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par

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Sujet de la thèse

Fonctions de Green et support de diffusions hypoelliptiques

Soutenue le 27 juin 1995 devant la commission d'examen composée de :

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à *Dor*,

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Title: Green functions and the support of hypoelliptic diffusions

Abstract:

The first part contains a precise description of the singularity near the diagonal of the Green function associated to a hypoelliptic operator. Our approach is probabilistic and relies on the stochastic Taylor expansion of paths of the associated diffusion and on a priori estimates of the Green function. Examples and applications to potential theory are given.

In the second part one extends the Stroock-Varadhan support theorem for Hölder norms. The central tool is an estimate of the probability that the Brownian motion has a large Hölder norm conditionally on the fact that it has a small uniform norm.

Key words: hypoelliptic operator - Green function - degenerate diffusion - Taylor stochastic expansion - capacity - Hölder norm - support theorem - correlation inequality

1991 Mathematics Subject Classification: 60J60, 35H05, 60H10, 60J45, 60J65, 26A16, 46E15, 60G15

Table de matières

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I Introduction générale

Résumé. On fait la présentation des deux parties indépendantes de cette thèse, complétée d'un aperçu bibliographique.

Sommaire:

1. La fonction de Green hypoelliptique

Soit L un opérateur différentiel du second ordre à coefficients réels de classe C^∞, défini sur $\mathbb{R}^d$, $d \geq 3$. L est dit *hypoelliptique* dans un ouvert borné connexe régulier, Ω, si, toute distribution u dans Ω est une fonction C^∞, dans tout ouvert de Ω où Lu est une fonction C^∞.

Soient $X_0, X_1, \dots, X_m$ des champs de vecteurs (opérateurs différentiels homogènes de premier ordre) de classe C^∞. Nous supposerons que l'algèbre de Lie engendrée par les champs $X_1, \dots, X_m$ est de rang plein en tout point:

$$\forall x \in \mathbb{R}^d\,,\ \dim \operatorname{Lie}(X_1, \dots, X_m)(x) = d\,. \tag{1.1}$$

Cette hypothèse assure que l'opérateur

$$L = \frac{1}{2}\sum_{j=1}^{m} X_j^2 + X_0 \tag{1.2}$$

est hypoelliptique.

Les opérateurs de cette forme ont été introduits et étudiés par Hörmander [H] en 1967, sous l'hypothèse que $X_0, \dots, X_m$, avec leurs crochets de longueur au plus r engendrent $\mathbb{R}^d$ en tout point. La condition (1.1) est connue comme l'hypothèse d'Hörmander forte.

Nous noterons par $G(x, y)$ la solution fondamentale de L sur Ω. G est la *fonction de Green de L sur* Ω, et, par l'hypothèse (1.1), on sait que G est C^∞ hors de la diagonale.

La première partie de cette thèse contient une description précise de la singularité de la fonction de Green, par une approche probabiliste.

Il y a très peu de situations où la fonction de Green hypoelliptique est connue explicitement. Ainsi, en 1973 Folland [F], p. 375, (voir aussi Folland et Stein [F-S], p. 440) indique l'expression exacte de G sur l'espace entier, pour le cas du groupe d'Heisenberg H_{2n+1} (voir p. 66). En 1990 Greiner [Gr2], p. 136, donne la formule de G dans un cas légèrement modifié (voir p. 68). Récemment, Beals, Gaveau et Greiner [B-Ga-Gr] ont fait un calcul dans une situation plus générale. Dans tous ces cas, la longueur maximale des crochets de champs de vecteurs utilisés pour engendrer l'espace est deux. On connaît un cas où on utilise les crochets de longueur quatre: il s'agit d'un opérateur sur $\mathbb{R}^3$ qui parait de l'étude de la frontière du complexe de Cauchy-Riemann.

La fonction de Green a été trouvée en 1979 par Greiner [Gr1], p. 1108.

Faute d'expression exacte de la fonction de Green, on peut se contenter de connaître le comportement asymptotique sur la diagonale, dont l'intérêt a été souligné, en 1986, par Jerison et Sánchez-Calle [J-Sa], p. 51.

Auparavant, Nagel, Stein et Wainger [N-S-W], p. 114, et Sánchez-Calle [Sa], p. 143, ont obtenu des majorations de la fonction de Green et de ses dérivées. Ils ont utilisé une notion fondamentale de distance, associée à L. Leurs bornes n'impliquent pas la distance sous-riemannienne seule, mais aussi le volume des boules construites avec cette distance:

$$|G(x,y)| \leq c \frac{\rho(x,y)^2}{\mathrm{vol}\,(B_\rho(x,\rho(x,y)))} \,.$$

Dans le cas auto-adjoint on obtient aussi des minorations de même type (voir Fefferman et Sánchez-Calle [Fe-Sa], p. 248). Cela prouve qu'il s'agit du bon ordre de grandeur (voir [J-Sa], p. 51).

De point de vue probabiliste , G est la densité de la mesure d'occupation de la diffusion (x_t) associée à L. Précisément, soit $(B^1, \ldots, B^m)$ un mouvement brownien m-dimensionel. Alors, (x_t) est la solution de l'équation stochastique de Stratonovich

$$dx_t = \sum_{j=1}^{m} X_j(x_t) \circ dB_t^j + X_0(x_t)dt \,,\; x_0 = x \,, \tag{1.3}$$

tuée au premier temps de sortie de Ω, $\tau = \inf\{t > 0 : x_t \notin \Omega\}$. La loi de x_t a, sur Ω, une densité par rapport à la mesure de Lebesgue, $p_t^\Omega(x,y)$, le noyau de la chaleur associé à L. La fonction de Green (probabiliste) s'écrit alors

$$G(x,y) = \int_0^\infty p_t^\Omega(x,y)dt \,,\; x,y \in \Omega \,, \tag{1.4}$$

satisfaisant,

$$E_x \int_0^\tau f(x_t)dt = \int_\Omega f(y)G(x,y)dy \,, \tag{1.5}$$

quelque soit la fonction positive mesurable, f.

À l'aide de cette interprétation probabiliste, Gaveau [Ga], p. 101, retrouve en 1977 le résultat de Folland [F]. Par la formule de P. Lévy pour l'aire stochastique, on peut calculer la transformée de Fourier du noyau de la chaleur. Par inversion de Fourier et en intégrant ensuite en t on déduit l'expression de G.

Gaveau [Ga] donne aussi une formule de la transformée de Fourier du noyau de la chaleur sur un groupe nilpotent libre de pas deux. On pourrait essayer de trouver G, mais déjà pour le cas de H_{2n+1}, $n \geq 2$, le calcul devient difficile.

Pour les cas de pas plus grand que deux, on devrait disposer des lois d'intégrales stochastiques triples. Il n'y a pas de calcul de loi à notre connaissance.

En 1989, Chaleyat-Maurel et Le Gall [CM-LG] ont considéré la situation suivante: soient les champs de vecteurs X_1, X_2 sur $\mathbb{R}^3$, tels que pour tout $x \in \Omega$, $X_1(x)$, $X_2(x)$ et $[X_1, X_2](x)$ engendrent $\mathbb{R}^3$. On décrit le comportement de $G(x, y)$, lorsque $\|y - x\|$ est petite. On prouve qu'il existe une constante positive, c_x, telle que

$$\lim_{\varepsilon \downarrow 0} \sup_{\|y-x\|<\varepsilon} \left| G(x,y)\, d(x,y)^2 - c_x \right| = 0\,. \tag{1.6}$$

Ici, $d(x, y)$ est une pseudo-distance localement équivalente à la distance sous-riemannienne. D'abord, on obtient des estimations (locales) de la fonction de Green qui font intervenir la pseudo-distance seule. Ensuite, l'idée est de comparer la diffusion engendrée par X_1, X_2 à la diffusion invariante sur H_3, dont on connaît la fonction de Green (Folland [F] ou Gaveau [Ga]).

La comparaison est faite en utilisant le développement de Taylor stochastique. Cette idée a l'origine dans les travaux de Azencott [A] et de Ben Arous [BA1]. La forme finale a été donnée par Castell [Ca], travail où on apprend comment développer explicitement les flots stochastiques.

Ces techniques, combinées avec d'autres méthodes ont été appliquées pour l'étude du noyau de la chaleur par Ben Arous [BA2], Léandre [Le], lorsque il n'y a pas de drift, et par Ben Arous et Léandre [BA-Le] pour le cas où $X_0 \neq 0$.

Dans la première partie de ce travail nous avons suivi et étendu la stratégie donnée par Chaleyat-Maurel et Le Gall [CM-LG], au cas des champs de vecteurs réguliers $X_1, \ldots, X_m$ sur $\mathbb{R}^d$, $d \geq 3$, satisfaisant l'hypothèse (1.1).

Pour énoncer le résultat central nous allons introduire quelques notations. Pour un multi-indice $J = (j_1, \ldots, j_p) \in \{1, \ldots, m\}^p$, de longueur $|J| = p$, on notera $X^J = [X_{j_1}, [X_{j_2}, \ldots, [X_{j_{p-1}}, X_{j_p}] \ldots]]$, le crochet de Lie des champs $X_{j_1}, \ldots, X_{j_p}$. Pour tout $x \in \mathbb{R}^d$, on considère

$$C_k(x) = \mathrm{Vect}\,\{X^J(x)\,,\ |J| \leq k\}\,,\ k \in \mathbb{N}^*\,,$$

et le pas,

$$r(x) = \inf\{k : \dim C_k(x) = d\}\,.$$

Nous noterons par $Q(x)$, la dimension graduée en x:

$$Q(x) = \sum_{k=1}^{r(x)} k\,(\dim C_k(x) - \dim C_{k-1}(x))\,. \tag{1.7}$$

Nous allons supposer que la géométrie des crochets est localement constante au voisinage de x, c'est à dire que pour tout $k \in \mathbb{N}^*$ et tout y dans un voisinage $A(x)$ de x, $\dim C_k(y) = \dim C_k(x)$. Sur ce voisinage, $r(y)$ et $Q(y)$ sont constantes, r et Q. Nous supposerons que $Q \geq 4$. Dans le cas de Chaleyat-Maurel et Le Gall [CM-LG] la dimension graduée est quatre, constante sur tout Ω.

Nous allons introduire à présent une norme localement homogène. Pour cela on utilise une "bonne" carte en $x \in \Omega$ (voir aussi Ben Arous [BA2], p. 81). Soit $B = \{J_1, \ldots, J_d\}$, une famille de multi-indices telle que $\{X^J(x) : J \in B\}$ est une base triangulaire. C'est à dire, pour tout $k \leq r$, $\{X^J(x) : J \in B\,,\ |J| \leq k\}$ engendre $C_k(x)$. Il existe un voisinage W de 0, tel que, l'application

$$u \mapsto \varphi_x(u) = \exp\left(\sum_{j=1}^{d} u_j X^{J_j}\right)(x)$$

définisse un difféomorphisme de W sur son image $\varphi_x(W)$. Il existe un voisinage U de x tel que $U \subset \varphi_x(W) \cap A(x)$. On note $|J_j| = l_j$, $j = 1, \ldots, d$.

Alors, $y \in U$, $y = \varphi_x(u)$, a la norme en x:

$$|y|_x = \left[\sum_{k=1}^{r}\left(\sum_{j, l_j = k} u_j^2\right)^{\frac{Q}{2k}}\right]^{\frac{1}{Q}}\,. \tag{1.8}$$

$|y|_x$ est équivalente à la pseudo-distance $d(x, y)$, utilisée par Chaleyat-Maurel et Le Gall [CM-LG].

Nous avons démontré (voir § II.6) que l'estimation à priori de Nagel, Stein et Wainger [N-S-W] s'écrit:

$$|G(x,y)| \le \frac{c}{|y|_x^{Q-2}}\,. \tag{1.9}$$

Notre résultat central est le suivant: il existe une fonction regulière $\Phi_x > 0$, qu'on va décrire plus bas, telle que

$$\lim_{\varepsilon \downarrow 0} \sup_{\|x-y\|<\varepsilon} \Big| G(x,y)\,|y|_x^{Q-2} - \Phi_x(\theta_x(y)) \Big| = 0\,, \tag{1.10}$$

où la variable angulaire homogène, $\theta_x(y)$, est définie par:

$$\theta_x(y) = \left(\frac{u_1}{|y|_x^{l_1}}, \dots, \frac{u_d}{|y|_x^{l_d}} \right)\,,\ y = \varphi_x(u) \in U \setminus \{x\}\,. \tag{1.11}$$

Dans le cas du groupe d'Heisenberg ou dans celui de Chaleyat-Maurel et Le Gall [CM-LG], Φ_x est une constante.

Par (1.11) on voit que, en général, la limite

$$\lim_{y \to x} G(x,y)\,|y|_x^{Q-2}$$

n'existe pas; elle existe seulement d'une façon radiale, à savoir, lorsque y s'approche de x tel que la variable angulaire $\theta_x(y)$ a une limite. Il s'agit d'un comportement différent par rapport à la situation elliptique, à celle du groupe d'Heisenberg ou à celle étudiée par Chaleyat-Maurel et Le Gall [CM-LG].

Nous allons décrire le nouveau coefficient géométrique, Φ_x, en termes de la densité d'occupation d'un processus qu'on appellera processus tangent.

Pour introduire ce processus nous allons rappeler encore quelques notations (voir aussi Castell [Ca], 227). Pour un multi-indice J, on note par B_t^J l'intégrale stochastique itérée de Stratonovich,

$$B_t^J = \int_{0<t_1<\dots<t_p<t} dB_{t_1}^{j_1} \circ \dots \circ dB_{t_p}^{j_p}$$

et par c_t^J, une combinaison linéaire explicite d'intégrales itérées,

$$c_t^J = \sum_{\tau \in \sigma_{|J|}} \frac{(-1)^{e(\tau)}}{|J|^2 \begin{pmatrix} |J|-1 \\ e(\tau) \end{pmatrix}} B_t^{J\circ\tau^{-1}}\,.$$

Ici, pour une permutation $\tau \in \sigma_p$, d'ordre p, on a noté $e(\tau)$ le nombre d'erreurs dans l'ordre de $\tau(1), \ldots, \tau(p)$, et $J \circ \tau = (j_{\tau(1)}, \ldots, j_{\tau(p)})$.

Puisque $\{X^J(y) : J \in B\}$ est une base triangulaire pour y proche de x, quelque soit le multi-indice L, il existe des fonctions C^∞, définies au voisinage de x, $(a_J^L)_{J\in B}$, telles que

$$X^L = \sum_{J\in B} a_J^L X^J \,.$$

Alors, le processus tangent est défini par:

$$u_t^{(x)} = \left(\sum_{L,|L|=|J|} a_J^L(x) c_t^L \right)_{J\in B} . \tag{1.12}$$

Ce processus est en général non-markovien. Comment peut-on alors étudier sa densité d'occupation? Sa loi, a-t-elle une densité, par rapport à la mesure de Lebesgue? Pour répondre à ces questions, nous démontrons qu'on peut regarder ce processus, comme la projection de la diffusion invariante sur un groupe de Lie nilpotent:

$$u_t^{(x)} = \pi_x(\mathcal{G}_t) \,. \tag{1.13}$$

Ici, $(\mathcal{G}_t)$ est la diffusion invariante sur le groupe de Lie nilpotent $\mathcal{N}(m,r)$ associé à $g(m,r)$, l'algèbre de Lie libre nilpotente de pas r à m générateurs $Y_1, \ldots, Y_m$. Précisément,

$$d\mathcal{G}_t = \sum_{j=1}^m Y_j(\mathcal{G}_t) \circ dB_t^j \,,\; \mathcal{G}_0 = e \,, \tag{1.14}$$

où e est l'élément unité du groupe. Cette diffusion a la fonction de Green, $G^{(N)}$, strictement positive.

En utilisant (1.13), on montre que la matrice de Malliavin de $u_t^{(x)}$, $t > 0$, est non-dégénérée. Alors, sa loi admet une densité regulière par rapport à la mesure de Lebesgue, $q_t^{(x)}(0,u)$. On note, pour $u \in \mathbb{R}^d \setminus \{0\}$,

$$g^{(x)}(0,u) = \int_0^\infty q_t^{(x)}(0,u)dt \,, \tag{1.15}$$

la densité d'occupation du processus $(u_t^{(x)})$. On se sert de cette interprétation, pour écrire cette fonction régulière, comme l'intégrale de $G^{(N)}$ sur une fibre de la projection π_x. On déduit que $g^{(x)}$ est strictement positive (voir § II.3).

La fonction Φ_x est définie par:

$$\Phi_x(\theta) = \frac{1}{J_x} g^{(x)}(0,\theta)\,,\ \theta \in \mathbb{R}^d \setminus \{0\}\,, \tag{1.16}$$

où $J_x = |\det(X^{J_1}(x), \ldots, X^{J_d}(x))|$.

On peut comparer notre résultat (1.10), au résultat de comportement du noyau de la chaleur sur la diagonale:

$$p_t^{\Omega}(x,x) \sim \frac{c_0(x)}{\sqrt{t}^{\,Q}}\,.$$

Le coefficient $c_0(x)$ est la densité de la loi du processus tangent $(u_t^{(x)})$, prise au temps 1, $q_1^{(x)}(0,0)$ (voir Ben Arous [BA2], p. 97).

La démonstration de (1.10) repose sur le développement de Taylor stochastique et sur les estimations à priori de la fonction de Green et de ses dérivées.

La diffusion associée à L, en temps petit, a la même loi que la solution (x_t^{ε}), $\varepsilon > 0$, de l'équation

$$dx_t^{\varepsilon} = \varepsilon \sum_{j=1}^{m} X_j(x_t^{\varepsilon}) \circ dB_t^j\,,\ x_0^{\varepsilon} = x\,, \tag{1.17}$$

tuée au premier instant de sortie de Ω, $\tau_\varepsilon = \tau/\varepsilon^2$. On ramène cette diffusion dans la "bonne" échelle:

$$v_t^{(\varepsilon,x)} = (T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(x_t^{\varepsilon})\,,\ t < \tau_\varepsilon\,, \tag{1.18}$$

où T_λ, $\lambda > 0$, est la dilatation sur $\mathbb{R}^d$,

$$T_\lambda(u) = (\lambda^{l_1} u_1, \ldots, \lambda^{l_d} u_d)\,.$$

En utilisant le développement de Taylor stochastique on compare le processus tangent et la diffusion $(v_t^{(\varepsilon,x)})$ (voir § II.2). Si f est une fonction lipschitzienne bornée et $T > 0$, alors, pour $\varepsilon > 0$ suffisamment petit, il existe une constante $c > 0$, telle que

$$\left| E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)} \int_0^T f(v_t^{(\varepsilon,x)})dt\right) - E_0 \int_0^T f(u_t^{(x)})dt \right| \le c\,\|f\|_{\mathrm{Lip}}\, T\,\varepsilon\,.$$

Par l'estimation à priori (1.9), on vérifie que G est localement intégrable. On démontre ensuite que, si f est une fonction continue, bornée par 1, à support dans une boule $B(0,\rho)$, alors

$$\lim_{\varepsilon\downarrow 0, T\uparrow\infty} E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_T^{\tau_\varepsilon} f(v_t^{(\varepsilon,x)})dt\right) = 0$$

(voir § II.4).

En utilisant encore l'estimation à priori dans le cadre nilpotent, ainsi qu'un autre résultat d'intégrabilité, on déduit que le temps passé par $(u_t^{(x)})$ dans la boule $B(0,\rho)$ est fini. D'où, si f est une fonction continue, bornée par 1, à support dans $B(0,\rho)$, alors,

$$\lim_{T\uparrow\infty} E_0\int_T^\infty f(u_t^{(x)})dt = 0\,,$$

(voir p. 48).

De cette façon, on obtient le plus important pas de la démonstration:

$$\lim_{\varepsilon\downarrow 0}\sup_{u\in H}|G^{(\varepsilon,x)}(0,u) - g^{(x)}(0,u)| = 0\,. \tag{1.19}$$

Ici $G^{(\varepsilon,x)}$ est la fonction de Green de $(v_t^{(\varepsilon,x)})$ et $H\subset \mathbb{R}^d\setminus\{0\}$. Pour cela, on se sert aussi des estimations des dérivées de G,

$$|X_{i_1}\dots X_{i_q}G(x,y)| \le \frac{c}{|y|_x^{Q-2+q}}\,,\ y\neq x\,\text{proches}$$

(voir § II.5).

Enfin, pour conclure, on voit que $G^{(\varepsilon,x)}$ et G sont liées par

$$G^{(\varepsilon,x)}\left(0,\left(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1}\right)(y)\right) = J_x\,\varepsilon^{Q-2}G(x,y)\,,$$

et que

$$g^{(x)}\left(0,T_{\frac{1}{\varepsilon}}(u)\right) = \varepsilon^{Q-2}g^{(x)}(0,u)\,.$$

On doit remarquer qu'en général il n'est pas facile de calculer Φ_x. Sa valeur peut être calculée dans certains cas, par exemple sur le groupe d'Heisenberg.

Nous avons donné des exemples, pour illustrer le résultat, par des calculs explicites.

Le premier exemple consiste à généraliser la situation considérée par Chaleyat-Maurel et Le Gall [CM-LG] sur $\mathbb{R}^{2n+1}$. L'espace est engendré en tout point par $X_1, \ldots, X_{2n}$ et le seul crochet non-nul

$$[X_1, X_2] = [X_{2k-1}, X_{2k}]\,,\; k = 1, \ldots, n\,.$$

On calcule Φ_x grâce au résultat de Folland [F]. Φ_x a une forme simple, mais elle est loin d'être constante, ainsi qu'elle est dans le cas (très particulier) $n = 1$.

On complique ensuite les relations entre crochets:

$$[X_{2k-1}, X_{2k}] = a_k [X_1, X_2]\,,\; a_k \in \mathbb{R}^*\,,\; k = 1, \ldots, n\,.$$

En utilisant le "cas modèle" de Greiner [Gr2] à la place du cas d'Heisenberg on trouve Φ_x. On pourrait écrire des formules similaires en utilisant les résultats de Beals, Gaveau et Greiner [B-Ga-Gr].

Nous avons considéré ensuite un exemple qui vient de l'analyse complexe et où le pas est plus grand que deux. Soit le champ de vecteurs holomorphe,

$$Z = \frac{\partial}{\partial z} + i\,p\,z^{p-1}\bar{z}^p \frac{\partial}{\partial x_3}\,,\; z = x_1 + i\,x_2\,,\; p \in \mathbb{N}^*\,.$$

On considère l'opérateur de type $\Box_b$ (voir Greiner et Stein [Gr-S]):

$$L = Z\bar{Z} + \bar{Z}Z = \frac{1}{2}(X_1^2 + X_2^2)\,,\; Z = \frac{1}{2}\,X_1 - \frac{i}{2}\,X_2\,.$$

Il est facile à voir que pour $p = 1$ on obtient le cas d'Heisenberg. Pour $p > 1$ il n'y a pas de structure de groupe de Lie sur $\mathbb{R}^3$, par rapport à laquelle Z soit invariant. Greiner [Gr1] a étudié le cas $p = 2$.

Nous avons considéré $p > 1$ arbitraire. L'opérateur L est hypoelliptique sur tout $\mathbb{R}^3$. Hors de l'axe $\{x_1 = x_2 = 0\}$, $X_1(x)$, $X_2(x)$ et $[X_1, X_2](x)$ engendrent $\mathbb{R}^3$ et on est dans la situation traitée par Chaleyat-Maurel et Le Gall [CM-LG]. Pour des points sur l'axe on a besoin d'aller jusqu'aux crochets de l'ordre $2p$ pour engendrer $\mathbb{R}^3$. La dimension graduée dans un tel point est $2p + 2$. On constate que, autour de ces points, la géométrie des crochets n'est pas localement constante, donc on ne peut pas appliquer le résultat (1.10).

Pourtant, nous avons calculé la fonction de Green de pôle $(0, 0, x_3)$:

$$(1.20)\qquad G((0,0,x_3),(y_1,y_2,y_3)) = \frac{1/(4\,p\,\pi)}{\sqrt{(y_1^2+y_2^2)^{2p}+(y_3-x_3)^2}}$$

(voir p. 70). Le calcul de G de pôle arbitraire semble plus difficile.

Nous avons utilisé (1.20) pour étudier la loi d'une fonctionnelle d'un mouvement brownien plan, (w_t). Pour retrouver l'expression de la fonction de Green sur H_3, Chaleyat-Maurel et Le Gall [CM-LG], p. 228, utilisent une formule obtenue par Pitman et Yor [P-Y], p. 432, de la transformée de Laplace jointe de $(|w_t|^2, \int_0^t |w_s|^2 ds)$. Inversement, nous avons pensé que, en partant de (1.20), on pourrait trouver la transformée de Laplace jointe de $(|w_t|^2, \int_0^t |w_s|^{4p-2} ds)$. On obtient une relation qui la fait intervenir (voir § II.11). Une dernière inversion de Laplace semble plus délicate.

Enfin, nous avons considéré le cas de l'opérateur de Grushin sur $\mathbb{R}^2$,

$$L = \frac{1}{2}(\partial_{x_1}^2 + x_1^2 \partial_{x_2}^2)\,,$$

qui est hypoelliptique sur l'axe $\{x_1 = 0\}$. Nous avons calculé la fonction de Green de pôle $(0,0)$, en partant de la diffusion associée à L (voir p. 75). Le résultat obtenu est de même nature. Pourtant, dans ce cas, plusieurs de nos hypothèses cessent d'être vraies: $d = 2$, cas où l'estimation à priori (1.9) n'est plus valide, la géométrie des crochets n'est pas localement constante autour des points de l'axe $\{x_1 = 0\}$ et, enfin, la dimension graduée dans un tel point est trois.

Une extension possible de cette étude serait de supposer $X_0 \neq 0$. Si X_0 peut s'écrire

$$(1.21)\qquad X_0 = \sum_{j=1}^{m} f_j X_j + \sum_{j,k} f_{j,k}[X_j, X_k]\,,$$

où les fonctions f_j, $f_{j,k}$ sont C^∞ au voisinage de x, alors le résultat sur le comportement du noyau de la chaleur sur la diagonale reste valide (voir Ben Arous et Léandre [BA-Le], p. 378). De même, l'estimation à priori reste vraie (voir Nagel, Stein et Wainger [N-S-W], p. 107). Le résultat de Chaleyat-Maurel et Le Gall [CM-LG] a été prouvé sous la même hypothèse. Il est très plausible que le résultat (1.10) reste vrai en dimension d, sous l'hypothèse forte d'Hörmander et avec le drift de la forme (1.21).

Par ailleurs, on sait que, pour le comportement du noyau de la chaleur sur la diagonale en présence d'un drift, on peut avoir des phénomène surprenants.

Il s'agit du cas où $X_0(x) \notin C_2(x)$ ou même dans des cas encore plus délicats où $X_0(x) \in C_2(x)$, mais ne peut s'écrire sous la forme (1.21) (voir Ben Arous et Léandre [BA-Le]).

Nous avons donné un exemple où la fonction de Green se comporte très différemment sur la diagonale. Dans le cas du groupe nilpotent $\mathcal{N}(m,r)$ on suppose que le drift se trouve dans le centre de l'algèbre de Lie. Alors, lorsqu'on s'approche de la diagonale par la direction opposée au drift, la fonction de Green reste bornée (voir § II.10).

Le résultat de comportement de la fonction de Green sur la diagonale s'applique à la théorie du potentiel et à l'étude de la trajectoire de la diffusion associée. Ainsi, Sznitmann [Sz] en 1987, après avoir étudié la fonction de Green elliptique, analyse le volume de la saucisse de Wiener de petit rayon associée au processus (x_t).

Chaleyat-Maurel et Le Gall [CM-LG] menent le même travail pour certaines diffusions dégénérées. Ils obtiennent des résultats sur la capacité et la probabilité d'atteinte des ensembles petits compacts, sur le volume de la saucisse de Wiener, ainsi que certaines propriétés trajectorielles.

Nous avons décrit aussi des applications de (1.11). D'abord nous avons étudié le comportement de la capacité des petits compacts, répondant ainsi à l'affirmation de Chaleyat-Maurel et Le Gall [CM-LG], selon laquelle "l'estimation de la capacité semble plus délicate en dimension plus grande".

Pour un $\lambda > 0$ suffisamment grand, soit $(x_t^{(\lambda)})$ la diffusion tuée au temps ξ exponentiel indépendant, de paramètre λ.

Soit un compact $H \subset \Omega$. La *λ-capacité* de H, $c_\lambda(H)$, est la masse totale de la mesure d'équilibre $\mu_H^{(\lambda)}$ de H, à savoir, de l'unique mesure supportée par H, telle que

$$P_x(T_H^{(\lambda)} < \infty) = G_\lambda \mu_H^{(\lambda)}(x) = \int_{\mathbb{R}^d} G_\lambda(x,y)\mu_H^{(\lambda)}(dy)\,.$$

Ici,

$$T_H^{(\lambda)} = \inf\{t > 0 : x_t^{(\lambda)} \in H\}$$

et

$$G_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p_t^\Omega(x,y)dt$$

est la fonction de Green de $(x_t^{(\lambda)})$. On peut démontrer qu'elle satisfait aussi (1.10).

Pour énoncer le résultat, on introduit quelques notations. Si H est un compact de $\mathbb{R}^d$ qui contient 0, alors sa dilatation naturelle est:

$$H_\varepsilon^x = (\varphi_x \circ T_\varepsilon)(H) .$$

On pose aussi $u_\varepsilon^x = (\varphi_x \circ T_\varepsilon)(u)$, pour $u \in \mathbb{R}^d$. On note, pour $u \neq v \in \mathbb{R}^d \backslash \{0\}$,

$$\lim_{\varepsilon \downarrow 0} \frac{|v_\varepsilon^x|_{u_\varepsilon^x}}{\varepsilon} = \alpha(u,v) > 0 \, , \; \lim_{\varepsilon \downarrow 0} \theta_{u_\varepsilon^x}(v_\varepsilon^x) = \beta(u,v) \neq 0 \, .$$

Enfin, soient

$$r_x(u,v) = \frac{\Phi_x(\beta(u,v))}{\alpha(u,v)^{Q(x)-2}} \, , \; q_x(H) = \frac{m(H)}{\max_{u \in \partial H} \int_H r_x(u,v) dv} \, .$$

Alors, le résultat sur la capacité s'écrit:

$$\lim_{\varepsilon \downarrow 0} \frac{c_\lambda \left(H_\varepsilon^x\right)}{\varepsilon^{Q(x)-2}} = q_x(H) \tag{1.22}$$

(voir § II.7).

On pourrait étudier le même problème pour un compact dilaté d'une façon arbitraire, (par exemple comme le fait Chaleyat-Maurel et Le Gall [CM-LG], avec la dilatation usuelle de $\mathbb{R}^d$) et pas avec la dilatation naturelle T_ε.

