\boldsymbol{w} \mapsto \|\mathbf{curl}\ \boldsymbol{w}\|_{L^2(\Omega)^3} + \|\mathrm{div}\ \boldsymbol{w}\|_{L^2(\Omega)} + \sum_{j=1}^{J} |<\boldsymbol{w}\cdot\boldsymbol{n}\,,\,1>_{\Sigma_j}|, \tag{3.20}$$

is equivalent to the norm $\|\cdot\|_{X(\Omega)}$.

Proof: Due to the compactness result of Theorem 2.8 and to the previous characterization of $K_T(\Omega)$, this corollary can be derived from the Peetre–Tartar well-known proposition (see Peetre [27], Tartar [29] or also [19, Chapter I, Thm 2.1]).

Proof of Theorem 3.12:
1) Assume that (3.1) holds and let $\boldsymbol{\psi}_0$ denote the function associated with $\boldsymbol{u}$ by Lemma 3.5. We introduce the solution χ in $H^1(\Omega)$, unique up to an additive constant, of the problem

$$\begin{cases} -\Delta\chi = 0 & \text{in } \Omega,\\ \partial_n\chi = \boldsymbol{\psi}_0\cdot\boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

Finally, we set

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{grad}\ \chi - \sum_{j=1}^{J} <(\boldsymbol{\psi}_0 - \mathbf{grad}\ \chi)\cdot\boldsymbol{n}\,,\,1>_{\Sigma_j} \widetilde{\mathbf{grad}}\ q_j^T\,.$$

It is easy to check that the function $\boldsymbol{\psi}$ belongs to $X(\Omega)$ and satisfies (3.16).
2) The uniqueness of this function $\boldsymbol{\psi}$ is a straightforward consequence of Proposition 3.14.
3) The necessity of conditions (3.1) was established in the proof of Lemma 3.5.

3.d. Normal vector potentials

Now, we study the case of a vector potential in $X_N(\Omega)$. The proofs are very similar to the previous ones.

Theorem 3.17. *A function $\boldsymbol{u}$ in $H(\mathrm{div},\Omega)$ satisfies*

$$\begin{aligned} &\mathrm{div}\ \boldsymbol{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega,\\ &<\boldsymbol{u}\cdot\boldsymbol{n}\,,\,1>_{\Sigma_j} = 0, \quad 1 \le j \le J, \end{aligned} \tag{3.21}$$

if and only if there exists a vector potential $\boldsymbol{\psi}$ in $X(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi} \quad \text{in } \Omega \quad \text{and} \quad \mathrm{div}\ \boldsymbol{\psi} = 0 \quad \text{in } \Omega,\\ &\boldsymbol{\psi}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \partial\Omega,\\ &<\boldsymbol{\psi}\cdot\boldsymbol{n}\,,\,1>_{\Gamma_i} = 0, \quad 0 \le i \le I. \end{aligned} \tag{3.22}$$

This function $\boldsymbol{\psi}$ is unique.

Note that the quantity $< \boldsymbol{u}\cdot\boldsymbol{n}\,, 1 >_{\Sigma_j}$ makes sense since $\boldsymbol{u}$ belongs to $H_0(\text{div}, \Omega)$, which implies by Lemma 3.10 that $\boldsymbol{u}\cdot\boldsymbol{n}$ belongs to $H^{\frac{1}{2}}(\Sigma_j)'$.

As previously, the uniqueness result is linked to the characterization of the kernel

$$K_N(\Omega) = \{\boldsymbol{w} \in X_N(\Omega);\ \mathbf{curl}\ \boldsymbol{w} = \mathbf{0} \text{ in } \Omega \text{ and div } \boldsymbol{w} = 0 \text{ in } \Omega\}. \tag{3.23}$$

The following result is due to Dominguez [11].

Proposition 3.18. *The dimension of the space $K_N(\Omega)$ is equal to I. It is spanned by the functions* $\mathbf{grad}\ q_i^N$*, $1 \le i \le I$, where each q_i^N is the unique solution in $H^1(\Omega)$ of the problem*

$$\begin{cases} -\Delta q_i^N = 0 \quad \textit{in } \Omega, \\ q_i^N\big|_{\Gamma_0} = 0 \quad \textit{and} \quad q_i^N\big|_{\Gamma_k} = \text{constant}, \quad 1 \le k \le I, \\ < \partial_n q_i^N\,, 1 >_{\Gamma_k} = \delta_{ik}, \quad 1 \le k \le I, \quad \textit{and} \quad < \partial_n q_i^N\,, 1 >_{\Gamma_0} = -1. \end{cases} \tag{3.24}$$

Proof: Let Θ^0 denote the space

$$\Theta^0 = \{r \in H^1(\Omega);\ r\big|_{\Gamma_0} = 0 \text{ and } r\big|_{\Gamma_i} = \text{constant},\ 1 \le i \le I\}. \tag{3.25}$$

For $1 \le i \le I$, the problem: *find q_i^N in Θ^0 such that*

$$\forall r \in \Theta^0, \quad \int_\Omega \mathbf{grad}\ q_i^N \cdot \mathbf{grad}\ r\, d\boldsymbol{x} = r\big|_{\Gamma_i}, \tag{3.26}$$

has a unique solution. An argument similar (but simpler) to that used in proving Proposition 3.14 shows that q_i^N satisfies (3.24). The functions $\mathbf{grad}\ q_i^N$, $1 \le i \le N$, are obviously independent and belong to $K_N(\Omega)$; it remains to prove that they span $K_N(\Omega)$. Take any function $\boldsymbol{w}$ in $K_N(\Omega)$ and consider the function

$$\boldsymbol{u} = \boldsymbol{w} - \sum_{i=1}^{I} < \boldsymbol{w}\cdot\boldsymbol{n}\,, 1 >_{\Gamma_i} \mathbf{grad}\ q_i^N\,.$$

Noting that

$$\int_\Omega \text{div}\ \boldsymbol{w}\, d\boldsymbol{x} = < \boldsymbol{w}\cdot\boldsymbol{n}\,, 1 >_{\partial\Omega} = 0,$$

it is easy to prove that $\boldsymbol{u}$ satisfies (3.1), so that it can be written $\mathbf{curl}\ \boldsymbol{\psi}_0$, for some $\boldsymbol{\psi}_0$ in $H^1(\Omega)^3$. This allows to compute

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{u}\, d\boldsymbol{x} = \int_\Omega \boldsymbol{u}\cdot\mathbf{curl}\ \boldsymbol{\psi}_0\, d\boldsymbol{x} = \int_\Omega \mathbf{curl}\ \boldsymbol{u}\cdot\boldsymbol{\psi}_0\, d\boldsymbol{x} + < \boldsymbol{u}\times\boldsymbol{n}\,, \boldsymbol{\psi}_0 >_{\partial\Omega} = 0,$$

so that $\boldsymbol{u}$ is equal to $\mathbf{0}$. That ends the proof.

As previously, this proposition has a corollary about equivalent norms, which is proven by the same arguments as Corollary 3.16.

Corollary 3.19. *On the space $X_N(\Omega)$, the seminorm*

$$\boldsymbol{w} \mapsto \|\mathbf{curl}\ \boldsymbol{w}\|_{L^2(\Omega)^3} + \|\mathrm{div}\ \boldsymbol{w}\|_{L^2(\Omega)} + \sum_{i=1}^{I} |< \boldsymbol{w}\cdot\boldsymbol{n}\,,\, 1 >_{\Gamma_i}|, \tag{3.27}$$

is equivalent to the norm $\|\cdot\|_{X(\Omega)}$.

Proof of Theorem 3.17: The proof is divided into three steps.
1) We assume that (3.22) holds. A function of the form $\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}$ is obviously divergence-free; moreover, for any function χ in $H^2(\Omega)$, formulas (2.6) and (2.5) yield respectively

$$\int_\Omega \mathbf{curl}\ \boldsymbol{\psi}\cdot\mathbf{grad}\ \chi\, d\boldsymbol{x} = < \boldsymbol{u}\cdot\boldsymbol{n}\,,\, \chi >_{\partial\Omega},$$

$$\int_\Omega \mathbf{curl}\ \boldsymbol{\psi}\cdot\mathbf{grad}\ \chi\, d\boldsymbol{x} = - < \boldsymbol{\psi}\times\boldsymbol{n}\,,\, \mathbf{grad}\ \chi >_{\partial\Omega}.$$

Therefore if $\boldsymbol{\psi}\times\boldsymbol{n}$ vanishes on $\partial\Omega$, a density argument gives $\mathbf{curl}\ \boldsymbol{\psi}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. Hence, $\mathbf{curl}\ \boldsymbol{\psi}$ belongs to $H_0(\mathrm{div},\Omega)$ and it follows from Lemma 3.10 that $\mathbf{curl}\ \boldsymbol{\psi}\cdot\boldsymbol{n}$ belongs to $H^{\frac{1}{2}}(\Sigma_j)'$, for $1 \le j \le J$. Then the density of $\mathscr{D}(\Sigma_j)$ in $H^{\frac{1}{2}}(\Sigma_j)'$ and the choice $\mu = 1$ in (3.15) give

$$< \mathbf{curl}\ \boldsymbol{\psi}\cdot\boldsymbol{n}\,,\, 1 >_{\Sigma_j} = 0,\ 1 \le j \le J\,.$$

Hence, all the conditions of (3.21) hold.
2) Conversely, with any function $\boldsymbol{u}$ satisfying (3.21), we associate the function $\boldsymbol{\psi}_0$ of Lemma 3.5. We define the space

$$V_T^*(\Omega) = \{\boldsymbol{w} \in X_T(\Omega);\ \mathrm{div}\ \boldsymbol{w} = 0 \text{ in } \Omega \text{ and } < \boldsymbol{w}\cdot\boldsymbol{n}\,,\, 1 >_{\Sigma_j} = 0,\ 1 \le j \le J\}.$$

