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DISTRIBUTION TAILS FOR SOLUTIONS OF SDE DRIVEN BY AN ASYMMETRIC STABLE LÉVY PROCESS

BY

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Abstract. The behaviour of the tails of the invariant distribution for stochastic differential equations driven by an asymmetric stable Lévy process is obtained. One generalizes a result by Samorodnitski and Grigoriu [8] where the stable driven noise was supposed symmetric.

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1. INTRODUCTION

The goal of this paper is to extend a result obtained by Samorodnitski and Grigoriu in [8]. The authors consider the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}L_t - f(X_t)\mathrm{d}t, \quad X_0 = x, \tag{1.1}$$

where f is a quickly increasing to infinity function and L is a symmetric Lévy motion and they study the exact rate of decay of the tail probabilities of the random variables X_t, $t > 0$. The proof in [8] is technical and in Remark 3.2, p. 76, the authors conjecture that their main result remains true without the assumption of symmetry of the Lévy process. The present paper (Section 2) contains a proof of this conjecture and we reduce the technical difficulties announced in the cited remark by assuming that the Lévy process is α-stable. More precisely, we assume that X is a solution of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}\ell_t - f(X_t)\mathrm{d}t, \quad X_0 = x, \tag{1.2}$$

where ℓ is the asymmetric α-stable Lévy process having its Lévy measure given by

$$\nu(\mathrm{d}z) = |z|^{-1-\alpha}\left[a_- \mathbb{1}_{\{z<0\}} + a_+ \mathbb{1}_{\{z>0\}}\right] \mathrm{d}z. \tag{1.3}$$

Here $\alpha \in (0,2) \setminus \{1\}$, $a_+ \neq a_-$ and x is a real number.

Dynamics of some integrated processes driven by Lévy noises appears in financial mathematics models or in physics. Moreover, diffusions in heterogeneous materials or prices in finance could be modelled by using stochastic differential equations driven by asymmetric Lévy noises (see for instance [9]). In [3] a scaling limit of the position process whose speed satisfies a one-dimensional stochastic differential equation driven by a small α-stable Lévy process in a potential of the form a power function of exponent $\beta + 1$ was studied. Precisely, one considers the stochastic differential equation

$$\mathrm{d}v_t^\varepsilon = \varepsilon \mathrm{d}\ell_t - |v_t^\varepsilon|^\beta \mathrm{sgn}(v_t^\varepsilon)\mathrm{d}t, \quad v_0^\varepsilon = 0, \tag{1.4}$$

and assume that ℓ is an α-stable Lévy noise. It was proved that, when the driving noise ℓ is a symmetric stable process and by taking a natural scaling of the position process $x_t^\varepsilon = \int_0^t v_t^\varepsilon \mathrm{d}t$, there is convergence in distribution toward a Brownian motion. One can wonder if this still stay true when ℓ is an asymmetric α-stable Lévy noise. To get the limit in distribution as $\varepsilon \to 0$, of the position process one needs to know the exact rate of decay of the tail probabilities for the speed process (see also [2], §4, pp. 70-80).

Let us end this section by introducing some notations and by stating our results. In the following we will always assume that ℓ is the asymmetric α-stable

Lévy process having its Lévy measure given by (1.3), with $\alpha \in (0,2) \setminus \{1\}$, $a_+ \neq a_-$ and $a_+ \neq 0$ and $a_- \neq 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function with $f(0) = 0$ which is regularly varying at $+\infty$ with exponent $\beta > 1$: for all $a > 0$, $\lim_{x \to +\infty} \frac{f(ax)}{f(x)} = a^\beta$. The function f could be supposed equally regularly varying at $-\infty$ with exponent $\beta_1 > 1$, but one can only assume, that for all $x \;\; 1$, $f(-x) \;\; -\kappa x^{\beta_1}$, for some constants $\kappa > 0$ and $\beta - 1 > 1$ (see also Remark 5 and Step 9 in the proof of Theorem 1 below).

Recall that the process X satisfies

$$X_t = x + \ell_t - \int_0^t f(X_s)\mathrm{d}s,\; t \;\; 0. \tag{1.5}$$

Let us note that the existence and the uniqueness of a global solution for the equation (1.5) is justified in [8] for a general Lévy driven noise and it is a consequence of Theorem 6.2.11, p. 376 in [1] (see also Proposition 1.2.10, p. 27 in [2]). The statement of our main result is the following:

THEOREM 1. Assume furthermore that f is locally Lipschitz and denote, for all $u > 0$,

$$h(u) := \int_u^{+\infty} \frac{\nu((y, +\infty))}{f(y)} \mathrm{d}y\,. \tag{1.6}$$

Then

$$\lim_{u \to +\infty} \frac{\mathbb{P}_x(X_t > u)}{h(u)} = 1 \tag{1.7}$$

uniformly with respect to $x \in \mathbb{R}$ and $t \geqslant 1$.