Comme il a été affirmé par Chaleyat-Maurel et Le Gall [CM-LG], p. 222, dès qu'on dispose du résultat sur la capacité des petits compacts, on peut déduire d'autres propriétés trajectorielles. Les methodes générales utilisées par Chaleyat-Maurel et Le Gall [CM-LG] §7 − 8 s'appliquent (voir § II.8). Ainsi, on étend les résultats concernant la probabilité d'atteinte des petits compacts et la saucisse de Wiener de petit rayon associée à (x_t).

Pour démontrer la non-existence des points doubles de la trajectoire il suffit d'utiliser les estimations à priori (majoration et minoration) de la fonction de Green. Il est de même, pour vérifier le test de Wiener.

2. Le support d'une diffusion

Dans cette partie, on considère que les coefficients des champs de vecteurs X_j dépendent aussi de t, $X_j = X_j(t,x)$, $j = 0, \ldots, m$. L'opérateur sera noté par L_t et la loi de la diffusion (x_t) par P_x.

Le *support* de la diffusion (x_t) est le plus petit fermé de l'espace des trajectoires, de probabilité 1. Ainsi donc, on suppose définie une topologie sur cet espace.

En 1972, Stroock et Varadhan [S-V], p. 349, énoncent le théorème de support pour la topologie uniforme $\|\cdot\|_0$:

$$\operatorname{supp}_0(P_x) = \overline{\Phi_x(L^2)}^{\,0} . \tag{2.1}$$

Ici, on a noté par Φ_x l'application qui associe à $h \in L^2 = L^2([0,1], \mathbb{R}^m)$ la solution de l'équation

$$dy_t = \sum_{j=1}^{m} X_j(t, y_t)\, h_t^j\, dt + X_0(t, y_t)\, dt\,,\; y_0 = x\,.$$

Il y a déjà beaucoup des travaux où l'espace de trajectoires est muni d'autres topologies, plus fortes que la topologie uniforme. Par exemple, dans Ben Arous et Léandre [BA-Le], Baldi et Roynette [B-R], Baldi, Ben Arous et Kerkyacharian [B-BA-K] on considère la topologie hölderienne. La norme hölderienne d'indice α, $0 < \alpha < \frac{1}{2}$, d'une fonction $f \in C_0([0,1]; \mathbb{R})$, définie sur $[0,1]$, $f(0) = 0$, est:

$$\|f\|_\alpha = \sup_{0 \le s \ne t \le 1} \frac{|f(s) - f(t)|}{|s-t|^\alpha} . \tag{2.2}$$

Dans la deuxième partie de cette thèse, nous avons démontré que le théorème de support peut être étendu pour la topologie α-hölderienne:

$$\operatorname{supp}_\alpha(P_x) = \overline{\Phi_x(L^2)}^{\,\alpha}\,,\; 0 < \alpha < \frac{1}{2} . \tag{2.3}$$

Ce résultat a été obtenu indépendamment par Millet et Sanz-Solé [M-S] et par Aida, Kusuoka et Stroock [A-K-S], en utilisant d'autres méthodes.

Nous avons suivi la stratégie de Stroock et Varadhan [S-V] (voir aussi Ikeda et Watanabe [I-W], § VI.8). Une question naturelle est: quelle est la probabilité que le mouvement brownien ait une grande norme α-hölderienne, conditionnellement au fait qu'il ait une petite norme uniforme (ou β-hölderienne, $\beta < \alpha$)?

Nous avons commencé par répondre à cette question (pour d'autres applications voir Ben Arous et Ledoux [BA-L]).

On traite d'abord le problème pour la norme de Ciesielski, équivalente à la norme hölderienne (voir Ciesielski [C], p. 218):

$$\|x\|'_\alpha = \sup_{m\geq 1} |m^{\alpha-\frac{1}{2}}\xi_m(x)|\,,\; x\in C_0([0,1];\mathbb{R})\,. \tag{2.4}$$

On a noté par $\xi_m(x)$ la m-ème coordonnée de x par rapport à la base de Schauder sur $C_0([0,1];\mathbb{R})$:

$$\xi_m(x) = \xi_{2^n+k}(x) = 2^{\frac{n}{2}}\left(2x\left(\frac{2k-1}{2^{n+1}}\right) - x\left(\frac{k}{2^n}\right) - x\left(\frac{k-1}{2^n}\right)\right)\,,$$

où $n\geq 0$ et $k = 1,\ldots,2^n$.

Soient $R, r > 0$, $\alpha,\beta \in]0,\frac{1}{2}[$ et w un mouvement brownien réel issu de 0. On va noter $a = \frac{1}{2}-\alpha$, $b = \frac{1}{2}-\beta$, $v = (\frac{R^b}{r^a})^{\frac{1}{b-a}}$, $\varphi(t) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{t^2}{2})$ et

$$\Lambda_{\alpha,\beta}(r,R) = \frac{\varphi(v)}{v} + \frac{1}{a}R^{-\frac{1}{a}}\int_v^\infty \varphi(t)t^{\frac{1}{a}-2}dt\,.$$

Alors, nous avons montré que

$$P(\|w.\|'_\alpha > R\,|\,\|w.\|'_\beta < r) \leq \frac{\Lambda_{\alpha,\beta}(r,R)}{\int_0^v \varphi(t)\,dt}\,.$$

Le calcul est direct, en remarquant que $g_m = \xi_m(w)$ est une suite de variables aléatoires indépendantes gaussiennes centrées réduites (voir p. 95).

Pour écrire des estimations qui concernent la norme hölderienne, nous avons utilisé une *inégalité de corrélation*.

En 1972, Das Gupta *et al.* [DG-E-...] énoncent la conjecture suivante: si C et C' sont deux convexes symétriques et si γ_d est la mesure gaussienne sur $\mathbb{R}^d$, alors

$$\gamma_d(C\bigcap C') \geq \gamma_d(C)\gamma_d(C')\,. \tag{2.5}$$

On sait que cette inégalité est vraie en dimension $d=2$ (voir Pitt [P]), ainsi que pour d arbitraire mais avec C' une bande symétrique (voir Scott [Sc] et Sidak [Si]). Le cas général est toujours ouvert (voir aussi Ledoux [L], p. 86).

En prenant pour un ensemble la bande $\{|g_m| > Rm^{\frac{1}{2}-\alpha}\}$, on obtient

$$P(\|w.\|'_\alpha > R\,|\,\|w.\|_\beta \leq r) \leq \Lambda_{\alpha,\beta}(kr,R)\,,\; k>0$$

(voir p. 96).

Enfin, pour les normes hölderiennes,

$$P(\|w.\|_\alpha > R \,|\, \|w.\|_\beta \leq r) \leq \Lambda_{\alpha,\beta}(kr, KR)\,,\; k, K > 0\,.$$

Nous avons raffiné ces estimations. Par exemple, il existe des constantes positives, k, k', k'', telles que

$$P(\|w.\|_\alpha > R \,|\, \|w.\|_\beta < r) \leq k \left(1 + k' \left(\frac{R^\beta}{r^\alpha}\right)^{\frac{2}{\alpha-\beta}}\right) \exp\left(-k'' \left(\frac{R^{1-2\beta}}{r^{1-2\alpha}}\right)^{\frac{1}{\alpha-\beta}}\right)$$

(voir p. 98).

Si on prend dans l'inégalité précédente $r = 1$ et R suffisamment grand, alors, il existe deux constantes positives c, c', telles que

$$P(\|w.\|_\alpha > R \,|\, \|w.\|_\beta \leq 1) \leq c' \exp\left(-c\, R^{\frac{1-2\beta}{\alpha-\beta}}\right)\,,$$

pour tout $0 \leq \beta < \alpha < \frac{1}{2}$. On compare ceci à l'estimation gaussienne classique, pour R suffisamment grand,

$$P(\|w.\|_\alpha > R) \sim e^{-c\,R^2}$$

(voir § III.2).

Ainsi, nous avons obtenu un cas particulier de l'inégalité de corrélation:

$$P(B_\alpha(R) \,|\, B_\beta(1)) \geq P(B_\alpha(R))\,, \tag{2.6}$$

pour R assez grand, où par $B_\alpha(\rho)$ on a noté la boule en norme α-hölderienne $\{\|w.\|_\alpha \leq \rho\}$.

O. Zeitouni m'a fait remarquer qu'en utilisant encore une fois l'inégalité de corrélation pour une bande, on peut démontrer que les paires des boules $(B'_\alpha(R), B'_\beta(r))$ et $(B'_\alpha(R), B_\beta(r))$ sont positivement corrélées (voir p. 100).

On pense que, en général, on devrait chercher une inégalité de corrélation plus faible, avec une constante devant qui dépend de la géométrie.

Par la suite, on utilise l'application suivante de nos inégalités sur les probabilités conditionnelles. Si $0 \leq \beta < \alpha < \frac{1}{2}$, alors quelque soit $u \in [0, (1-2\alpha)/(1-2\beta)[$, il existe $M_0, k_1, k_2 > 0$, telles que, pour tout $M \geq M_0$,

$$\sup_{0<\delta\leq 1} P(\|w.\|_\alpha > M\delta^u \,|\, \|w.\|_\beta < \delta) \leq k_1 M^{\frac{2\beta}{\alpha-\beta}} \exp(-k_2 M^{\frac{1-2\beta}{\alpha-\beta}})$$

(voir p. 98).

On note par $\mathcal{M}_u^{\alpha,\beta}$ la classe des processus tels que

$$\lim_{M\uparrow\infty} \sup_{0<\delta\le 1} P(\|Y.\|_\alpha > M\delta^u \mid \|B.\|_\beta < \delta) = 0\,. \tag{2.7}$$

Par calcul stochastique élémentaire et en utilisant des inégalités simples concernant les normes höderiennes, on vérifie succesivement que, certains processus sont dans $\mathcal{M}_u^{\alpha,\beta}$. Ainsi, l'aire stochastique et les intégrales stochastiques suivantes $\xi.^{ij} = \int_0^\cdot B_s^i \circ dB_s^j$ et $\int_0^\cdot f(x_s) d\xi_s^{ij}$ sont dans $\mathcal{M}_u^{\alpha,0}$, $\alpha \in]0, \frac{1}{2}[$, $u \in [0,1]$, avec f une fonction régulière sur $\mathbb{R}^d$. Ensuite, on prouve que $\int_0^\cdot f(x_s) \circ dB_s^i \in \mathcal{M}_u^{\alpha,0}$ pour $\alpha \in [0, \frac{1}{2}[$ et $u \in [0, 1-2\alpha[$ (voir § III.3).

De là on déduit que, pour tout $\varepsilon > 0$,

$$\lim_{\delta\downarrow 0} P\left(\left\|\int_0^\cdot X_k(s, x_s) \circ dB_s^k\right\|_\alpha > \varepsilon \mid \|B.\|_0 < \delta\right) = 0 \tag{2.8}$$

(voir p. 108).

Nous avons prouvé une version du lemme de Gronwall en norme hölderienne (voir p. 108). Alors on obtient que, pour tout $\varepsilon > 0$,

$$\lim_{\delta\downarrow 0} P(\|x. - \Phi_x(0)\|_\alpha < \varepsilon \mid \|B.\|_0 < \delta) = 1\,. \tag{2.9}$$

Enfin, on applique la formule de Girsanov pour voir que, pour tout $\varepsilon > 0$,

$$P(\|\Phi_x(10.) - \Phi_x(h.)\|_\alpha < \varepsilon) > 0\,. \tag{2.10}$$

Ceci prouve l'inclusion

$$\mathrm{supp}_\alpha(P_x) \supseteq \overline{\Phi_x(L^2)}^{\,\alpha}\,,\ 0 < \alpha < \frac{1}{2}\,. \tag{2.11}$$

L'inclusion inverse est simple. Elle est obtenue en utilisant l'approximation polygonale du mouvement brownien (voir p. 110).

On peut obtenir l'estimation de $P(\|w.\|_\alpha > R \| \|w.\|_0 < r)$ sans utiliser l'inégalité de corrélation et le théorème de Ciesielski, mais une inégalité de concentration pour la mesure gaussienne (voir III. Appendix).

Bibliographie:

.[A] Azencott, R.: Formule de Taylor stochastique et développements asymptotiques d'intégrales de Feynmann, Dans: Azéma, J., Yor, M. (eds.) Seminaire de Probabilités XVI. Supplément: Géométrie différentielle stochastique (Lect. Notes Math. vol. 921, pp. 237-284), Berlin Heidelberg New York: Springer 1982

.[A-K-S] Aida, S., Kusuoka, S., Stroock, D.: On the support of Wiener functionals, Asymptotic problems in probability theory : Wiener functionals and asymptotics, Longman Sci. and Tech., Pitman Research Notes in Math. Series 294, New York, pp. 3-34 (1993)

.[BA1] Ben Arous, G.: Flots et séries de Taylor stochastiques, Probab. Th. Rel. Fields **81**, pp. 29-77 (1989)

.[BA2] Ben Arous, G.: Développement asymptotique du noyau de la chaleur hypoelliptique sur la diagonale, Ann. Inst. Fourier **39**, pp. 73-99 (1989)

.[BA-G1] Ben Arous, G., Gradinaru, M.: Normes hölderiennes et support des diffusions, C. R. Acad. Sci. Paris **316**, pp. 283-286 (1993)

.[BA-G-L] Ben Arous, G., Gradinaru, M., Ledoux, M.: Hölder norms and the support theorem for diffusions, Ann. Inst. "H. Poincaré" **30**, pp. 415-436 (1994)

.[BA-G2] Ben Arous, G., Gradinaru, M.: Singularities of hypoelliptic Green functions, en préparation

.[BA-L] Ben Arous, G., Ledoux, M.: Grandes déviations de Freidlin-Wentzell en norme hölderienne, In: Azéma, J., Meyer, P.-A., Yor, M. (eds.) Seminaire de Probabilités XXVIII (Lect. Notes Math. vol. 1583, pp. 293-299), Berlin Heidelberg New York: Springer 1994

.[BA-Le] Ben Arous, G., Léandre, R.: Décroissance exponentielle du noyau de la chaleur sur la diagonale I,II, Probab. Th. Rel. Fields **90**, pp. 175-202, 377-402 (1991)

.[B-BA-K] Baldi, P., Ben Arous, G., Kerkyacharian, G.: Large deviations and Strassen law in Hölder norm, Stoch. Proc. Appl. **42**, pp. 171-180 (1992)

.[B-Ga-Gr] Beals, R., Gaveau, B., Greiner, P.C.: Lecture at the Seminar on Analysis, Université Paris VI, "Pierre et Marie Curie" (1993)

.[B-R] Baldi, P., Roynette, B.: Some exact equivalents for the Brownian motion in Hölder norm, Probab. Th. Rel. Fields **93**, pp. 457-484 (1992)

.[C] Ciesielski, Z.: On the isomorphisms of the spaces H_α and m, Bull. Acad. Pol. Sci. **8**, pp. 217-222 (1960)

.[Ca] Castell, F.: Asymptotic expansion of stochastic flows, Probab. Th. Rel. Fields **96**, pp. 225-239 (1993)

.[CM-LG] Chaleyat-Maurel, M., Le Gall, J.-F.: Green function, capacity and sample paths properties for a class of hypoelliptic diffusions processes, Probab. Th. Rel. Fields **83**, pp. 219-264 (1989)

.[DG-E-...] Das Gupta, S., Eaton, M.L., Olkin, I., Perlman, M., Savage, L.J., Sobel, M.: Inequalities on the probability content of convex regions for elliptically contoured distributions, Proceedings of the Sixth Berkeley Symposium of Math. Statist. Prob. II 1970, pp. 241-267, University of California Press, Berkeley 1972

.[F] Folland, G.B.: A fundamental solution for a subelliptic operator, Bull. Amer. Math.

Soc. **79**, pp. 373-376 (1973)
.[F-S] Folland, G.B., Stein, E.M.: Estimates for the $\bar{\partial}_b$-complex and analysis on the Heisenberg group, Comm. Pure Appl. Math. **27**, pp. 429-522 (1974)
.[Fe-Sa] Fefferman, C.L., Sánchez-Calle, A.: Fundamental solutions for second order subelliptic operators, Ann. Math. **124**, pp. 247-272 (1986)
.[Ga] Gaveau, B.: Principe de moindre action, propagation de la chaleur et estimées sous-elliptiques sur certains groupes nilpotents, Acta Math. **139**, pp. 96-153 (1977)
.[Gr1] Greiner, P.C.: A fundamental solution for a nonelliptic partial differential operator, Canad. Jour. Math. **31**, pp. 1107-1120 (1979)
.[Gr2] Greiner, P.C.: On second order hypoelliptic differential operators and the $\bar{\partial}$-Neumann problem, In: Diedrich, K. (ed.) Complex analysis, Proceedings of Workshop at Wuppertal 1990, pp. 134-142, Braunschweig : Vieweg 1991
.[Gr-S] Greiner, P.C., Stein, E.M.: On the solvability of some differential operators of type $\Box_b$, Dans: Several complex variables, Proceedings of the conference at Cortona 1976-1977, pp. 106-165, Pisa: Scuola Normale Superiore 1978
.[H] Hörmander, L.: Hypoelliptic second order differential equations,Acta Math, **119**, pp. 147-171 (1967)
.[I-W] Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes, Amsterdam Oxford New York Tokyo: North-Holland Kodansha 1989
.[J-Sa] Jerison, D., Sánchez-Calle, A.: Subelliptic second order differential operators, Dans: Berenstein, C.A. (ed.) Complex analysis III, Proceedings of the Special Year at University of Maryland 1985-1986 (Lect. Notes Math. vol. 1277, pp. 46-77), Berlin Heidelberg New York: Springer 1987
.[Le] Léandre, R.: Développement asymptotique de la densité d'unediffusion dégénérée, Forum Math. **4**, pp. 45-75 (1992)
.[L] Ledoux, M.: Isoperimetry and gaussian analysis, Ecole d'été de probabilités de Saint-Flour, 1994
.[M-S] Millet, A., Sanz-Solé, M.: A simple proof of the support theorem for diffusion processes, Dans: Azéma, J., Meyer, P.-A., Yor, M. (eds.) Seminaire de Probabilités XXVIII (Lect. Notes Math. vol. 1583, pp. 36-48), Berlin Heidelberg New York: Springer 1994
.[N-S-W] Nagel, A., Stein, E.M., Wainger, S.: Balls and metrics defined by vector fields I. Basic properties, Acta Math. **155**, pp. 103-147 (1985)
.[P] Pitt, L.: A Gaussian correlation inequality for symmetric convex sets, Ann. Probab. **5**, pp. 470-474 (1977)
.[Sa] Sánchez-Calle, A.: Fundamental solutions and geometry of the sum of square of vector fields, Invent. math. **78**, pp. 143-160 (1984)
.[Sc] Scott, A.: A note on conservative confidence regions for the mean value of multivariate normal, Ann. Math. Stat. **38**, pp. 278-280 (1967)
.[Si] Sidak, Z.: Rectangular confidence regions for the means of multivariate normal distributions, J. Amer. Stat. Assoc. **62**, pp. 626-633 (1967)
.[S-V] Stroock, D.W., Varadhan, S.R.S.: On the support of diffusion processes with applications to the strong maximum principle, Proceedings of Sixth Berkeley Symposium of Math. Statist. Prob. III 1970, pp. 333-359, University of California

Press, Berkeley 1972
.[Sz] Sznitman, A.S.: Some bounds and limiting results for the measure of the Wiener sausage of small radius associated with elliptic diffusions, Stoch. Proc. Appl. **25**, pp. 1-25 (1987).

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II Singularities of hypoelliptic Green functions

Summary. This chapter is devoted to a precise description of the singularity near the diagonal of the Green function associated to a hypoelliptic operator using a probabilistic approach. Examples and some applications to potential theory are given. The present part contains a work made in collaboration with G. Ben Arous.

Contents:

1. Introduction

Let $X_1, \dots, X_m$ be smooth vector fields on $\mathbb{R}^d$, $d \geq 3$ such that the Lie algebra generated by $X_1, \dots, X_m$ is of full rank at every point:

$$\forall x \in \mathbb{R}^d \,, \, \dim \operatorname{Lie}(X_1, \dots, X_m)(x) = d \,. \tag{1.1}$$

We are interested on the behaviour of the Green function G of the hypoelliptic operator

$$L = \frac{1}{2} \sum_{j=1}^{m} X_j^2 \tag{1.2}$$

on a smooth bounded domain Ω of $\mathbb{R}^d$. G is smooth off the diagonal and we give in this paper a precise description of its singularity near the diagonal.

From the work of Nagel, Stein and Wainger [N-S-W], it is known that the Green function can be estimated in terms of the natural sub-Riemannian distance ρ:

$$|G(x,y)| \leq c \, \frac{\rho(x,y)^2}{\operatorname{vol}(B_\rho(x, \rho(x,y))} \,. \tag{1.3}$$

To state a more precise form of these upper bound, let us introduce some notations. For a multi-index $J = (j_1, \dots, j_p) \in \{1, \dots, m\}^p$, we shall write $|J| = p$, and

$$X^J = [X_{j_1}, [X_{j_2}, \dots, [X_{j_{p-1}}, X_{j_p}] \dots]]$$

will denote the Lie bracket of the vector fields $X_{j_1}, \dots, X_{j_p}$. For any $k \in \mathbb{N}^*$ and any $x \in \mathbb{R}^d$, we consider

$$C_k(x) = \operatorname{Span} \{X^J(x) \,, \, |J| \leq k\}$$

and

$$r(x) = \inf\{k \,:\, \dim C_k(x) = d\} \,. \tag{1.4}$$

By (1.1), $r(x)$ is finite.

Let us denote by $Q(x)$ the graded dimension at x:

$$Q(x) = \sum_{k=1}^{r(x)} k \left(\dim C_k(x) - \dim C_{k-1}(x)\right) . \tag{1.5}$$

We shall assume that the geometry of the brackets is locally constant near x, that is, for every $k \in \mathbb{N}^*$ and every y in a neighbourhood $A(x)$ of x, $\dim C_k(y) = \dim C_k(x)$. Then, of course, $r(y)$ and $Q(y)$ are constant on this neighbourhood. Since we want to exclude the trivial elliptic cases where $d = Q = 2$ and $d = Q = 3$, we assume $Q \geq 4$.

Following Ben Arous [BA2], we shall introduce a useful coordinate chart. For a fixed $x \in \Omega$ we choose a family of multi-indices $B = \{J_1, \dots, J_d\}$, such that $\{X^J(x) : J \in B\}$ is a triangular basis. That is, for every $k \leq r$, $\{X^J(x) : J \in B , |J| \leq k\}$ generates $C_k(x)$. We shall denote the length $|J_j| = l_j$, $j = 1, \dots, d$. There exists a neighbourhood W of 0 such that the mapping

$$u \mapsto \varphi_x(u) = \exp\left(\sum_{j=1}^{d} u_j X^{J_j}\right)(x) \tag{1.6}$$

defines a diffeomorphism of W on $\varphi_x(W)$. There exists a neighbourhood U of x such that $U \subset \varphi_x(W) \cap A(x)$.

For $y \in U$, $y = \varphi_x(u)$ we shall denote

$$|y|_x = \left[\sum_{k=1}^{r}\left(\sum_{j, l_j = k} u_j^2\right)^{\frac{Q}{2k}}\right]^{\frac{1}{Q}} \tag{1.7}$$

and we shall show that the estimate of Nagel, Stein and Wainger [N-S-W], can be written as

$$|G(x,y)| \leq \frac{c}{|y|_x^{Q(x)-2}} . \tag{1.8}$$

We want to give a sharper description of the singularity of $G(x,y)$ when $y \to x$. For this purpose we introduce the homogeneous angular variable, for $y \in U \setminus \{x\}$, $y = \varphi_x(u)$,

$$\theta_x(y) = \left(\frac{u_1}{|y|_x^{l_1}}, \dots, \frac{u_d}{|y|_x^{l_d}}\right) . \tag{1.9}$$

Then our main result will be:

(1.10) **Theorem.** *There exists a smooth function* $\Phi_x > 0$ *such that,*

$$\lim_{\varepsilon \downarrow 0} \sup_{\|x-y\|<\varepsilon} \left| G(x,y)\, |y|_x^{Q(x)-2} - \Phi_x(\theta_x(y)) \right| = 0\,. \tag{1.11}$$

Here and elsewhere $\|\cdot\|$ denotes the Euclidian norm on $\mathbb{R}^d$.

This new geometric coefficient Φ will be described in §5 as the density of the occupation measure for a process (u_t), that we call tangent process. This process, non-markovian in general, will be seen as a projection of a left invariant diffusion process on a free nilpotent Lie group.

It must be noticed that, in general, computing Φ is not easy. The value of Φ is computable in some examples (see §9), for instance on Heisenberg groups.

Theorem (1.10) shows that, in general, the limit

$$\lim_{y \to x} G(x,y)\, |y|_x^{Q(x)-2}$$

does not exist; it exists only "radially", that is, if y approaches x in such a way that the angular variable $\theta_x(y)$ tends to a limit. This is in contrast with the elliptic situation, the Heisenberg group situation or the "curved" Heisenberg group situation studied by Chalyat-Maurel and Le Gall [CM-LG], where Φ_x is constant.

Our approach for the proof of the Theorem (1.10) is probabilistic. It relies on results on stochastic Taylor expansion of paths of the diffusion generated by L and on the a priori estimate given by Nagel, Stein and Wainger [N-S-W]. We follow and extend the strategy given by Chalyat-Maurel and Le Gall [CM-LG] in a simple context.

One must also notice that the behaviour of the heat kernel $p_t^\Omega(x,y)$ on the diagonal has been studied using the same probabilistic tools in Ben Arous [BA2]. The results can be compared with the Theorem (1.10):

$$p_t^\Omega(x,x) \sim \frac{c_0(x)}{\sqrt{t}^{\,Q(x)}}\,, \tag{1.12}$$

where $c_0(x)$ is the density of the law of the tangent process (u_t) taken at time 1.

The plan of the chapter is as follows: in §2 we introduce the stochastic Taylor expansion and the tangent process, which we study in §3. In §4 − 6 we prove the Theorem (1.10) except for some technical lemmas postponed to the Appendix. We then apply our results to some potential theoretical problems in §7 − 8: estimates of the capacities of small sets, of the volume of the Wiener sausage of small radius, double points. In §9 we give examples where direct computations illustrate our general theorem and even two examples where the conclusion of the theorem is valid though our hypothesis of locally constant geometry fail. Sections 10 and 11 contain, respectively, an example of behaviour near the diagonal of the Green function in presence of a drift, and the study of the law of a functional of planar Brownian motion (independent work).

2. Taylor stochastic expansion

Let $(B^1, \ldots, B^m)$ be a m-dimensional Brownian motion and consider (x_t) the solution of the Stratonovich equation

$$dx_t = \sum_{j=1}^{m} X_j(x_t) \circ dB_t^j \, , \; x_0 = x \, , \tag{2.1}$$

killed at the first exit time from Ω, $\tau = \inf\{t > 0 \, , \, x_t \notin \Omega\}$. It is known that $\tau < \infty$, P_x-a.s., for every $x \in \mathbb{R}^d$.

By hypoellipticity, for every $x \in \Omega$, $t > 0$, the law of x_t under P_x has, on Ω, a density with respect to the Lebesgue measure, $p_t^\Omega(x, y)$. It is the heat kernel associated to L on Ω and the Green function is

$$G(x, y) = \int_0^\infty p_t^\Omega(x, y) dt \, , \; x, y \in \Omega \, . \tag{2.2}$$

G is the density of occupation measure of (x_t), that is, for every positive measurable function f,

$$E_x \int_0^\tau f(x_t) dt = \int_\Omega f(y) G(x, y) dy \, . \tag{2.3}$$

(2.4) *Remark.* We note that for the study of the singularity of G near the diagonal it suffices to consider Ω as a bounded neighbourhood of x, $\Omega \subset U$. Indeed, if we denote by G_V the Green function of L on a neighbourhood V

of x, then the singular behaviour near the diagonal of G and G_V is the same, because $L(G - G_V) = 0$. From now on we shall assume that $\Omega \subset U$.

We are interested in the study of the process in short time, $(x_{\varepsilon^2 t})$, $\varepsilon > 0$. For every $x \in \Omega$, it has the same law under P_x as the solution of the equation

$$dx_t^\varepsilon = \varepsilon \sum_{j=1}^{m} X_j(x_t^\varepsilon) \circ dB_t^j \, , \; x_0^\varepsilon = x \, , \tag{2.5}$$

killed at the first exit time from Ω, $\tau_\varepsilon = \tau/\varepsilon^2$.

Let us consider, for $\lambda > 0$, the dilation defined on $\mathbb{R}^d$,

$$T_\lambda(u) = (\lambda^{l_1} u_1, \ldots, \lambda^{l_d} u_d) \, . \tag{2.6}$$

For $0 \le t < \tau_\varepsilon$, we define the diffusion $(v_t^{(\varepsilon,x)})$, starting from 0,

$$v_t^{(\varepsilon,x)} = (T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(x_t^\varepsilon) \, . \tag{2.7}$$

We shall introduce a new process, called tangent process. For a multi-index $J = (j_1, \ldots, j_p)$, we denote by B_t^J the Stratonovich iterated integral

$$B_t^J = \int_{0 < t_1 < \ldots < t_p < t} dB_{t_1}^{j_1} \circ \ldots \circ dB_{t_p}^{j_p} \tag{2.8}$$

and by c_t^J the completely explicit linear combination of Stratonovich iterated integrals

$$c_t^J = \sum_{\tau \in \sigma_{|J|}} \frac{(-1)^{e(\tau)}}{|J|^2 \begin{pmatrix} |J| - 1 \\ e(\tau) \end{pmatrix}} B_t^{J \circ \tau^{-1}} \, . \tag{2.9}$$

Here, for a permutation $\tau \in \sigma_p$, of order p, we denoted $e(\tau)$ the number of errors in ordering $\tau(1), \ldots, \tau(p)$ and

$$J \circ \tau = (j_{\tau(1)}, \ldots, j_{\tau(p)}) \, .$$

Recall that $\{X^J(y) : J \in B\}$ is a triangular basis for y close to x. So, for any multi-index L, there exists smooth functions, defined on a neighbourhood of x, $(a_J^L)_{J \in B}$, such that

$$(2.10)\qquad X^L = \sum_{J\in B} a_J^L X^J\,.$$

Definition. We shall call *tangent proces*, the process,

$$(2.11)\qquad u_t^{(x)} = \left(\sum_{L,|L|=|J|} a_J^L(x) c_t^L \right)_{J\in B}\,.$$

(2.12) **Proposition.** *Let fix $T > 0$. Then, for any bounded Lipschitz continuous function f on $\mathbb{R}^d$ and for sufficiently small $\varepsilon > 0$, there exists a positive constant c, such that,*

$$(2.13)\qquad \left| E_0 \left(\mathbb{1}_{(T<\tau_\varepsilon)} \int_0^T f(v_t^{(\varepsilon,x)})dt \right) - E_0 \int_0^T f(u_t^{(x)})dt \right| \le c\, \|f\|_{\mathrm{Lip}}\, T\, \varepsilon\,.$$

Here we denoted

$$\|f\|_{\mathrm{Lip}} = \sup_{x\in\mathbb{R}^d} |f(x)| + \sup_{x\neq y} \frac{|f(x)-f(y)|}{\|x-y\|}\,.$$

To prove this result, we shall use the results of [BA1] or [Ca] on the asymptotic expansion in small time of x_t in terms of Lie brackets and iterated Stratonovich integrals. According to the Theorem 4.1 [Ca], p. 234, for $t \le T$,

$$(2.14)\qquad x_t^\varepsilon = \exp\left(\sum_{k=1}^{r} \varepsilon^k \sum_{L,|L|=k} c_t^L X^L \right)(x) + \varepsilon^{r+1} R_{r+1}(\varepsilon,t)\,.$$

Here $R_{r+1}(\varepsilon,t)$ is bounded in probability. More precisely, there exists $\alpha, c > 0$ such that, for every $R > c$

$$(2.15)\qquad \lim_{\varepsilon\downarrow 0} P\left(\sup_{0\le t\le T} \|R_{r+1}(\varepsilon,t)\| \ge R \right) \le \exp\left(-\frac{R^\alpha}{cT} \right)\,.$$

Proof of the Proposition (2.12). We can write

$$\left| E_0 \left(\mathbb{1}_{(T<\tau_\varepsilon)} \int_0^T f(v_t^{(\varepsilon,x)})dt \right) - E_0 \int_0^T f(u_t^{(x)})dt \right| \le$$

$$\left| E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_0^T f(v_t^{(\varepsilon,x)})dt\right) - E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_0^T f(u_t^{(x)})\right)dt\right| +$$
$$\|f\|_{\text{Lip}}\, T\, P(T\geq\tau_\varepsilon)\,.$$

By the classical exponential inequality we know that, there exists two positive constants c, c', such that

$$P(T\geq\tau_\varepsilon)\leq c\,e^{-\frac{c'}{\varepsilon^2 T}}\,.$$

We shall study only the first term.