Due to Corollary 3.16, the problem: *find $\boldsymbol{\zeta}$ in $V_T^*(\Omega)$ such that*

$$\forall \boldsymbol{\varphi} \in V_T^*(\Omega),\quad \int_\Omega \mathbf{curl}\ \boldsymbol{\zeta}\cdot\mathbf{curl}\ \boldsymbol{\varphi}\, d\boldsymbol{x} = \int_\Omega \boldsymbol{\psi}_0\cdot\mathbf{curl}\ \boldsymbol{\varphi}\, d\boldsymbol{x} - \int_\Omega \mathbf{curl}\ \boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\, d\boldsymbol{x}, \tag{3.28}$$

has a unique solution. Next, we want to extend (3.28) to any test function in $X_T(\Omega)$. For any function $\widetilde{\boldsymbol{\varphi}}$ in $X_T(\Omega)$, defining the function χ in $H^1(\Omega)$ as a solution of the Neumann problem

$$\begin{cases} \Delta\chi = \mathrm{div}\ \widetilde{\boldsymbol{\varphi}} & \text{in } \Omega, \\ \partial_n\chi = 0 & \text{on } \partial\Omega, \end{cases}$$

we set

$$\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\ \chi - \sum_{j=1}^{J} < (\widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\ \chi)\cdot\boldsymbol{n}\,,\, 1 >_{\Sigma_j} \widetilde{\mathbf{grad}}\ q_j^T\,.$$

We note that $\boldsymbol{\varphi}$ belongs to $V_T^*(\Omega)$ and we observe from (3.21) that

$$\int_\Omega \mathbf{curl}\ \boldsymbol{\psi}_0\cdot\mathbf{grad}\ \chi\, d\boldsymbol{x} = \int_\Omega \boldsymbol{u}\cdot\mathbf{grad}\ \chi\, d\boldsymbol{x} = 0,$$

and from (3.21) and (3.11) that

$$\int_\Omega \mathbf{curl}\ \boldsymbol{\psi}_0 \cdot \widetilde{\mathbf{grad}}\ q_j^T\, d\boldsymbol{x} = \int_{\Omega^\circ} \boldsymbol{u} \cdot \mathbf{grad}\ q_j^T\, d\boldsymbol{x}$$

$$= \sum_{k=1}^{J} [q_j^T]_k \ < \boldsymbol{u} \cdot \boldsymbol{n}\, , 1 >_{\Sigma_k} = 0.$$

Hence, equation (3.28) with this test function $\boldsymbol{\varphi}$ becomes

$$\forall \widetilde{\boldsymbol{\varphi}} \in X_T(\Omega), \quad \int_\Omega \mathbf{curl}\ \boldsymbol{\zeta} \cdot \mathbf{curl}\ \widetilde{\boldsymbol{\varphi}}\, d\boldsymbol{x} = \int_\Omega \boldsymbol{\psi}_0 \cdot \mathbf{curl}\ \widetilde{\boldsymbol{\varphi}}\, d\boldsymbol{x} - \int_\Omega \mathbf{curl}\ \boldsymbol{\psi}_0 \cdot \widetilde{\boldsymbol{\varphi}}\, d\boldsymbol{x}.$$

It follows from this equation that the function

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 - \mathbf{curl}\ \boldsymbol{\zeta} - \sum_{i=1}^{I} < (\boldsymbol{\psi}_0 - \mathbf{curl}\ \boldsymbol{\zeta}) \cdot \boldsymbol{n}\, , 1 >_{\Gamma_i} \mathbf{grad}\ q_i^N,$$

belongs to $X(\Omega)$ and satisfies (3.22).
3) The uniqueness of the function $\boldsymbol{\psi}$ satisfying (3.22) is an immediate consequence of Proposition 3.18.

3.e. Miscellaneous

In Theorems 3.12 and 3.17, we have exhibited special vector potentials for divergence-free vector fields. The question is now to split any vector field $\boldsymbol{u}$ in $L^2(\Omega)^3$ into the sum of a **grad** q and a **curl** $\boldsymbol{\psi}$, with $\boldsymbol{\psi}$ satisfying some boundary conditions. We present three different choices for q and $\boldsymbol{\psi}$.

(i) With any function $\boldsymbol{u}$ in $L^2(\Omega)^3$, we associate the solution q_N of the Neumann problem: *find* q_N *in* $H^1(\Omega)/\mathbb{R}$ *such that*

$$\forall p \in H^1(\Omega), \quad \int_\Omega \mathbf{grad}\ q_N \cdot \mathbf{grad}\ p\, d\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\ p\, d\boldsymbol{x}. \tag{3.29}$$

Applying Theorem 3.12 to the function $\boldsymbol{u} - \mathbf{grad}\ q_N$ leads to the following decomposition: for any function $\boldsymbol{u}$ in $L^2(\Omega)^3$, there exists a function q (here equal to q_N) in $H^1(\Omega)$ and a function $\boldsymbol{\psi}$ in $X(\Omega)$ such that

$$\boldsymbol{u} = \mathbf{grad}\ q + \mathbf{curl}\ \boldsymbol{\psi} \quad \text{in } \Omega, \tag{3.30}$$

with the following properties:

$$\begin{aligned} &\operatorname{div} \boldsymbol{\psi} = 0 \quad \text{in } \Omega,\\ &\boldsymbol{\psi} \cdot \boldsymbol{n} = \mathbf{curl}\ \boldsymbol{\psi} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega,\\ &< \boldsymbol{\psi} \cdot \boldsymbol{n}\, , 1 >_{\Sigma_j} = 0, \quad 1 \le j \le J. \end{aligned} \tag{3.31}$$

(ii) With any function $\boldsymbol{u}$ in $L^2(\Omega)^3$, we associate the solution $\tilde{q}_N$ of the quasi-Neumann problem — recall that Θ is the space defined in (3.10):

find $\tilde{q}_N$ *in* $\Theta/\mathbb{R}$ *such that*

$$\forall p \in \Theta, \quad \int_\Omega \widetilde{\mathbf{grad}}\ \tilde{q}_N \cdot \widetilde{\mathbf{grad}}\ p\, d\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \widetilde{\mathbf{grad}}\ p\, d\boldsymbol{x}. \tag{3.32}$$

Applying Theorem 3.17 to the function $\boldsymbol{u} - \widetilde{\mathbf{grad}}\ \tilde{q}_N$ leads to the following decomposition: for any function $\boldsymbol{u}$ in $L^2(\Omega)^3$, there exists a function q in Θ and a function $\boldsymbol{\psi}$ in $X(\Omega)$ such that

$$\boldsymbol{u} = \widetilde{\mathbf{grad}}\ q + \mathbf{curl}\ \boldsymbol{\psi} \quad \text{in } \Omega, \tag{3.33}$$

with the following properties:

$$\begin{aligned} &\text{div}\ \boldsymbol{\psi} = 0 \quad \text{in } \Omega, \\ &\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \partial\Omega \quad (\text{which implies } \mathbf{curl}\ \boldsymbol{\psi} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega), \\ &< \boldsymbol{\psi} \cdot \boldsymbol{n}\, , 1 >_{\Gamma_i} = 0, \quad 0 \le i \le I. \end{aligned} \tag{3.34}$$

Note that the solutions q_N and $\tilde{q}_N$ of problems (3.29) and (3.32) are linked by the equation (up to an additive constant)

$$\tilde{q}_N = q_N + \sum_{j=1}^{J} < \boldsymbol{u} \cdot \boldsymbol{n} - \partial_n q_N,\, 1 >_{\Sigma_j} q_j^T\, .$$

(iii) Furthermore, solving for any function $\boldsymbol{u}$ in $L^2(\Omega)^3$ the quasi-Dirichlet problem — recall that Θ^0 is the space defined in (3.25):

find q_D *in* Θ^0 *such that*

$$\forall p \in \Theta^0, \quad \int_\Omega \mathbf{grad}\ q_D \cdot \mathbf{grad}\ p\, d\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\ p\, d\boldsymbol{x}, \tag{3.35}$$

and applying Theorem 3.12 to the function $\boldsymbol{u} - \mathbf{grad}\ q_D$, we obtain the third decomposition: for any function $\boldsymbol{u}$ in $L^2(\Omega)^3$, there exists a function q in Θ^0 and a function $\boldsymbol{\psi}$ in $X(\Omega)$ such that

$$\boldsymbol{u} = \mathbf{grad}\ q + \mathbf{curl}\ \boldsymbol{\psi} \quad \text{in } \Omega, \tag{3.36}$$

with the following properties:

$$\begin{aligned} &\text{div}\ \boldsymbol{\psi} = 0 \quad \text{in } \Omega, \\ &\boldsymbol{\psi} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \\ &< \boldsymbol{\psi} \cdot \boldsymbol{n}\, , 1 >_{\Sigma_j} = 0, \quad 1 \le j \le J. \end{aligned} \tag{3.37}$$

There also, the solution q_D of problem (3.35) is linked to the solution q_D^0 of the pure Dirichlet problem (*i.e.* with Θ^0 replaced by $H_0^1(\Omega)$) by the equation

$$q_D = q_D^0 + \sum_{i=1}^{I} < \boldsymbol{u} \cdot \boldsymbol{n} - \partial_n q_D^0,\, 1 >_{\Gamma_i} q_i^N\, .$$