As a consequence we obtain the behaviour of the tail for the invariant probability measure. According to Proposition 0.1, p. 604 in [5], and under the assumptions on the function f, the exponential ergodicity of the solution X of (1.1) is insured. Moreover its unique invariant probability measure, denoted by $m_{\alpha,\beta}$, satisfies

$$\forall x \in \mathbb{R}, \quad \|\mathbb{P}_x^t - m_{\alpha,\beta}\|_{\mathrm{TV}} = \mathrm{O}(\exp(-Ct)), \quad \text{as } t \to \infty, \tag{1.8}$$

where $\mathbb{P}_x^t$ is the distribution of X_t under $\mathbb{P}_x$ and $\|\cdot\|_{\mathrm{TV}}$ is the norm in total variation. Therefore letting t goes to infinity in Theorem 1, we get:

COROLLARY 2. Under the same assumptions as in Theorem 1, we have

$$\lim_{u\to+\infty}\frac{m_{\alpha,\beta}((u,+\infty))}{h(u)}=1. \tag{1.9}$$

2. PROOF OF THEOREM 1

We split the proof of Theorem 1 in several steps.

Step 1. Introduce, for $\sigma>0$ and for some $c>0$ to be chosen, the Lévy process $\ell^{(\sigma)}$ with the following small jumps prescribed by the Lévy measure

$$\nu^{(\sigma)}(\mathrm{d}z)=|z|^{-1-\alpha}\big[a_-\mathbb{1}_{\{z<-\sigma\}}+a_+\mathbb{1}_{\{z>c\sigma\}}\big]\,\mathrm{d}z. \tag{2.1}$$

The process $\ell^{(\sigma)}$ has a finite number of jumps on each finite interval of time. Denote by T_j the time when the j-th jump occurs (with the convention $T_0=0$) and by $W_j^{(\sigma)}$ its size. The random variables $(W_j^{(\sigma)})$ are i.i.d. We will choose the constant c such that, for all y and σ,

$$\mathbb{E}\big(W_1^{(\sigma)}\mathbb{1}_{\{-y\leqslant W_1^{(\sigma)}\leqslant cy\}}\big)=0.$$

Since the probability density of $W_1^{(\sigma)}$ is given by

$$z\mapsto\frac{1}{\lambda_\sigma}|z|^{-1-\alpha}\big[a_-\mathbb{1}_{\{z<-\sigma\}}+a_+\mathbb{1}_{\{z>c\sigma\}}\big]\,,\ \text{with } \lambda_\sigma:=\frac{\sigma^{-\alpha}}{\alpha}(a_-+a_+c^{-\alpha}), \tag{2.2}$$

we deduce that the only possible value of the constant is

$$c=\big(\frac{a_-}{a_+}\big)^{1/(1-\alpha)}. \tag{2.3}$$

Let us point out that, by the definition of $\nu^{(\sigma)}$, for $u>c\sigma>0$,

$$\nu^{(\sigma)}\big((u,+\infty)\big)=\nu\big((u,+\infty)\big)=:\rho(u). \tag{2.4}$$

Step 2. Let us denote

$$X_t^{(\sigma)} = x + \ell_t^{(\sigma)} - \int_0^t f(X_s^{(\sigma)})\,\mathrm{d}s,\ t \geqslant 0. \tag{2.5}$$

According to Theorem 19.25 in [4], p. 385, $X^{(\sigma)}$ converges in distribution to X, as σ tends to 0. To get (1.7) it is enough to prove that there exists σ_0, such that,

$$\Big|\frac{\mathbb{P}_x(X_t^{(\sigma)} > u)}{h(u)} - 1\Big| \leqslant \mathrm{o}(1), \quad \text{as } u \to +\infty, \tag{2.6}$$

uniformly in $x \in \mathbb{R}$, $\sigma \leqslant \sigma_0$ and $t \geqslant 1$.

Step 3. The ordinary differential equation

$$x(t) = x - \int^t f(x(s))\mathrm{d}s, \quad t \geqslant 0 \tag{2.7}$$

has a unique solution. As in [8], p. 93, we introduce, for all $u > 0$

$$g(u) := \int_u^{+\infty} \frac{1}{f(y)}\mathrm{d}y\,. \tag{2.8}$$

This function is clearly finite, non-negative, continuous and strictly decreasing for large u. Let us fix $1 \leqslant s \leqslant t$. It is no difficult to see that the solution of (2.7) satisfies $g(x(t)) = g(x(s)) + t - s$ and in particular, for any $u > 0$, if $x(t) > u$ then $g(u) > g(x(t)) \geqslant t - s$. We deduce that the solution of (2.7) on $[t - g(u), t]$ will end up, at time t, not higher than u.