Let us consider ψ_x the diffeomorphism

$$\psi_x((v_L)_{|L|\leq r}) = \exp\left(\sum_{L,|L|\leq r} v_L X_L\right)(x)$$

and we denote

$$T_{\bar\Omega}^\varepsilon = \inf\{t>0 : (\varepsilon^{|L|}c_t^L)_{|L|\leq r}\notin\psi_x^{-1}(\bar\Omega)\}\,. \tag{2.16}$$

We can write,

$$\left| E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_0^T f(v_t^{(\varepsilon,x)})dt\right) - E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_0^T f(u_t^{(x)})\right)dt\right| \leq$$

$$\left| E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T_{\bar\Omega}^\varepsilon)}\int_0^T f(v_t^{(\varepsilon,x)})dt\right) - E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T_{\bar\Omega}^\varepsilon)}\int_0^T f(u_t^{(x)})\right)dt\right| +$$
$$2\,\|f\|_{\text{Lip}}\, T\, P(T\geq T_{\bar\Omega}^\varepsilon)\,.$$

As in [Ca], p. 238, we have that, for sufficiently small ε,

$$P(T\geq T_{\bar\Omega}^\varepsilon)\leq\sum_{L,|L|\leq r}\exp\left(-\frac{c_L}{\varepsilon^{2|L|}T}\right)\,. \tag{2.17}$$

So, it remains to consider the first term:

$$\left| E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T_{\bar\Omega}^\varepsilon)}\int_0^T f(v_t^{(\varepsilon,x)})dt\right) - E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T_{\bar\Omega}^\varepsilon)}\int_0^T f(u_t^{(x)})\right)dt\right| \leq$$

$$\|f\|_{\text{Lip}}\, T\, E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T_{\bar\Omega}^\varepsilon)}\sup_{0\leq t\leq T}\|v_t^{(\varepsilon,x)}-u_t^{(x)}\|\right)\,.$$

Hence, to finish the proof of (2.13), it suffices to prove the following:

(2.18) **Lemma.** *There exists a positive constant* c, *such that for any sufficiently small* $\varepsilon > 0$,

$$E_0\left(1\!\!1_{(T<\tau_\varepsilon\wedge T^\varepsilon_{\bar\Omega})}\sup_{0\le t\le T}\|v_t^{(\varepsilon,x)}-u_t^{(x)}\|\right)\le c\,\varepsilon\,. \tag{2.19}$$

Proof. For $J\in B$ and $t<T^\varepsilon_{\bar\Omega}$, we denote

$$u_J(\varepsilon,t,x)=(\varphi_x^{-1}\circ\psi_x)_J((\varepsilon^{|L|}c_t^L)_{|L|\le r})\,.$$

We have that, for $J\in B$ and $t<T^\varepsilon_{\bar\Omega}$,

$$(\partial_\varepsilon)^k u_J(\varepsilon,t,x)_{|\varepsilon=0}=0\,,\ \text{if}\, k<|J|\,.$$

Indeed, by the triangularity of the basis $\{X^J(y):J\in B\}$, for y close to x, we have, for $J\in B$,

$$a_J^L\equiv 0\,,\ \text{if}\,|L|<|J|\,,$$

on a neighbourhood of x. So, for $J\in B$,

$$[\partial(\varphi_x^{-1}\circ\psi_x)_J/\partial v_L]_{|v=0}=a_J^L(x)=0\,,\ \text{if}\,|L|<|J|\,.$$

Moreover, by the last equality we also have that, for $J\in B$ and $t<T^\varepsilon_{\bar\Omega}$,

$$(\partial_\varepsilon)^{|J|}u_J(\varepsilon,t,x)_{|\varepsilon=0}=\sum_{L,|L|=|J|}a_J^L(x)(\partial_\varepsilon)^{|L|}(\varepsilon^{|L|}c_t^L)_{|\varepsilon=0}\,,$$

because the terms corresponding to L, with $|L|>|J|$, are zero having the factor $\varepsilon^{|L|}$ (see also [BA2], pp. 93-94).

Hence, the Taylor expansion around $\varepsilon=0$ of $u_J(\varepsilon,t,x)$, for $J\in B$ and $t<T^\varepsilon_{\bar\Omega}$, can be written,

$$u_J(\varepsilon,t,x)=\frac{\varepsilon^{|J|}}{|J|!}(\partial_\varepsilon)^{|J|}u_J(\varepsilon,t,x)_{|\varepsilon=0}+\varepsilon^{|J|+1}R_{J,|J|+1}(\varepsilon,t,x)\,,$$

or

$$\frac{1}{\varepsilon^{|J|}}u_J(\varepsilon,t,x)=\sum_{L,|L|=|J|}c_t^La_J^L(x)+\varepsilon\,R_{J,|J|+1}(\varepsilon,t,x)\,.$$

Here, for $J \in B$ and $t < T^{\varepsilon}_{\bar{\Omega}}$,

$$R_{J,|J|+1}(\varepsilon, t, x) = \int_0^1 (\partial_\varepsilon)^{|J|+1} u_J(\varepsilon\,\xi, t, x) \frac{(1-\xi)^{|J|}}{|J|!}\, d\xi \,.$$

Using properties (P1), (P2) in [Ca], p. 238 (see also [A], p. 252), we see that, for every $J \in B$, there exists $\alpha_J, c_J > 0$, such that, for any $R > c_J$ and for $\varepsilon > 0$ sufficiently small,

$$P\left(\sup_{0 \le t \le T} |R_{J,|J|+1}(\varepsilon, t, x)| \ge R\,;\, T < T^{\varepsilon}_{\bar{\Omega}}\right) \le \exp\left(-\frac{R^{\alpha_J}}{c_J T}\right) . \tag{2.20}$$

Indeed, B^J_t satisfies (2.20) and we get the same thing for $u_J(\varepsilon, t, x)$, using its definition in terms of $(\varepsilon^{|L|} c^L_t)$. Then we obtain (2.20).

By (2.14), we have, for $t < T \wedge T^{\varepsilon}_{\bar{\Omega}}$,

$$(T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(x^\varepsilon_t - \varepsilon^{r+1} R_{r+1}(\varepsilon, t)) = \left(\sum_{L,|L|=|J|} c^L_t a^L_J(x) + \varepsilon\, R_{J,|J|+1}(\varepsilon, t, x)\right)_{J \in B} .$$

We note that, for any $0 < \varepsilon < 1$ and any $u \in \mathbb{R}^d$, $\|T_{\frac{1}{\varepsilon}}(u)\| \le \frac{1}{\varepsilon^r}\|u\|$. Therefore, by the Lipschitz property of φ_x^{-1}, we can write, for $t \le T \wedge \tau_\varepsilon \wedge T^{\varepsilon}_{\bar{\Omega}}$,

$$\|v^{(\varepsilon,x)}_t - u^{(x)}_t\| \le \|v^{(\varepsilon,x)}_t - (T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(x^\varepsilon_t - \varepsilon^{r+1} R_{r+1}(\varepsilon, t))\| +$$

$$\|(T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(x^\varepsilon_t - \varepsilon^{r+1} R_{r+1}(\varepsilon, t)) - u^{(x)}_t\| \le c\, \frac{1}{\varepsilon^r}\, \|\varepsilon^{r+1} R_{r+1}(\varepsilon, t)\| +$$

$$\|\varepsilon\,(R_{J,|J|+1}(\varepsilon, t, x))_{J \in B}\| \,.$$

Hence, for $t \le T \wedge \tau_\varepsilon \wedge T^{\varepsilon}_{\bar{\Omega}}$,

$$\|v^{(\varepsilon,x)}_t - u^{(x)}_t\| \le \varepsilon\, R(\varepsilon, t) \,, \tag{2.21}$$

where, for $t \le T \wedge \tau_\varepsilon \wedge T^{\varepsilon}_{\bar{\Omega}}$,

$$R(\varepsilon, t) = c\, \|R_{r+1}(\varepsilon, t)\| + \|(R_{J,|J|+1}(\varepsilon, t, x))_{J \in B}\| \,. \tag{2.22}$$

Using (2.15) and (2.20) we prove the existence of positive constants α', c', such that, for any $R > c'$ and for $\varepsilon > 0$ sufficiently small,

$$(2.23) \qquad P\left(\sup_{0\le t\le T} R(\varepsilon,t) \ge R\,;\, T<\tau_\varepsilon \wedge T^\varepsilon_{\bar\Omega}\right) \le \exp\left(-\frac{R^{\alpha'}}{c'T}\right).$$

Finally, by (2.21), we can write,

$$E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T^\varepsilon_{\bar\Omega})}\sup_{0\le t\le T}\|v_t^{(\varepsilon,x)}-u_t^{(x)}\|\right) \le \varepsilon\, E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T^\varepsilon_{\bar\Omega})}\sup_{0\le t\le T} R(\varepsilon,t)\right) =$$

$$\varepsilon\int_0^\infty P\left(\sup_{0\le t\le T} R(\varepsilon,t)\ge R\,;\, T<\tau_\varepsilon\wedge T^\varepsilon_{\bar\Omega}\right) dR\,.$$

Now, (2.19) follows from this, using (2.23).

q.e.d. Lemma (2.18)

This also ends the proof of the Proposition (2.12).

3. Study of the tangent process

The process $(u_t^{(x)})$ is not necessarily a diffusion process. However, we shall prove that it is the image by a projection of a left invariant diffusion on a nilpotent group.

We denote by $g(m,r)$ the free r-nilpotent Lie algebra with m generators $Y_1,\dots,Y_m$. We shall identifie $g(m,r)$ and the associated simple connected nilpotent Lie group $\mathcal{N}(m,r)$, which is nothing but $g(m,r)$ with the multiplication given by the Campbell-Hausdorff formula. We denote, by a clear abuse of notation, Y_j the left invariant vector field on $\mathcal{N}(m,r)$ defined by the generator Y_j of $g(m,r)$.

Let us consider $(\mathcal{G}_t)$ the invariant diffusion on $\mathcal{N}(m,r)$. That is the solution, starting from the unit element, $e\in\mathcal{N}(m,r)$, of the Stratonovich equation

$$(3.1) \qquad d\mathcal{G}_t = \sum_{j=1}^m Y_j(\mathcal{G}_t)\circ dB_t^j\,,\ \mathcal{G}_0 = e\,.$$

(3.2) **Proposition.** *There exists a unique linear projection, π_x, such that*

$$(3.3) \qquad u_t^{(x)} = \pi_x(\mathcal{G}_t)\,.$$

Proof. According to the result of the Proposition 3.1 [Ca], p. 228,

$$\mathcal{G}_t = \exp\left(\sum_{L,|L|\le r} c_t^L Y^L\right)(e)\,.$$

Let $\{Y^K : K \in A\}$ be a Hall basis of $g(m,r)$. Then, for every multi-index L,

$$Y^L = \sum_{K\in A,|K|=|L|} c_K^L Y^K\,, \tag{3.4}$$

with universal constants c_K^L. Let us denote, for $K \in A$,

$$b_t^K = \sum_{L,|L|=|K|} c_K^L c_t^L\,.$$

and then, by a simple calculation, we get that

$$\mathcal{G}_t = \exp\left(\sum_{K\in A} b_t^K Y^K\right)(e)\,.$$

We note that, by the properties of vector fields, (3.4) it is also true with X_j instead Y_j. By the fact that $\{X^J(x) : J \in B\}$ is a basis, we see that $u_t^{(x)}$ can be written:

$$u_t^{(x)} = \left(\sum_{K\in A,|K|=|J|} a_J^K(x) b_t^K\right)_{J\in B}\,.$$

Put $n = \dim g(m,r) - d$ and $A = \{K_i : i = 1,\ldots,d+n\}$. There exists a diffeomorphism between $\mathbb{R}^{d+n}$, and $\mathcal{N}(m,r)$,

$$w \mapsto \phi_e(w) = \exp\left(\sum_{i=1}^{d+n} w_i Y^{K_i}\right)(e)\,.$$

Let us denote by $p_x : \mathbb{R}^{d+n} \to \mathbb{R}^d$ the projection

$$p_x(w) = \left(\sum_{i,|K_i|=|J_j|} a_j^i(x)\, w_i\right)_{j=1,\ldots,d} = \tilde{M}(x)\, w\,.$$

Here we denoted $a_j^i(x) = a_{J_j}^{K_i}(x)$, $j = 1,\ldots,d$, $i = 1,\ldots,d+n$ and $\tilde{M}(x)$ is the matrix with elements $a_j^i(x)$ if $|J_j = |K_i|$ and zero otherwise.

Hence, taking

$$\pi_x = p_x \circ \phi_e^{-1} , \tag{3.5}$$

we obtain (3.3).

q.e.d.

(3.6) **Corollary.** *For every $t > 0$, the law of $u_t^{(x)}$ has a smooth density with respect to the Lebesgue measure, $q_t^{(x)}(0,u)$.*

Proof. We show that the Malliavin covariance matrix of $u_t^{(x)}$ is not degenerate for every $t > 0$. It is known that the Malliavin covariance matrix of $\mathcal{G}_t$ is not degenerate for $t > 0$. The same thing is true for $b_t = \phi_e^{-1}(\mathcal{G}_t)$. But, by (3.3),

$$u_t^{(x)} = \tilde{M}(x)\ b_t ,$$

and we conclude, noting that $\tilde{M}(x)$ is a full rank matrix.

q.e.d.

Let us denote, for $u \in \mathbb{R}^d \setminus \{0\}$,

$$g^{(x)}(0,u) = \int_0^\infty q_t^{(x)}(0,u)dt , \tag{3.7}$$

the density of the occupation measure of the process $(u_t^{(x)})$. That is, for every positive measurable function f,

$$E_0 \int_0^\infty f(u_t^{(x)})dt = \int_{\mathbb{R}^d} g^{(x)}(0,u)f(u)du . \tag{3.8}$$

(3.9) **Proposition.** *$g^{(x)}(0,\cdot)$ is a strictly positive smooth function on $\mathbb{R}^d \setminus \{0\}$.*

Proof. The fact that $g^{(x)}$ is smooth follows from (3.7). We show now that $g^{(x)}$ is a strictly positive function. We denote by $G^{(N)}$ the Green function of the diffusion $(\mathcal{G}_t)$. Then, for every positive measurable function f,

$$E_e \int_0^\infty f(\mathcal{G}_t)dt = \int_{\mathcal{N}(m,r)} G^{(N)}(e,g)f(g)dg ,$$

where dg denotes the Haar measure on $\mathcal{N}(m,r)$.

It is known that $G^{(N)}$ is a strictly positive function (see for instance [G], p. 102). Using again (3.3), we shall write $g^{(x)}$ in terms of $G^{(N)}$ as an integral on a fiber of the projection map π_x, and we shall conclude. We prove:

$$(3.10) \qquad g^{(x)}(0,u) = c\int_{\mathbb{R}^n} G^{(N)}(\phi_e(0,0), \phi_e(u - M(x)h, h)dh\,.$$

Here $c > 0$ and $M(x)$ is the block of the matrix $\tilde{M}(x)$, having d lines indexed by B and n columns indexed by $A \setminus B$. Indeed, we have

$$\int_{\mathbb{R}^d} g^{(x)}(0,u)f(u)du = E_0\int_0^\infty f(u_t^{(x)})dt =$$

$$E_e\int_0^\infty (f\circ\pi_x)(\mathcal{G}_t)dt = \int_{\mathcal{N}(m,r)} G^{(N)}(e,g)(f\circ\pi_x)(g)dg =$$

$$c\int_{\mathbb{R}^d\times\mathbb{R}^n} G^{(N)}(\phi_e(0,0), \phi_e(u,h)(f\circ\pi_x)(\phi_e(u,h)du\,dh\,,$$

where $c > 0$ is the absolute value of the jacobian of ϕ_e. In the latter integral we perform the change of variables $v = u + D(x)h$. Since f was an arbitrary function we get (3.10).

q.e.d.

We show now that the time spent by $(u_t^{(x)})$ in a Euclidian ball is finite:

(3.11) **Proposition.** *For every* $\rho > 0$,

$$(3.12) \qquad E_0\int_0^\infty \mathbb{1}_{B(0,\rho)}(u_t^{(x)})dt < \infty\,.$$

Before proving this result we shall make a useful remark. We note that, in this nilpotent context, the estimate of the Green function (1.8), can be written:

$$(3.13) \qquad |G^{(N)}(e,g)| \leq \frac{c}{|g|_N^{Q_N-2}}\,,\; g\neq e\,,$$

where the homogeneous norm of $g = \phi_e(w)$, $w\in\mathbb{R}^{d+n}$, is

(3.14) $$|g|_N = \left[\sum_{k=1}^{r} \left(\sum_{i,|K_i|=k} w_i^2 \right)^{\frac{Q_N}{2k}} \right]^{\frac{1}{Q_N}} .$$

Here Q_N is the homogeneous dimension of $\mathcal{N}(m,r)$,

(3.15) $$Q_N = \sum_{k=1}^{r} k \, \mathrm{dim} V_k \, ,$$

with

$$V_k = \mathrm{Span}\{Y^J : |J| = k\} \, , k = 1, \ldots, r \, .$$

V_k's form the natural graduation of the Lie algebra, $g(m,r) = V_1 \oplus \ldots \oplus V_r$.

Proof of the Proposition (3.11). By (3.10), we can write

$$E_0 \int_0^\infty \mathbb{1}_{B(0,\rho)}(u_t^{(x)}) dt = \int_{B(0,\rho)} g^{(x)}(0,u) f(u) du =$$

$$c \int_{B(0,\rho)} du \int_{\mathbb{R}^n} G^{(N)}(\phi_e(0,0), \phi_e(u - M(x)h, h) dh =$$

$$c \int_{F_x(B(0,\rho) \times \mathbb{R}^n)} G^{(N)}(\phi_e(0,0), \phi_e(v,h) dv \, dh \, ,$$

where we denoted $F_x(u,h) = (p_x(u,h), h)$. So, by (3.13),

$$E_0 \int_0^\infty \mathbb{1}_{B(0,\rho)}(u_t^{(x)}) dt \le c \int_{\|\pi_x(g)\| < \rho} \frac{dg}{|g|_N^{Q_N - 2}} \, .$$

The right hand side of this last inequality is finite (see Lemma (A.7)).

q.e.d. Proposition (3.11)

(3.16) **Corollary.** *For every* $\rho > 0$, *for every continuous function* f *on* $\mathbb{R}^d$, *bounded by* 1, *with support in* $B(0,\rho)$, *and for every* $\delta > 0$, *there exists* $T(\delta) > 0$ *such that,*

(3.17) $$\left| E_0 \int_{T(\delta)}^\infty f(u_t^{(x)}) dt \right| \le \delta \, .$$

(3.18) **Corollary.** *For* $t > 0$, *we denote by* $\mu_t^{(x)}$ *the law of* $u_t^{(x)}$. *Then, for* *every $\rho > 0$ and for every $\delta > 0$, there exists $T(\delta) > 0$ such that,*

$$\mu^{(x)}_{T(\delta)}(B(0,\rho)) \leq \delta\,. \tag{3.19}$$

Proof. We get the convergence of the integral $\int_0^\infty P_0(u^{(x)}_t \in B(0,\rho))dt$, using (3.12). Hence, $\lim_{t\uparrow\infty} \mu^{(x)}_t(B(0,\rho)) = 0$.

q.e.d.

4. Study of the rescaled diffusion

We shall analyse now the diffusion $(v^{(\varepsilon,x)}_t)$. We shall prove the following:

(4.1) **Proposition.** *For every $0 < \rho < 1$ and for every continuous function f on $\mathbb{R}^d$, bounded by 1, with support in $B(0,\rho)$,*

$$\lim_{\varepsilon\downarrow 0, T\uparrow\infty} E_0\left(1\!\!1_{(T<\tau_\varepsilon)}\int_T^{\tau_\varepsilon} f(v^{(\varepsilon,x)}_t)dt\right) = 0\,. \tag{4.2}$$

Proof. Let $G^{(\varepsilon,x)}$ be the Green function of $(v^{(\varepsilon,x)}_t)$. For every positive measurable f,

$$E_0\int_0^{\tau_\varepsilon} f(v^{(\varepsilon,x)}_t)dt = \int_{(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)} G^{(\varepsilon,x)}(0,u)f(u)du\,. \tag{4.3}$$

We can write

$$E_0\left(1\!\!1_{(T<\tau_\varepsilon)}\int_T^{\tau_\varepsilon} f(v^{(\varepsilon,x)}_t)dt\right) = E_0\left(1\!\!1_{(T<\tau_\varepsilon)}E_{v^{(\varepsilon,x)}_T}\int_0^{\tau_\varepsilon} f(v^{(\varepsilon,x)}_t)dt\right) =$$

$$E_0\left(1\!\!1_{(T<\tau_\varepsilon)}\int_{(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)} G^{(\varepsilon,x)}(v^{(\varepsilon,x)}_T,u)f(u)du\right) =$$

$$\int_{(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)} d\mu^{(\varepsilon,x)}_T(v)\int_{(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)} G^{(\varepsilon,x)}(v,u)f(u)du\,.$$

Here $\mu^{(\varepsilon,x)}_T$ denotes the measure having the density $1\!\!1_{(T<\tau_\varepsilon)}$ with respect to the law of $v^{(\varepsilon,x)}_T$. We shall estimate the integral of $G^{(\varepsilon,x)}$.

It is a simple calculation to show that, for $v,u \in (T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)$,

$$G^{(\varepsilon,x)}(v,u) = J_x\,\varepsilon^{Q-2}G(v^x_\varepsilon,u^x_\varepsilon)\,, \tag{4.4}$$

Here we denoted $u_\varepsilon^x = (\varphi_x \circ T_\varepsilon)(u)$, for $x \in \Omega$, $\varepsilon > 0$ sufficiently small and $u \in \mathbb{R}^d$ and $J_x = |\mathrm{Jac}\,\varphi_x| = |\det(X^{J_1}(x), \ldots, X^{J_d}(x))|$.

Therefore, by (1.8), we get

$$\int_{B(0,\rho)} G^{(\varepsilon,x)}(v,u)du \le \int_{B(0,\rho)} \frac{c\, J_x\, \varepsilon^{Q-2} du}{|u_\varepsilon^x|_{v_\varepsilon^x}^{Q-2}}\,, \tag{4.5}$$

for $\varepsilon > 0$ sufficiently small and $u \in (T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(\Omega)$.

(4.6) **Lemma.** *For any $v \in \mathbb{R}^d$ and for $\varepsilon > 0$ sufficiently small, there exists a constant $c > 0$, such that*

$$\int_{B(0,\rho)} \frac{\varepsilon^{Q-2} du}{|u_\varepsilon^x|_{v_\varepsilon^x}^{Q-2}} < c\,. \tag{4.7}$$

Moreover,

$$\lim_{\|v\|\uparrow\infty} \int_{B(0,\rho)} \frac{\varepsilon^{Q-2} du}{|u_\varepsilon^x|_{v_\varepsilon^x}^{Q-2}} = 0\,, \tag{4.8}$$

uniformly in $\varepsilon > 0$.

Proof. For the first part we write the integral as

$$\varepsilon^{-2} \int_{(\varphi_x \circ T_\varepsilon)(B(0,\rho))} \frac{dy''}{|y''|_{z_\varepsilon^x}^{Q-2}}\,.$$

(4.7) is a particular case of the following estimate: there exists a positive constant c such that, for $\varepsilon > 0$ sufficiently small

$$\sup_z \int_{|y|_z < \varepsilon} \frac{dy}{|y|_z^{Q-2}} < c\,\varepsilon^2\,, \tag{4.9}$$

with the supremum taken for z in a neighbourhood of x. We shall now prove (4.9). Firstly, by the change of variables $v = (T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(y)$, we get that

$$\int_{|y|_x < \varepsilon} \frac{dy}{|y|_x^{Q-2}} = c\,\varepsilon^2 \int_{|v|_h < 1} \frac{dv}{|v|_h^{Q-2}} < c\,\varepsilon^2\,,$$

as follows from the Lemma (A.1) of the Appendix. Here and elsewhere $|u|_h$

denotes the homogeneous norm of $u \in \mathbb{R}^d$:

$$|u|_h = \left[\sum_{k=1}^{r} \left(\sum_{j, l_j = k} u_j^2 \right)^{\frac{Q}{2k}} \right]^{\frac{1}{Q}} .$$

To get (4.9) it suffices to note that the bound in Lemma (A.1) depends only on the radius of the homogeneous ball (here equal to 1). Since $\{X^{J_j}(z) : j = 1, \ldots, d\}$, is a triangular basis, for z close enough to x, we conclude by a smooth change of coordinates.

In proving (4.8) we use some simple properties of the locally homogeneous norm (see (6.9), (6.11)). There exists some constants $c_0, c', c'' > 0$, such that

$$\sup_{\|u\|<\rho} \frac{\varepsilon^{Q-2}}{|u_\varepsilon^x|_{v_\varepsilon^x}^{Q-2}} \leq \sup_{\|u\|<\rho} \frac{1}{\left(\frac{1}{c_0}|v|_h - |u|_h\right)^{Q-2}} \leq \frac{1}{\left(\frac{c'}{c_0}\|v\| - c''\rho^{\frac{1}{r}}\right)^{Q-2}} .$$

From this, (4.8) is easily obtained.

q.e.d. Lemma (4.6)

Now we can complete the proof of the Proposition (4.1). By (4.5) and (4.7) we can write, for every $R > 0$,

$$E_0 \left(\mathbb{I}_{(T<\tau_\varepsilon)} \int_T^{\tau_\varepsilon} f(v_t^{(\varepsilon,x)}) dt \right) \leq \tag{4.10}$$

$$. \qquad c\, \mu_T^{(\varepsilon,x)}(B(0,R)) + \sup_{\|v\| \geq R} \int_{B(0,\rho)} G^{(\varepsilon,x)}(v,u) du$$

(with the convention that $G(z,y) = 0$ if z or $y \notin \Omega$).

We can make small the second term in (4.10) by choosing a large R, as follows from (4.5) and (4.8). Hence, to finish the proof of (4.2), it suffices to prove the following:

(4.11) **Lemma.** *For every* $R > 0$,

$$\lim_{\varepsilon \downarrow 0, T \uparrow \infty} \mu_T^{(\varepsilon,x)}(B(0,R)) = 0 . \tag{4.12}$$

Proof. Noting the result of the Corollary (3.18), the conclusion is obtained as soon as we show that, for every $R > 0$, (4.13) $$\lim_{\varepsilon\downarrow 0} \mu_T^{(\varepsilon,x)}(B(0,R)) = \mu_T^{(x)}(B(0,R)) \,.$$

For this, we write

$$\Big| E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\mathbb{1}_{B(0,R)}(v_T^{(\varepsilon,x)})\right) - E_0\mathbb{1}_{B(0,R)}(u_T^{(x)})\Big| \le$$

$$E_0\left(\mathbb{1}_{(T<\tau_\varepsilon\wedge T^\varepsilon_{\tilde\Omega})}\Big|\mathbb{1}_{B(0,R)}(v_T^{(\varepsilon,x)}) - \mathbb{1}_{B(0,R)}(u_T^{(x)})\Big|\right) + P(T\ge\tau_\varepsilon) + 2\,P(T\ge T^\varepsilon_{\tilde\Omega}) \,.$$

As in the proof of the Proposition (2.12), it suffices to study the first term. But, the result of the Lemma (2.18) allows us to control this term, using the fact that u_T^x does not charge the boundary of the ball, and (4.13) follows.

q.e.d. Lemma (4.11)

This also ends the proof of the Proposition (4.1).

5. Proof of the Theorem (1.10)

To prove the Theorem (1.10) we need the following important:

(5.1) **Proposition.** *Let H be a compact subset of $\mathbb{R}^d\setminus\{0\}$. Then*

(5.2) $$\lim_{\varepsilon\downarrow 0}\sup_{u\in H} |G^{(\varepsilon,x)}(0,u) - g^{(x)}(0,u)| = 0 \,.$$

Proof. We shall show that, for $\varepsilon\downarrow 0$,

(5.3) $$G^{(\varepsilon,x)}(0,u)du \to g^{(x)}(0,u)du \,, \text{ vaguely} \,,$$

and then, that there exists $\varepsilon_0 > 0$, such that $\{G^{(\varepsilon,x)}(0,\cdot), \varepsilon\in(0,\varepsilon_0]\}$ is a relatively compact subset of the set of continuous functions on H.

For the proof of (5.3), we denote $\mathrm{Lip}_\rho(\mathbb{R}^d)$ the set of all bounded Lipschitz continuous functions f on $\mathbb{R}^d$, with support in $B(0,\rho)$, such that $\|f\|_{\mathrm{Lip}}\le 1$.