The question of mixing the boundary conditions on the normal component and the tangential component of the vector potential is not completely solved, however we can extend the previous results and proofs to the following simple cases. Assuming that the boundary $\partial\Omega$ is divided into two open Lipschitz parts Γ^N and Γ^T, *i.e.*,

$$\partial\Omega = \overline{\Gamma}^N \cup \overline{\Gamma}^T \quad \text{and} \quad \Gamma^N \cap \Gamma^T = \emptyset, \tag{3.38}$$

we denote by Γ^N_ℓ, $0 \le \ell \le L_N$, the connected components of Γ^N and similarly by Γ^T_ℓ, $0 \le \ell \le L_T$, the connected components of Γ^T. Also, $\partial\Sigma$ stands for the union of the boundaries $\partial\Sigma_j$ of an admissible set of cuts Σ_j. Then, the following properties hold:
(i) if $\partial\Sigma$ is included in Γ^T: a function $\boldsymbol{u}$ in $H(\mathrm{div},\Omega)$ satisfies

$$\begin{aligned} &\mathrm{div}\, \boldsymbol{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}\,.\,\boldsymbol{n} = 0 \quad \text{on } \Gamma^N, \\ &< \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1 >_{\Gamma^T_\ell} = 0, \quad 0 \le \ell \le L_T, \end{aligned} \tag{3.39}$$

if and only if there exists a function $\boldsymbol{\psi}$ in $X(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad \text{in } \Omega \quad \text{and} \quad \mathrm{div}\, \boldsymbol{\psi} = 0 \quad \text{in } \Omega, \\ &\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma^N \quad \text{and} \quad \boldsymbol{\psi}\,.\,\boldsymbol{n} = 0 \quad \text{on } \Gamma^T, \\ &< \boldsymbol{\psi}\cdot\boldsymbol{n}\,,\,1 >_{\Gamma^N_\ell} = 0, \quad 0 \le \ell \le L_N, \\ &< \boldsymbol{\psi}\cdot\boldsymbol{n}\,,\,1 >_{\Sigma_j} = 0, \quad 1 \le j \le J; \end{aligned} \tag{3.40}$$

(ii) if $\partial\Sigma$ is included in Γ^N: a function $\boldsymbol{u}$ in $H(\mathrm{div},\Omega)$ satisfies

$$\begin{aligned} &\mathrm{div}\, \boldsymbol{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}\,.\,\boldsymbol{n} = 0 \quad \text{on } \Gamma^N, \\ &< \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1 >_{\Gamma^T_\ell} = 0, \quad 0 \le \ell \le L_T, \\ &< \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1 >_{\Sigma_j} = 0, \quad 1 \le j \le J, \end{aligned} \tag{3.41}$$

if and only if there exists a function $\boldsymbol{\psi}$ in $X(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad \text{in } \Omega \quad \text{and} \quad \mathrm{div}\, \boldsymbol{\psi} = 0 \quad \text{in } \Omega, \\ &\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma^N \quad \text{and} \quad \boldsymbol{\psi}\,.\,\boldsymbol{n} = 0 \quad \text{on } \Gamma^T, \\ &< \boldsymbol{\psi}\cdot\boldsymbol{n}\,,\,1 >_{\Gamma^N_\ell} = 0, \quad 0 \le \ell \le L_N. \end{aligned} \tag{3.42}$$

In both cases, this function $\boldsymbol{\psi}$ is unique. However, the condition $\partial\Sigma \subset \Gamma^T$ or $\partial\Sigma \subset \Gamma^N$ cannot be satisfied in all geometries: take the case of a tyre Ω where Γ^T is the rim.

3.f. Other vector potentials

The third vector potential which is presented in the next theorem is less standard; however it turns out to be useful in special cases.

Theorem 3.20. *Assume that the domain Ω is of class $\mathscr{C}^{1,1}$. A function $\boldsymbol{u}$ in $H(\mathrm{div},\Omega)$ satisfies*

$$\begin{aligned} &\mathrm{div}\, \boldsymbol{u} = 0 \quad \textit{in } \Omega \quad \textit{and} \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \textit{on } \partial\Omega, \\ &< \boldsymbol{u}\cdot\boldsymbol{n}\,,\,1 >_{\Sigma_j} = 0, \quad 1 \le j \le J, \end{aligned} \tag{3.43}$$

if and only if there exists a vector potential $\boldsymbol{\psi}$ in $H^1(\Omega)^3$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi} \quad \text{in } \Omega \quad \text{and} \quad \text{div}\ (\Delta \boldsymbol{\psi}) = 0 \quad \text{in } \Omega, \\ &\boldsymbol{\psi} = \mathbf{0} \quad \text{on } \partial\Omega, \\ &< \partial_n(\text{div}\ \boldsymbol{\psi})\,,\, 1 >_{\Gamma_i} = 0, \quad 0 \le i \le I. \end{aligned} \tag{3.44}$$

This function $\boldsymbol{\psi}$ is unique.

Note that the condition div $(\Delta\boldsymbol{\psi}) = 0$, *i.e.* $\Delta(\text{div}\ \boldsymbol{\psi}) = 0$, implies that the quantity $< \partial_n(\text{div}\ \boldsymbol{\psi})\,,\, 1 >_{\Gamma_i}$ makes sense.

Here also, we have to characterize the kernel

$$K_0(\Omega) = \{\boldsymbol{w} \in H_0^1(\Omega)^3;\ \mathbf{curl}\ \boldsymbol{w} = \mathbf{0} \text{ in } \Omega \text{ and div } (\Delta \boldsymbol{w}) = 0 \text{ in } \Omega\}. \tag{3.45}$$

Proposition 3.21. *Assume that the domain Ω is of class $\mathscr{C}^{1,1}$. The dimension of the space $K_0(\Omega)$ is equal to I. It is spanned by the functions* **grad** q_i^0, $1 \le i \le I$, *where each q_i^0 is the unique solution in $H^2(\Omega)$ of the problem*

$$\begin{cases} \Delta^2 q_i^0 = 0 \quad \text{in } \Omega, \\ q_i^0\big|_{\Gamma_0} = 0 \quad \text{and} \quad q_i^0\big|_{\Gamma_k} = \text{constant}, \quad 1 \le k \le I, \\ \partial_n q_i^0 = 0 \quad \text{on } \partial\Omega, \\ < \partial_n(\Delta q_i^0)\,,\, 1 >_{\Gamma_k} = \delta_{ik}, \quad 1 \le k \le I, \quad \text{and} \quad < \partial_n(\Delta q_i^0)\,,\, 1 >_{\Gamma_0} = -1. \end{cases} \tag{3.46}$$

Proof: Let Θ^{00} denote the space

$$\Theta^{00} = \{r \in H^2(\Omega);\ r\big|_{\Gamma_0} = 0 \text{ and } r\big|_{\Gamma_i} = \text{ constant},\ 1 \le i \le I,\ \ \partial_n r = 0 \text{ on } \partial\Omega\}. \tag{3.47}$$

For $1 \le i \le I$, the problem: *find q_i^0 in Θ^{00} such that*

$$\forall r \in \Theta^{00}, \quad \int_\Omega \Delta q_i^0 \cdot \Delta r\, d\boldsymbol{x} = -r\big|_{\Gamma_i}, \tag{3.48}$$

has a unique solution. Furthermore, the following Green's formula can be proven by a density argument, for any functions q and r in Θ^{00} with $\Delta^2 q$ in $L^2(\Omega)$:

$$\int_\Omega (\Delta^2 q) r\, d\boldsymbol{x} = \int_\Omega (\Delta q)\,(\Delta r)\, d\boldsymbol{x} + \sum_{i=1}^{I} r\big|_{\Gamma_i} < \partial_n(\Delta q), 1 >_{\Gamma_i}.$$

Using this formula yields that the solution q_i^0 of (3.48) satisfies (3.46). Here also, the functions **grad** q_i^0, $1 \le i \le N$, are obviously independent and they belong to $K_0(\Omega)$.

Finally, let us prove that any function $\boldsymbol{w}$ in $K_0(\Omega)$ can be written

$$\boldsymbol{w} = \sum_{i=1}^{I} < \partial_n(\text{div } \boldsymbol{w})\,,\,1 >_{\Gamma_i} \textbf{grad } q_i^0.$$

Indeed, set $\boldsymbol{u} = \boldsymbol{w} - \sum_{i=1}^{I} < \partial_n(\text{div } \boldsymbol{w})\,,\,1 >_{\Gamma_i} \textbf{grad } q_i^0$. On one hand, we have by construction $< \partial_n(\text{div } \boldsymbol{u})\,,\,1 >_{\Gamma_i} = 0$ for $0 \le i \le I$. Therefore, $\textbf{grad }(\text{div } \boldsymbol{u})$ satisfies conditions (3.1) and Theorem 3.12 implies that there exists $\boldsymbol{\mu}$ in $X(\Omega)$ such that

$$\textbf{grad }(\text{div } \boldsymbol{u}) = \textbf{curl } \boldsymbol{\mu}, \quad \text{div } \boldsymbol{\mu} = 0 \text{ in } \Omega \quad \text{and} \quad \boldsymbol{\mu}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega.$$

Then

$$\begin{aligned}\int_\Omega (\text{div } \boldsymbol{u})^2\,d\boldsymbol{x} &= -\int_\Omega \boldsymbol{u}\cdot\textbf{grad }(\text{div } \boldsymbol{u})\,d\boldsymbol{x} \\ &= -\int_\Omega \boldsymbol{u}\cdot\textbf{curl } \boldsymbol{\mu}\,d\boldsymbol{x} = -\int_\Omega \boldsymbol{\mu}\cdot\textbf{curl } \boldsymbol{u}\,d\boldsymbol{x} = 0,\end{aligned}$$

because $\boldsymbol{u}$ belongs to $K_0(\Omega)$. So div $\boldsymbol{u}$ is equal to 0 and $\boldsymbol{u}$ belongs to $K_T(\Omega) \cap K_N(\Omega)$. Thus $\boldsymbol{u}$ is $\mathbf{0}$.

Proof of Theorem 3.20: Using exactly the same argument as in Theorem 3.17, we note that (3.44) implies (3.43). The uniqueness result follows from Proposition 3.21. Finally, with any function $\boldsymbol{u}$ satisfying (3.43), we associate the vector potential of Theorem 3.17, which we denote for the moment by $\widetilde{\boldsymbol{\psi}}$. The fact that Ω is of class $\mathscr{C}^{1,1}$ has two consequences: first, due to Theorem 2.12, the function $\widetilde{\boldsymbol{\psi}}$ belongs to $H^1(\Omega)^3$ and second the normal trace $\widetilde{\boldsymbol{\psi}}\cdot\boldsymbol{n}$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$. Hence, the fourth-order problem

$$\begin{cases} \Delta^2\chi = 0 & \text{in } \partial\Omega, \\ \chi = 0 \quad \text{and} \quad \partial_n\chi = \widetilde{\boldsymbol{\psi}}\cdot\boldsymbol{n} & \text{on } \partial\Omega, \end{cases}$$

has a unique solution χ in $H^2(\Omega)$. The function

$$\boldsymbol{\psi} = \widetilde{\boldsymbol{\psi}} - \textbf{grad } \chi + \sum_{i=1}^{I} < \partial_n(\Delta\chi)\,,\,1 >_{\Gamma_i} \textbf{grad } q_i^0,$$

satisfies (3.44) (note that the quantities $< \partial_n(\Delta\chi), 1 >_{\Gamma_i}$ are well defined since $\Delta^2\chi$ is zero).