At this level let us recall an important result from [8] (see Lemma 5.1, p. 94). Let $A > 0$ and denote by y, the solution of the deterministic equation (2.7) on each interval of the form (S_{i-1}, S_i) with $0 = S_0 < \cdots < S_n < A$ but with jumps at time S_i of a size j_i. More precisely

$$y'(t) = -f(y(t)), \quad \text{on } (S_{i-1}, S_i) \text{ and } \quad y(S_i) = y(S_i^-) + j_i, \quad y(0) = x. \tag{2.9}$$

As previously, it is not difficult to see that y satisfies $g(y(A)) = g(y(S_n)) + A - S_n$ and in particular, for any $u > 0$, if $y(A) > u$, then $A - S_n \leqslant g(u)$. Moreover, one can compare the solution x of (2.7) with y:

$$-\max_{k=1,\dots,n}\big(\sum_{i=k}^{n} j_i\big)_- \leqslant y(A)-x(A)\leqslant \max_{k=1,\dots,n}\big(\sum_{i=k}^{n} j_i\big)_+ .$$

In particular, if we set, for $a>0$, $N(a)=\sup\{i\leqslant n:\ j_i+\cdots+j_n>a\}$ (=0 if the set is empty), then

$$\text{for } t\in[S_{N(a)},A] \text{ such that } y(t)\leqslant b, \text{ we have } y(A)\leqslant a+b. \tag{2.10}$$

Step 4. For $t\geqslant 1$, denote by $N_t^{(\sigma)}$ the number of jumps of $\ell^{(\sigma)}$ during the interval $[0,t]$ and define, for all $a<0$ and $b>0$,

(2.11)

$$M_1^{(\sigma)}(a,b):=\sup\{j\leqslant N_t^{(\sigma)}: W_j^{(\sigma)}\notin[a,b]\},\quad \text{and } =0 \text{ if the set is empty.}$$

To simplify notations we will denote by $\tau_1:=T_{M_1^{(\sigma)}(-\varepsilon u,c\varepsilon u)}$ the time of the jump with index $M_1^{(\sigma)}(-\varepsilon u,c\varepsilon u)$. We can write

$$\begin{aligned}\mathbb{P}_x(X_t^{(\sigma)}>u)&=\mathbb{P}_x\left(X_t^{(\sigma)}>u,\,\tau_1<t-g(\delta u)\right)\\&\quad+\mathbb{P}_x\left(X_t^{(\sigma)}>u,\,\tau_1\in[t-g(\delta u),t]\right):=p_1(u)+p_2(u).\end{aligned}\tag{2.12}$$

Let us fix $s\leqslant t$ and for $\varepsilon,\gamma,\delta>0$ and $u>0$, introduce the event

$$A_{\varepsilon,\gamma,\delta,u,s}:=\left\{\sup_{\substack{1\leqslant i\leqslant N_t^{(\sigma)}\\ s-g(\delta u)\leqslant T_i\leqslant s}}\ \sum_{i\leqslant j\leqslant N_t^{(\sigma)}} W_j^{(\sigma)}\mathbb{1}_{\{-\varepsilon u\leqslant W_j^{(\sigma)}\leqslant c\varepsilon u\}}\geqslant \gamma u\right\}.\tag{2.13}$$

We can state the following lemma:

LEMMA 3. If $(1\vee c)\varepsilon\leqslant\gamma/4$ then there exist $u_0(\varepsilon,\gamma,\delta)$, σ_0 and a positive constant $C(\beta,\gamma)$ such that, for all $u\geqslant u_0(\varepsilon,\gamma,\delta)$ and $\sigma\leqslant\sigma_0$,

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s})\leqslant C(\varepsilon,\gamma)g(\delta u)\rho(u)^{\gamma/(4\varepsilon(1\vee c))}.\tag{2.14}$$

REMARK 4. Let us point out that all the constants in (2.14) do not depend on t.

We postpone the proof of Lemma 3 and we proceed with the proof of our main result.

Step 5. To begin with, we study the term p_1 in (2.12). We can write

$$p_1(u) \leqslant \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) + \mathbb{P}_x(A^c_{\varepsilon,\gamma,\delta,u,t} \cap \{X^{(\sigma)}_t > u,\ \tau_1 < t - g(\delta u)\}). \tag{2.15}$$

Since the solution of (2.7) on $[t - g(\delta u), t]$ will end up, at time t, not higher than δu, by using (2.10) we get,

$$X^{(\sigma)}_t \leqslant \delta u + \gamma u \quad \text{on the event} \quad A^c_{\varepsilon,\gamma,\delta,u,t} \cap \big\{\tau_1 < t - g(\delta u)\big\}.$$