By (4.3) and (3.8), for every $f\in\mathrm{Lip}_\rho(\mathbb{R}^d)$,

$$\left|\int_{(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)} G^{(\varepsilon,x)}(0,u)f(u)du - \int_{\mathbb{R}^d} g^{(x)}(0,u)f(u)du\right| \le$$

$$\left|E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_0^T f(v_t^{(\varepsilon,x)})dt\right) - E_0\int_0^T f(u_t^{(x)})dt\right| +$$

$$\left|E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_T^{\tau_\varepsilon} f(v_t^{(\varepsilon,x)})dt\right)\right| + T\,P(T\geq\tau_\varepsilon) + \left|E_0\int_T^\infty f(u_t^{(x)})dt\right| \leq$$
$$c\,T\,\varepsilon + \left|E_0\left(\mathbb{1}_{(T<\tau_\varepsilon)}\int_T^{\tau_\varepsilon} f(v_t^{(\varepsilon,x)})dt\right)\right| + c\,T\,e^{-\frac{c'}{\varepsilon^2 T}} + \left|E_0\int_T^\infty f(u_t^{(x)})dt\right|,$$

as follows from (2.13) and from the classical exponential inequality. We can make small the last term by choosing a large T, as in (3.17). To control the second term we use (4.2). Doing so we get (5.3).

Now, we shall show that there exists $\varepsilon_0>0$, such that the functions $G^{(\varepsilon,x)}(0,\cdot)$, $\varepsilon\in(0,\varepsilon_0]$, are uniformly equicontinuous, provided they are restricted to the compact set H.

We prove the existence of a constant $c>0$, such that, for every $u\in H$, and $\varepsilon\in(0,\varepsilon_0]$,

(5.4) $$|X^{J_j}G^{(\varepsilon,x)}(0,u)|\leq c\,,\ j=1,\ldots,d\,.$$

By (4.4), for $u\in(T_{\frac{1}{\varepsilon}}\circ\varphi_x^{-1})(\Omega)$, we have

$$X^{J_j}G^{(\varepsilon,x)}(0,u) = J_x\,\varepsilon^{Q-2}X^{J_j}G(x,u_\varepsilon^x) = J_x\,\varepsilon^{Q-2+l_j}(X^{J_j}G)(x,u_\varepsilon^x)\,.$$

To obtain (5.4), we use another important estimate. It is similar to (1.8), but on the derivatives of G (see §6):

(5.5) $$|X_{i_1}\ldots X_{i_q}G(x,y)|\leq\frac{c}{|y|_x^{Q-2+q}}\,,\ y\neq x\text{ close enough}\,.$$

Hence, for $j=1,\ldots,d$,

$$\varepsilon^{Q-2+l_j}(X^{J_j}G)(x,u_\varepsilon^x)\leq\frac{c\,\varepsilon^{Q-2+l_j}}{|u_\varepsilon^x|_x^{Q-2+l_j}}=\frac{c}{|u|_h^{Q-2+l_j}}\,,$$

which is bounded when u lies in a compact set, and (5.4) is verified.

Using the weak convergence in (5.3) and the relatively compactness of $\{G^{(\varepsilon,x)}(0,\cdot),\varepsilon\in(0,\varepsilon_0]\}$ on H, we can identifie the limit of $G^{(\varepsilon,x)}(0,\cdot)$. This ends the proof of (5.2).

q.e.d.

Proof of the Theorem (1.10). We take

$$H=\{u\in\mathbb{R}^d:\sup(|u_j|:j=1,\ldots,d)=1\}$$

and

$$\varepsilon_y = \sup(|u_j|^{\frac{1}{l_j}} : j = 1, \dots, d)\,,$$

with $y \in \Omega$, $y = \varphi_x(u)$. Clearly,

$$\left(T_{\frac{1}{\varepsilon_y}} \circ \varphi_x^{-1}\right)(y) \in \left(T_{\frac{1}{\varepsilon_y}} \circ \varphi_x^{-1}\right)(\Omega) \cap H\,.$$

For every $\delta > 0$ and for every y sufficiently close to x, there exists $\varepsilon(\delta) > 0$, such that $\varepsilon_y \leq \varepsilon(\delta)$ and, by (5.1),

$$\left| G^{(\varepsilon,x)}\left(0, \left(T_{\frac{1}{\varepsilon_y}} \circ \varphi_x^{-1}\right)(y)\right) - g^{(x)}\left(0, \left(T_{\frac{1}{\varepsilon_y}} \circ \varphi_x^{-1}\right)(y)\right)\right| \leq \delta\,.$$

We note that, $u^{(x)}_{\varepsilon^2 t}$ and $T_\varepsilon(u^{(x)}_t)$ have the same law. Hence, by (3.8), we get

$$g^{(x)}\left(0, T_{\frac{1}{\varepsilon}}(u)\right) = \varepsilon^{Q-2} g^{(x)}(0,u)\,. \tag{5.6}$$

Then, using (4.4) and (5.6), for every $\delta > 0$ and for every y sufficiently close to x, $y = \varphi_x(u)$,

$$\left| J_x\, \varepsilon_y^{Q-2} G(x,y) - \varepsilon_y^{Q-2} g^{(x)}(0,u)\right| \leq \delta\,. \tag{5.7}$$

Moreover, we can replace here ε_y by $|y|_x$ because, there exists $c > 0$ such that,

$$|y|_x \leq c\,\varepsilon_y\,. \tag{5.8}$$

Finally, let us denote, for $\theta \in \mathbb{R}^d \setminus \{0\}$,

$$\Phi_x(\theta) = \frac{1}{J_x} g^{(x)}(0,\theta)\,, \tag{5.9}$$

where J_x is as in (4.4). As a consequence of the Proposition (3.9), Φ_x is a strictly positive smooth function on $\mathbb{R}^d \setminus \{0\}$.

By (1.9), for $y \neq x$,

$$\theta_x(y) = \left(T_{\frac{1}{|y|_x}} \circ \varphi_x^{-1}\right)(y)\,.$$

So, we conclude that, for every $\delta > 0$ and for every y sufficiently close to x,

$$\Big| |y|_x^{Q-2} G(x,y) - \Phi_x(\theta_x(y)) \Big| \leq \delta \,,$$

that is, (1.11).

The proof of the Theorem (1.10) is complete, except for the proof of Lemmas (A.1) and (A.7) of the Appendix and of the estimates (1.8), (3.13) and (5.5), which are simple consequences of [N-S-W] estimates, as we show in the following section.

6. Locally homogeneous norm associated to L

In this section we shall study the locally homogeneous norm $|\cdot|_x$ and we shall then justify the estimates (1.8), (3.13) and (5.5). It suffices to prove the following:

(6.1) **Proposition.** *There exists some positive constants c, c', such that, for $y \neq x$ close enough,*

$$|G(x,y)| \leq \frac{c\,|y|_x^2}{m(B_h(x,|y|_x))} \,, \; |X_{i_1} \dots X_{i_q} G(x,y)| \leq \frac{c'\,|y|_x^{2-q}}{m(B_h(x,|y|_x))} \,. \tag{6.2}$$

The estimates are then obtained using the simple calculation of the volume of a small homogeneous ball, $B_h(x,\varepsilon) = \{y : |y|_x < \varepsilon\}$:

$$m(B_h(x,\varepsilon)) = \int_{|y|_x<\varepsilon} dy = c\,\varepsilon^Q \int_{|v|_h<1} dv = c'\,\varepsilon^Q \,. \tag{6.3}$$

Here we performed the change of variables $v = (T_{\frac{1}{\varepsilon}} \circ \varphi_x^{-1})(y)$ and c' denotes a positive constant.

Proof of the Proposition (6.1). Noting the result of the Corollary in [N-S-W], p. 117, it is enough to show that there exists a positive constant c, such that, for y sufficiently close to x,

$$\rho(x,y) \leq c\,|y|_x \,. \tag{6.4}$$

Recall that $\rho(x,y)$ is the distance introduced by [N-S-W], p. 107.

But by the Theorem 3 in [N-S-W], p. 112, ρ is locally equivalent to the pseudo-distance ρ_3. So, there exists a positive constant c, such that, for y sufficiently close to x,

$$\rho(x,y) \le c\,\rho_3(x,y)\,. \tag{6.5}$$

Recall that,

$$\rho_3(x,y) = \inf\{\delta > 0 : \exists f \in C_3(\delta), f(0) = x, f(1) = y\}\,.$$

Here $C_3(\delta) = \cup_D C_3(\delta, D)$, where, for each d-tuple D of multi-indices J, with $|J| \le r$, $C_3(\delta, D)$ denote the class of smooth curves $f : [0,1] \to \mathbb{R}^d$, such that

$$f'(t) = \sum_{J\in D} c_J X^J(f(t))\,, \text{ with } |c_J| < \delta^{|J|}\,,\ J \in D\,.$$

We shall introduce a slight modification of the pseudo-distance ρ_3. We denote by $\mathcal{C}(\delta, B)$ the set of C^1-functions $f : [0,1] \to \mathbb{R}^d$, such that

$$f'(t) = \sum_{j=1,\dots,d} c_j X^{J_j}(f(t))\,, \text{ with } \sum_{k=1}^{r}\left(\sum_{j,l_j=k} c_j^2\right)^{\frac{Q}{2k}} < \delta^Q\,.$$

Then we define,

$$d_B(x,y) = \inf\{\delta > 0 : \exists f \in \mathcal{C}(\delta, B), f(0) = x, f(1) = y\} \wedge 1\,.$$

But

$$\sum_{k=1}^{r}\left(\sum_{j,l_j=k} c_j^2\right)^{\frac{Q}{2k}} < \delta^Q \Rightarrow |c_j| < \delta^{l_j}\,,\ j = 1,\dots,d\,,$$

so, $\mathcal{C}(\delta, B) \subset C_3(\delta)$. It follows that, for y sufficiently close to x,

$$\rho_3(x,y) \le d_B(x,y)\,. \tag{6.6}$$

Moreover, by the definitions of $|y|_x$ and of $d_B(x,y)$, and by our assumptions on Ω, it is a simple observation that, for $x, y \in \Omega$,

$$d_B(x,y) = |y|_x\,. \tag{6.7}$$

This ends the proof of (6.4).

q.e.d.

(6.8) *Remark.* Clearly, $d_B(x,y)$ is a pseudo-distance in the sense of [N-S-W], p. 109. From this, by (6.7), we see that there exists a constant $c_0 \geq 1$, such that, for every $x, y, z \in \Omega$,

$$|y|_x \leq c_0 \left(|z|_x + |z|_y\right) . \tag{6.9}$$

(6.10) *Remark.* We can check another simple property of $|\cdot|_x$. For every $x, y \in \Omega$, $y = \varphi_x(u)$, there exists two positive constants, c', c'', such that

$$c' \, \|u\| \leq |y|_x \leq c'' \, \|u\|^{\frac{1}{r}} . \tag{6.11}$$

7. Capacity of small compact sets

In this section we shall estimate the capacity (relative to the kernel G) of small compact sets.

To apply the theory of Blumenthal and Getoor for Markov processes in duality, we must consider the process (x_t) killed at an independent exponential random time ξ, of parameter $\lambda > 0$, which we denote by $(x_t^{(\lambda)})$.

The Green function of $(x_t^{(\lambda)})$ is the λ-potential of (x_t):

$$G_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p_t^\Omega(x,y) dt . \tag{7.1}$$

(7.2) *Remark.* The result of the Theorem (1.12) still holds with G replaced by G_λ. Indeed, we have

$$\left| G_\lambda(x,y) \, |y|_x^{Q-2} - \Phi_x(\theta_x(y)) \right| \leq$$

$$\left| \frac{G_\lambda(x,y)}{G(x,y)} - 1 \right| \cdot \left| G(x,y) \, |y|_x^{Q-2} \right| + \left| G(x,y) \, |y|_x^{Q-2} - \Phi_x(\theta_x(y)) \right| .$$

The conclusion follows as soon as we show that,

$$\lim_{\varepsilon \downarrow 0} \sup_{\|y-x\| < \varepsilon} \left| \frac{G_\lambda(x,y)}{G(x,y)} - 1 \right| = 0 ,$$

which can be done as in [CM-LG], p. 241.

Therefore, for every $\eta > 0$ and for every $y \neq x$ close enough,

$$\frac{-\eta + \Phi_x(\theta_x(y))}{|y|_x^{Q-2}} \leq G_\lambda(x,y) \leq \frac{\eta + \Phi_x(\theta_x(y))}{|y|_x^{Q-2}} . \tag{7.3}$$

Now, let us recall some definitions. By choosing $\lambda > 0$ large enough, we can apply the theory of [B-G] to the process $(x_t^{(\lambda)})$. For a compact subset H in Ω, we denote

$$T_H^{(\lambda)} = \inf\{t > 0 : x_t^{(\lambda)} \in H\} .$$

Let $\mu_H^{(\lambda)}$ the equilibrium measure of H, that is the unique finite measure supported by H such that, for every $x \in \Omega$,

$$P_x(T_H^{(\lambda)} < \infty) = G_\lambda \mu_H^{(\lambda)}(x) = \int_{\mathbb{R}^d} G_\lambda(x,y)\mu_H^{(\lambda)}(dy) . \tag{7.4}$$

The λ-capacity of H will be denoted by $c_\lambda(H)$, and is the total mass of $\mu_H^{(\lambda)}$, or, equivalently

$$c_\lambda(H) = \sup\{|\mu| : \mu \in \mathcal{M}(H), G_\lambda \mu \leq 1 \text{ on } \Omega\} . \tag{7.5}$$

Here $\mathcal{M}(H)$ is the set of all positive finite measures supported on H.

Let H be a compact subset of $\mathbb{R}^d$ containing 0. We shall describe the capacity of a small compact set. The natural dilation of H is $H_\varepsilon^x = (\varphi_x \circ T_\varepsilon)(H)$. We shall study the asymptotic behaviour of $c_\lambda(H_\varepsilon^x)$ as $\varepsilon \to 0$.

To write down the statement we need the following:

(7.6) **Lemma.** *There exists*

$$\lim_{\varepsilon \downarrow 0} \frac{|v_\varepsilon^x|_{u_\varepsilon^x}}{\varepsilon} = \alpha(u,v) > 0 \tag{7.7}$$

and

$$\lim_{\varepsilon \downarrow 0} \theta_{u_\varepsilon^x}(v_\varepsilon^x) = \beta(u,v) \neq 0 , \tag{7.8}$$

for $u \neq v \in \mathbb{R}^d \setminus \{0\}$.

Proof. We have to calculate $|v_\varepsilon^x|_{u_\varepsilon^x}$. Since $\{X^{J_j}(y) : j = 1, \dots, d\}$ is a basis for y close to x, we have,

$$v_\varepsilon^x = \exp(Z_\varepsilon)(x) = \exp(W_\varepsilon)(u_\varepsilon^x)\,,\; u_\varepsilon^x = \exp(Y_\varepsilon)(x)\,,$$

with,

$$Z_\varepsilon = \sum_{j=1,\dots,d} \varepsilon^{l_j} v_j X^{J_j}\,,\; Y_\varepsilon = \sum_{j=1,\dots,d} \varepsilon^{l_j} u_j X^{J_j}\,,\; W_\varepsilon = \sum_{j=1,\dots,d} w_j^\varepsilon X^{J_j}\,.$$

By the Campbell-Hausdorff formula we get,

$$Z_\varepsilon = W_\varepsilon + Y_\varepsilon + \frac{1}{2}[W_\varepsilon, Y_\varepsilon] + \dots\,,$$

so,

$$w_j^\varepsilon = \varepsilon^{l_j} b_j(u,v) + O(\varepsilon^{l_j+1})\,,\; j = 1, \dots, d\,,\; b_j(u,v) \neq 0\,.$$

Using (1.6), we get

$$|v_\varepsilon^x|_{u_\varepsilon^x} = \varepsilon\, \alpha(u,v) + O(\varepsilon^{1+\delta})\,,\; \delta \in (0,1)\,. \tag{7.9}$$

with,

$$\alpha(u,v) = \left[\sum_{k=1}^{r} \left(\sum_{j, l_j = k} b_j(u,v)^2 \right)^{\frac{Q}{2k}} \right]^{\frac{1}{Q}}\,. \tag{7.10}$$

This proves (7.7).

On the other hand, by (1.10) and the preceding calculation, we can write,

$$\theta_{u_\varepsilon^x}(v_\varepsilon^x) = \left(\frac{w_j^\varepsilon}{|v_\varepsilon^x|_{u_\varepsilon^x}^{|J|}} \right)_{j=1,\dots,d} = \left(\frac{b_j(u,v) + O(\varepsilon)}{\alpha(u,v)^{l_j} + O(\varepsilon^\delta)} \right)_{j=1,\dots,d}\,,\; \delta \in (0,1)\,.$$

Taking,

$$\beta(u,v) = \left(\frac{b_j(u,v)}{\alpha(u,v)^{l_j}} \right)_{j=1,\dots,d} \tag{7.11}$$

we get (7.8).

q.e.d.

We denote

$$r_x(u,v) = \frac{\Phi_x(\beta(u,v))}{\alpha(u,v)^{Q(x)-2}}\,,\ q_x(H) = \frac{m(H)}{\max_{u\in\partial H}\int_H r_x(u,v)dv}\,. \tag{7.12}$$

We can state now the main result of this section:

(7.13) **Proposition.** *Let H be the closure of a bounded domain in $\mathbb{R}^d$ containing* 0, *and $x \in \Omega$. Then*

$$\lim_{\varepsilon\downarrow 0} \frac{c_\lambda\left(H_\varepsilon^x\right)}{\varepsilon^{Q(x)-2}} = q_x(H)\,. \tag{7.14}$$

Proof. We consider ν, the measure with the density $\mathbb{I}_H$ with respect to the Lebesgue measure and ν_ε^x, the image measure of ν through $\varphi_x \circ T_\varepsilon$.

A lower bound for $c_\lambda\left(H_\varepsilon^x\right)$ is obtained as soon as we can obtain a uniform bound on $G_\lambda \nu_\varepsilon^x$. By the maximum principle of Bony [Bo], for hypoelliptic operators, it suffices to bound $G_\lambda \nu_\varepsilon^x$ on H_ε^x.

Take $u_\varepsilon^x \in H_\varepsilon^x$. Then,

$$G_\lambda \nu_\varepsilon^x(u_\varepsilon^x) = \int_{\mathbb{R}^d} G_\lambda(u_\varepsilon^x, v)\nu_\varepsilon^x(dv) = \int_H G_\lambda(u_\varepsilon^x, v_\varepsilon^x)dv\,.$$

Then, by (7.3) and (7.9),

$$G_\lambda \nu_\varepsilon^x(u_\varepsilon^x) \le \int_H \frac{\eta + \Phi_{u_\varepsilon^x}(\theta_{u_\varepsilon^x}(v_\varepsilon^x))}{|v_\varepsilon^x|_{u_\varepsilon^x}^{Q-2}}dv = \int_H \frac{\eta + \Phi_{u_\varepsilon^x}(\theta_{u_\varepsilon^x}(v_\varepsilon^x))}{\varepsilon^{Q-2}(\alpha(u,v)^{Q-2} + O(\varepsilon^\delta))}dv\,.$$

Using (7.5), for all $u \in H$,

$$\frac{c_\lambda\left(H_\varepsilon^x\right)}{\varepsilon^{Q-2}} \ge \frac{m(H)}{\int_H \frac{\eta+\Phi_{u_\varepsilon^x}(\theta_{u_\varepsilon^x}(v_\varepsilon^x))}{\alpha(u,v)^{Q-2}+O(\varepsilon^\delta)}dv}\,.$$

Hence, by the continuity of $\Phi_x(\theta)$ and by (7.11), we get,

$$\liminf_{\varepsilon\downarrow 0} \frac{c_\lambda\left(H_\varepsilon^x\right)}{\varepsilon^{Q-2}} \ge q_x(H)\,. \tag{7.15}$$

On the other hand,

$$\nu_\varepsilon^x G_\lambda(u_\varepsilon^x) = \int_{\mathbb{R}^d} G_\lambda(v, v_\varepsilon^x)\nu_\varepsilon^x(dv) = \int_H G_\lambda(v_\varepsilon^x, u_\varepsilon^x)dv = \int_H G_\lambda(u_\varepsilon^x, v_\varepsilon^x)dv\,,$$

so, again by (7.3) and (7.9),

$$\nu_\varepsilon^x G_\lambda(u_\varepsilon^x) \geq \int_H \frac{-\eta + \Phi_{u_\varepsilon^x}(\theta_{u_\varepsilon^x}(v_\varepsilon^x))}{|v_\varepsilon^x|_{u_\varepsilon^x}^{Q-2}} dz = \int_H \frac{-\eta + \Phi_{u_\varepsilon^x}(\theta_{u_\varepsilon^x}(v_\varepsilon^x))}{\varepsilon^{Q-2}(\alpha(u,v)^{Q-2} + O(\varepsilon^\delta))} dv \,.$$

We denote by $\mu_{\varepsilon,H}^{\lambda,x}$, the equilibrium measure of H_ε^x. We can write,

$$|\mu_{\varepsilon,H}^{\lambda,x}| \cdot \int_H \frac{-\eta + \Phi_{u_\varepsilon^x}(\theta_{u_\varepsilon^x}(v_\varepsilon^x))}{\varepsilon^{Q-2}(\alpha(u,v)^{Q-2} + O(\varepsilon^\delta))} dv \leq \int_{\mathbb{R}^d} \mu_{\varepsilon,H}^{\lambda,x}(du_\varepsilon^x) \nu_\varepsilon^x G_\lambda(u_\varepsilon^x) =$$

$$\int_{\mathbb{R}^d} \nu_\varepsilon^x(dv) G_\lambda \mu_{\varepsilon,H}^{\lambda,x}(v) \leq |\nu_\varepsilon^x| = m(H) \,.$$

Hence, for all $u \in H$,

$$\frac{c_\lambda\left(H_\varepsilon^x\right)}{\varepsilon^{Q-2}} \cdot \int_H \frac{-\eta + \Phi_{u_\varepsilon^x}(\theta_{u_\varepsilon^x}(v_\varepsilon^x))}{\alpha(u,v)^{Q-2} + O(\varepsilon^\delta)} dv \leq m(H) \,,$$

from which we get, by (7.11),

$$\limsup_{\varepsilon \downarrow 0} \frac{c_\lambda\left(H_\varepsilon^x\right)}{\varepsilon^{Q-2}} \leq q_x(H) \,. \tag{7.16}$$

q.e.d.

8. Applications: various sample path properties

As was said in [CM-LG], p. 222, as soon as we dispose of the results on the Green function and on the capacity of small compact sets, we can derive some sample path properties. The general methods used in [CM-LG], §7 and §8, can be applied.

We note that, for certain properties we do not need the exact behaviour of G, but only the estimates

$$\frac{c'}{|y|_x^{Q-2}} \leq G(x,y) \leq \frac{c}{|y|_x^{Q-2}} \,, \text{ with } x \neq y \text{ close enough} \,, \tag{8.1}$$

c, c' being positive constants. The right hand is (1.8) and the left hand can be obtained in a similar way as (1.8), that is, using the estimate on the volume of homogeneous small balls, (6.3) and the Theorem I (ii) in [Fe-Sa], p. 248 (or [J-Sa2], p. 51).

We shall emphasize only the differences with respect to the case considered by [CM-LG].

(a) Hitting probabilities of small compact sets.

For $\varepsilon > 0$ sufficiently small, we denote,

$$T_{H_\varepsilon^x} = \inf\{t > 0 : x_t \in H_\varepsilon^x\} .$$

(8.2) **Proposition.** *For $n \geq 1$ integer, for $x_0, x_1, \ldots, x_n$ distinct points of Ω and for $t \geq 0$,*

$$(8.3) \qquad \lim_{\varepsilon \downarrow 0} \left(\frac{1}{\varepsilon^{Q(x)-2}} \right)^n P_{x_0}(T_{H_\varepsilon^{x_1}} \leq \ldots \leq T_{H_\varepsilon^{x_n}} \leq t) = q_{x_1}(H) \ldots q_{x_n}(H) \cdot$$

$$\cdot \quad \int_{0 \leq s_1 \leq \ldots \leq s_n \leq t} ds_1 \ldots ds_n \, p_{s_1}^\Omega(x_0, x_1) p_{s_2 - s_1}^\Omega(x_1, x_2) \ldots p_{s_n - s_{n-1}}^\Omega(x_{n-1}, x_n) .$$

Moreover, there exists constants $c, c_{n,t} > 0$, independent of $x_0, x_1, \ldots, x_n$, such that, whenever $|x_j|_{x_{j-1}} \geq c\,\varepsilon$, $j = 1, \ldots, n$

$$(8.4) \qquad \left(\frac{1}{\varepsilon^{Q(x)-2}} \right)^n P_{x_0}(T_{H_\varepsilon^{x_1}} \leq \ldots \leq T_{H_\varepsilon^{x_n}} \leq t) \leq c_{n,t} \prod_{j=1}^n \frac{1}{|x_j|_{x_{j-1}}^{Q(x)-2}} .$$

For the proof we use the result on the capacity (7.11) and we repeat the arguments in pp. 250-252, [CM-LG].

(b) Wiener sausage.

We shall analyse the asymptotic behaviour of the volume of the Wiener sausage of small radius. For $0 \leq t < \tau$, let us denote

$$(8.5) \qquad \mathcal{S}_{H_\varepsilon^x}(0, t) = \bigcup_{0 \leq s \leq t} H_\varepsilon^{x_s} ,$$

the "sausage" associated to (x_t) and to H_ε^x, $H \subset \mathbb{R}^d$, containing 0.

By a similar proof as in [CM-LG], pp. 253-257, we could obtain:

(8.6) **Proposition.** *Let $\mu(dx) = f(x)\,dx$, where f is a bounded measurable function on Ω. Then, for every $p \geq 1$, $0 < T < \tau$, $x_0 \in \Omega$,*

$$\text{(8.7)} \qquad \lim_{\varepsilon\downarrow 0} E_{x_0}\left[\sup_{0\le t\le T}\left|\frac{1}{\varepsilon^{Q(x)-2}}\mu\left(\mathcal{S}_{H^x_\varepsilon}(0,t)\right)-\int_0^t f(x_s)q_{x_s}(H)ds\right|^p\right]=0\,.$$

(8.8) *Remark.* Recall that $(\mathcal{G}_t)$ denote the invariant diffusion on $\mathcal{N}(m,r)$. Let us denote, for $\varepsilon > 0$, $t \ge 0$,

$$\text{(8.9)} \qquad \mathcal{S}^N_\varepsilon(0,t) = \{g \in \mathcal{N}(m,r) : |g\cdot\mathcal{G}_s^{-1}|_N \le \varepsilon\,,\ \text{for some}\, s\le t\}\,.$$

If μ denotes the Haar measure on the group, by the Theorem (4.9) in [G], we get,

$$\text{(8.10)} \qquad \lim_{t\uparrow\infty}\frac{1}{t}\,\mu(\mathcal{S}^N_1(0,t)) = c\,,\ P_e - \text{a.s.}$$

From this we obtain a similar result as (7.q) in [CM-LG], p. 258:

$$\text{(8.11)} \qquad \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{Q_N-2}}\,\mu(\mathcal{S}^N_\varepsilon(0,1)) = c, \quad \text{in probability}\,.$$

Indeed, if δ_ε denotes the image on $\mathcal{N}(m,r)$ of the dilation on the algebra $g(m,r)$ (see [BA2], p. 88), then $(\delta_\varepsilon(\mathcal{G}_s))$ and $(\mathcal{G}_{\varepsilon^2 s})$ have the same law. By scaling and homogeneity properties we can show that $\mu(\mathcal{S}^N_\varepsilon(0,1))$ and $\varepsilon^{Q_N}\mu(\mathcal{S}^N_1(0,\frac{1}{\varepsilon^2}))$ have the same law.

(c) Double points.

We could prove the same result as the Theorem 8.2 in [CM-LG], p. 261:

(8.12) **Proposition.** *For every $x\in\Omega$, with P_x probability one, the process $\{x_s : 0\le s<\tau\}$ does not have double points.*

In proving this, we use the Hausdorff measure with respect to the homogeneous norm $|\cdot|_x$ and the estimates for G, (8.1). The difference with respect to [CM-LG] is that, instead (8.d), p. 262, we prove:

$$E_x\left(\sup_{s<\delta\wedge\tau}|x_s|_x^{Q(x)-2}\right)\le c\,\delta^{\frac{Q(x)-2}{2}}\,,$$

for every $x \in \Omega$, $\delta \in (0,1)$, c being a positive constant. For this, we use the Taylor stochastic expansion and the fact that, for every multi-index J, there exists a constant $c(J) > 0$, such that, $E(|B_t^J|^2) \leq c(J)\, t^{|J|}$ (see [BA1], p. 34).

(d) Wiener and Poincaré tests.

The result which we formulate is similar to the classical Wiener test (see [W], p. 130). For another form we refer to [Bi], p. 98.

Let us consider a constant α greather than the constant $c_0 \geq 1$, which appears in the triangular inequality for the homogeneous norm $|\cdot|_x$, (6.9). For B a Borel set contained in U we denote

$$B_n = \{y \in B : \frac{1}{\alpha^{n+1}} \leq |y|_x \leq \frac{1}{\alpha^n}\}\,,\; n \geq 1\,. \tag{8.13}$$

(8.14) **Proposition.** *The probability* $P_x(T_B^{(\lambda)} = 0) = 0$ *or* 1 *according as the series* $\sum_n \alpha^{n(Q(x)-2)} c_\lambda(B_n)$ *converges or diverges.*

We show that, for $n \geq 1$,

$$c'\,\alpha^{n(Q-2)} c_\lambda(B_n) \leq P_x(T_{B_n}^{(\lambda)} < \infty) \leq c\,\alpha^{(n+1)(Q-2)} c_\lambda(B_n)\,,$$

using the estimates in (8.1). Then we conclude as in [G], pp. 108-110 (see also [La]).

This result could be applied to obtain the cone test of Poincaré. A homogeneous cone with vertex 0 is a Borel set C with non-empty interior, which is stable for the dilations T_α and such that $0 \in \partial C$.

(8.15) **Corollary.** *Consider* C *a homogeneous cone with vertex* 0 *and* N *a neighbourhood of* 0. *If* B *is a Borel set such that* $\varphi_x(N \cap C) \subset B \subset U$, *then* $P_x(T_B^{(\lambda)} = 0) = 1$.

We note that $C_{n+1} = T_{1/\alpha}(C_n)$, $n \geq 1$, so, by a simple property of the capacity (see [G] Proposition (4.7)), we get, $c_\lambda(C_n) = c\alpha^{-n(Q-2)}$, $c > 0$. Then we can conclude, using the Proposition (8.14), since $\varphi_x((N \cap C)_n) \subset B_n$ and $c_\lambda(\varphi_x(N \cap C)) = c\, c_\lambda(N \cap C)$, $c > 0$, (see also the Corollary (5.4) [G]).

9. Examples

In this section we shall describe some concrete examples, where we can perform more calculations. Firstly, let us point out some simple cases.