Remark 3.22. We refer to Borchers & Sohr [6] for the completely general result: for any positive integer m, when the domain Ω is of class $\mathscr{C}^{m,1}$, for any function $\boldsymbol{u}$ in $H(\text{div}, \Omega)$ satisfying (3.43), there exists a vector potential $\boldsymbol{\psi}$ in $H_0^m(\Omega)$ satisfying:

$$\boldsymbol{u} = \textbf{curl } \boldsymbol{\psi} \quad \text{in } \Omega \quad \text{and} \quad \text{div }(\Delta^m\boldsymbol{\psi}) = 0 \quad \text{in } \Omega.$$

Remark 3.23. In the case of a cylinder, it is possible to construct another vector potential. Indeed, assume that Ω is a cylinder $\widetilde{\Omega} \times\,]a, b[$, where $\widetilde{\Omega}$ is a bounded Lipschitz domain in

$\mathbb{R}^2$ and a and b are real numbers. The Σ_j, $1 \le j \le J$, are taken of the form $\widetilde{\Sigma}_j \times]a, b[$. A function $\boldsymbol{u}$ in $H(\mathrm{div}, \Omega)$ satisfies

$$\mathrm{div}\, \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{3.49}$$

if and only if there exists a function $\boldsymbol{\psi} = (\vec{\psi}, 0)$, with $\vec{\psi}$ in $L^2(\Omega)^2$, such that

$$\begin{aligned}
&\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad \text{in } \Omega,\\
&\int_a^b (\mathrm{div}\, \boldsymbol{\psi})(., x_3)\, dx_3 = 0 \quad \text{in } \widetilde{\Omega},\\
&\int_a^b \boldsymbol{\psi}(., x_3) \cdot \boldsymbol{n}\, dx_3 = 0 \quad \text{on } \partial\widetilde{\Omega},\\
&\int_{\Sigma_j} \boldsymbol{\psi} \cdot \boldsymbol{n}\, d\boldsymbol{\tau} = 0, \quad 1 \le j \le J.
\end{aligned}$$

This function $\vec{\psi}$ is unique. However, the use of this new potential is severely limited by the geometry of the domain.

4. Applications.

We now present two examples of new formulations and/or discretizations where the potential vector representation of divergence-free functions is involved.

4.a. Approximation by divergence-free finite element functions

In most cases, discretizing the divergence-free constraint by finite element techniques does not lead to a discrete solution which is exactly divergence-free, see for instance the usual discretizations of the Stokes problem [19, Chapter II]. If this solution is used in the convection term of a coupled equation, this could yield a lack of stability of the time scheme (when constructed by the characteristics method for example). So, a natural idea is to construct a divergence-free finite element approximation of the form **curl** $\boldsymbol{\psi}_h$. However taking account of the gauge condition, which is necessary for the uniqueness of $\boldsymbol{\psi}_h$, makes the problem more complex. We quote El Dabaghi & Pironneau [14] and Verfürth [31] for solutions of this problem.

We propose here two methods of approximation of a divergence-free vector field $\boldsymbol{u}$ by a **curl** $\boldsymbol{\psi}_h$ — enforcing either zero normal values or zero tangential values on the boundary for the approximate vector potentials $\boldsymbol{\psi}_h$. More generally, our methods allow to approximate the **curl** part of any vector field $\boldsymbol{u}$, given either by the decomposition (3.36)(3.37) or by the decomposition (3.33)(3.34). A key argument to prove the convergence of the method is to establish that, for some spaces of nearly divergence-free vector fields, the L^2-norm of the **curl** is an equivalent norm.

Let Ω be a Lipschitz domain in $\mathbb{R}^3$, we assume that it has a polyhedral boundary. A triangulation of Ω is a set $\mathcal{T}_h$ of tetrahedra K such that $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$ and that the intersection of two different tetrahedra, if not empty, is either a corner or an edge or a face of both of them. As usual, h stands for the largest diameter of all K in $\mathcal{T}_h$. We introduce a regular family $(\mathcal{T}_h)_h$ of triangulations, in the sense that the ratio of the diameter of each element K in $\mathcal{T}_h$ to the diameter of its inscribed sphere is bounded by a constant σ independent of h. And we make the following assumption.

Hypothesis 4.1. For each h, there exists an admissible set of J cuts Σ_j, $1 \le j \le J$, such that each $\overline{\Sigma}_j$ is the union of the closure of faces of elements in $\mathcal{T}_h$.

Next, on each element K of $\mathcal{T}_h$, we define the space $\mathbb{P}_\ell(K)$ of all polynomials of total degree $\le \ell$, $\ell \ge 0$, together with its subspace $\overline{\mathbb{P}}_\ell(K)$ made of all homogeneous polynomials of degree ℓ. Now, consider the finite element introduced in [26]: for a fixed positive integer k, $\mathbf{P}_k(K)$ stands for the space

$$\mathbf{P}_k(K) = \big\{\boldsymbol{a}+\boldsymbol{b};\ \boldsymbol{a}\in\mathbb{P}_{k-1}(K)^3 \text{ and } \boldsymbol{b}\in\overline{\mathbb{P}}_k(K)^3,\ \boldsymbol{b}\,.\,\boldsymbol{x}=0\big\}. \tag{4.1}$$

In [19, Chapter III, §5.3], an explicit basis of this space is described for $k = 1$ and 2. It is also proven that this element is $H(\mathbf{curl})$-conforming, in the sense that an interpolation operator can be defined such that the tangential trace on each face F of an element K only depends on the tangential values on F of the function that is interpolated. So we set

$$Y_h = \big\{\boldsymbol{\varphi}_h \in H(\mathbf{curl},\Omega);\ \boldsymbol{\varphi}_h\big|_K \in \mathbf{P}_k(K),\ K\in\mathcal{T}_h\big\}. \tag{4.2}$$

Note that the elements of Y_h are piecewise-polynomial vector fields such that their tangential components are continuous through the faces. Owing to Hypothesis 4.1, we also introduce the space

$$\widetilde{\Theta}_h = \left\{\mu_h \in H^1(\Omega^\circ);\ \mu_h\big|_K \in \mathbb{P}_k(K),\ K \in \mathcal{T}_h\right\}. \tag{4.3}$$

So, the functions in $\widetilde{\Theta}_h$ are not necessarily continuous through the cuts Σ_j.

As in the continuous case, two types of boundary conditions can be enforced on the discrete vector potential $\boldsymbol{\psi}_h$. We begin with the case of zero normal values: for any function $\boldsymbol{u}$ satisfying (3.1), the associated vector potential in (3.16) is such that the pair $(\boldsymbol{\psi}, \theta)$, with $\theta = 0$, is the solution of the well-posed problem:

find $(\boldsymbol{\psi}, \theta)$ *in* $H(\mathbf{curl}, \Omega) \times \Theta/\mathbb{R}$ *such that*

$$\begin{aligned} \forall \boldsymbol{\varphi} \in H(\mathbf{curl}, \Omega), \quad & \int_\Omega \mathbf{curl}\ \boldsymbol{\psi} \cdot \mathbf{curl}\ \boldsymbol{\varphi}\, d\boldsymbol{x} + \int_\Omega \boldsymbol{\varphi} \cdot \widetilde{\mathbf{grad}}\ \theta\, d\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\ \boldsymbol{\varphi}\, d\boldsymbol{x}, \\ \forall \mu \in \Theta, \quad & \int_\Omega \boldsymbol{\psi} \cdot \widetilde{\mathbf{grad}}\ \mu\, d\boldsymbol{x} = 0, \end{aligned} \tag{4.4}$$

where Θ is the subspace of $H^1(\Omega^\circ)$ introduced in (3.10).

Remark 4.2. More generally, problem (4.4) is well-posed for any function $\boldsymbol{u}$ in $L^2(\Omega)^3$. In this case, θ is still zero and the function $\boldsymbol{\psi}$ coincides with the $\boldsymbol{\psi}$ in decomposition (3.36)(3.37). The following discretization can also be applied in this more general case.

Now, we define the discrete subspace of Θ:

$$\Theta_h = \{\mu_h \in \widetilde{\Theta}_h;\ [\mu_h]_j = \text{constant},\ 1 \le j \le J\}.$$

Let $\boldsymbol{u}$ be a function in $L^2(\Omega)^3$, and consider the problem:

find $(\boldsymbol{\psi}_h, \theta_h)$ *in* $Y_h \times \Theta_h/\mathbb{R}$ *such that*

$$\begin{aligned} \forall \boldsymbol{\varphi}_h \in Y_h, \quad & \int_\Omega \mathbf{curl}\ \boldsymbol{\psi}_h \cdot \mathbf{curl}\ \boldsymbol{\varphi}_h\, d\boldsymbol{x} + \int_\Omega \boldsymbol{\varphi}_h \cdot \widetilde{\mathbf{grad}}\ \theta_h\, d\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\ \boldsymbol{\varphi}_h\, d\boldsymbol{x}, \\ \forall \mu_h \in \Theta_h, \quad & \int_\Omega \boldsymbol{\psi}_h \cdot \widetilde{\mathbf{grad}}\ \mu_h\, d\boldsymbol{x} = 0. \end{aligned} \tag{4.5}$$

Of course, taking $\boldsymbol{\varphi}_h$ equal to $\widetilde{\mathbf{grad}}\ \theta_h$ (which belongs to Y_h) in problem (4.5) yields that θ_h is constant. So an equivalent formulation of this problem is:

find $\boldsymbol{\psi}_h$ *in* Y_h *such that*

$$\begin{aligned} \forall \boldsymbol{\varphi}_h \in Y_h, \quad & \int_\Omega \mathbf{curl}\ \boldsymbol{\psi}_h \cdot \mathbf{curl}\ \boldsymbol{\varphi}_h\, d\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\ \boldsymbol{\varphi}_h\, d\boldsymbol{x}, \\ \forall \mu_h \in \Theta_h, \quad & \int_\Omega \boldsymbol{\psi}_h \cdot \widetilde{\mathbf{grad}}\ \mu_h\, d\boldsymbol{x} = 0. \end{aligned} \tag{4.6}$$

We now present its numerical analysis. Firstly, we prove a useful lemma.