We consider on $\mathbb{R}^3$ the vector fields $X_1 = \partial_{x_1} + 2x_2\partial_{x_3}$, $X_2 = \partial_{x_2} - 2x_1\partial_{x_3}$. Then $[X_1, X_2] = -4\,\partial_{x_3}$ and the operator $L = \frac{1}{2}(X_1^2 + X_2^2)$ is hypoelliptic. This case is called the Heisenberg case and in [F], p. 375 (see also [Ga], p. 101) was calculated the Green function on $\mathbb{R}^3$ with pole 0:

$$G^H(0,y) = \frac{1/(4\pi)}{\sqrt{(y_1^2+y_2^2)^2+y_3^2}^{\,4-2}} = \frac{1/(4\pi)}{|y|_0^{Q_H-2}}\,. \tag{9.1}$$

In [CM-LG] a more general situation is treated. Consider two smooth vector fields X_1, X_2 on $\mathbb{R}^3$, such that for every $x \in \Omega$, $X_1(x)$, $X_2(x)$, $[X_1, X_2](x)$ span $\mathbb{R}^3$. Then the Green function satisfies:

$$\Big|G(x,y)\,d(x,y)^{4-2} - c\Big| \to 0\,, \text{ as } y \to x\,. \tag{9.2}$$

It is also shown that the pseudo-distance $d(x,y)$ is equivalent to $|y|_x$.

We firstly treat the following:

(a) Curved Heisenberg case.

For $n \geq 1$ integer, we take $m = 2n$ and $d = 2n+1$. Suppose that $X_1, \ldots, X_{2n}$ are smooth vector fields on $\mathbb{R}^{2n+1}$, such that,

$$[X_{2k-1}, X_{2k}] = [X_1, X_2]\,,\ k = 1, \ldots, n\,, \tag{9.3}$$

all other brackets being zero. Let us consider Ω a bounded domain in $\mathbb{R}^{2n+1}$. We shall suppose that, for every $x \in \Omega$, the vectors $X_1(x), \ldots, X_{2n}(x)$, $[X_1, X_2](x)$ span $\mathbb{R}^{2n+1}$.

It is a particular case because we consider only two order brackets and a single one is not zero. In this case $r = 2$ and $Q = 2n+2$. The basis is indexed by $B = \{1, 2, \ldots, 2n, (1,2)\}$.

The diffusion associated to the vector fields, starting from a fixed point $x \in \Omega$, is

$$(9.4)\quad x_t = \exp\left(\sum_{j=1}^{2n} B_t^j X_j - \frac{1}{2}\sum_{k=1}^{n}\int_0^t (B_s^{2k} dB_s^{2k-1} - B_s^{2k-1} dB_s^{2k})[X_1, X_2]\right)(x)$$

$$. \qquad\qquad +t^{\frac{3}{2}} R_3(t)\,,$$

as we can see by (2.14). We must compare (x_t) to the left invariant diffusion on the Heisenberg group, H_{2n+1}, with its usual structure on $\mathbb{R}^{2n+1}$. The left invariant vector fields are defined by

$$Y_{2k-1} = \partial_{x_{2k-1}} - 2\, x_{2k}\, \partial_{x_{2n+1}}\,,\; Y_{2k} = \partial_{x_{2k}} + 2\, x_{2k-1}\, \partial_{x_{2n+1}}\,,\; k = 1, \ldots, n\,,$$

so, the invariant diffusion started from 0 is

$$(9.5)\quad \mathcal{G}_t = \exp\left(\sum_{j=1}^{2n} B_t^j Y_j - \frac{1}{2}\sum_{k=1}^{n}\int_0^t (B_s^{2k} dB_s^{2k-1} - B_s^{2k-1} dB_s^{2k})[Y_1, Y_2]\right)(0)\,.$$

In this case we do not need any projection, and $(u_t^{(x)})$ is the diffusion

$$(9.6)\qquad \left(B_t^1, \ldots, B_t^{2n}, -\frac{1}{2}\sum_{k=1}^{n}\int_0^t (B_s^{2k} dB_s^{2k-1} - B_s^{2k-1} dB_s^{2k})\right)\,.$$

Its Green fonction, $g^{(x)}$, is the invariant Green function on the Heisenberg group. By the result of [F], p. 375, we get

$$(9.7)\qquad g^{(x)}(0, y) = \frac{1/c_n}{\left[\left(\sum_{j=1}^{2n} y_j^2\right)^2 + y_{2n+1}^2\right]^{\frac{n}{2}}}\,,\; c_n = \frac{2^{n-1}\Gamma(\frac{n}{2})}{\pi^{n+1}}\,.$$

For $y = \varphi_x(y_1, \ldots, y_{2n+1})$, we denote

$$(9.8)\qquad |y|_x = \left[\left(\sum_{j=1}^{2n} y_j^2\right)^{n+1} + |y_{2n+1}|^{n+1}\right]^{\frac{1}{2n+2}}\,.$$

Then, applying the Theorem (1.10), we obtain

$$(9.9)\qquad \lim_{\varepsilon\downarrow 0}\; \sup_{\|x-y\|<\varepsilon} \left| G(x, y)\, |y|_x^{2n} - \Phi_x(\theta_x(y))\right| = 0\,.$$

Here,

$$\theta_x(y) = \left(\frac{y_1}{|y|_x}, \ldots, \frac{y_{2n}}{|y|_x}, \frac{y_{2n+1}}{|y|_x^2} \right) \tag{9.10}$$

and

$$\Phi_x(t_1, \ldots, t_{2n+1}) = \frac{c_x/c_n}{\left[\left(\sum_{j=1}^{2n} t_j^2 \right)^2 + t_{2n+1}^2 \right]^{\frac{n}{2}}} \, , \; c_x > 0 \, . \tag{9.11}$$

(9.12) *Remark.* Noting the symmetry of the first $2n$ coordinates, we can write a simpler form of (9.9). Put

$$\vartheta(y) = \frac{|y_{2n+1}|}{\sum_{j=1}^{2n} y_j^2} \, , \; \Psi_x(t) = \frac{1}{c_n} \frac{(1+t^{n+1})^{\frac{n}{n+1}}}{(1+t^2)^{\frac{n}{2}}} \, . \tag{9.13}$$

Then, by (9.9), we get

$$\lim_{\varepsilon \downarrow 0} \sup_{\|y-x\|<\varepsilon} \left| G(x,y) \, |y|_x^{2n} - c_x \Psi(\vartheta(y)) \right| = 0 \, . \tag{9.14}$$

We also note that, for $n = 1$, $\Psi = \frac{1}{c_n}$ is constant and we can compare (9.14) with the result obtained by [CM-LG], (9.2).

(9.15) *Remark.* In this particular case we could easily write the result on the capacity of small compact sets.

Now, we shall study a slight extension of the last model. Let us replace (9.3) by the following assumption:

$$[X_{2k-1}, X_{2k}] = a_k [X_1, X_2] \, , \; a_k \in \mathbb{R}^* \, , \; k = 1, \ldots, n \, , \tag{9.16}$$

all other hypothesis on the vector fields being the same.

The associated diffusion can be written as in (9.4), using the Taylor stochastic expansion. It will be compared to the diffusion $(\tilde{u}_t^x)$ generated by the following vector fields:

$$Y_{2k-1} = \partial_{x_{2k-1}} + 2 \, a_k \, x_{2k} \, \partial_{x_{2n+1}} \, , \; Y_{2k} = \partial_{x_{2k}} - 2 \, a_k \, x_{2k-1} \, \partial_{x_{2n+1}} \, , \; k = 1, \ldots, n \, ,$$

that is,

$$(9.17)\qquad \left(B_t^1,\ldots,B_t^{2n},-\frac{1}{2}\sum_{k=1}^{n}a_k\int_0^t(B_s^{2k}dB_s^{2k-1}-B_s^{2k-1}dB_s^{2k})\right).$$

The Green function associated to $(\tilde{u}_t^x)$ was pointed out by [Gr2], p. 136:

$$(9.18)\qquad \tilde{g}^{(x)}(0,y)=c_n\int_{\mathbb{R}}\frac{A(s)\,ds}{\left(\sum_{k=1}^{n}b_k(s)(y_{2k-1}^2+y_{2k}^2)+i\,s\,y_{2n+1}\right)^n},$$

where $i=\sqrt{-1}$, $c_n=\frac{(n-1)!}{2\pi}$ and

$$(9.19)\qquad A(s)=\frac{1}{(4\pi)^n}\prod_{k=1}^{n}\frac{4a_ks}{\sinh(4a_ks)},\ b_k(s)=(a_ks)\coth(4a_ks).$$

(9.20) *Remark.* When $(y_1,\ldots,y_{2n+1})=(0,\ldots,0,y_{2n+1})$, with $y_{2n+1}\neq 0$, we must integrate in (9.19) on $\mathbb{R}+i\,q$, $q>0$.

We can obtain the behaviour of the Green function $\tilde{G}$, associated to the vector fields X_j, as in the first case. We use the same homogeneous norm, given by (9.8), and we get the same relation as (9.9), with Φ_x replaced by:

$$(9.21)\qquad \tilde{\Phi}_x(t_1,\ldots,t_{2n+1})=c_x\,c_n\int_{\mathbb{R}}\frac{A(s)\,ds}{\left(\sum_{k=1}^{n}b_k(s)(t_{2k-1}^2+t_{2k}^2)+i\,s\,t_{2n+1}\right)^n}.$$

(9.22) *Remark.* We can simplify the result again, using the symmetry of the pairs of coordinates. Denoting,

$$(9.23)\qquad \vartheta^k(y)=\sqrt{\frac{y_{2k-1}^2+y_{2k}^2}{\sum_{j=1}^{2n}y_j^2}},\ k=1,\ldots,n$$

and

$$(9.24)\qquad \tilde{\Psi}(t_1,\ldots,t_n,t)=c_n\int_{\mathbb{R}}\frac{(1+t^{n+1})^{\frac{n}{n+1}}A(s)\,ds}{\left(\sum_{k=1}^{n}b_k(s)t_k^2+i\,s\,t\right)^n},$$

we get

$$(9.25)\qquad \lim_{\varepsilon\downarrow 0}\sup_{\|y-x\|<\varepsilon}\left|\tilde{G}(x,y)\,|y|_x^{2n}-c_x\tilde{\Psi}(\vartheta^1(y),\ldots,\vartheta^n(y),\vartheta(y))\right|=0,$$

where $\vartheta(y)$ was given in (9.13).

(9.26) *Remark.* We can find again the result of [CM-LG], for $n=1$. Also, we could formulate the result on the capacity.

(9.27) *Remark.* A more general situation can be obtained assuming that $m=2n$, $d=2n+p$ (p missing directions, $p\geq 1$, integer) and $r=2$. Using some recent results of [B-Ga-Gr] we could write similar results.

As was said, we shall describe a case when the condition that the geometry of the brackets is locally constant fails:

(b) A case at step larger than two.

Let us consider on $\mathbb{R}^3$ the vector fields

$$(9.28)\quad X_1=\partial_{x_1}+2\,p\,x_2(x_1^2+x_2^2)^{p-1}\partial_{x_3}\,,\; X_2=\partial_{x_2}-2\,p\,x_1(x_1^2+x_2^2)^{p-1}\partial_{x_3}\,,$$

with $p\geq 1$, integer, and $L=\frac{1}{2}(X_1^2+X_2^2)$. The case $p=1$ is the classical Heisenberg case $H_3=\mathcal{N}(2,2)$.

The operator L is nowhere elliptic, but is hypoelliptic. Indeed, for $p>1$ and for $x\notin\{x_1=x_2=0\}$, we have

$$[X_1,X_2]=-8\,p\,(x_1^2+x_2^2)^{p-1}\partial_{x_3}\,.$$

So, for $p>1$ and for $x\notin\{x_1=x_2=0\}$, $X_1(x),X_2(x)$ and $[X_1,X_2](x)$ span $\mathbb{R}^3$. This situation was already treated. On the other hand we see that for the points on the axis $\{x_1=x_2=0\}$, to span $\mathbb{R}^3$ we need to go up to the brackets of order $2\,p$ in this points. This time $r(0,0,x_3)=2\,p$ and $Q(0,0,x_3)=2\,p+2$. Clearly, the geometry of the brackets is not locally constant around the point $(0,0,x_3)$.

Operators like L occur in the study of the boundary of the Cauchy-Riemann complex (see [Gr-S]). Precisely, let us consider the domain

$$\mathcal{D}=\{(z_1,z)\in\mathbb{C}^2\,,\,\mathrm{Im}\,z_1>|z|^{2p}\}\,.$$

If $p=1$, $\mathcal{D}$ is the generalized upper half plane in $\mathbb{C}^2$. The vector field $\partial_z-2\,i\,\bar{z}\,\partial_{z_1}$ is the unique holomorphic vector field which is tangent to the boundary $b\mathcal{D}$ of $\mathcal{D}$. In the tangential coordinate system (see [Gr-S]: coordinates $\rho=\mathrm{Im}\,z_1-|z|^2$, $z,\bar{z}$ and $x_3=\mathrm{Re}\,z_1$) this vector field takes the form

$$Z = \partial_z + i\,\bar{z}\,\partial_{x_3}\,.$$

Z is left-invariant with respect to the nilpotent group structure, the Heisenberg group, on $\mathbb{R}^3 = b\mathcal{D}$.

In the case $p > 1$, we have

$$Z = \partial_z + i\,p\,z^{p-1}\bar{z}^p\,\partial_{x_3}$$

and there is no group structure on $\mathbb{R}^3$ with respect to which Z is left-invariant.

We also note that $Z = \frac{1}{2}\,X_1 - \frac{i}{2}\,X_2$ and L is of the type $-\square_b$, precisely,

$$L = Z\,\bar{Z} + \bar{Z}\,Z\,.$$

Recall that in the Heisenberg case, the Green function on $\mathbb{R}^3$ is known. By left-invariance it suffices to know the Green function with pole $(0,0,0)$ (see (9.1)).

In [Gr1] the case $p = 2$ is considered and the expression of the Green function on $\mathbb{R}^3$ with arbitrary pole is given.

Here we consider an arbitrary p. As was said, the case when the pole is outside of the axis $\{x_1 = x_2 = 0\}$ was treated. It is plausible that the method of [Gr1] can give an exact formula for the Green function with arbitrary pole. However, the calculation seems to be more delicate (see also [Gr-S], p. 157). Nevertheless, we can give an exact formula for the Green function with pole on the axis $\{x_1 = x_2 = 0\}$:

(9.29) **Proposition.** *The Green function on* $\mathbb{R}^3$, *associated to the vector fields* X_1, X_2, *with pole* $(0,0,x_3)$, *is*

$$G((0,0,x_3),(y_1,y_2,y_3)) = \frac{1/(4\,p\,\pi)}{\sqrt{(y_1^2+y_2^2)^{2p} + (y_3 - x_3)^2}}\,. \tag{9.30}$$

Proof. We denote $w = y_1 + i\,y_2$, $\sigma^2 = |w|^{4p} + (y_3 - x_3)^2$ and we must show that the Green function is

$$G((0,x_3),(w,y_3)) = \frac{1}{4\,p\,\pi\,\sigma}\,.$$

Clearly, this function is a C^∞-function of (w,y_3), as long as $(w,y_3) \neq (0,x_3)$.

We consider, for $\epsilon > 0$, the C^∞-function on $\mathbb{R}^3$,

$$G_\epsilon((0,x_3),(w,y_3)) = \frac{1}{4\,p\,\pi\,\sigma_\epsilon}\,,$$

where $\sigma_\epsilon^2 = (|w|^{2p} + \epsilon^{2p})^2 + (y_3 - x_3)^2$.

Then, $G_\epsilon((0,x_3),(w,y_3)) \to G((0,x_3),(w,y_3))$, pointwise as $\epsilon \downarrow 0$, as long as $(w,y_3) \neq (0,x_3)$. In fact, we can show that

$$G((0,x_3),(w,y_3)) = \lim_{\epsilon\downarrow 0} G_\epsilon((0,x_3),(w,y_3))\,, \text{ as a distribution in } \mathbb{R}^3\,.$$

Indeed, we see that there exists a positive constant c, independent of ϵ, such that $|G_\epsilon| \leq \frac{c}{\sigma}$. If we show that $\frac{1}{\sigma}$ is locally integrable, then, by the Lebesgue dominated convergence theorem we get

$$G_\epsilon((0,x_3),\cdot) \to G((0,x_3),\cdot)\,, \text{ in } \mathcal{D}'(\mathbb{R}^3)\,, \text{ as } \epsilon \downarrow 0\,.$$

We study the integrability at $(w,y_3) = (0,x_3)$ and we may suppose that $x_3 = 0$. We shall estime $\frac{1}{\sigma}$ on the domain $|w| \leq 1$, $|y_3| \leq 1$. We have,

$$\int_{-1}^{1} \frac{dy_3}{\sigma} = 2\,\log\left[1 + (1 + |w|^{4p})^{1/2}\right] - 4\,p\,\log|w|\,.$$

The first term is clearly integrable on $|w| \leq 1$, as for the second,

$$\int_{|w|\leq 1} |\log|w||dv(w) = 2\,\pi \int_0^1 r\,\log r dr < \infty\,.$$

After some calculations, we get

$$L\left(\frac{1}{4\,p\,\pi\,\sigma_\epsilon}\right) = \frac{p}{2\,\pi} \cdot \frac{\epsilon^{2p}\,|w|^{2p-2}}{\sigma_\epsilon^3}\,.$$

Hence, we have,

$$LG = 0\,, \text{ as long as } (w,y_3) \neq (0,x_3)$$

and

$$LG_\epsilon((0,x_3),(w,y_3)) \to 0\,, \text{ as } \epsilon \downarrow 0\,,$$

uniformly on compact subsets of $\mathbb{R}^3$ which do not contain the point $(0,x_3)$.

We show that

$$\int_{\mathbb{R}^3} LG_\epsilon((0,x_3),(w,y_3))dv(w,y_3) = 1\,.$$

Indeed,

$$\frac{p}{2\,\pi}\epsilon^{2p}\,|w|^{2p-2}\int_{\mathbb{R}}\frac{dy_3}{\left[(|w|^{2p}+\epsilon^{2p})^2+(y_3-x_3)^2\right]^{3/2}} =$$

$$\frac{p}{2\,\pi}\epsilon^{2p}\,|w|^{2p-2}\int_{\mathbb{R}}\frac{du}{\left[(|w|^{2p}+\epsilon^{2p})^2+u^2\right]^{3/2}} = \frac{p}{\pi}\cdot\frac{\epsilon^{2p}\,|w|^{2p-2}}{(|w|^{2p}+\epsilon^{2p})^2}$$

and then,

$$\frac{p}{\pi}\,\epsilon^{2p}\int_{\mathbb{R}^2}\frac{|w|^{2p-2}\,dv(w)}{(|w|^{2p}+\epsilon^{2p})^2} = 2\,p\,\epsilon^{2p}\int_0^\infty\frac{r^{2p-1}\,dr}{(r^{2p}+\epsilon^{2p})^2} = \epsilon^{2p}\int_0^\infty\frac{dt}{(t+\epsilon^{2p})^2} = 1\,.$$

Now we consider an arbitrary $\phi\in C_0^\infty(\mathbb{R}^3)$. Then, for any neighbourhood U of $(0,x_3)$, we can write,

$$< G((0,x_3),\cdot), L\,\phi >= \int_{\mathbb{R}^3} G((0,x_3),(w,y_3))L\,\phi(w,y_3)dv(w,y_3) =$$

$$\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^3} G_\epsilon((0,x_3),(w,y_3))L\,\phi(w,y_3)dv(w,y_3) =$$

$$\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^3} L\,G_\epsilon((0,x_3),(w,y_3))\phi(w,y_3)dv(w,y_3) =$$

$$\lim_{\epsilon\downarrow 0}\phi(0,x_3)\int_{\mathbb{R}^3} L\,G_\epsilon((0,x_3),(w,y_3))dv(w,y_3)+$$

$$\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^3} L\,G_\epsilon((0,x_3),(w,y_3))(\phi(w,y_3)-\phi(0,x_3))dv(w,y_3) =$$

$$\phi(0,x_3)+\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}^3\setminus U} L\,G_\epsilon((0,x_3),(w,y_3))(\phi(w,y_3)-\phi(0,x_3))dv(w,y_3)+$$

$$\lim_{\epsilon\downarrow 0}\int_{U} L\,G_\epsilon((0,x_3),(w,y_3))(\phi(w,y_3)-\phi(0,x_3))dv(w,y_3) =$$

$$\phi(0,x_3)+0+\sup_{(w,y_3)\in U}|\phi(w,y_3)-\phi(0,x_3)|\cdot 1 = \phi(0,x_3)\,.$$

This proves the fact that G is the Green function of L on $\mathbb{R}^3$ with pole $(0,x_3)$.
q.e.d.

(9.31) *Remark.* In the Heisenberg case, the Green function with arbitrary pole is given by (9.30). For the case treated in [Gr1], $p = 2$, the Green function with arbitrary pole has two terms, the first being the right hand of (9.30). In the general case we should attempt to find p terms for the Green function with arbitrary pole, the first being the right hand of (9.30).

The diffusion started from $(0,0,0) \in \{x_1 = x_2 = 0\}$, generated by X_1, X_2 is

$$x_t = \left(B_t^1, B_t^2, 4\,p \int_0^t R_s^{2(p-1)} dS_s \right) , \tag{9.32}$$

where

$$R_t^2 = (B_t^1)^2 + (B_t^2)^2 \, , \; S_t = \frac{1}{2} \int_0^t B_s^2 dB_s^1 - B_s^1 dB_s^2 \, . \tag{9.33}$$

We denote, for $y = \varphi_{(0,0,0)}(y_1, y_2, y_3)$,

$$|y|_0 = \left[(y_1^2 + y_2^2)^{p+1} + |y_3|^{\frac{p+1}{p}} \right]^{\frac{1}{2p+2}} , \tag{9.34}$$

$$\theta_0(y) = \left(\frac{y_1}{|y|_0}, \frac{y_2}{|y|_0}, \frac{y_3}{|y|_0^{2p}} \right) , \tag{9.35}$$

and

$$\Phi_0(t_1, t_2, t_3) = \frac{1}{4\,p\,\pi} \cdot \frac{1}{\sqrt{(t_1^2 + t_2^2)^{2p} + t_3^2}} \, . \tag{9.36}$$

Then, by (9.30),

$$G(0, y) = \frac{\Phi_0(\theta_0^1(y), \theta_0^2(y), \theta_0^3(y))}{|y|_0^{2p}} \, . \tag{9.37}$$

(9.38) *Remark.* Using the symmetry of the first two coordinates, and denoting

$$\vartheta_0(y) = \frac{|y_3|}{(y_1^2 + y_2^2)^p} \, , \; \Psi_0(t) = \frac{1}{4\,p\,\pi} \cdot \frac{(1 + t^{\frac{p+1}{p}})^{\frac{p}{p+1}}}{(1 + t^2)^{\frac{1}{2}}} \, , \tag{9.39}$$

we can write, by (9.30),

$$G(0,y) = \frac{\Psi_0(\vartheta_0^1(y))}{|y|_0^{2p}} \,. \tag{9.40}$$

Finally, we shall consider the:

(c) Grushin case.

Let us consider on $\mathbb{R}^2$ the vector fields

$$X_1 = \partial_{x_1} \,,\; X_2 = x_1 \partial_{x_2} \,. \tag{9.41}$$

Then $[X_1, X_2] = \partial_{x_2}$ and the operator $L = \frac{1}{2}(X_1^2 + X_2^2)$ is hypoelliptic on the axis $\{x_1 = 0\}$ and elliptic elsewhere.

We consider the point $x = (0,0)$, which lies on the axis $\{x_1 = 0\}$. Clearly, $r(0,0) = 2$, $Q(0,0) = 3$ and $B = \{1, (12)\}$.

The diffusion started from x is

$$x_t = \left(B_t^1, \int_0^t B_s^1 dB_s^2\right) = \left(B_t^1, \frac{B_t^1 B_t^2}{2} - S_t\right) \,, \tag{9.42}$$

where, again

$$S_t = \frac{1}{2}\int_0^t B_s^2 dB_s^1 - B_s^1 dB_s^2 \,.$$

The left invariant diffusion started from 0 on the Heisenberg group H_3 is

$$\mathcal{G}_t = (B_t^1, B_t^2, -S_t) \,. \tag{9.43}$$

Therefore,

$$x_t = \pi_x(\mathcal{G}_t) \,,\; \pi_x(a,b,c) = (a, \frac{a\,b}{2} + c) \,. \tag{9.44}$$

From this it is not difficult to see that the Green function of (x_t) is

$$G((0,0),(y_1,y_2)) = \int_{\mathbb{R}} G^H\left((0,0,0),(y_1,h,y_2 - \frac{y_1\,h}{2})\right) dh \,, \tag{9.45}$$

or, by (9.1)

$$(9.46) \qquad G((0,0),(y_1,y_2)) = \frac{1}{4\pi}\int_{\mathbb{R}} \frac{dh}{\sqrt{(y_1^2+h^2)^2+(y_2-\frac{y_1 h}{2})^2}}\,.$$

Let us denote

$$(9.47) \qquad |(y_1,y_2)|_0 = \sqrt[3]{|y_1|^3+|y_2|^{\frac{3}{2}}}\,.$$

If we take

$$(9.48) \qquad \theta_0(y_1,y_2) = \left(\frac{y_1}{|(y_1,y_2)|_0}, \frac{y_2}{|(y_1,y_2)|_0^2}\right)$$

and

$$(9.49) \qquad \Phi_0(t_1,t_2) = \frac{1}{4\pi}\int_{\mathbb{R}} \frac{dh}{\sqrt{(t_1^2+h^2)^2+(t_2-\frac{t_1 h}{2})^2}}\,,$$

then

$$(9.50) \qquad G((0,0),(y_1,y_2)) = \frac{\Phi_0(\theta_0(y_1,y_2))}{|(y_1,y_2)|_0^{3-2}}\,.$$

(9.51) *Remark.* Denoting,

$$(9.52) \qquad \vartheta_0(y_1,y_2) = \frac{y_2}{y_1^2}\,,$$

and

$$(9.53) \qquad \Psi_0(t) = \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\sqrt{1+|t|^{\frac{3}{2}}}}{\sqrt{(1+h^2)^2+(t-\frac{h}{2})^2}}\,dh\,,$$

we get

$$(9.54) \qquad G((0,0),(y_1,y_2)) = \frac{\Psi_0(\vartheta_0(y_1,y_2))}{|(y_1,y_2)|_0^{3-2}}\,.$$

(9.55) *Remark.* In this case several of our hypothesis fail: $d=2$, $Q(x)=3$, the geometry of the brackets is not locally constant in x and the estimates of [N-S-W] are not proved. Nevertheless, the result obtained by a direct calculation, (9.50) is quite close to the result of the Theorem (1.10).

10. Influence of a drift on the behaviour of the Green function

We give an example where the behaviour near the diagonal of the Green function is quite different in presence of a drift.

We denote $g(m,r)$ the free r-nilpotent Lie algebra with m generators $Y_1,\ldots,Y_m$. Its natural graduation is $g(m,r) = V_1 \oplus \ldots \oplus V_r$, where $V_k = \mathrm{Span}\{Y^J : |J| = k\}$, $k = 1,\ldots,r$. Assume that $r \geq 3$ and take $Y_0 \in V_r \setminus \{0\}$. The associated simple connected nilpotent Lie group is $\mathcal{N}(m,r)$, with the multiplication given by the Campbell-Hausdorff formula.

The left invariants vector fields on $\mathcal{N}(m,r)$ defined by Y_j will be also denoted by Y_j, $j = 0,\ldots,m$. It is known that the operators

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{m} Y_j^2\,,\ \bar{\mathcal{L}} = \mathcal{L} + Y_0 \tag{10.1}$$

are hypoelliptic. Their associated heat kernels will be denoted by $p_t^{(N)}$ and $\bar{p}_t^{(N)}$.

The Green function on $\mathcal{N}(m,r)$ associated to $\bar{\mathcal{L}}$, with pole the unit element $e \in \mathcal{N}(m,r)$, is

$$\bar{G}^{(N)}(e,g) = \int_0^\infty \bar{p}_t^{(N)}(e,g)dt\,. \tag{10.2}$$

(10.3) **Proposition.** *There exists a positive constant k such that*

$$\limsup_{s\downarrow 0} \bar{G}^{(N)}\left(e, \exp(-sY_0)(e)\right) \leq k\,. \tag{10.4}$$

(10.5) *Remark.* We obtain that, when the drift is in the center of the Lie algebra $g(m,r)$, the Green function $\bar{G}^{(N)}(e,g)$ remains bounded when g approachs e against the drift vector field.

We can compare this with the result obtained by [BA-Le], p. 177, for the heat kernel:

$$\limsup_{t\downarrow 0} t^{1-\frac{2}{r}} \log \bar{p}_t^{(N)}(e,e) < 0\,, \tag{10.6}$$

that is an exponential decay on the diagonal.

Proof. To study $\bar{G}^{(N)}$ we shall use the estimates for the heat kernel.

Since $Y_0 \in V_r$, it commutes with all Y_j, $j = 1, \dots, m$. So, Y_0 commutes with $\mathcal{L}$. Therefore, for $t > 0$ and $g \in \mathcal{N}(m, r)$,

$$\bar{p}_t^{(N)}(e, g) = p_t^{(N)}(\exp(tY_0)(e), g) = p_t^{(N)}\left(e, \exp(tY_0)(e)^{-1} \cdot g\right), \tag{10.7}$$

also using the left invariance.

By the results of [J-Sa1], p. 843, and [K-S2], p. 182, it is known that, for any $g \in \mathcal{N}(m, r)$,

$$p_t^{(N)}(e, g) \le \frac{M \exp\left(-\frac{|g|_N^2}{Mt}\right)}{t^{\frac{Q_N}{2}}}, \; t \in (0, 1], \tag{10.8}$$

and

$$p_t^{(N)}(e, g) \le \frac{M' \exp\left(-\frac{\|g-e\|^2}{M't}\right)}{t^{\frac{d+n}{2}}}, \; t \in (1, \infty). \tag{10.9}$$

Here M, M' are positive constants, $|\cdot|_N$ is the homogeneous norm on the group, $\|\cdot\|$ is the Euclidian norm, Q_N the homogeneous dimension of $\mathcal{N}(m, r)$ and $d + n = \dim g(m, r)$.

We know that, for any $g \in \mathcal{N}(m, r)$, with $\|g - e\|$ bounded,

$$|g|_N \le c\|g - e\|^{\frac{1}{r}},$$

(see (6.11)). So, (10.9) can be written

$$p_t^{(N)}(e, g) \le \frac{M' \exp\left(-\frac{|g|_N^{2r}}{c^r M't}\right)}{t^{\frac{d+n}{2}}}, \; t \in (1, \infty). \tag{10.9'}$$

We shall put (10.8) and (10.9') in (10.7). But firstly note, that

$$|\exp(tY_0)(e)^{-1} \cdot g|_N = |\exp(-tY_0)(e) \cdot g|_N.$$

Taking $g = \exp(-sY_0)(e)$, $s > 0$, we get

$$|\exp(tY_0)(e)^{-1} \cdot \exp(-sY_0)(e)|_N^2 = |\exp((-t-s)Y_0)(e)|_N^2 = (t+s)^{\frac{2}{r}} c_0^2.$$

The last equality is a direct calculation and we denoted $c_0 = |\exp(Y_0)(e)|_N$.

By this simple remark, using (10.2), (10.7), (10.8) and (10.9') we get, for $g = \exp(-sY_0)(e)$, $s > 0$ small enough,

$$\text{(10.10)} \qquad \bar{G}^{(N)}(e,g) \leq \int_0^1 \frac{M \exp\left(-\frac{c_0^2 (t+s)^{\frac{2}{r}}}{Mt}\right)}{t^{\frac{Q_N}{2}}} dt + \int_1^\infty \frac{M' \exp\left(-\frac{c_0^{2r}(t+s)^2}{c^r M' t}\right)}{t^{\frac{d+n}{2}}} dt\,.$$

The first integral in (10.10) can be written as

$$\int_0^{\frac{1}{s}} \frac{M \exp\left(-\frac{c_0^2}{Ms^{1-\frac{2}{r}}} \frac{(1+u)^{\frac{2}{r}}}{u}\right)}{s^{\frac{Q_N}{2}-1} u^{\frac{Q_N}{2}}} du \leq \frac{M}{s^{\frac{Q_N}{2}-1}} \int_0^\infty \frac{\exp\left(-\frac{c_0^2}{Ms^{1-\frac{2}{r}}} \frac{(1+u)^{\frac{2}{r}}}{u}\right)}{u^{\frac{Q_N}{2}}} du =$$

$$k' \sigma^{\frac{r}{r-2}(\frac{Q_N}{2}-1)} \int_0^\infty v^{\frac{Q_N}{2}-2} \exp\left(-\sigma\, v^{1-\frac{2}{r}} (1+v)^{\frac{2}{r}}\right) dv\,,$$

where we denoted $\sigma = \frac{c_0^2}{Ms^{1-\frac{2}{r}}}$.

Applying the Laplace method, we get an equivalent for the last integral, for $s \downarrow 0$ (that is $\sigma \uparrow \infty$):

$$\int_0^\infty v^{\frac{Q_N}{2}-2} \exp\left(-\sigma v^{1-\frac{2}{r}} (1+v)^{\frac{2}{r}}\right) dv \sim k'' \cdot \sigma^{-\frac{r}{r-2}(\frac{Q_N}{2}-1)}\,.$$

So, there exists positive constants k', k'', such that

$$\text{(10.11)} \qquad \int_0^1 \frac{M \exp\left(-\frac{c_0^2 (t+s)^{\frac{2}{r}}}{Mt}\right)}{t^{\frac{Q_N}{2}}} dt \sim k'k''\,, \text{ when } s \downarrow 0\,.$$

For the second integral in (10.10), by the Lebesgue dominated convergence theorem, we see that

$$\text{(10.12)} \qquad \lim_{s \downarrow 0} \int_1^\infty \frac{M' \exp\left(-\frac{c_0^{2r}(t+s)^2}{c^r M' t}\right)}{t^{\frac{d+n}{2}}} dt = M' \int_1^\infty \frac{e^{-\frac{c_0^{2r}}{c^r M'} t}}{t^{\frac{d+n}{2}}} dt\,,$$

the last integral being convergent.

Combining (10.10), (10.11) and (10.12), we get the conclusion.

q.e.d.

(10.13) *Remark.* We note that in this case the a priori estimates on the Green function are not proved (see [N-S-W], p. 107), Y_0 being in V_r and r

being larger than 2 (see also [Le]). So, we can not say that, estimating the Green function on the diagonal is a small time behaviour problem as in the case without drift.

We could use the natural sub-Riemannian distance ρ_N, instead $|\cdot|_N$. We then apply the general result in [J-Sa1], p. 849, or in [K-S1], p. 427. Also, we must use (1.7) in [BA-Le], p. 178 together with (6.4) and (6.3).

The result of the Proposition (10.3) could give an idea of the pathologies for the behaviour near the diagonal of the Green function, which could appear in presence of a drift.

11. Law of a functional of planar Brownian motion

In this section we use the result of the Proposition (9.29) to get a result involving the joint Laplace transform of some functionals of the planar Brownian motion.

We consider (B_t^1, B_t^2) a two-dimensional Brownian motion and we denote $w_t = B_t^1 + i\,B_t^2$. Let $p \geq 1$ be an integer.

(11.1) **Proposition.** *The joint Laplace transform of*

$$\left(|w_t|^2, \int_0^t |w_s|^{4p-2} ds\right) \tag{11.2}$$

satisfies, for $\lambda, \mu > 0$,

$$\int_0^\infty E_{(0,0)} \exp\left(-\lambda|w_t|^2 - \mu \int_0^t |w_s|^{4p-2} ds\right) dt = \tag{11.3}$$

$$. \qquad \frac{2}{p}\int_0^\infty \frac{1}{\cosh 2t}\frac{\sqrt{p}}{(2\pi)^{\frac{p}{2}-1}}\,\frac{1}{\lambda}\,\mathrm{G}_{p,1}^{1,p}\left(\frac{p^{p-1}\sqrt{\frac{\mu}{2}}\tanh 2t}{\lambda^p}\,\middle|\,\begin{matrix} 0, \frac{1}{p}, \dots, \frac{p-1}{p} \\ 0 \end{matrix}\right) dt\,.$$

Here we used the Meijer G*-function (see* §9.3 *in* [G-Ry][1]*). If* $p = 1$ *or* $p = 2$ *the integrands in the right hand of* (11.3) *are, respectively,*

$$\frac{2}{\lambda\cosh 2t + \sqrt{\frac{\mu}{2}}\sinh 2t}$$

[1]Gradshteyn, I.S., Ryzhik, I.M.: Table of integrals, series and products, New York London: Academic Press, 1980

and

$$\frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{\lambda}\cosh 2t}\exp\left(\frac{\lambda^2}{\sqrt{2\mu}\tanh 2t}\right)\operatorname{erfc}\left(\frac{\lambda}{\sqrt{\sqrt{2\mu}\tanh 2t}}\right),$$

with $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-y^2}dy$.

Proof. We shall use the set-up and the notations in § 9(b). We denote

$$y_t = 4p\int_0^t R_s^{2(p-1)}dS_s\,, \tag{11.4}$$

where

$$R_t^2 = (B_t^1)^2 + (B_t^2)^2\,,\ S_t = \frac{1}{2}\int_0^t B_s^2 dB_s^1 - B_s^1 dB_s^2\,. \tag{11.5}$$

So, by (9.32), the diffusion started from $(0,0)$, generated by X_1, X_2 from (9.28) is

$$x_t = (w_t, y_t)\,. \tag{11.6}$$

The law of x_t is invariant under rotations $(w,y) \mapsto (e^{i\varphi}w, y)$. This implies that

$$G((0,0),(w,y_3)) = G((0,0),(|w|,y_3))\,. \tag{11.7}$$

If we denote

$$A_t = 4p^2\int_0^t |w_s|^{4p-2}ds\,, \tag{11.8}$$

y_t can be written as a one-dimensional Brownian motion W at time A_t:

$$y_t = W(A_t)\,. \tag{11.9}$$

Moreover, W is independent of the process (A_t), so, of the process $(|w_t|^2)$.

For any $\lambda > 0$ and $\mu \in \mathbb{R} \setminus \{0\}$, we can write as in [CM-LG], p. 228:

$$\mathcal{I} = \pi \int_{\mathbb{R}_+ \times \mathbb{R}} e^{-\lambda\rho + i\,\mu y} G((0,0),(\sqrt{\rho},y))d\rho dy =$$

$$\int_{\mathbb{C}\times\mathbb{R}} e^{-\lambda|w|^2 + i\,\mu y} G((0,0),(w,y))dwdy =$$

$$E_{(0,0)} \int_0^\infty e^{-\lambda|w_t|^2 + i\,\mu y_t} dt = \int_0^\infty E_{(0,0)} e^{-\lambda|w_t|^2 + i\,\mu W(A_t)} dt =$$

$$\int_0^\infty E_{(0,0)} \exp(-\lambda|w_t|^2 - \frac{\mu^2}{2} A_t) dt\,,$$

as follows from the independence of W and $|w_t|^2$.

On the other hand, by (9.30), after some calculations (see also [G-Ry], 3.518.3, 3.723.2), we have

$$\mathcal{I} = \pi \int_{\mathbb{R}_+ \times \mathbb{R}} e^{-\lambda\rho + i\,\mu y} d\rho dy \cdot \frac{2}{p\pi^2} \int_0^\infty \frac{\rho^p \cosh 2t}{\rho^{2p}(\cosh 2t)^2 + y^2(\sinh 2t)^2} dt =$$

$$\frac{2}{p\pi} \int_0^\infty dt \int_0^\infty e^{-\lambda\rho} \frac{\pi}{\cosh 2t} \exp(-\mu\, \tanh 2t \cdot \rho^p) d\rho =$$

$$\frac{2}{p} \int_0^\infty \frac{dt}{\cosh 2t} \int_0^\infty e^{-\lambda\rho} e^{-\mu\, \tanh 2t \cdot \rho^p} d\rho\,.$$

Therefore, we get

$$\int_0^\infty E_{(0,0)} \exp\left(-\lambda|w_t|^2 - 2\mu^2 p^2 \int_0^t |w_s|^{4p-2} ds\right) dt = \tag{11.10}$$

.
$$\frac{2}{p} \int_0^\infty \frac{dt}{\cosh 2t} \int_0^\infty e^{-\lambda\rho} e^{-\mu\, \tanh 2t \cdot \rho^p} d\rho\,.$$

From this, the conclusion is obtained using formulas 2.2.1.1, 2.2.1.5, 2.2.1.22 in [P-B-M][2].

q.e.d.

(11.11) *Remark.* For $p = 1$, (11.10) is

$$\int_0^\infty E_{(0,0)} \exp\left(-\lambda|w_t|^2 - 2\mu^2 \int_0^t |w_s|^2 ds\right) dt = \int_0^\infty \frac{2\,dt}{\lambda \cosh 2t + \mu \sinh 2t}\,.$$

[2]Prudnikov, Brychkov, Marichev: Integrals and series, vol. 4, Direct Laplace transforms, New York Reading Paris Montreux Tokyo Melbourne: Gordon and Breach SciencePublishers, 1992

The joint Laplace transform of

$$\left(|w_t|^2, \int_0^t |w_s|^2 ds\right)$$

was obtained in [P-Y], p. 432 using other method. One could obtain the joint Laplace transform of (11.2), for arbitrary p, using (11.3) and inverting another Laplace transform in t:

$$\int_0^\infty e^{-\nu t} E_{(0,0)} \exp\left(-\lambda |w_t|^2 - 2\mu^2 p^2 \int_0^t |w_s|^{4p-2} ds\right) dt \,, \; \nu > 0 \,,$$

This involves to consider the ν-potential of (x_t), $G_\nu((0,0),(w,y_3))$ (see (7.1)), although obtaining its expresion seem to be more delicate.

Appendix

We prove here the integral estimates which we used in the proof of the Theorem (1.10).

We shall denote $d_k = \operatorname{card}\{j : l_j = k\}$, $k = 1, \ldots, r$. So, $d = \sum_{k=1}^r d_k$ and $Q = \sum_{k=1}^r k d_k$. We assume that $r \geq 2$, $d_1 \geq 2$ and $d_k \geq 1$, $k = 2, \ldots, r$.

(A.1) **Lemma.** *There exists two positive constants c_0, c_1, such that, for every $S > 0$,*

$$\mathcal{I} = \int_{|u|_h < S} \frac{du}{|u|_h^{Q-2}} < c\, S^{\frac{2}{r^2}} \,, \tag{A.2}$$

where $c = c_0 (2\pi)^{l-r} c_1^{r-1}$ except for $r = 2$, $d_2 = 1$ where $c = \sqrt{2}(2\pi)^2$.

Proof. In estimating $\mathcal{I}$ we shall use the following simple observation. Let us denote, for $n \geq 1$, $p, q > 0$ and $\sigma > 0$,

$$\Lambda_{n,p,q}(\sigma) = \int_0^\sigma \frac{\rho^{n-1}}{(\rho^q + 1)^p} d\rho \,. \tag{A.3}$$

Clearly, $\Lambda_{n,p,q}$ is increasing and we see that, there exists $c_1 > 0$, depending only on n, p, q, such that

$$\lim_{\sigma \uparrow \infty} \Lambda_{n,p,q}(t) < c_1 \,, \text{ provided } pq - n > 0 \,. \tag{A.4}$$

Also, for $S, R > 0$, we have

$$\int_0^S \frac{\rho^{n-1}}{(\rho^q + R)^p} d\rho = R^{\frac{n-pq}{q}} \Lambda_{n,p,q} \left(\frac{S}{R^{\frac{1}{q}}} \right) . \tag{A.5}$$

We shall denote, for $k = 1, \ldots, r$,

$$s_k^2 = \sum_{j, l_j = k} u_j^2 \, , \; Q_k = \sum_{i=k}^{r} i \, d_i \, . \tag{A.6}$$

Then

$$\mathcal{I} \leq \int_{\{u : |s_k| < S^{\frac{1}{k}}, k=1,\ldots,r\}} \frac{du}{\left(\sum_{k=1}^r s_k^{\frac{Q}{k}} \right)^{\frac{Q-2}{Q}}} =$$

$$\int_{\{|s_k| < S^{\frac{1}{k}}, k=2,\ldots,r\}} du'' \int_{\{|s_1| < S\}} \frac{du'}{(s_1^Q + R_1)^{\frac{Q_1 - 2}{Q}}} \, ,$$

where $du' = \prod_{j, l_j = 1} du_j$, $du'' = \prod_{j, 2 \leq l_j \leq r} du_j$ and $R_1 = \sum_{k=2}^r s_k^{\frac{Q}{k}}$. By a simple change of variables and by (A.5), we get

$$\int_{\{|s_1| < S\}} \frac{du'}{(s_1^Q + R_1)^{\frac{Q_1 - 2}{Q}}} = (2\pi)^{d_1 - 1} \int_0^S \frac{\rho^{d_1 - 1} d\rho}{(\rho^Q + R_1)^{\frac{Q_1 - 2}{Q}}} =$$

$$(2\pi)^{d_1 - 1} R_1^{\frac{d_1 - Q_1 + 2}{Q}} \Lambda_{d_1, \frac{Q_1 - 2}{Q}, Q} \left(\frac{S}{R_1^{\frac{1}{Q}}} \right) .$$

We have $Q \cdot \frac{Q_1 - 2}{Q} - d_1 = Q_2 - 2$. The case $r = 2$, $d_2 = 1$ will be considered separately.

For $r = 2$ and $d_2 > 1$, by (A.4) we get

$$\mathcal{I} < c \int_{\{|s_2| < S^{\frac{1}{2}}\}} \frac{du''}{s_2^{\frac{Q}{2} \cdot \frac{Q_2 - 2}{Q}}} = c \int_0^{S^{\frac{1}{2}}} \frac{\rho^{d_2 - 1} d\rho}{\rho^{\frac{2 \, d_2 - 2}{2}}} = c \, S^{\frac{1}{2}} \, ,$$

where $c = (2\pi)^{d_1 + d_2 - 2} c_1$.

For $r \geq 3$, again by (A.4), we can write

$$\mathcal{I} < c \int_{\{|s_k| < S^{\frac{1}{k}}, k=3,\ldots,r\}} du'' \int_{\{|s_2| < S^{\frac{1}{2}}\}} \frac{du'}{(s_2^{\frac{Q}{2}} + R_2)^{\frac{Q_2 - 2}{Q}}} \, ,$$

where, this time $du' = \prod_{j,l_j=2} du_j$, $du'' = \prod_{j,3\le l_j\le r} du_j$ and $R_2 = \sum_{k=3}^r s_k^{\frac{Q}{k}}$. By a similar calculation:

$$\int_{\{|s_2|<S^{\frac12}\}} \frac{du'}{(s_2^{\frac{Q}{2}} + R_2)^{\frac{Q_2-2}{Q}}} = (2\,\pi)^{d_2-1} \int_0^{S^{\frac12}} \frac{\rho^{d_2-1} d\rho}{(\rho^{\frac{Q}{2}} + R_2)^{\frac{Q_2-2}{Q}}} =$$

$$(2\,\pi)^{d_2-1} R_2^{\frac{2\,d_2-Q_2+2}{Q}} \Lambda_{d_2,\frac{Q_2-2}{Q},\frac{Q}{2}} \left(\frac{S^{\frac12}}{R_2^{2/Q}} \right).$$

Since $Q \cdot \frac{Q_2-2}{Q} - 2\,d_2 = Q_3 - 2 > 0$, we get

$$\mathcal{I} < c \int_{\{|s_k|<S^{\frac1k}, k=3,\ldots,r\}} R_2^{-\frac{Q_3-2}{Q}} du'',$$

where $c = (2\pi)^{d_1+d_2-2} c_1^2$.

For $r = 3$, $d_3 = 1$, we have $\mathcal{I} < c\,S^{\frac29}$, with $c = 3(2\pi)^{d_1+d_2-2} c_1^2$, and for $r = 3$, $d_3 > 1$,

$$\mathcal{I} < c \int_{\{|s_3|<S^{\frac13}\}} \frac{du''}{s_3^{\frac{Q}{3}\cdot\frac{Q_3-2}{Q}}} = (2\pi)^{d_3-1} c \int_0^{S^{\frac13}} \frac{\rho^{d_3-1} d\rho}{\rho^{\frac{3\,d_3-2}{3}}} = c\,S^{\frac29},$$

with $c = \frac32 (2\pi)^{d_1+d_2+d_3-3} c_1^2$.

For $r \ge 4$ we repeat the reasoning and (A.2) is obtained in a finite number of steps.

To finish the proof we must treat the case $r = 2$, $d_2 = 1$. We have

$$\mathcal{I} \le \int_{\{|s_1|<S, |s_2|<S^{\frac12}\}} \frac{du_1 du_2 du_3}{\sqrt{s_1^4 + s_2^2}} = 2\pi \int_{(0,S)\times(0,S^{\frac12})} \frac{\rho\, d\rho\, dz}{\sqrt{\rho^4 + z^2}} =$$

$$2\pi \int_{(0,S^{\frac12})\times(0,S^{\frac12})} \frac{d\sigma\, dz}{\sqrt{\sigma^2+\rho^2}} \le 2\pi \int_{\sigma^2+\rho^2\le 2\,S} \frac{d\sigma\, dz}{\sqrt{\sigma^2+\rho^2}} = (2\pi)^2 \sqrt{2S}.$$

q.e.d.

Before stating the second result of this section we introduce some notations. Recall that $n = \dim g(m,r) - d = \operatorname{card} A - \operatorname{card} B$. Put $A \setminus B = \{L_1, \ldots, L_n\}$, $m_i = |L_i|$, $i = 1, \ldots, n$ and $e_k = \operatorname{card}\{i : m_i = k\}$, $k = 1, \ldots, r$.

So, $n = \sum_{k=1}^r e_k$ and $Q_N = \sum_{k=1}^r k(d_k + e_k)$. For a point $(u,h) \in \mathbb{R}^d \times \mathbb{R}^n$ we denote

$$|(u,h)|_N = \left[\sum_{k=1}^r \left(\sum_{j,l_j=k} u_j^2 + \sum_{i,m_i=k} h_i^2\right)^{\frac{Q_N}{2k}}\right]^{\frac{1}{Q_N}} .$$

(A.7) **Lemma.** *For every* $S > 0$, *there exists a positive constant* c, *such that*

$$\int_{\{|u|_h < S\}\times\mathbb{R}^n} \frac{du\,dh}{|(u,h)|_N^{Q_N-2}} < c \tag{A.8}$$

Proof. Let us denote, for $k = 1,\ldots,r$

$$t_k^2 = \sum_{i,m_i=k} h_i^2\,,\; Q_{N,k} = \sum_{i=k}^r i\,(d_i + e_i)\,. \tag{A.9}$$

Replacing in (A.2), d by $d+n$ and $|u|_h$ by $|(u,h)|_N$, we get the existence of a constant $c > 0$, such that for every $U > 0$,

$$\int_{|(u,h)|_N < U} \frac{du\,dh}{|(u,h)|_N^{Q_N-2}} < c\,U^{\frac{2}{r^2}}\,. \tag{A.10}$$

So, it suffices to prove that, for every $S, T > 0$, there exists a constant $c > 0$, such that

$$\mathcal{J} = \int_{\{|s_k| < S^{\frac{1}{k}},|t_k| \geq T^{\frac{1}{k}},k=1,\ldots,r\}} \frac{du\,dh}{\left[\sum_{k=1}^r (s_k^2 + t_k^2)^{\frac{Q_N}{2k}}\right]^{\frac{Q_N-2}{Q_N}}} < c \tag{A.11}$$

We see that, for $S, T > 0$, $b \geq 1$ and for $a \geq 2$ and $p \geq 2$ or $a = 1$ and $p \geq 3$, there exists a constant $c_2 > 0$, such that

$$\int_0^S ds \int_T^\infty dt\, \frac{s^{a-1}t^{b-1}}{(s^2+t^2)^{\frac{p(a+b)-2}{2p}}} \leq c_2\,. \tag{A.12}$$

Indeed, we have to study only the integral in t and, clearly,

$$\frac{t^{b-1}}{(s^2+t^2)^{\frac{p(a+b)-2}{2p}}} \sim \frac{1}{t^{1+a-\frac{2}{p}}}\,,\ \text{as}\, t \uparrow \infty\,.$$

We proceed as in the proof of the Lemma (A.1):

$$\mathcal{J} = \int_{\{|s_k|<S^{\frac{1}{k}},|t_k|\geq T^{\frac{1}{k}},k=2,\ldots,r\}} du''\,dh'' \int_{|s_1|<S,|t_1|\geq T} \frac{du'\,dh'}{\left[(s_1^2+t_1^2)^{\frac{Q_N}{2}}+R_1\right]^{\frac{Q_{N,1}-2}{Q_N}}}\,,$$

where $du' = \prod_{j,l_j=1} du_j$, $dh' = \prod_{i,m_i=1} h_i$, $du'' = \prod_{j,2\leq l_j\leq r} du_j$, $dh'' = \prod_{i,2\leq m_i\leq r} dh_i$ and $R_1 = \sum_{k=2}^{r}(s_k^2+t_k^2)^{\frac{Q_N}{2k}}$. Using again (A.5) and (A.4), we get

$$\mathcal{J} < c\int_{\{|s_k|<S^{\frac{1}{k}},|t_k|\geq T^{\frac{1}{k}},k=2,\ldots,r\}} du''\,dh'' \int_0^\infty \frac{\rho^{d_1+e_1-1}d\rho}{(\rho^{Q_N}+R_1)^{\frac{Q_{N,1}-2}{Q_N}}} <$$

$$c\,c_1\int_{\{|s_k|<S^{\frac{1}{k}},|t_k|\geq T^{\frac{1}{k}},k=2,\ldots,r\}} du''\,dh'' R_1^{\frac{d_1+e_1-Q_{N,1}+2}{Q_N}}\,.$$

Here we used the fact that $Q_N\cdot\frac{Q_{N,1}-2}{Q_N}-(d_1+e_1) = Q_{N,2}-2>0$, excepting the case when $r=2$, $d_2=1$ and $e_2=0$ which will be treated separately.

For $r=2$, $d_2>1$ and $e_2\geq 1$, we can write

$$\mathcal{J} < c\,c_1\int_{|s_2|<S^{\frac{1}{2}},|t_2|\geq T^{\frac{1}{2}}} \frac{du''\,dh''}{(s_2^2+t_2^2)^{\frac{Q_N}{4}\cdot\frac{Q_{N,2}-2}{Q_N}}} =$$

$$c\,c_1\int_0^{S^{\frac{1}{2}}} ds\int_{T^{\frac{1}{2}}}^\infty dt\frac{s^{d_2-1}t^{e_2-1}}{(s^2+t^2)^{\frac{2\,(d_2+e_2)-2}{4}}} < c\,c_1c_2\,,$$

by (A.12).

For $r\geq 3$ we repeat the reasoning:

$$\mathcal{J} < c\,c_1\int_{\{|s_k|<S^{\frac{1}{k}},|t_k|\geq T^{\frac{1}{k}},k=3,\ldots,r\}} du''\,dh''\cdot$$

$$\cdot\int_{|s_2|<S^{\frac{1}{2}},|t_2|\geq T^{\frac{1}{2}}} \frac{du'\,dh'}{\left[(s_2^2+t_2^2)^{\frac{Q_N}{4}}+R_2\right]^{\frac{Q_{N,2}-2}{Q_N}}}\,,$$

where $du' = \prod_{j,l_j=2} du_j$, $dh' = \prod_{i,m_i=2} h_i$, $du'' = \prod_{j,3\leq l_j\leq r} du_j$, $dh'' = \prod_{i,3\leq m_i\leq r} dh_i$ and $R_2 = \sum_{k=3}^{r}(s_k^2+t_k^2)^{\frac{Q_N}{2k}}$. Then, by (A.5) and (A.4), we get

$$\mathcal{J} < c\, c_1 \int_{\{|s_k|<S^{\frac{1}{k}},|t_k|\geq T^{\frac{1}{k}},k=3,\ldots,r\}} du''\, dh'' \int_0^\infty \frac{\rho^{d_2+e_2-1} d\rho}{(\rho^{\frac{Q_N}{2}} + R_2)^{\frac{Q_{N,2}-2}{Q_N}}} <$$

$$c\, c_1^2 \int_{\{|s_k|<S^{\frac{1}{k}},|t_k|\geq T^{\frac{1}{k}},k=3,\ldots,r\}} du''\, dh''\, R_2^{\frac{2(d_2+e_2)-Q_{N,2}+2}{Q_N}} ,$$

since $\frac{Q_N}{2} \cdot \frac{Q_{N,2}-2}{Q_N} - (d_2+e_2) = Q_{N,3} - 2 > 0$.

If $r = 3$, we have, by (A.12),

$$\mathcal{J} < c\, c_1^2 \int_{|s_3|<S^{\frac{1}{3}},|t_3|\geq T^{\frac{1}{3}}} \frac{du''\, dh''}{(s_3^2 + t_3^2)^{\frac{Q_N}{6} \cdot \frac{Q_{N,3}-2}{Q_N}}} =$$

$$c\, c_1^2 \int_0^{S^{\frac{1}{3}}} ds \int_{T^{\frac{1}{3}}}^\infty dt \frac{s^{d_3-1} t^{e_3-1}}{(s^2+t^2)^{\frac{3(d_3+e_3)-2}{6}}} < c\, c_1^2 c_2 \,.$$

For $r \geq 4$ we repeat the calculation and (A.11) is obtained in a finite number of steps.

Finally we treat the case $r = 2$, $d_2 = 1$, $e_2 = 0$:

$$\mathcal{J} = \int_{|s_1|<S,|s_2|<S^{\frac{1}{2}},|t_1|\geq T} \frac{du\, dh}{[(s_1^2+t_1^2)^{\frac{Q_N}{2}} + (s_2^2)^{\frac{Q_N}{4}}]^{\frac{Q_N-2}{Q_N}}} =$$

$$c \int_0^S \int_T^\infty \int_0^{S^{\frac{1}{2}}} \frac{s^{d_1-1} t^{e_1-1} ds\, dt\, dz}{[(s^2+t^2)^{\frac{Q_N}{2}} + z^{\frac{Q_N}{2}}]^{\frac{Q_N-2}{Q_N}}} < c\, c_2 S^{\frac{1}{2}} \,.$$

This ends the proof of (A.8).

q.e.d.

References:

.[A] Azencott, R.: Formule de Taylor stochastique et développements asymptotiques d'intégrales de Feynmann, In: Azéma, J., Yor, M. (eds.) Seminaire de Probabilités XVI. Supplément: Géométrie différentielle stochastique (Lect. Notes Math. vol. 921, pp. 237-284), Berlin Heidelberg New York: Springer 1982

.[BA1] Ben Arous, G.: Flots et séries de Taylor stochastiques, Probab. Th. Rel. Fields **81**, pp. 29-77 (1989)

.[BA2] Ben Arous, G.: Développement asymptotique du noyau de la chaleur

hypoelliptique sur la diagonale, Ann. Inst. Fourier **39**, pp. 73-99 (1989)
.[BA-G] Ben Arous, G., Gradinaru, M.: Singularities of hypoelliptic Green functions, in progress
.[BA-Le] Ben Arous, G., Léandre, R.: Décroissance exponentielle du noyau de la chaleur sur la diagonale I,II, Probab. Th. Rel. Fields **90**, pp. 175-202, 377-402 (1991)
.[B-G] Blumenthal, R.M., Getoor, R.K.: Markov processes and potential theory, New York London: Academic Press, 1968
.[B-Ga-Gr] Beals, R., Gaveau, B., Greiner, P.C.: Lecture at the Seminar on Analysis, Université Paris VI, "Pierre et Marie Curie" (1993)
.[Bi] Biroli, M.: The Wiener Test for Poincaré-Dirichlet Forms, In: GowriSankaran K. et al. (eds.), Classical and Modern Potential Theory and Applications, pp. 93-104, Dordrecht Boston London: Kluwer Academic Publishers
.[Bo] Bony, J.M.: Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés, Ann. Inst. Fourier **19**, pp. 277-304 (1969)
.[Ca] Castell, F.: Asymptotic expansion of stochastic flows, Probab. Th. Rel. Fields **96**, pp. 225-239 (1993)
.[CM-LG] Chaleyat-Maurel, M., Le Gall, J.-F.: Green function, capacity and sample paths properties for a class of hypoelliptic diffusions processes, Probab. Th. Rel. Fields **83**, pp. 219-264 (1989)
.[F] Folland, G.B.: A fundamental solution for a subelliptic operator, Bull. Amer. Math. Soc. **79**, pp. 373-376 (1973)
.[Fe-Sa] Fefferman, C.L., Sánchez-Calle, A.: Fundamental solutions for second order subelliptic operators, Ann. Math. **124**, pp. 247-272 (1986)
.[G] Gallardo, L.: Capacités, mouvement brownien et problème de l'épine de Lebesgue sur les groupes de Lie nilpotents, In: Heyer, H. (ed.) Probability measures on groups, Proceedings of the Conference at Oberwolfach 1981 (Lect. Notes Math. vol. 928, pp. 96-120), Berlin Heidelberg New York: Springer 1982
.[Ga] Gaveau, B.: Principe de moindre action, propagation de la chaleur et estimées sous-elliptiques sur certains groupes nilpotents, Acta Math. **139**, pp. 96-153 (1977)
.[Gr1] Greiner, P.C.: A fundamental solution for a nonelliptic partial differential operator, Canad. Jour. Math. **31**, pp. 1107-1120 (1979)
.[Gr2] Greiner, P.C.: On second order hypoelliptic differential operators and the $\bar{\partial}$-Neumann problem, In: Diedrich, K. (ed.) Complex analysis, Proceedings of Workshop at Wuppertal 1990, pp. 134-142, Braunschweig: Vieweg 1991
.[Gr-S] Greiner, P., Stein, E.M.: On the solvability of some differential operators of type $\Box_b$, In: Several complex variables, Proceedings of the conference at Cortona 1976-1977, pp. 106-165, Pisa: Scuola Normale Superiore 1978
.[K-S1] Kusuoka, S., Stroock, D.W.: Applications of the Malliavin calculus, Part III, J. Fac. Sci. Univ. Tokyo **34**, pp. 391-442 (1987)
.[K-S2] Kusuoka, S., Stroock, D.W.: Long time estimates for the heat kernel associated with a uniformly subelliptic symmetric second order operator, Ann. Math. **127**, pp. 165-189 (1988)
.[J-Sa1] Jerison, D., Sánchez-Calle, A.: Estimates for the heat kernel for a sum of squares

of vector fields, Indiana Univ. Math. J. **35**, pp. 835-854 (1986)

.[J-Sa2] Jerison, D., Sánchez-Calle, A.: Subelliptic second order differential operators, In: Berenstein, C.A. (ed.) Complex analysis III, Proceedings of the Special Year at University of Maryland 1985-1986 (Lect. Notes Math. vol. 1277, pp. 46-77), Berlin Heidelberg New York: Springer 1987

.[La] Lamperti, J.: Wiener's test and Markov chains, Jour. Math. Anal. Appl. **6**, pp. 58-66 (1963)

.[Le] Léandre, R.: Volume des boules sous-riemanniennes et explosion du noyau de la chaleur au sens de Stein, , In: Azéma, J., Meyer, P.-A., Yor, M. (eds.) Seminaire de Probabilités XXIII (Lect. Notes Math. vol. 1372, pp. 426-447), Berlin Heidelberg New York: Springer 1989

.[N-S-W] Nagel, A., Stein, E.M., Wainger, S.: Balls and metrics defined by vector fields I. Basic properties, Acta Math. **155**, pp. 103-147 (1985)

.[P-Y] Pitman, J.W., Yor, M.: A decomposition of Bessel bridges, Z. Wahrscheinlichkeitstheor. Verw. Geb. **59**, pp. 425-457 (1982)

.[Sa] Sánchez-Calle, A.: Fundamental solutions and geometry of the sum of square of vector fields, Invent. math. **78**, pp. 143-160 (1984)

.[W] Wiener, N.: The Dirichlet Problem, J. Math. Phys. **3**, pp. 127-146 (1924).