Lemma 4.3. *A function* $\boldsymbol{\varphi}_h$ *in* Y_h *such that* $\mathbf{curl}\ \boldsymbol{\varphi}_h$ *is* $\mathbf{0}$*, is the gradient of a function in* Θ_h.

Proof: If the curl of $\boldsymbol{\varphi}_h$ in Y_h is zero, since Ω° is simply connected, it is well-known [19, Chapter I, Thm 2.9] that it is the gradient of a function p in $H^1(\Omega^\circ)/\mathbb{R}$. Then, it can be observed that, on each element K of $\mathcal{T}_h$, since $\boldsymbol{\varphi}_h$ belongs to $\mathbf{P}_k(K)$, p belongs to $\mathbb{P}_{k+1}(K)$ and, even, to $\mathbb{P}_k(K)$, see [19, Chapter III, Lemma 5.3]; this is due to Euler's formula:

$$\forall q \in \overline{\mathbb{P}}_{k+1}(K), \quad \mathbf{grad}\ q \cdot \boldsymbol{x} = (k+1)\, q.$$

Hence, p belongs to $\widetilde{\Theta}_h$. Moreover, since the tangential components of $\boldsymbol{\varphi}_h$ are continuous through the Σ_j, the tangential components of $\mathbf{grad}\ p$ are continuous through Σ_j, hence the jump of p through each Σ_j is constant and p belongs to Θ_h.

Remark 4.4. As a consequence, it can be checked that the functions $\boldsymbol{\varphi}_h$ in Y_h such that $\mathbf{curl}\ \boldsymbol{\varphi}_h = \mathbf{0}$, are of degree $\leq k-1$ on each element K of $\mathcal{T}_h$.

Proposition 4.5. *For any data $\boldsymbol{u}$ in $L^2(\Omega)^3$, problem* (4.5) *has a unique solution.*

Proof: Problem (4.5) is a square linear system, so it suffices to check that the only solution for $\boldsymbol{u} = \mathbf{0}$ is zero. So, if $\boldsymbol{u}$ is equal to $\mathbf{0}$, taking $\boldsymbol{\varphi}_h$ equal to $\boldsymbol{\psi}_h$ in the first line and using the second line yield that $\mathbf{curl}\ \boldsymbol{\psi}_h$ is $\mathbf{0}$. Thus, from Lemma 4.3, $\boldsymbol{\psi}_h$ is the gradient of a function p_h in Θ_h and taking μ_h equal to p_h in the second line of (4.5) implies that $\boldsymbol{\psi}_h$ is $\mathbf{0}$. Moreover, as previously, θ_h is constant. This concludes the proof.

Of course, equation (4.5) is a saddle-point problem. The two bilinear forms that are involved in this problem, are continuous respectively on $H(\mathbf{curl}, \Omega) \times H(\mathbf{curl}, \Omega)$ and on $H(\mathbf{curl}, \Omega) \times H^1(\Omega^\circ)$. Moreover, the following inf-sup condition holds

$$\forall \mu_h \in \Theta_h, \quad \sup_{\boldsymbol{\varphi}_h \in Y_h} \frac{\int_\Omega \boldsymbol{\varphi}_h \cdot \widetilde{\mathbf{grad}}\ \mu_h\, d\boldsymbol{x}}{\|\boldsymbol{\varphi}_h\|_{H(\mathbf{curl},\Omega)}} \geq \frac{\int_\Omega \widetilde{\mathbf{grad}}\ \mu_h \cdot \widetilde{\mathbf{grad}}\ \mu_h\, d\boldsymbol{x}}{\|\widetilde{\mathbf{grad}}\ \mu_h\|_{L^2(\Omega)^3}} = |\mu_h|_{H^1(\Omega^\circ)}.$$

Thus, it remains to check the ellipticity of the first bilinear form on the discrete kernel

$$V_h = \big\{\boldsymbol{\varphi}_h \in Y_h;\ \forall \mu_h \in \Theta_h,\ \int_\Omega \boldsymbol{\varphi}_h \cdot \widetilde{\mathbf{grad}}\ \mu_h\, d\boldsymbol{x} = 0\big\}. \tag{4.7}$$

It is a consequence of the next proposition.

Proposition 4.6. *There exists a constant c independent of h such that, for all $\boldsymbol{\varphi}_h$ in V_h,*

$$\|\boldsymbol{\varphi}_h\|_{L^2(\Omega)^3} \leq c\, \|\mathbf{curl}\ \boldsymbol{\varphi}_h\|_{L^2(\Omega)^3}. \tag{4.8}$$

The proof is given in [19, Chapter III, Prop. 5.1] with further restrictions on the regularity of the domain and of the triangulation. It relies on the use of an interpolation operator defined as follows: for any tetrahedron K, there exists [19, Chapter III, Thm 5.3] a unique operator r_K with values in $\mathbf{P}_k(K)$ such that, for any smooth enough function $\boldsymbol{v}$ on K, the following moments of $r_K \boldsymbol{v}$ coincide with those of $\boldsymbol{v}$:

$$M_K(\boldsymbol{v}, \boldsymbol{\varphi}) = \int_K \boldsymbol{v} \cdot \boldsymbol{\varphi}\, d\boldsymbol{x}, \quad \boldsymbol{\varphi} \in \mathbb{P}_{k-3}(K)^3,$$

$$\forall F \text{ face of } K, \quad M_F(\boldsymbol{v}, \boldsymbol{\varphi}) = \int_F (\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{\varphi}\, d\boldsymbol{\tau}, \quad \boldsymbol{\varphi} \in \mathbb{P}_{k-2}(F)^2,$$

$$\forall E \text{ edge of } K, \quad M_E(\boldsymbol{v}, \varphi) = \int_E (\boldsymbol{v} \cdot \boldsymbol{\tau})\, \varphi\, d\tau, \quad \varphi \in \mathbb{P}_{k-1}(E).$$

Then, a global interpolation operator r_h is defined for smooth enough functions $\boldsymbol{v}$ on Ω by

$$\forall K \in \mathcal{T}_h, \quad (r_h \boldsymbol{v})\big|_K = r_K \boldsymbol{v},$$

with values in Y_h. It has the further property [19, Chapter III, Lemma 5.10] that for any function p in $H^1(\Omega^\circ)$ such that $r_h(\mathbf{grad}\ p)$ is defined, it coincides with the gradient of a function p_h in $\widetilde{\Theta}_h$ and if moreover, p belongs to Θ, then p_h belongs to Θ_h. The next lemma states a continuity property of the operator r_K.

Lemma 4.7. *For any $p > 2$ and for any tetrahedron K, the operator r_K is continuous on the space*

$$\big\{\boldsymbol{v} \in L^p(K)^3;\ \mathbf{curl}\ \boldsymbol{v} \in L^p(K)^3 \text{ and } \boldsymbol{v} \times \boldsymbol{n} \in L^p(\partial K)^2\big\}. \tag{4.9}$$

Proof: Let $\boldsymbol{v}$ be any function in this space for a fixed $p > 2$. The moments $M_K(\boldsymbol{v}, \boldsymbol{\varphi})$, resp. $M_F(\boldsymbol{v}, \boldsymbol{\varphi})$, are clearly continuous since $\boldsymbol{v}$ belongs to $L^p(K)^3$, resp. $\boldsymbol{v} \times \boldsymbol{n}$ belongs to $L^p(\partial K)^2$, so it remains to consider the moments $M_E(\boldsymbol{v}, \varphi)$.

Let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. For an edge E of a face F of K, we combine the extension by zero from $W^{1-\frac{1}{p'},p'}(E)$ into $W^{1-\frac{1}{p'},p'}(\partial F)$ with a lifting from $W^{1-\frac{1}{p'},p'}(\partial F)$ into $W^{1,p'}(F)$ and we denote by $\overline{\varphi}$ the image by this product of operators of a polynomial φ of $\mathbb{P}_{k-1}(E)$. Thus, if $(\mathbf{grad}\ \overline{\varphi})_F$ denotes the tangential gradient of $\overline{\varphi}$ on F, we have

$$M_E(\boldsymbol{v}, \varphi) = \int_F (\mathbf{curl}\ \boldsymbol{v}) \cdot \boldsymbol{n}\, \overline{\varphi}\, d\boldsymbol{\tau} + \int_F (\boldsymbol{v} \times \boldsymbol{n}) \cdot (\mathbf{grad}\ \overline{\varphi})_F\, d\boldsymbol{\tau}.$$

Next, we extend $\overline{\varphi}$ by 0 on the other faces of K and we denote by $\overline{\overline{\varphi}}$ its image by a lifting operator from $W^{1-\frac{1}{p'},p'}(\partial K)$ onto $W^{1,p'}(K)$. This gives

$$M_E(\boldsymbol{v}, \varphi) = \int_K (\mathbf{curl}\ \boldsymbol{v}) \cdot (\mathbf{grad}\ \overline{\overline{\varphi}})\, d\boldsymbol{x} + \int_F (\boldsymbol{v} \times \boldsymbol{n}) \cdot (\mathbf{grad}\ \overline{\varphi})_F\, d\boldsymbol{\tau},$$

whence

$$|M_E(\boldsymbol{v}, \varphi)| \leq c\left(\|\mathbf{curl}\ \boldsymbol{v}\|_{L^p(K)^3} + \|\boldsymbol{v} \times \boldsymbol{n}\|_{L^p(F)^2}\right) \|\varphi\|_{W^{1-\frac{1}{p'},p'}(E)},$$

with a constant c that only depends on K and p.

Proof of Proposition 4.6: With any function $\boldsymbol{\varphi}_h$ in V_h, we associate the solution p in $\Theta/\mathbb{R}$ of the problem

$$\forall q \in \Theta, \quad \int_\Omega \widetilde{\mathbf{grad}}\ p \cdot \widetilde{\mathbf{grad}}\ q\, d\boldsymbol{x} = \int_\Omega \boldsymbol{\varphi}_h \cdot \widetilde{\mathbf{grad}}\ q\, d\boldsymbol{x},$$

where Θ is the subspace of $H^1(\Omega^\circ)$ defined in (3.10). Thus, the function $\boldsymbol{v} = \boldsymbol{\varphi}_h - \widetilde{\mathbf{grad}}\ p$ satisfies:

$$\begin{aligned} &\mathbf{curl}\ \boldsymbol{v} = \mathbf{curl}\ \boldsymbol{\varphi}_h \quad \text{and} \quad \text{div}\ \boldsymbol{v} = 0 \quad \text{in } \Omega, \\ &\qquad \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad <\boldsymbol{v} \cdot \boldsymbol{n}, 1>_{\Sigma_j} = 0, \quad 1 \leq j \leq J. \end{aligned}$$

Hence, from Proposition 3.7, it belongs to $H^{\frac{1}{2}+\delta}(\Omega)^3$ for a real number $\delta > 0$, so it has traces in $L^p(F)^3$ for all faces F of elements of $\underset{\sim}{\mathcal{T}_h}$ for a real number $p > 2$. As a consequence, we can apply the operator r_h to $\boldsymbol{v}$, hence to $\widetilde{\mathbf{grad}}\ p$. Thus $r_h(\widetilde{\mathbf{grad}}\ p)$ is the gradient of a function p_h in $\widetilde{\Theta}_h$ and even in Θ_h since the tangential component of $\widetilde{\mathbf{grad}}\ p$ is continuous through each Σ_j. Then, since $\boldsymbol{\varphi}_h$ belongs to V_h, we have

$$\int_\Omega \boldsymbol{\varphi}_h^2\, d\boldsymbol{x} = \int_\Omega \boldsymbol{\varphi}_h \cdot r_h \boldsymbol{\varphi}_h\, d\boldsymbol{x} = \int_\Omega \boldsymbol{\varphi}_h \cdot (\widetilde{\mathbf{grad}}\ p_h + r_h \boldsymbol{v})\, dx = \int_\Omega \boldsymbol{\varphi}_h \cdot r_h \boldsymbol{v}\, d\boldsymbol{x},$$

so that

$$\|\boldsymbol{\varphi}_h\|_{L^2(\Omega)^3} \leq \|r_h \boldsymbol{v}\|_{L^2(\Omega)^3}.$$

To bound $\|r_h \boldsymbol{v}\|_{L^2(\Omega)^3}$, we fix an element K of $\mathcal{T}_h$ and we denote by B_K the Jacobian matrix of the affine transformation F_K that maps a reference tetrahedron $\hat{K}$ onto K. We write

$$\|r_h \boldsymbol{v}\|_{L^2(K)^3} \leq c\, h_K^{\frac{3}{2}}\, \|(r_h \boldsymbol{v}) \circ F_K\|_{L^2(\hat{K})^3}.$$

Next, denoting by $\hat{r}$ the operator $r_{\hat{K}}$ on the reference element $\hat{K}$, we recall the following formula, see Nedelec [25, Thm 4]:

$$(r_h \boldsymbol{v}) \circ F_K = (B_K^{-1})^t\, \hat{r}\big(B_K^t\, (\boldsymbol{v} \circ F_K)\big),$$

where the superscript t denotes the transpose of the matrix. Using this formula, we derive

$$\|r_h \boldsymbol{v}\|_{L^2(K)^3} \leq c\, h_K^{\frac{1}{2}}\, \|\hat{r}\big(B_K^t\, (v \circ F_K)\big)\|_{L^2(\hat{K})^3}.$$

Next, we apply Lemma 4.7 on $\hat{K}$ (note that the norm of $\hat{r}$ does not depend on h): for the same δ and p as previously,

$$\|r_h \boldsymbol{v}\|_{L^2(K)^3} \leq c\, h_K^{\frac{1}{2}}\, \Big(\|\mathbf{curl}\ \big(B_K^t\, (\boldsymbol{v} \circ F_K)\big)\|_{L^p(\hat{K})^3} + \|B_K^t\, (\boldsymbol{v} \circ F_K)\|_{H^{\frac{1}{2}+\delta}(\hat{K})^3}\Big).$$

We also recall [25] that, if $\mathbf{Curl}\ \boldsymbol{v}$ denotes the matrix with coefficients $(\partial_{x_i} v_j - \partial_{x_j} v_i)_{1\leq i,j\leq 3}$,

$$(B_K^{-1})^t\, \mathbf{Curl}\ \big(B_K^t(\boldsymbol{v} \circ F_K)\big)\, B_K^{-1} = (\mathbf{Curl}\ \boldsymbol{v}) \circ F_K,$$

which yields

$$\|r_h \boldsymbol{v}\|_{L^2(K)^3} \leq c\, h_K^{\frac{5}{2}}\, \|(\mathbf{curl}\ \boldsymbol{v}) \circ F_K\|_{L^p(\hat{K})^3} + c'\, h_K^{\frac{3}{2}}\, \|\boldsymbol{v} \circ F_K\|_{H^{\frac{1}{2}+\delta}(\hat{K})^3}.$$

Going back to K leads to

$$\|r_h \boldsymbol{v}\|_{L^2(K)^3} \leq c\, h_K^{\frac{5}{2}-\frac{3}{p}}\, \|\mathbf{curl}\ \boldsymbol{v}\|_{L^p(K)^3} + c'\, \|\boldsymbol{v}\|_{H^{\frac{1}{2}+\delta}(K)^3}.$$

Next, since $\mathbf{curl}\ \boldsymbol{v}$ coincides with $\mathbf{curl}\ \boldsymbol{\varphi}_h$, applying an inverse inequality gives

$$\|r_h \boldsymbol{v}\|_{L^2(K)^3} \leq c\, \Big(h_K\, \|\mathbf{curl}\ \boldsymbol{\varphi}_h\|_{L^2(K)^3} + \|\boldsymbol{v}\|_{H^{\frac{1}{2}+\delta}(K)^3}\Big).$$

And by summing the square of this inequality on all K, we deduce

$$\|r_h \boldsymbol{v}\|_{L^2(\Omega)^3} \leq c\left(h \,\|\mathbf{curl}\ \boldsymbol{\varphi}_h\|_{L^2(\Omega)^3} + \|\boldsymbol{v}\|_{H^{\frac{1}{2}+\delta}(\Omega)^3}\right).$$

The desired result follows by applying the imbedding of $X_T(\Omega)$ into $H^{\frac{1}{2}+\delta}(\Omega)^3$ (see Proposition 3.7), and finally Corollary 3.16.

We can now state the approximation theorem.

Theorem 4.8. *Let $\boldsymbol{u}$ be a function in $H^\sigma(\Omega)^3$, $0 < \sigma \leq k$, which satisfies* (3.1). *The following error estimate holds for the solution $\boldsymbol{\psi}_h$ of problem* (4.6)*:*

$$\|\boldsymbol{u} - \mathbf{curl}\ \boldsymbol{\psi}_h\|_{L^2(\Omega)^3} \leq c\, h^\sigma \,\|\boldsymbol{u}\|_{H^\sigma(\Omega)^3}. \tag{4.10}$$

Proof: Let $\boldsymbol{\psi}$ denote the function associated with $\boldsymbol{u}$ by (3.16). By comparing problems (4.4) (with $\theta = 0$) and (4.6), we derive

$$\forall \boldsymbol{\varphi}_h \in Y_h, \qquad \int_\Omega \mathbf{curl}\ (\boldsymbol{\psi} - \boldsymbol{\psi}_h) \cdot \mathbf{curl}\ \boldsymbol{\varphi}_h \, d\boldsymbol{x} = 0,$$

which yields the estimate

$$\|\mathbf{curl}\ \boldsymbol{\psi} - \mathbf{curl}\ \boldsymbol{\psi}_h\|_{L^2(\Omega)^3} \leq \inf_{\boldsymbol{\varphi}_h \in Y_h} \|\mathbf{curl}\ \boldsymbol{\psi} - \mathbf{curl}\ \boldsymbol{\varphi}_h\|_{L^2(\Omega)^3}.$$

Next, we observe that the **curl** operator maps the space Y_h into the space

$$Z_h = \left\{\boldsymbol{v}_h \in H(\mathrm{div}, \Omega);\ \boldsymbol{v}_h\big|_K \in \mathbb{P}_{k-1}(K)^3,\ K \in \mathcal{T}_h\right\}$$

and moreover [19, Chapter III, Lemma 5.11] that it is onto the subspace Z_h^* of functions in Z_h satisfying (3.1). This implies

$$\|\mathbf{curl}\ \boldsymbol{\psi} - \mathbf{curl}\ \boldsymbol{\psi}_h\|_{L^2(\Omega)^3} \leq \inf_{\boldsymbol{v}_h \in Z_h^*} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{L^2(\Omega)^3}.$$

And the approximation properties of the space Z_h^*, leading to the desired estimate, are given in [19, Chapter III, Prop. 5.3] for $\sigma \geq 1$ and can be extended to any $\sigma > 0$ by the arguments of Lemma 4.7.