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III Hölder norms and the support theorem for diffusions

Summary. In this chapter we show that the Stroock-Varadhan support theorem is valid in α-Hölder norm. The central tool is an estimate of the probability that the Brownian motion has a large Hölder norm conditionally on the fact that it has a small uniform norm. This part contains a work made in collaboration with G. Ben Arous and M. Ledoux (see [BA-G-L]).

Contents:

Introduction

What is the probability that the Brownian motion oscillates rapidly conditionally on the fact that it is small in uniform norm? More precisely, what is the probability that the α-Hölder norm of the Brownian motion is large conditionally on the fact that its uniform norm (or more generally its β-Hölder norm with $\beta < \alpha$) is small?

This is the kind of question that naturally appears if one wants to extend the Stroock-Varadhan [S-V] characterization of the support of the law of diffusion processes to sharper topologies than the one induced by the uniform norm.

We deal with this question in §1 and show that those tails are much smaller than the gaussian tails one would get without the conditioning. This gives a family of examples where the conjecture (stated in Das Gupta *et al.* [DG-E-...]) that two convex symmetric bodies are positively correlated (for gaussian measures) is true.

Our proofs are based on the Ciesielski isomorphism [C] (see Baldi and Roynette [B-R] for other applications of this theorem) and on the correlation inequality. We give in appendix a proof which avoids these tools.

This enables us to control in §3 the probability that a Brownian stochastic integral oscillates rapidly conditionally on the fact that the Brownian motion is small in uniform norm. This is the tool to extend the Stroock-Varadhan support theorem to α-Hölder norms.

1. Conditional tails for oscillations of the Brownian motion

If x is a continuous real function on $[0,1]$, vanishing at zero, we define the sequence $(\,\xi_m(x)\,)_{m\geq 1}$ by the formula:

$$\xi_m(x) = \xi_{2^n+k}(x) = 2^{\frac{n}{2}}\left(2x\left(\frac{2k-1}{2^{n+1}}\right) - x\left(\frac{k}{2^n}\right) - x\left(\frac{k-1}{2^n}\right)\right) ,$$

for $n \geq 0$ and $k = 1, \ldots, 2^n$. Denote

$$\|x\|_0 = \sup_{0\leq t\leq 1} |x_t| \ , \tag{1.1}$$

$$\|x\|_\alpha = \sup_{0\leq s\neq t\leq 1} \frac{|x_t - x_s|}{|t-s|^\alpha} \ , \ \ \alpha \in \,]0,1]\,, \tag{1.2}$$

$$(1.3) \qquad \|x\|'_\alpha = \sup_{m\geq 1} |m^{\alpha-\frac{1}{2}}\xi_m(x)| \ , \ \alpha \in \ [0,1] \, .$$

It is now classical that, for $\alpha \in]0,1[$, the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{\alpha'}$ are equivalent (see [C], p. 218):

$$(1.4) \qquad 2^{\alpha-1}\,\|x\|'_\alpha \leq \|x\|_\alpha \leq 2^{-\frac{1}{2}}\,k_\alpha\,\|x\|'_\alpha\,,\ \alpha \in]0,1[\,,$$

where

$$k_\alpha = \frac{2}{(2^\alpha-1)(2^{1-\alpha}-1)}\,,$$

and

$$(1.5) \qquad 2^{-2}\,\|x\|'_0 \leq \|x\|_0\,.$$

Let w be a liniar Brownian motion started from zero. We want to estimate the probability that $\|w.\|_\alpha$ is large conditionally on the fact that $\|w.\|_\beta$ is small. We shall first tackle the same problem with the norms $\|\cdot\|'$.

(1.6) **Theorem.** *Let (r,R) be a couple of real positive numbers, $v = \left(\frac{R^b}{r^a}\right)^{\frac{1}{b-a}}$ and denote*

$$(1.7) \qquad \Lambda_{\alpha,\beta}(r,R) = \frac{\varphi(v)}{v} + \frac{1}{a}\,R^{-\frac{1}{a}}\int_v^\infty \varphi(t)t^{\frac{1}{a}-2}\,dt\,,$$

where $\varphi(t) = \dfrac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}$, $a = \dfrac{1}{2}-\alpha$, $b = \dfrac{1}{2}-\beta$. Then,

$$(1.8) \qquad P(\,\|w.\|'_\alpha > R\,|\,\|w.\|'_\beta < r\,) \leq \frac{1}{\int_0^v \varphi(t)\,dt}\,\Lambda_{\alpha,\beta}(r,R)\,;$$

$$(1.9) \qquad P(\,\|w.\|'_\alpha > R\,|\,\|w.\|_\beta \leq r\,) \leq \Lambda_{\alpha,\beta}(p_\beta r,R)\,,$$

where $p_\beta = 2^{1-\beta}$, if $\beta > 0$ and $p_0 = 4$;

$$(1.10) \qquad P(\,\|w.\|_\alpha > R \,|\, \|w.\|_\beta \le r\,) \le \Lambda_{\alpha,\beta}(p_\beta r, 2^{\frac{1}{2}} k_\alpha^{-1} R)\,.$$

To prove the theorem we need the following:

(1.11) **Lemma.** *Let us denote* $n_0 = \left[\left(\frac{R}{r}\right)^{\frac{1}{b-a}} \right]$. *Then*

$$(1.12) \qquad \sum_{n \ge n_0+1} \int_{Rn^a}^{\infty} \varphi(t)\, dt \le \Lambda_{\alpha,\beta}(r, R)\,.$$

Proof. By the classical bound:

$$\int_t^\infty \varphi(s)\, ds \le \frac{\varphi(t)}{t} \equiv \psi(t)\,,\; t > 0\,,$$

and the fact that ψ is decreasing, we get:

$$\sum_{n \ge n_0+1} \int_{Rn^a}^\infty \varphi(t)\, dt \le \sum_{n \ge n_0+1} \psi(Rn^a) =$$

$$\psi(R(n_0+1)^a) + \sum_{n \ge n_0+2} \psi(Rn^a) \le \psi(v) + \int_{n_0+1}^\infty \psi(Rt^a)\, dt =$$

$$\frac{\varphi(v)}{v} + \frac{1}{a} R^{-\frac{1}{a}} \int_{R(n_0+1)^a}^\infty \psi(t)\, t^{\frac{1}{a}-1}\, dt\,.$$

From this the conclusion follows.

q.e.d.

We make another essential observation. If C, C' are two symmetric convex sets in $\mathbb{R}^d$, a general conjecture stated in [DG-E-...] predicts that they are positively correlated for the canonical Gaussian measure γ_d, that is,

$$(1.13) \qquad \gamma_d(C \cap C') \ge \gamma_d(C)\, \gamma_d(C')\,.$$

This is true for $d = 2$ (see [P]), and for arbitrary d provided C' is a symmetric strip (see [Sc] or [Si]). The general case is still open.

Proof of the Theorem (1.6).

Proof of (1.8). We note that $g_n = \xi_n(w)$ is a sequence of independent identically distributed standard Gaussian random variables. Then,

$$P(\,\|w.\|'_\alpha > R \,|\, \|w.\|'_\beta < r\,) = P(\sup_{n\ge1} |n^{-a} g_n| > R \,|\, \sup_{m \ge 1} |m^{-b} g_m| < r\,) =$$

$$\frac{P(\cup_{n\geq 1}(\,|g_n| > Rn^a) \cap \cap_{m\geq 1}(\,|g_m| < rm^b))}{P(\cap_{m\geq 1}|g_m| < rm^b)} \leq$$

$$\frac{\sum_{n\geq 1} P((Rn^a < |g_n| < rn^b) \cap \cap_{m\geq 1, m\neq n}(\,|g_m| < rm^b))}{\prod_{m\geq 1} P(\,|g_m| < rm^b)} =$$

$$\sum_{n\geq 1} \frac{P(Rn^a < |g_n| < rn^b)}{P(\,|g_n| < rn^b)} \cdot 1_{(\,Rn^a < rn^b\,)} = \sum_{n\geq 1} \frac{2\int_{Rn^a}^{rn^b} \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\, ds}{2\int_0^{rn^b} \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\, ds} \cdot 1_{(\,Rn^a < rn^b\,)} =$$

$$\sum_{n\geq n_0+1} \frac{\displaystyle\int_{Rn^a}^{rn^b} \varphi(t)\, dt}{\int_0^{rn^b} \varphi(t)\, dt} \leq \frac{1}{\int_0^v \varphi(t)\, dt} \sum_{n\geq n_0+1} \int_{Rn^a}^{\infty} \varphi(t)\, dt\,.$$

Clearly $rn^b \geq r(n_0+1)^b \geq v$ so the last inequality is true. Then (1.8) is a consequence of the Lemma (1.11).

Proof of (1.9). We can write again

$$P(\,\|w.\|'_\alpha > R\,|\,\|w.\|_\beta \leq r\,) = P(\sup_{n\geq 1} |n^{-a} g_n| > R\,|\,\|w.\|_\beta \leq r\,) =$$

$$P(\cup_{n\geq 1}(\,|g_n| > Rn^a)\,|\,\|w.\|_\beta \leq r\,) \leq \sum_{n\geq 1} P(\,|g_n| > Rn^a\,|\,\|w.\|_\beta \leq r\,)\,.$$

But for $\|w.\|_\beta \leq r$, by (1.4) or (1.5) we get

$$|g_n| \leq 2^{1-\beta} rn^b \ , \text{ if } \ \beta > 0$$

or

$$|g_n| \leq 4rn^{\frac{1}{2}} \ , \text{ if } \ \beta = 0\,.$$

So, the preceding sum is taken over all integer $n \geq 1$ such that $2^{1-\beta} rn^b \geq Rn^a$, if $\beta > 0$, or $4rn^{\frac{1}{2}} \geq Rn^a$, if $\beta = 0$, that is,

$$n \geq 2^{\frac{\beta-1}{\alpha-\beta}} \left(\frac{R}{r}\right)^{\frac{1}{b-a}} \ \text{ or } \ n \geq 2^{-\frac{2}{\alpha}} \left(\frac{R}{r}\right)^{\frac{1}{\alpha}}\,.$$

On the other hand, $g_n = \xi_n(w)$ is a linear form on the Wiener space. By the correlation inequality with one of sets a symmetric strip and by a simple finite dimensional approximation (see also [S-Z], p. 654), we obtain

$$P(\,|g_n| > Rn^a\,,\, \|w.\|_\beta \leq r\,) \geq P(\,|g_n| > Rn^a\,)\, P(\,\|w.\|_\beta \leq r\,)$$

or

$$P(\,|g_n| > Rn^a \,|\, \|w.\|_\beta \le r\,) \le P(\,|g_n| > Rn^a\,)\,.$$

Therefore

$$P(\,\|w.\|'_\alpha > R \,|\, \|w.\|_\beta \le r\,) \le \sum_{n \ge n_1+1} P(\,|g_n| > Rn^a\,) = \sum_{n \ge n_1+1} \int_{Rn^a}^\infty \varphi(t)\,dt\,,$$

where $n_1 = \left[p_\beta^{\frac{1}{b-a}} \left(\frac{R}{r}\right)^{\frac{1}{b-a}} \right]$. By the Lemma (1.11) we get (1.9).

Proof of (1.10). It is a consequence of (1.4) or (1.5) and (1.9).

q.e.d.

We shall estimate $\Lambda_{\alpha,\beta}(r,R)$:

(1.14) **Lemma.** *With the notations of the* Theorem (1.6), *there exists a polynomial function* Ψ_a, *increasing on* $]0,\infty[$, *such that*

$$\Lambda_{\alpha,\beta}(r,R) \le \frac{\varphi(v)}{v} \left(1 + \frac{1}{a} R^{-\frac{1}{a}} v^{\frac{1}{a}-2} \Psi_a\left(\frac{1}{v}\right) \right)\,. \tag{1.15}$$

Proof. We shall simply give an upper bound for $\int_v^\infty \varphi(t)\, t^{\frac{1}{a}-2}\,dt$. Noting that $\varphi'(t) = -t\,\varphi(t)$ and integrating by parts, we get

$$\int_v^\infty \varphi(t)\, t^{\frac{1}{a}-2}\,dt = -\int_v^\infty \varphi'(t)\, t^{\frac{1}{a}-3}\,dt = \varphi(v)\, v^{\frac{1}{a}-3} + \left(\frac{1}{a}-3\right) \int_v^\infty \varphi(t)\, t^{\frac{1}{a}-4}\,dt\,.$$

If $a \ge \frac{1}{3}$,

$$\int_v^\infty \varphi(t)\, t^{\frac{1}{a}-2}\,dt \le \varphi(v)\, v^{\frac{1}{a}-3}\,,$$

which gives (1.14) with $\Psi_a(x) \equiv 1$. If $a < \frac{1}{3}$, similarly,

$$\int_v^\infty \varphi(t)\, t^{\frac{1}{a}-4}\,dt = \varphi(v)\, v^{\frac{1}{a}-5} + \left(\frac{1}{a}-5\right) \int_v^\infty \varphi(t)\, t^{\frac{1}{a}-6}\,dt\,.$$

So, if $\frac{1}{5} \le a < \frac{1}{3}$,

$$\int_v^\infty \varphi(t)\, t^{\frac{1}{a}-2}\,dt \le \varphi(v)\, v^{\frac{1}{a}-3} + \left(\frac{1}{a}-3\right) \varphi(v)\, v^{\frac{1}{a}-5}\,,$$

which is exactly (1.14) with $p = 1$ in the following expression:

$$\Psi_a(x) = 1 + \left(\frac{1}{a} - 3\right) x^2 + \ldots + \left(\frac{1}{a} - 3\right) \left(\frac{1}{a} - 5\right) \ldots \left(\frac{1}{a} - 2p - 1\right) x^{2p} .$$

Repeating the same reasoning the result is easily obtained for any p and any a such that $\frac{1}{2p+3} \le a < \frac{1}{2p+1}$. Ψ_a has positive coefficients, it is therefore increasing on $]0, \infty[$.

q.e.d.

Combining this result with the Theorem (1.6) we obtain the following:

(1.16) **Corollary.** *Let (R, r) be such that $v \ge \varepsilon > 0$. Then,*

(1.17) $$P(\|w.\|'_\alpha > R \,|\, \|w.\|'_\beta < r) \le c(\varepsilon) \frac{\varphi(v)}{\varepsilon} \left(1 + \Psi_a \left(\frac{1}{\varepsilon}\right) \left(\frac{R^\beta}{r^\alpha}\right)^{\frac{2}{\alpha-\beta}} \right) ;$$

(1.18) $$P(\|w.\|'_\alpha > R \,|\, \|w.\|_\beta < r) \le \frac{\varphi(q_\beta v)}{q_\beta \varepsilon} \left(1 + q_\beta^{\frac{1}{a}-2} \Psi_a \left(\frac{1}{\varepsilon}\right) \left(\frac{R^\beta}{r^\alpha}\right)^{\frac{2}{\alpha-\beta}} \right) ;$$

(1.19) $$P(\|w.\|_\alpha > R \,|\, \|w.\|_\beta < r) \le \frac{\varphi(c_{\alpha,\beta} v)}{c_{\alpha,\beta} \varepsilon} (1 + (2^{\frac{1}{2}} k_\alpha^{-1})^{-\frac{1}{a}} c_{\alpha,\beta}^{\frac{1}{a}-2} \cdot$$

. $$\Psi_a \left(\frac{1}{\varepsilon}\right) \left(\frac{R^\beta}{r^\alpha}\right)^{\frac{2}{\alpha-\beta}}) .$$

Here $q_\beta = p_\beta^{-\frac{a}{b-a}}$, $c_{\alpha,\beta} = q_\beta\, (2^{\frac{1}{2}} k_\alpha^{-1})^{\frac{b}{b-a}}$ and $c(\varepsilon) = \dfrac{1}{\int_0^\varepsilon \varphi(t)\, dt}$. Note that if $\varepsilon \to \infty$ then $c(\varepsilon) \to 2$ and $\Psi_a(\frac{1}{\varepsilon}) \to 1$.

We prove now a stronger result:

(1.20) **Theorem.** *Let $\alpha\,, \beta$ be two real numbers such that $0 \le \beta < \alpha < \frac{1}{2}$. There exists a positive number $u_{\alpha,\beta} = \frac{1-2\alpha}{1-2\beta}$, such that, for every $u \in [0, u_{\alpha,\beta}[$, there exists $M_0(\alpha, \beta, u)$ and positive constants $k_i(\alpha, \beta, u)\,,\ i = 1, 2$, such that, for every $M \ge M_0$,*

$$(1.21) \quad \sup_{0<\delta\le 1} P(\,\|w.\|_\alpha > M\delta^u \,|\, \|w.\|_\beta < \delta) \le k_1 M^{\frac{2\beta}{\alpha-\beta}} \exp\left(-k_2\, M^{\frac{1-2\beta}{\alpha-\beta}}\right) .$$

Proof. First of all we take in the Corollary (1.16), $R = M\delta^u$ and $r = \delta$. So, for every $\delta \in]0,1]$,

$$P(\,\|w.\|_\alpha > M\delta^u \,|\, \|w.\|_\beta < \delta) \le c_{\alpha,\beta}\, M^{\frac{1-2b}{b-a}}\, \delta^{\frac{u(1-2b)-(1-2a)}{b-a}} \cdot$$

$$\exp\left(-c'_{\alpha,\beta}\, M^{\frac{2b}{b-a}} \delta^{2\frac{ub-a}{b-a}}\right) .$$

It is clear that, when $M \ge \left(\frac{2}{c'_{\alpha,\beta}} \cdot \frac{u(1-2b)-(1-2a)}{b-a} \cdot \frac{b-a}{a-ub}\right)^{\frac{b-a}{2b}}$ the right hand side of the last inequality is an increasing function of δ, for $\delta \in]0,1]$. So,

$$\sup_{0<\delta\le 1} P(\,\|w.\|_\alpha > M\delta^u \,|\, \|w.\|_\beta < \delta) \le c_{\alpha,\beta}\, M^{\frac{1-2b}{b-a}} \exp\left(-c'_{\alpha,\beta}\, M^{\frac{2b}{b-a}}\right) ,$$

namely the conclusion.

q.e.d.

2. Hölder balls of different exponent are positively correlated

We show here that the conjecture on the correlation inequality is true for Hölder balls. We denote $B_\alpha(\rho) = \{\,\|w.\|_\alpha \le \rho\,\}$ and $B'_\alpha(\rho) = \{\,\|w.\|'_\alpha \le \rho\,\}$.

(2.1) **Theorem.** *If R is sufficient large and if r is fixed, then $B_\alpha(R)$ and $B_\beta(r)$ are positively correlated.*

Proof. We proved in Corollary (1.16), for example when $r = 1$, that, for large R,

$$(2.2) \qquad P(\,B_\alpha(R)^c \,|\, B_\beta(1)\,) \le c_{\alpha,\beta} \exp\left(-c'_{\alpha,\beta}\, R^{\frac{1-2\beta}{\alpha-\beta}}\right) ,$$

for every $0 \le \beta < \alpha < \frac{1}{2}$. We compare this estimate with the classical gaussian estimate, for large R,

$$(2.3) \qquad P(\,\|w.\|_\alpha > R\,) \le \exp\left(-c_\alpha\, R^2\right)$$

(see [BA-Le] or [B-BA-K] for other consequences of this inequality).

By large deviations principle we obtain in fact,

$$P(B_\alpha(R)^c) \sim e^{-c_\alpha R^2} ,$$

provided R is sufficiently large. Therefore, by (2.2), for large R,

$$P(B_\alpha(R) \mid B_\beta(1)) \geq P(B_\alpha(R)) . \tag{2.4}$$

So, in this particular case, the general conjecture is valid: the two symmetric convex sets $B_\alpha(R)$ and $B_\beta(1)$ are positively correlated, for large R.

q.e.d.

(2.5) *Remark.* We see that, for any $R, r > 0$, the pairs of balls $(B'_\alpha(R), B'_\beta(r))$ and $(B'_\alpha(R), B_\beta(r))$ are positively correlated. Indeed, by (1.3),

$$B'_\alpha(R) = \cap_{m\geq 1} \left(|g_m| \leq R m^{\frac{1}{2}-\alpha} \right) = \cap_{m\geq 1} S_m ,$$

so, it is an intersection of independent symmetric strips. Then, with the same argument as in the proof of (1.9), we get, for any convex symmetric C,

$$P(C \cap B'_\alpha(R)) = P\left(C \cap \cap_{m\geq 1} S_m\right) \geq$$

$$P\left(C \cap \cap_{m\geq 2} S_m\right) P(S_1) \geq \ldots \geq P(C) \prod_{m\geq 1} P(S_m) =$$

$$P(C)\, P\left(\cap_{m\geq 1} S_m\right) = P(C)\, P(B'_\alpha(R)) .$$

Here we used the independence of S_m. The conclusion is obtained taking $C = B'_\beta(r)$ or $C = B_\beta(r)$.

3. Conditional tails for oscillations of stochastic integrals

We shall estimate the Hölder norm of some stochastic integrals. Let $X_j(t,x)$, $j = 1, \ldots, m$, $X_0(t,x)$ be smooth vector fields on $\mathbb{R}^{d+1}$. Denote $(B^1, \ldots, B^m)$ a m-dimensional Brownian motion. Let P_x be the law of the diffusion (x_t), the solution of the Stratonovich equation

$$dx_t = \sum_{j=1}^{m} X_j(t, x_t) \circ dB_t^j + X_0(t, x_t)\, dt \,,\; x_0 = x . \tag{3.1}$$

Let us introduce the following class of stochastic processes:

(3.2) **Definition.** *For* $\alpha\,,\,\beta \in [0,\frac{1}{2}[$ *and* $u \in [0,1]$, *we shall denote by* $\mathcal{M}_u^{\alpha\,,\beta}$ *the set of stochastic processes* Y, *such that*

$$\lim_{M\uparrow\infty}\sup_{0<\delta\leq 1} P(\,\|Y.\|_\alpha > M\delta^u \mid \|B.\|_\beta < \delta\,) = 0\,. \tag{3.3}$$

Here and elsewhere $\|B.\|_\alpha = \max_{1\leq i\leq m}\|B.^i\|_\alpha$. We collect our results in the following:

(3.4) **Lemma.** *Let* $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ *be a smooth function and, for* $i,j \in \{1,\ldots,r\}$, *denote*

$$\eta_t^{ij} = \frac{1}{2}\int_0^t (B_s^i\,dB_s^j - B_s^j\,dB_s^i)\,,\; \xi_t^{ij} = \int_0^t B_s^i \circ dB_s^j\,. \tag{3.5}$$

Then,

(i) $B.^i \in \mathcal{M}_u^{\alpha\,,\beta}$, *for* $0\leq\beta<\alpha<\frac{1}{2}$ *and* $u \in \left[0,\frac{1-2\alpha}{1-2\beta}\right[$.

(ii) $\eta.^{ij} \in \mathcal{M}_u^{\alpha\,,0}$, *for* $\alpha \in [0,\frac{1}{2}[$ *and* $u \in [0,1]$.

(iii) $\xi.^{ij} \in \mathcal{M}_u^{\alpha\,,0}$, *for* $\alpha \in [0,\frac{1}{2}[$ *and* $u \in [0,1]$.

(iv) $\int_0^{\cdot} f(x_s)\,d\xi_s^{ij} \in \mathcal{M}_u^{\alpha\,,0}$, *for* $\alpha \in [0,\frac{1}{2}[$ *and* $u \in [0,1]$.

(v) $\int_0^{\cdot} f(x_s)\circ dB_s^i \in \mathcal{M}_u^{\alpha\,,0}$, *for* $\alpha \in [\,0\,,\,\frac{1}{2}[$ *and* $u \in [0,1-2\alpha[$.

Proof. Clearly, (i) is proved in the Theorem (1.20).

(ii) We proceed as in [S-V]. There exists a one dimensional Brownian motion w, such that, when $i\neq j$,

$$\eta_t^{ij} = w(a(t))\,,\; a(t) = \frac{1}{4}\int_0^t ((B_s^i)^2 + (B_s^j)^2)\,ds\,,$$

where w is independent of the process $(B_t^i)^2 + (B_t^j)^2$ and so, independent of $\|B.\|_0$. There exists a positive constant c, such that $\|a\|_0\,,\,\|a\|_1$ are bounded by $c\,\|B.\|_0$. Then we can write

$$P(\,\|\eta.^{ij}\|_\alpha > M\delta^u \,|\, \|B.\|_0 < \delta\,) =$$

$$P(\,\|B.\|_0 < \delta\,)^{-1} \cdot P(\,\|w(a(\cdot))\|_\alpha > M\delta^u\,,\ \|B.\|_0 < \delta\,)\,.$$

If z is α-Hölder, $\tilde{z}$ is β-Hölder then $z \circ \tilde{z}$ is $\alpha\beta$-Hölder and

$$\|z \circ \tilde{z}\|_{\alpha\beta} \le \|z\|_{\alpha\,,\,\|\tilde{z}\|_0} \cdot \|\tilde{z}\|_\beta^\alpha\,.$$

Here and elsewhere $\|\cdot\|_{\alpha,T}$ denotes the Hölder norm on $[0,T\,]$.

So,

$$\|w(a(\cdot))\|_\alpha \le \|w\|_{\alpha\,,\,\|a\|_0} \cdot \|a\|_1^\alpha\,.$$

Therefore

$$P(\,\|w(a(\cdot))\|_\alpha > M\delta^u\,,\ \|B.\|_0 < \delta\,) \le$$
$$P(\,\|w\|_{\alpha\,,\,c\,\|B.\|_0^2}\, c\,\|B.\|_0^{2\alpha} > M\delta^u\,,\ \|B.\|_0 < \delta\,)\,.$$

A scaling in Hölder norm shows that $\|w\|_{\alpha,\tau^2}$ and $\tau^{1-2\alpha}\,\|w\|_{\alpha,1}$ have the same law. Then we can write

$$P(\,\|\eta.^{ij}\|_\alpha > M\delta^u \,|\, \|B.\|_0 < \delta\,) \le$$

$$P(\,\|B.\|_0 < \delta\,)^{-1} \cdot P(\,\|w.\|_\alpha\, c\,\|B.\|_0^{1-2\alpha}\|B.\|_0^{2\alpha} > M\delta^u\,,\ \|B.\|_0 < \delta\,)\,.$$

Finally,

$$P(\,\|\eta.^{ij}\|_\alpha > M\delta^u \,|\, \|B.\|_0 < \delta\,) \le P(\,\|w.\|_\alpha\, c\,\delta > M\delta^u) \le \exp\left(-\frac{c_\alpha\, M^2}{\delta^{2(1-u)}}\right)\,,$$

by the independence of w and $\|B.\|_0$, and by the gaussian inequality (2.3).

(iii) We note another trivial inequality: if $z\,,\,\tilde{z}$ are α-Hölder then $z\,\tilde{z}$ is α-Hölder and

$$\|z\,\tilde{z}\|_\alpha \le \|z\|_\alpha\,\|\tilde{z}\|_0 + \|z\|_0\,\|\tilde{z}\|_\alpha\,.$$

In particular

$$\|B.^i B.^j\|_\alpha \le 2\,\|B.\|_0\,\|B.\|_\alpha\,.$$

But

$$P(\,\|B.\|_0\,\|B.\|_\alpha > M\delta^u \,|\, \|B.\|_0 < \delta\,) = P(\,\|B.\|_\alpha > M\delta^{u-1} \,|\, \|B.\|_0 < \delta\,)\,.$$

The conclusion follows at once from (i), (ii) and

$$\|\xi.^{ij}\|_\alpha \le \|\eta.^{ij}\|_\alpha + \frac{1}{2}\,\|B.^i B.^j\|_\alpha \le \|\eta.^{ij}\|_\alpha + \|B.\|_0\,\|B.\|_\alpha\,.$$

(iv) We apply Ito's formula several times (using the usual convention that repeated indices are summed):

$$\int_0^t f(x_s)\,d\xi_s^{ij} = f(x_t)\,\xi_t^{ij} - \int_0^t f_l(x_s)\,X_k^l(x_s)\,\xi_s^{ij}\,dB_s^k -$$

$$\int_0^t (L_s f)\,(x_s)\,\xi_s^{ij}\,ds - \int_0^t f_l(x_s)\,X_j^l(x_s)\,B_s^i\,ds = I_1 + I_2 + I_3 + I_4\,.$$

Here L_t is the generator of the diffusion (x_t) and X_j^l denotes the l component of X_j. It is sufficient to verify (iv) for each I_i , $i = 1, 2, 3, 4$. We readily see that

(a) $$I_3\,,\ I_4 \in\ \mathcal{M}_u^{\alpha\,,\,0}\,,$$

because

$$\|I_4\| \le c\,\|B.\|_0 \quad \text{and} \quad \|I_3\| \le c\,\|\xi.^{ij}\|_0\,,$$

so, we consider only I_1 and I_2.