Remark 4.9. By using the complete formulations (4.4) and (4.5), we deduce the modified estimate from the ellipticity property of Proposition 4.6 and the inf-sup condition:

$$\begin{aligned}\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{H(\mathbf{curl},\Omega)} &\leq c\,\big(\inf_{\boldsymbol{\varphi}_h \in Y_h} \|\boldsymbol{\psi} - \boldsymbol{\varphi}_h\|_{H(\mathbf{curl},\Omega)} + \inf_{\mu_h \in \Theta_h} \|\mu_h\|_{H^1(\Omega^\circ)}\big) \\ &\leq c \inf_{\boldsymbol{\varphi}_h \in Y_h} \|\boldsymbol{\psi} - \boldsymbol{\varphi}_h\|_{H(\mathbf{curl},\Omega)},\end{aligned}$$

Thus, from the approximation result in $H(\mathbf{curl}, \Omega)$, relying on the use of the operator r_h (see [19, Chapter III, Thm 5.4]), we obtain the following bound: if the function $\boldsymbol{u}$ satisfies (3.1) and is such that its vector potential $\boldsymbol{\psi}$ given in (3.16) belongs to $H^{\sigma+1}(\Omega)^3$, $1 \leq \sigma \leq k$,

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{H(\mathbf{curl},\Omega)} \leq c\, h^\sigma \,\|\boldsymbol{\psi}\|_{H^{\sigma+1}(\Omega)^3}. \tag{4.11}$$

However, even for smooth data $\boldsymbol{u}$, the function $\boldsymbol{\psi}$ is not so regular in general, as explained in Remark 3.8, so that this estimate is much less interesting.

Theorem 4.8 relies on the existence of the vector potential in Theorem 3.12. In order to obtain an analogous result for Theorem 3.17, we observe that, for any function $\boldsymbol{u}$ in $H(\mathrm{div}, \Omega)$ satisfying (3.21), the corresponding vector potential $\boldsymbol{\psi}$ in (3.22) solves, with $\theta = 0$, the variational problem:

find $(\boldsymbol{\psi}, \theta)$ *in* $H_0(\mathbf{curl}, \Omega) \times \Theta^0$ *such that*

$$\begin{aligned} \forall \boldsymbol{\varphi} \in H_0(\mathbf{curl}, \Omega), \quad & \int_\Omega \mathbf{curl}\, \boldsymbol{\psi} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, d\boldsymbol{x} + \int_\Omega \boldsymbol{\varphi} \cdot \mathbf{grad}\, \theta\, d\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi}\, d\boldsymbol{x}, \\ \forall \mu \in \Theta^0, \quad & \int_\Omega \boldsymbol{\psi} \cdot \mathbf{grad}\, \mu\, d\boldsymbol{x} = 0, \end{aligned} \tag{4.12}$$

for the space Θ^0 introduced in (3.25).

Remark 4.10. There also, problem (4.12) and its discretization are well-posed for any function $\boldsymbol{u}$ in $L^2(\Omega)^3$: in this case, the function $\boldsymbol{\psi}$ coincides with the $\boldsymbol{\psi}$ in decomposition (3.33)(3.34).

So, we set: $Y_h^0 = Y_h \cap H_0(\mathbf{curl}, \Omega)$ and we define Θ_h^0 as the space

$$\Theta_h^0 = \{\mu_h \in \widetilde{\Theta}_h \cap H^1(\Omega);\ \mu_h\big|_{\Gamma_0} = 0 \text{ and } \mu_h\big|_{\Gamma_i} = \text{constant},\ 1 \le i \le I\}$$

(hence it is made of functions continuous through the Σ_j). We consider the problem:

find $(\boldsymbol{\psi}_h, \theta_h)$ *in* $Y_h^0 \times \Theta_h^0$ *such that*

$$\begin{aligned} \forall \boldsymbol{\varphi}_h \in Y_h^0, \quad & \int_\Omega \mathbf{curl}\, \boldsymbol{\psi}_h \cdot \mathbf{curl}\, \boldsymbol{\varphi}_h\, d\boldsymbol{x} + \int_\Omega \boldsymbol{\varphi}_h \cdot \mathbf{grad}\, \theta_h\, d\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi}_h\, d\boldsymbol{x}, \\ \forall \mu_h \in \Theta_h^0, \quad & \int_\Omega \boldsymbol{\psi}_h \cdot \mathbf{grad}\, \mu_h\, d\boldsymbol{x} = 0. \end{aligned} \tag{4.13}$$

There also, it is readily checked that θ_h is zero, so that this problem can equivalently be written:

find $\boldsymbol{\psi}_h$ *in* Y_h^0 *such that*

$$\begin{aligned} \forall \boldsymbol{\varphi}_h \in Y_h^0, \quad & \int_\Omega \mathbf{curl}\, \boldsymbol{\psi}_h \cdot \mathbf{curl}\, \boldsymbol{\varphi}_h\, d\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi}_h\, d\boldsymbol{x}, \\ \forall \mu_h \in \Theta_h^0, \quad & \int_\Omega \boldsymbol{\psi}_h \cdot \mathbf{grad}\, \mu_h\, d\boldsymbol{x} = 0. \end{aligned} \tag{4.14}$$

Its numerical analysis relies on the same arguments as for problem (4.5), together with the fact that the operator r_h preserves the nullity of tangential traces on $\partial\Omega$ [19, Chapter III, Lemma 5.7]. In particular we need the uniform ellipticity property on the discrete kernel

$$V_h^0 = \big\{\boldsymbol{\varphi}_h \in Y_h^0;\ \forall \mu_h \in \Theta_h^0,\ \int_\Omega \boldsymbol{\varphi}_h \cdot \mathbf{grad}\, \mu_h\, d\boldsymbol{x} = 0\big\}. \tag{4.15}$$

Here we only state the results.

Proposition 4.11. *For any data* $\boldsymbol{u}$ *in* $L^2(\Omega)^3$*, problem* (4.13) *has a unique solution.*

Proposition 4.12. *There exists a constant* c *independent of* h *such that, for all* $\boldsymbol{\varphi}_h$ *in* V_h^0*,*

$$\|\boldsymbol{\varphi}_h\|_{L^2(\Omega)^3} \leq c\,\|\mathbf{curl}\ \boldsymbol{\varphi}_h\|_{L^2(\Omega)^3}. \tag{4.16}$$

Theorem 4.13. *Let* $\boldsymbol{u}$ *be a function in* $H^\sigma(\Omega)^3$*,* $0 < \sigma \leq k$*, which satisfies* (3.21)*. The following error estimate holds for the solution* $\boldsymbol{\psi}_h$ *of problem* (4.14)*:*

$$\|\boldsymbol{u} - \mathbf{curl}\ \boldsymbol{\psi}_h\|_{L^2(\Omega)^3} \leq c\,h^\sigma\,\|\boldsymbol{u}\|_{H^\sigma(\Omega)^3}. \tag{4.17}$$

4.b. The Stokes problem in vector potential formulation

In a bounded Lipschitz domain Ω in $\mathbb{R}^3$, we consider the Stokes problem in the primitive variables: the velocity $\boldsymbol{u}$ and the pressure p. For the sake of brevity, we limit ourselves to the case of homogeneous boundary conditions on the velocity. For a positive viscosity ν and given data $\boldsymbol{f}$ in $L^2(\Omega)^3$, it is written:

$$\begin{cases} -\nu\,\Delta\boldsymbol{u} + \mathbf{grad}\ p = \boldsymbol{f} & \text{in } \Omega, \\ \text{div}\ \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases} \tag{4.18}$$

If V stands for the space

$$V = \big\{\boldsymbol{v} \in H_0^1(\Omega)^3;\ \text{div}\ \boldsymbol{v} = 0 \,\text{in}\, \Omega\big\},$$

it admits the following equivalent variational formulation:

find $\boldsymbol{u}$ *in* V *such that*

$$\forall \boldsymbol{v} \in V, \quad \nu \int_\Omega \mathbf{grad}\ \boldsymbol{u} \cdot \mathbf{grad}\ \boldsymbol{v}\,d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,d\boldsymbol{x}. \tag{4.19}$$

Using either Theorem 3.12 or Theorem 3.17 to handle the divergence-free condition leads to new variational formulations of this problem where the unknown is the vector potential $\boldsymbol{\psi}$. Like in the previous section §4.a, the main advantage of this new formulations is that the corresponding approximate velocity $\mathbf{curl}\ \boldsymbol{\psi}_h$ is exactly divergence-free, in contrast to most discretizations of the Stokes problem in the primitive variables of velocity and pressure (there is no direct discretization of formulation (4.19)). This property is important, since it makes easier:

- the way of handling the convection term either in the full Navier-Stokes system or in the heat equation when coupled with the Navier-Stokes system by the Boussinesq approximation (indeed, many time schemes are more stable when the convection term is exactly divergence-free),

- the way of coupling the Navier-Stokes equations with the Euler's equations on adjacent subdomains.

Such formulations were studied in [3] [4] for a domain Ω of class $\mathscr{C}^{2,1}$ in order that the vector potential belongs to $H^2(\Omega)^3$. Our purpose is to extend their analysis to general Lipschitz domains, which allows for discretizing the Stokes problem in Lipschitz polyhedra.

We begin with the case of the tangential potential vector. In view of Theorem 3.12, we define the space

$$W_T(\Omega) = \big\{\boldsymbol{\varphi} \in X_T(\Omega);\ \Delta\boldsymbol{\varphi} \in L^2(\Omega)^3 \text{ and } \mathbf{curl}\ \mathbf{curl}\ \boldsymbol{\varphi} \in L^2(\Omega)^3, \\ <\boldsymbol{\varphi}\cdot\boldsymbol{n}, 1>_{\Sigma_j} = 0,\ 1 \le j \le J, \text{ and } \mathbf{curl}\ \boldsymbol{\varphi} = \mathbf{0} \text{ on } \partial\Omega\big\},$$

provided with the natural norm

$$\|\boldsymbol{\varphi}\|_{W(\Omega)} = \Big(\|\boldsymbol{\varphi}\|_{L^2(\Omega)^3} + \|\mathbf{curl}\ \boldsymbol{\varphi}\|_{L^2(\Omega)^3} + \|\text{div}\ \boldsymbol{\varphi}\|_{L^2(\Omega)} \\ + \|\Delta\boldsymbol{\varphi}\|_{L^2(\Omega)^3} + \|\mathbf{curl}\ \mathbf{curl}\ \boldsymbol{\varphi}\|_{L^2(\Omega)^3}\Big). \tag{4.20}$$

And we consider the problem:

find $\boldsymbol{\psi}$ in $W_T(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in W_T(\Omega), \quad \nu \int_\Omega \Delta\boldsymbol{\psi}\cdot\Delta\boldsymbol{\varphi}\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\mathbf{curl}\ \boldsymbol{\varphi}\, d\boldsymbol{x}. \tag{4.21}$$

Remark 4.14. It can be noted that all functions $\boldsymbol{\varphi}$ in $W_T(\Omega)$ are such that $\mathbf{grad}\ (\text{div}\ \boldsymbol{\varphi}) = \Delta\boldsymbol{\varphi} + \mathbf{curl}\ \mathbf{curl}\ \boldsymbol{\varphi}$ belongs to $L^2(\Omega)^3$, hence their divergences are in $H^1(\Omega)$. However, this does not imply a further regularity on the solution $\boldsymbol{\psi}$ of (4.21), which does not belong to $H^2(\Omega)^3$ in the general case. We shall see further on that it is divergence-free.