Firstly,

$$I_1 = f(x)\,\xi_t^{ij} + \left(\int_0^t (L_s f)\,(x_s)\,ds\right)\,\xi_t^{ij} + \left(\int_0^t f_l(x_s)\,X_k^l(x_s)\,\xi_s^{ij}\,dB_s^k\right)\,\xi_t^{ij} =$$

$$I_{10} + I_{11} + I_{12}\,.$$

Again

(b) $$I_{10}\,,\ I_{11} \in\ \mathcal{M}_u^{\alpha\,,\,0}\,,$$

because

$$\|I_{10}\|_\alpha = c\,\|\xi.^{ij}\|_\alpha\,,\ \|I_{11}\|_\alpha \le c\,\|\xi.^{ij}\|_\alpha\,.$$

Setting $\alpha_k = -f_l\,X_k^l$, $\alpha_{k,m} = \frac{\partial \alpha_k}{\partial x^m}$ we can write

$$I_{12} = -\alpha_k(x_t)\,B_t^k\,\xi_t^{ij} - \left(\int_0^t B_s^k(L_s\alpha_k)\,(x_s)\,ds\right)\,\xi_t^{ij} -$$

$$\left(\int_0^t B_s^k\,\alpha_{k,m}(x_s)X_n^m(x_s)\,dB_s^n\right)\,\xi_t^{ij}+\left(\int_0^t(\alpha_{k,p})^2(x_s)\,(X_k^p)^2(x_s)\,ds\right)\,\xi_t^{ij}=$$
$$I_{121}+I_{122}+I_{123}+I_{124}\,.$$

There is no problem to see that

$$\|I_{122}\|_\alpha\le c\,\|B.\|_0\,\|\xi.^{ij}\|_\alpha\ \text{ and }\ \|I_{124}\|_\alpha\le c\,\|\xi.^{ij}\|_\alpha$$

and so,

(c) $$I_{122}\,,\,I_{124}\in\,\mathcal{M}_u^{\alpha\,,\,0}\,.$$

There exists a one-dimensional Brownian motion w, such that

$$I_{123}=w(a(t))\,\xi_t^{ij}\ \ ,\ a(t)=\int_0^t(\alpha_{k,m}\alpha_{k',m'}a^{mm'})(x_s)B_s^kB_s^{k'}\,ds\,,$$

where $a^{ij}=\sum_{k=1}^m X_k^i\,X_k^j$. We obtain

$$P(\,\|I_{123}\|_\alpha>M\delta^u\,|\,\|B.\|_0<\delta\,)\le P(\,\|\xi.^{ij}\|_\alpha>M^{\frac12}\delta\,|\,\|B.\|_0<\delta\,)+$$
$$P(\,\|w.\|_{\alpha\,,\,c\,\|B.\|_0^2}\,c\,\|B.\|_0^{2\alpha}\,\|\xi.^{ij}\|_\alpha>M\delta^u\,,\,\|\xi.^{ij}\|_\alpha\le M^{\frac12}\delta\,|\,\|B.\|_0<\delta\,)\,.$$

By (iii), we have to consider only the second term:

$$P(\,\|w.\|_\alpha>c\,M^{\frac12}\delta^{u-2})\cdot P(\,\|B.\|_0<\delta\,)^{-1}\le\exp\left(-\frac{c_\alpha\,M}{\delta^{2(2-u)}}+\frac{c}{\delta^2}\right)\,.$$

This yields

(d) $$I_{123}\in\,\mathcal{M}_u^{\alpha\,,\,0}\,.$$

Then,

$$I_2=\int_0^t\alpha_k(x_s)\,\xi_s^{ij}\,dB_s^k=$$
$$\alpha_k(x_t)\,\xi_t^{ij}\,B_t^k-\int_0^t\alpha_{k,l}(x_s)\,(X_m^l)(x_s)\,\xi_s^{ij}\,B_s^k\,dB_s^m-\int_0^t(L_s\alpha_k)\,(x_s)\,\xi_s^{ij}\,B_s^k\,ds$$
$$-\int_0^t\alpha_k(x_s)\,B_s^k\,d\xi_s^{ij}-\int_0^t\alpha_j(x_s)\,B_s^i\,ds-\int_0^t\xi_s^{ij}\,\alpha_{k,l}(x_s)\,X_m^l(x_s)\,\delta^{km}\,ds-$$

$$\int_0^t B_s^k\, \alpha_{k,l}(x_s)\, X_j^l(x_s)\, B_s^i\, ds = J_1 + \cdots + J_7\,.$$

Clearly,

(e) $$I_{121} + J_1 = 0$$

and

$$\|J_3\|_\alpha \le c\,\|B.\|_0\,\|\xi.^{ij}\|_0\ ,\ \|J_5\|_\alpha \le c\,\|B.\|_0\,,$$
$$\|J_6\|_\alpha \le c\,\|\xi.^{ij}\|_0\ ,\ \|J_7\|_\alpha \le c\,\|B.\|_0^2\,.$$

So,

(f) $$J_3\,,\ J_5\,,\ J_6\,,\ J_7 \in \mathcal{M}_u^{\alpha\,,\,0}\,.$$

By the same reasoning,

$$J_2 = w(a(t))\ ,\ a(t) = \int_0^t (\xi_s^{ij})^2 (\alpha_{k,l}\,\alpha_{k',l'}\,a^{ll'})(x_s)\, B_s^k\, B_s^{k'}\, ds\,,$$

so, it suffices to estimate

$$P(\,\|J_2\|_\alpha > M\delta^u\,,\ \|\xi.^{ij}\|_0 \le M^{\frac12}\delta \,|\, \|B.\|_0 < \delta\,) \le$$
$$P(\,\|w.\|_{\alpha\,,\,c\,\|\xi.^{ij}\|_0^2\,\|B.\|_0^2}\, c\,\|\xi.^{ij}\|_0^{2\alpha}\,\|B.\|_0^{2\alpha} > M\delta^u\,,\ \|\xi.^{ij}\|_0 < M^{\frac12}\delta \,|\, \|B.\|_0 < \delta\,)$$
$$\le P(\,\|w.\|_\alpha > c\,M^{\frac12}\delta^{u-2}) \cdot P(\,\|B.\|_0 < \delta\,)^{-1} \le \exp\left(-\frac{c_\alpha\, M}{\delta^{2(2-u)}} + \frac{c}{\delta^2}\right)\,.$$

Again

(g) $$J_2 \in \mathcal{M}_u^{\alpha\,,\,0}\,.$$

Finally we have to study the martingale part of J_4, the bounded variation being obviously controlled. We can write as above,

$$\int_0^t \alpha_k(x_s)\, B_s^k\, B_s^i\, dB_s^j = w(a(t))\ ,\ a(t) = \int_0^t \alpha_k^2(x_s)\,(B_s^k\, B_s^i)^2\, ds\,.$$

Obviously,

$$P(\,\|\int_0^\cdot \alpha_k(x_s)\, B_s^k\, B_s^i\, dB_s^j\|_\alpha > M\delta^u \,|\, \|B.\|_0 < \delta\,) \le$$

$$P(\,\|w.\|_\alpha > c\,M\delta^{u-2})\cdot P(\,\|B.\|_0 < \delta\,)^{-1} \le \exp{(-\frac{c_\alpha\,M^2}{\delta^{2(2-u)}} + \frac{c}{\delta^2})}\,.$$

So,

(h) $$J_4 \in \mathcal{M}_u^{\alpha\,,0}\,.$$

Using formulas (a)-(h) we can conclude that $\int_0^{\cdot} f(x_s)\,d\xi_s^{ij} \in \mathcal{M}_u^{\alpha\,,0}$.

(v) We use the same idea, namely we shall apply Ito's formula several times. Firstly, denoting $\frac{\partial f}{\partial x^l} = f_l$,

$$\int_0^t f(x_s)\,dB_s^i = f(x)\,B_t^i + \int_0^t dB_s^i \int_0^s (L_u f)(x_u)\,du +$$

$$\int_0^t dB_s^i \int_0^s f_l(x_u) X_j^l(x_u)\,dB_u^j = S_1 + S_2 + S_3\,.$$

But $\|S_1\|_\alpha \le c\,\|B.\|_\alpha$ and

$$S_2 = B_t^i \int_0^t (L_s f)(x_s)\,ds - \int_0^t B_s^i\,(L_s f)(x_s)\,ds = S_{21} + S_{22}\,,$$

where $\|S_{21}\|_\alpha \le c\,\|B.\|_\alpha$ and $\|S_{22}\|_\alpha \le c\,\|B.\|_0$.

Clearly,

$$S_1\,,\;S_{21}\,,\;S_{22} \in \mathcal{M}_u^{\alpha\,,0}\,.$$

Then, with the same notation as in (iv),

$$S_3 = -B_t^i \int_0^t \alpha_j(x_s)\,dB_s^j + \int_0^t B_s^i\,\alpha_j(x_s)\,dB_s^j + \int_0^t \alpha_j(x_s)\,ds =$$

$$S_{31} + S_{32} + S_{33}\,.$$

By (iv), it is clear that

$$S_{32} = \int_0^t \alpha_j(x_s)\,d\xi_s^{ij} \in \mathcal{M}_u^{\alpha\,,0}\,,\ \text{if } i \ne j\,.$$

For $i = j$ we get a term with the same form as S_{33}, terms which are bounded in Hölder norm by a constant. To prove (v), it is sufficient to prove that $S_{31} \in \mathcal{M}_u^{\alpha\,,0}$. Note that

$$S_{31} = -B_t^i\,B_t^j\,\alpha_j(x) - B_t^i \int_0^t dB_s^j \int_0^s (L_u\alpha_j)(x_u)\,du -$$

$$B_t^i \int_0^t dB_s^j \int_0^s \alpha_{j,l}(x_u) X_k^l(x_u)\, dB_u^k = S_{311} + S_{312} + S_{313}\,.$$

But $\|S_{311}\|_\alpha \le c\,\|B.\|_0\,\|B.\|_\alpha$ and

$$S_{312} = B_t^i\, B_t^j \int_0^t (L_s\alpha_j)(x_s)\, ds - B_t^i \int_0^t B_s^j\,(L_s\alpha_j)(x_s)\, ds = S_{3121} + S_{3122}\,,$$

where $\|S_{3121}\|_\alpha \le c\,\|B.\|_\alpha\,\|B.\|_0$ and $\|S_{3122}\|_\alpha \le c\,\|B.\|_\alpha\,\|B.\|_0$.

Again

$$S_{311}\,,\ S_{3121}\,,\ S_{3122} \in \mathcal{M}_u^{\alpha\,,0}\,.$$

We denote $\beta_k(x) = -\alpha_{j,l}(x) X_k^l(x)$. Then,

$$S_{313} = B_t^i\, B_t^j \int_0^t \beta_k(x_s)\, dB_s^k - B_t^i \int_0^t B_s^j\, \beta_k(x_s)\, dB_s^k - B_t^i \int_0^t \beta_k(x_s)\, ds =$$

$$S_{3131} + S_{3132} + S_{3133}\,.$$

Arguing as for S_{32}, S_{33} we see that $S_{3132} = -B_t^i \int_0^t \beta_k(x_s)\, d\xi_s^{jk}$, $j \ne k$ and S_{3133} are in $\mathcal{M}_u^{\alpha\,,0}$. We repeat with S_{3131} the computations which we already performed for S_{31} and we see that (with clear notations)

$$S_{31311}\,,\ S_{313121}\,,\ S_{313122}\,,\ S_{313133} \in \mathcal{M}_u^{\alpha\,,0}\,.$$

Then $S_{313132} = B_t^i B_t^j \int_0^t \gamma_l(x_s)\, d\xi_s^{kl}$, $l \ne k$, where $\gamma_l = \beta_m(x) X_l^m(x)$, so S_{313132} satisfies (v) as above.

To control the Hölder norm of S_{313131} we can write

$$S_{313131} = B_t^i B_t^j B_t^k \int_0^t \gamma_l(x_s)\, dB_s^l = B_t^i B_t^j B_t^k\, w(a(t))\ ,\ a(t) = \int_0^t \gamma_l^2(x_s)\, ds\,,$$

where w is a one-dimensional Brownian motion. So,

$$P(\,\|S_{313131}\|_\alpha > M\delta^u \,|\, \|B.\|_0 < \delta\,) \le P(\,\|B.\|_\alpha > M^{\frac{1}{2}}\delta^{u-\frac{1}{2}} \,|\, \|B.\|_0 < \delta\,) +$$

$$P(\,\|w.\|_\alpha\, c\,\|B.\|_\alpha\,\|B.\|_0^2 > M\delta^u\,,\ \|B.\|_\alpha \le M^{\frac{1}{2}}\delta^{u-\frac{1}{2}} \,|\, \|B.\|_0 < \delta\,) \le$$

$$P(\,\|B.\|_\alpha > M^{\frac{1}{2}}\delta^{u-\frac{1}{2}} \,|\, \|B.\|_0 < \delta\,) + \exp\left(-\frac{c_\alpha\, M}{\delta^3} + \frac{c}{\delta^2}\right)\,.$$

From this we see that S_{313131} satisfies (v).

The proof of the lemma is complete.

4. Support theorem in Hölder norm

Now we are able to extend the support theorem of Stroock-Varadhan for α-Hölder topology. Let us denote by Φ_x the mapping which associates to $h \in L^2 = L^2([0,1], \mathbb{R}^m)$ the solution of the differential equation

$$dy_t = \sum_{j=1}^{m} X_j(t, y_t)\, h_t^j\, dt + X_0(t, y_t)\, dt\,,\; y_0 = x\,. \tag{4.1}$$

(4.2) **Theorem.** *Let $\alpha \in [0, \frac{1}{2}[$. For the $\|\cdot\|_\alpha$-topology, the support of the probability P_x coincide with the closure of $\Phi_x(L^2)$, that is,*

$$\operatorname{supp}_\alpha(P_x) = \overline{\Phi_x(L^2)}^{\,\alpha}\,. \tag{4.3}$$

Proof. To begin with, we note that, for every $\varepsilon > 0$ and $\delta = (\frac{\varepsilon}{2^n})^{\frac{1}{u}}$, $u \in]0, 1-2\alpha[$, $n > 0$ integer,

$$P\left(\left\|\int_0^{\cdot} X_k(s, x_s) \circ dB_s^k\right\|_\alpha > \varepsilon \,|\, \|B.\|_0 < \delta\right) =$$

$$P\left(\left\|\int_0^{\cdot} X_k(s, x_s) \circ dB_s^k\right\|_\alpha > 2^n \delta^u \,|\, \|B.\|_0 < \delta\right) \leq$$

$$\sup_{0<\eta\leq 1} P\left(\left\|\int_0^{\cdot} X_k(s, x_s) \circ dB_s^k\right\|_\alpha > 2^n \eta^u \,|\, \|B.\|_0 < \eta\right)\,.$$

Letting $n \uparrow \infty$, by (v) of the Lemma (3.4), we obtain, for every $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} P\left(\left\|\int_0^{\cdot} X_k(s, x_s) \circ dB_s^k\right\|_\alpha > \varepsilon \,|\, \|B.\|_0 < \delta\right) = 0\,. \tag{4.4}$$

Then we prove that, for every $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} P(\, \|x. - \Phi_x(0)\|_\alpha < \varepsilon \,|\, \|B.\|_0 < \delta) = 1\,, \tag{4.5}$$

using (4.4) and the following variant of Gronwall's lemma:

(4.6) **Lemma.** *For m and l two functions, put*

$$z_t = z + m(t) + \int_0^t l(z_s)\, ds\,,\; \tilde{z}_t = z + \int_0^t l(\tilde{z}_s)\, ds\,.$$

Suppose that $\|m\|_\alpha \leq \eta$, $m(0) = 0$ and that l is a Lipschitz continuous function with constant L. Then

$$\|z - \tilde{z}\|_\alpha \leq (1 + L)\, e^L\, \eta\,.$$

Proof. By Gronwall's lemma we can immediately write

$$\|z - \tilde{z}\|_0 \leq \eta\, e^L\,.$$

Then,

$$\|z - \tilde{z}\|_{\alpha\,,\,t} \leq \eta + \| \int_0^{\cdot} (l(z_u) - l(\tilde{z}_u))\, du\|_{\alpha\,,\,t} \leq$$

$$\eta + \max_{0\,\leq\, p\,<\,q\,\leq\, t} \frac{L}{|p-q|^\alpha} | \int_q^p |z_u - \tilde{z}_u|\, du\,| \leq$$

$$\eta + \max_{0\,\leq\, p\,<\,q\,\leq\, t} \frac{L}{|p-q|^\alpha} \left| \int_q^p (\,|z_q - \tilde{z}_q| + |u-q|^\alpha \|z - \tilde{z}\|_{\alpha\,,\,u}\,)\, du \right| \leq$$

$$\eta + L\, \|z - \tilde{z}\|_0 + L \int_0^t \|z - \tilde{z}\|_{\alpha\,,\,u}\, du\,.$$

Gronwall's lemma ends up the proof of the Lemma (4.6).

We apply this with $z = \Phi_x(B.)$, $\tilde{z} = \Phi_x(0)$, $m(t) = \int_0^t X_k(s, x_s) \circ dB_s^k$ and $l(x_s) = X_0(s, x_s)$. So, there exists a positive constant K, such that

$$\|\Phi_x(B.) - \Phi_x(0)\,\|_\alpha < K\,\varepsilon\,,$$

provided

$$\left\| \int_0^{\cdot} X_k(s, x_s) \circ dB_s^k \right\|_\alpha \leq \varepsilon\,.$$

Thus we obtain

$$P(\,\|x. - \Phi_x(0)\|_\alpha > \varepsilon \,|\, \|B.\|_0 < \delta\,) =$$

$$P\left((\,\|x. - \Phi_x(0)\|_\alpha > \varepsilon\,) \cap \left(\left\| \int_0^{\cdot} X_k(s, x_s) \circ dB_s^k \right\|_\alpha > \frac{\varepsilon}{K} \right) \,|\, \|B.\|_0 < \delta \right)$$

$$\leq P\left(\left\| \int_0^{\cdot} X_k(s, x_s) \circ dB_s^k \right\|_\alpha > \frac{\varepsilon}{K} \,\mid\, \|B.\|_0 < \delta \right)\,.$$

Now (4.5) is a clear consequence of (4.4).

Finally, Girsanov's formula gives, for any $h \in L^2$ and $\varepsilon > 0$,

$$(4.7) \qquad \lim_{\delta \downarrow 0} P(\, \|\Phi_x(B.) - \Phi_x(h.)\,\|_\alpha < \varepsilon \mid \|B. - h.\|_0 < \delta\,) = 1$$

(as in [S-V], p. 353). But, (4.7) implies

$$(4.8) \qquad P(\, \|\Phi_x(B.) - \Phi_x(h.)\,\|_\alpha < \varepsilon) > 0\,, \text{ for every } \varepsilon > 0\,.$$

and, consequently, we obtain the inclusion

$$(4.9) \qquad \mathrm{supp}_\alpha(P_x) \supseteq \overline{\Phi_x(L^2)}^{\,\alpha}\,.$$

The converse inclusion is easily obtained using the polygonal approximation of the Brownian motion. For each $n \geq 0$ and $t \geq 0$, we consider

$$t_n = \frac{[2^n]}{2^n}\,,\ t_n^+ = \frac{[2^n]+1}{2^n}\,,\ \dot{B}_t^{(n)} = 2^n(B_{t_n^+} - B_{t_n})\,.$$

Let $(x_t^{(n)})$ be the solution of the equation (4.1) with $\dot{B}_t^{(n)k}$ instead h_t^k. If we denote $P_x^{(n)}$ the law of this solution, it is obvious that

$$x.^{(n)} \in \Phi_x(L^2) \ \text{ and } \ P_x^{(n)}(\,\overline{\Phi_x(L^2)}^{\,\alpha}\,) = 1\,.$$

It suffices to show that P_x is the weak limit of $(P_x^{(n)})$ or, that $(P_x^{(n)})$ is relatively weakly compact with respect to $\|\cdot\|_\alpha$-topology. By classical estimates, for every $p \geq 0$, there exists a positive constant c_p, such that, for every positive integer n and for every $s, t \in [\,0\,,\,1\,]$,

$$E|x_t^{(n)} - x_s^{(n)}|^{2p} \leq c_p|t-s|^p$$

(see for instance [Bi], p. 40). It is easy to see that

$$\sup_n E\left(\|x.^{(n)}\|_{\alpha'}^{2p}\right) < c\,, \text{ if } \alpha' < \frac{p-1}{2p}\,.$$

If we choose p large enough so that $\alpha < \frac{p-1}{2p}$, and if $\alpha' \in]\alpha, \frac{p-1}{2p}[$, it is then clear that the set $K(c) = \{\, z : \|z\|_{\alpha'} < c\,\}$ is compact in $\|\cdot\|_\alpha$-topology, and that, for every $\varepsilon > 0$, there exists a positive constant c_ε, such that,

$$\sup_n P_x^{(n)}(\,K(c_\varepsilon)\,) < \varepsilon\,.$$

So, $(P_x^{(n)})$ is tight.

The proof of the Theorem (4.2) is complete.

Appendix

We give now another proof of a variant of (1.10) (or (1.19)), when $\beta = 0$, which does not require the use of Ciesielski's theorem (that is (1.4) and (1.5)) nor the correlation inequality.

(A.1) **Theorem.** *Let (r, R) be a couple of real positive numbers. For every $a' < a$ and $b' > b$, there exists a constant c, such that, if $\frac{R^{a'}}{r^{b'}} > c$, then*

(A.2)
$$P\,(\,(\,\|w\|_\alpha > R\,)\,\cap\,(\,\|w\|_\beta < r\,)\,) \le \exp\left(-\frac{1}{2}\,\frac{R^{\frac{1-2\beta}{\alpha-\beta}}}{r^{\frac{1-2\alpha}{\alpha-\beta}}}\right),$$

for $0 \le \beta < \alpha < \frac{1}{2}$.

Proof. Put
$$\eta = \left(\frac{r}{R}\right)^{\frac{1}{\alpha-\beta}}.$$

Then, if $\|w\|_\beta < r$,
$$\sup_{s<t,t-s>\eta} \frac{|w_t - w_s|}{|t-s|^\alpha} \le R\,.$$

Thus we obtain
$$(\,(\,\|w\|_\alpha > R\,)\,\cap\,(\,\|w\|_\beta < r\,)\,) \subset$$
$$\left(\left(\sup_{s<t\le s+\eta} \frac{|w_t - w_s|}{|t-s|^\alpha} \ge R\right) \cap (\,\sup_t |w_t| < r\,)\right) \subset$$
$$\left(\sup_{s<t\le s+\eta} \frac{|w_t - w_s|}{|t-s|^\alpha} \ge R\right) = (\,\sup_{v\in D} |X_v^\alpha| \ge R\,)\,.$$

Here $v = (s,t)$, $D = \{v : s < t \le s + \eta\}$ and $X_v^\alpha = \frac{w_t - w_s}{|t-s|^\alpha}$ is a two-parameter gaussian variable.

Now, we have

$$P\,(\,(\,\|w\|_\alpha > R\,)\,\cap\,(\,\|w\|_\beta < r\,)\,) \le$$

$$P(\sup_{v\in D}|X_v^\alpha|\geq R\,)\leq \exp\left(-\frac{(R-M_\alpha)^2}{2X_\alpha^2}\right)\ ,$$

where the last inequality is valid when $R\geq M_\alpha$ (see [L-T], p. 57). Here

$$0<M_\alpha=E(\sup_{v\in D}|X_v^\alpha|\,)\leq E(\,\|w\|_\alpha\,)<\infty$$

and

$$X_\alpha^2=\sup_{v\in D}E(\,(X_v^\alpha)^2\,)=\eta^{1-2\alpha}\,.$$

So, we get

$$P\,(\,(\,\|w\|_\alpha>R\,)\cap(\,\|w\|_\beta<r\,)\,)\leq$$

$$\exp\left(-\frac{R^2}{2\eta^{1-2\alpha}}\right)=\exp\left(-\frac{1}{2}\,\frac{R^{\frac{1-2\beta}{\alpha-\beta}}}{r^{\frac{1-2\alpha}{\alpha-\beta}}}\right)\ ,\ \beta\geq 0\,.$$

The restriction $R\geq M_\alpha$ may be weakened as follows. Take $\alpha'>\alpha$ and write

$$\left(\sup_{s<t\leq s+\eta}\frac{|w_t-w_s|}{|t-s|^\alpha}\geq R\right)=\left(\sup_{s<t\leq s+\eta}\frac{|w_t-w_s|}{|t-s|^{\alpha'}}\cdot|t-s|^{\alpha'-\alpha}\geq R\right)\subset$$

$$\left(\sup_{s<t\leq s+\eta}\frac{|w_t-w_s|}{|t-s|^{\alpha'}}\geq R\,\eta^{\alpha-\alpha'}\right)=\left(\sup_{s<t\leq s+\eta}\frac{|w_t-w_s|}{|t-s|^{\alpha'}}\geq\frac{R^{\frac{\alpha'-\beta}{\alpha-\beta}}}{r^{\frac{\alpha'-\alpha}{\alpha-\beta}}}\right)\ .$$

Now, we need only

$$\frac{R^{\frac{\alpha'-\beta}{\alpha-\beta}}}{r^{\frac{\alpha'-\alpha}{\alpha-\beta}}}>E(\,\|w\|_{\alpha'}\,)=M_{\alpha'}\ ,$$

and the proof of the theorem is complete.

q.e.d.

Clearly, the Theorem (A.1) implies that

$$P(\,\|w.\|_\alpha>R\,|\,\|w.\|_0<r\,)=\frac{P\,(\,(\,\|w\|_\alpha>R\,)\bigcap(\,\|w\|_0<r\,))}{P(\,\|w\|_0<r\,)}\leq$$

$$\exp\left(-\frac{1}{2}\,\frac{R^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}-2}}\right)\,\exp\left(\frac{\pi^2}{8}\cdot\frac{1}{r^2}\right)\ .$$

If r is small we need the condition $\alpha < \frac{1}{4}$ for an interesting estimate.

References:

.[B-BA-K] Baldi, P., Ben Arous, G., Kerkyacharian, G.: Large deviations and Strassen law in Hölder norm, Stoch. Proc. Appl. **42**, pp. 171-180 (1992)

.[B-R] Baldi, P., Roynette, B.: Some exact equivalents for the Brownian motion in Hölder norm, Probab. Th. Rel. Fields **93**, pp. 457-484 (1992)

.[BA-G] Ben Arous, G., Gradinaru, M.: Normes hölderiennes et support des diffusions, C. R. Acad. Sci. Paris **316**, pp. 283-286 (1993)

.[BA-G-L] Ben Arous, G., Gradinaru, M., Ledoux, M.: Hölder norms and the support theorem for diffusions, Ann. Inst. "H. Poincaré" **30**, pp. 415-436 (1994)

.[BA-L] Ben Arous, G., Ledoux, M.: Grandes déviations de Freidlin-Wentzell en norme hölderienne, In: Azéma, J., Meyer, P.-A., Yor, M. (eds.) Seminaire de Probabilités XXVIII (Lect. Notes Math. vol. 1583, pp. 293-299), Berlin Heidelberg New York: Springer 1994

.[BA-Le] Ben Arous, G., Léandre, R.: Décroissance exponentielle du noyau de la chaleur sur la diagonale I,II, Probab. Th. Rel. Fields **90**, pp. 175-202, 377-402 (1991)

.[Bi] Bismut, J.M.: Mécanique aléatoire, (Lect. Notes Math. vol. 866) Berlin Heidelberg New York: Springer 1981

.[C] Ciesielski, Z.: On the isomorphisms of the spaces H_α and m, Bull. Acad. Pol. Sci. **8**, pp. 217-222 (1960)

.[DG-E-...] Das Gupta, S., Eaton, M.L., Olkin, I., Perlman, M., Savage, L.J., Sobel, M.: Inequalities on the probability content of convex regions for elliptically contoured distributions, Proceedings of the Sixth Berkeley Symposium of Math. Statist. Prob. II 1970, pp. 241-267, University of California Press, Berkeley 1972

.[L-T] Ledoux, M., Talagrand, M.: Probability in Banach spaces, Berlin Heidelberg New York : Springer 1991

.[M-S] Millet, A., Sanz-Solé, M.: A simple proof of the support theorem for diffusion processes, In: Azéma, J., Meyer, P.-A., Yor, M. (eds.) Seminaire de Probabilités XXVIII (Lect. Notes Math. vol. 1583, pp. 36-48), Berlin Heidelberg New York: Springer 1994

.[P] Pitt, L.: A Gaussian correlation inequality for symmetric convex sets, Ann. Probab. **5**, pp. 470-474 (1977)

.[Sc] Scott, A.: A note on conservative confidence regions for the mean value of multivariate normal, Ann. Math. Stat. **38**, pp. 278-280 (1967)

.[Si] Sidak, Z.: Rectangular confidence regions for the means of multivariate normal distributions, J. Amer. Stat. Assoc. **62**, pp. 626-633 (1967)

.[S-V] Stroock, D.W., Varadhan, S.R.S.: On the support of diffusion processes with applications to the strong maximum principle, Proceedings of Sixth Berkeley Symposium of Math. Statist. Prob. III 1970, pp. 333-359, University of California Press, Berkeley 1972

.[S-Z] Shepp, L.A., Zeitouni, O.: A note on conditional exponential moments and Onsager-Machlup functionals, Ann. Prob. **20**, pp. 652-654 (1992).

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Titre: Fonctions de Green et support de diffusions hypoelliptiques

Résumé:

La première partie contient une description précise de la singularité près de la diagonale de la fonction de Green associée à un opérateur hypoelliptique. L'approche est probabiliste et repose sur le développement de Taylor stochastique des trajectoires de la diffusion associée et sur les estimations à priori de la fonction de Green. On donne des exemples et des applications à la théorie du potentiel.

Dans la deuxième partie on étend le théorème de support de Stroock-Varadhan pour la norme hölderienne. L'outil central est l'estimation de la probabilité pour que le mouvement brownien ait une grande norme hölderienne, conditionnellement au fait qu'il ait une petite norme uniforme.

Mots clés: opérateur hypoelliptique - fonction de Green - diffusion dégénérée - développement de Taylor stochastique - capacité - norme hölderienne - théorème de support - inégalité de corrélation

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