Theorem 3.12 implies that the mapping: $\boldsymbol{u} \mapsto \boldsymbol{\psi}$, where $\boldsymbol{\psi}$ is defined in (3.16), is continuous from V into $W_T(\Omega)$. This leads to the first result.

Theorem 4.15. *If $(\boldsymbol{u}, p)$ denotes the solution in $H^1(\Omega)^3 \times L^2(\Omega)/\mathbb{R}$ of the Stokes problem* (4.18)*, the potential vector $\boldsymbol{\psi}$ introduced in* (3.16) *is a solution of problem* (4.21)*.*

Now, we study the well-posedness of problem (4.21). We begin with a lemma.

Lemma 4.16. *The mapping: $\boldsymbol{\varphi} \mapsto \|\Delta\boldsymbol{\varphi}\|_{L^2(\Omega)^3}$ is a norm on $W_T(\Omega)$, equivalent to the norm $\|\cdot\|_{W(\Omega)}$.*

Proof: Since one of the equivalence inequalities is obvious, we only have to prove the second one. Applying Corollary 3.16 to $\boldsymbol{\varphi}$, next Corollary 3.19 to $\mathbf{curl}\ \boldsymbol{\varphi}$, we obtain

$$\|\boldsymbol{\varphi}\|_{W(\Omega)} \le c\Big(\|\text{div}\ \boldsymbol{\varphi}\|_{L^2(\Omega)} + \|\Delta\boldsymbol{\varphi}\|_{L^2(\Omega)^3} + \|\mathbf{curl}\ \mathbf{curl}\ \boldsymbol{\varphi}\|_{L^2(\Omega)^3}\Big).$$

Next, we observe that $\int_\Omega \text{div}\ \boldsymbol{\varphi}\, d\boldsymbol{x}$ is equal to 0, so that

$$\|\boldsymbol{\varphi}\|_{W(\Omega)} \le c\Big(\|\mathbf{grad}\ (\text{div}\ \boldsymbol{\varphi})\|_{L^2(\Omega)^3} + \|\Delta\boldsymbol{\varphi}\|_{L^2(\Omega)^3} + \|\mathbf{curl}\ \mathbf{curl}\ \boldsymbol{\varphi}\|_{L^2(\Omega)^3}\Big).$$

And, from the formula

$$\|\mathbf{grad}\ (\mathrm{div}\ \boldsymbol{\varphi})\|^2_{L^2(\Omega)^3} + \|\mathbf{curl}\ \mathbf{curl}\ \boldsymbol{\varphi}\|^2_{L^2(\Omega)^3} = \|\Delta\boldsymbol{\varphi}\|^2_{L^2(\Omega)^3},$$

we derive

$$\|\boldsymbol{\varphi}\|_{W(\Omega)} \leq c\,\|\Delta\boldsymbol{\varphi}\|_{L^2(\Omega)^3},$$

which is the desired result.

Theorem 4.17. *For any data $\boldsymbol{f}$ in $L^2(\Omega)^3$, problem* (4.21) *has a unique solution in $W_T(\Omega)$.*

This solution gives back a solution of the Stokes problem: indeed, its uniqueness combined with Theorem 4.15, yields that $\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}$ is the solution of problem (4.19), and the equivalence with problem (4.18) is standard [19, Chapter I, §5].

Corollary 4.18. *The solution $\boldsymbol{\psi}$ of problem* (4.21) *is divergence-free. Moreover, there exists p in $L^2(\Omega)/\mathbb{R}$ such that the pair $(\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi}, p)$ is a solution of the Stokes problem* (4.18).

So, we have constructed a first equivalent problem where the new unknown is the tangential vector potential. The situation is slightly more complicated in the case of the normal vector potential, indeed Theorem 3.17 can only be applied to the function $\boldsymbol{u} - \boldsymbol{\vartheta}$, with

$$\boldsymbol{\vartheta} = \sum_{j=1}^{J} < \boldsymbol{u}\cdot\boldsymbol{n}, 1 >_{\Sigma_j} \widetilde{\mathbf{grad}}\ q_j^T.$$

It yields the existence of a vector potential $\boldsymbol{\psi}$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi} + \boldsymbol{\vartheta} \quad \text{in } \Omega \quad \text{and} \quad \mathrm{div}\ \boldsymbol{\psi} = 0 \quad \text{in } \Omega,\\ &\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \partial\Omega,\\ &< \boldsymbol{\psi}\cdot\boldsymbol{n}, 1 >_{\Gamma_i} = 0, \quad 0 \leq i \leq I. \end{aligned} \tag{4.22}$$

Of course, the function $\boldsymbol{\vartheta}$ belongs to $K_T(\Omega)$.

However, this function $\boldsymbol{\psi}$ can be used to write a new formulation. We firstly introduce the space

$$Z_N(\Omega) = \big\{\boldsymbol{\varphi} \in X_N(\Omega);\ \Delta\boldsymbol{\varphi} \in L^2(\Omega)^3 \text{ and } \mathbf{curl}\ \mathbf{curl}\ \boldsymbol{\varphi} \in L^2(\Omega)^3,\\ < \boldsymbol{\varphi}\cdot\boldsymbol{n}, 1 >_{\Gamma_i} = 0,\ 0 \leq i \leq I\big\},$$

provided with the same norm $\|\cdot\|_{W(\Omega)}$ as previously. Next, in view of (4.22), we need the following subspace

$$W_N(\Omega) = \big\{\boldsymbol{\varphi} \in Z_N(\Omega);\ \exists\boldsymbol{\vartheta} \in K_T(\Omega) \text{ such that } (\mathbf{curl}\ \boldsymbol{\varphi} + \boldsymbol{\vartheta}) \times \boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega\big\},$$

Since an element of $K_T(\Omega)$ with null tangential traces is zero, there exists at most one function $\boldsymbol{\vartheta}$ in $K_T(\Omega)$ such that $(\mathbf{curl}\ \boldsymbol{\varphi} + \boldsymbol{\vartheta}) \times \boldsymbol{n} = \mathbf{0}$ on $\partial\Omega$, so we denote it by $\boldsymbol{\vartheta}(\boldsymbol{\varphi})$. We now consider the problem:

find $\boldsymbol{\psi}$ *in* $W_N(\Omega)$ *such that*

$$\forall \boldsymbol{\varphi} \in W_N(\Omega), \quad \nu \int_\Omega \Delta\boldsymbol{\psi} \cdot \Delta\boldsymbol{\varphi} \, d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \big(\mathbf{curl}\ \boldsymbol{\varphi} + \boldsymbol{\vartheta}(\boldsymbol{\varphi})\big) \, d\boldsymbol{x}. \tag{4.23}$$

The following property is readily checked.

Theorem 4.19. *If* $(\boldsymbol{u}, p)$ *denotes the solution in* $H^1(\Omega)^3 \times L^2(\Omega)/\mathbb{R}$ *of the Stokes problem* (4.18)*, the potential vector* $\boldsymbol{\psi}$ *introduced in* (4.22) *is a solution of problem* (4.23)*.*

Proof: It follows from (4.22) that $\boldsymbol{u}$ is equal to $\boldsymbol{\psi} + \boldsymbol{\vartheta}(\boldsymbol{\psi})$ and that, more generally, each function in V can be written as $\mathbf{curl}\ \boldsymbol{\varphi} + \boldsymbol{\vartheta}(\boldsymbol{\varphi})$, with $\boldsymbol{\varphi}$ in $W_N(\Omega)$. A direct computation from (4.19) yields (4.23).

The same arguments as previously lead to the next statements.

Lemma 4.20. *The mapping:* $\boldsymbol{\varphi} \mapsto \|\Delta\boldsymbol{\varphi}\|_{L^2(\Omega)^3}$ *is a norm on* $W_N(\Omega)$*, equivalent to the norm* $\|\cdot\|_{W(\Omega)}$*.*

Theorem 4.21. *For any data* $\boldsymbol{f}$ *in* $L^2(\Omega)^3$*, problem* (4.23) *has a unique solution in* $W_N(\Omega)$*.*

Finally, we go back to the Stokes problem.

Corollary 4.22. *The solution* $\boldsymbol{\psi}$ *of problem* (4.23) *is divergence-free. Moreover, there exists* p *in* $L^2(\Omega)/\mathbb{R}$ *such that the pair* $(\boldsymbol{u} = \mathbf{curl}\ \boldsymbol{\psi} + \boldsymbol{\vartheta}(\boldsymbol{\psi}), p)$ *is a solution of the Stokes problem* (4.18)*.*

The two variational formulations (4.21) and (4.23) are fully appropriate for the construction of a discrete problem by the Galerkin method, with or without numerical integration: this can be achieved either by the finite or spectral element technique, leading to a discrete vector potential. Then, the velocity can be recovered by a discrete derivation process and, if needed, a discrete pressure can be computed as the solution of an uncoupled variational problem.

Of course, these formulations can be extended to the case of inhomogeneous Dirichlet conditions on the velocity. More important is the fact that they are also well appropriate for handling more complex and more physical boundary conditions such as the slip conditions or combining slip and no-slip conditions on different parts of the boundary, see for instance Girault [17]. The nonlinear term of the Navier–Stokes equations can be expressed as a function of the vector potential, which also leads to a new formulation of these equations. As a last application, the vector potential formulations can be extended to the Stokes equations in unbounded Lipschitz domains, as explained by Girault, Giroire & Sequeira [18].

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