By choosing $\delta + \gamma \leqslant 1$, the second term on the right hand side of (2.15) is equal to 0. Furthermore, assuming that $(1 \vee c)\varepsilon \leqslant \gamma/4$, using Lemma 3, we see that there exist $u_0(\varepsilon, \gamma, \delta)$ and σ_0 such that, for all $u \geqslant u_0(\varepsilon, \gamma, \delta)$ and $\sigma \leqslant \sigma_0$,

$$p_1(u) \leqslant \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) \leqslant C(\varepsilon, \gamma) g(\delta u) \rho(u)^{\gamma/(4\varepsilon(1\vee c))}. \tag{2.16}$$

We analyse now the term p_2 in (2.12). Let us introduce, for all $a < 0$ and $b > 0$,

$$M^{(\sigma)}_2(a, b) := \sup\{j < M^{(\sigma)}_1(a, b) : W^{(\sigma)}_j \notin [a, b]\}, \tag{2.17}$$

and again, to simplify, we set $\tau_2 := T_{M^{(\sigma)}_2(-\varepsilon u, c\varepsilon u)}$ the time of the jump with index $M^{(\sigma)}_2(-\varepsilon u, c\varepsilon u)$. We can write

$$\begin{aligned} p_2(u) &= \mathbb{P}_x\big(X^{(\sigma)}_t > u,\ \tau_1 \in [t - g(\delta u), t]\big) \\ &\leqslant \mathbb{P}\big(t - \tau_1 \leqslant g(\delta u),\ \tau_1 - \tau_2 \leqslant g(\delta u)\big) \\ &\quad + \mathbb{P}_x\big(X^{(\sigma)}_t > u,\ t - \tau_1 \leqslant g(\delta u),\ \tau_1 - \tau_2 > g(\delta u)\big) =: p_{21}(u) + p_{22}(u). \end{aligned} \tag{2.18}$$

Step 6. First, we estimate p_{21}. Since $N^{(\sigma)}_{g(\delta u)}$ has the same distribution as the number of jumps of $\ell^{(\sigma)}$ in the interval $[t - g(\delta u), t]$, we get

$$\mathbb{P}(\tau_1 \leqslant t - g(\delta u)) = \mathbb{P}\big(\forall j \in \{1, \dots, N^{(\sigma)}_{g(\delta u)}\},\ -\varepsilon u \leqslant W^{(\sigma)}_j \leqslant c\varepsilon u\big).$$

By using the fact that $N^{(\sigma)}_{g(\delta u)}$ is a Poisson distributed random variable of parameter $\lambda_\sigma g(\delta u)$ and is independent of the $W_i^{(\sigma)}$, we deduce

$$\mathbb{P}(\tau_1 \leqslant t - g(\delta u)) = e^{-\lambda_\sigma g(\delta u)} \sum_{n=0}^{+\infty} \frac{(\lambda_\sigma g(\delta u))^n}{n!} \mathbb{P}(-\varepsilon u \leqslant W_1^{(\sigma)} \leqslant c\varepsilon u)^n$$
$$= e^{-\lambda_\sigma g(\delta u)}(1 - \mathbb{P}(-\varepsilon u \leqslant W_1^{(\sigma)} \leqslant c\varepsilon u)) = e^{-\lambda_\sigma g(\delta u)} \mathbb{P}\big(W_1^{(\sigma)} \notin [-\varepsilon u, c\varepsilon u]\big).$$

Since

$$\mathbb{P}\big(W_1^{(\sigma)} \notin [-\varepsilon u, c\varepsilon u]\big) = \frac{c^{1-\alpha} + c^{-\alpha}}{\lambda_\sigma} \rho(\varepsilon u),$$

we get

$$\mathbb{P}(\tau_1 \leqslant t - g(\delta u)) = e^{-(c^{1-\alpha} + c^{-\alpha}) g(\delta u) \rho(\varepsilon u)}.$$

Since $t - \tau_1$ and $\tau_1 - \tau_2$ are independent and have the same distribution, we obtain

$$\begin{aligned} p_{21}(u) &= \mathbb{P}\big(t - \tau_1 \leqslant g(\delta u),\, \tau_1 - \tau_2 \leqslant g(\delta u)\big) \\ &= \big(1 - e^{-(c^{1-\alpha}+c^{-\alpha}) g(\delta u) \rho(\varepsilon u)}\big)^2 \leqslant (c^{1-\alpha} + c^{-\alpha})^2 \rho(\varepsilon u)^2 g(\delta u)^2. \end{aligned} \tag{2.19}$$

To estimate p_{22}, we fix η that will be chosen later. We can write

$$\begin{aligned} p_{22}(u) &\leqslant \mathbb{P}_x\big(X_t^{(\sigma)} > u,\, t - \tau_1 \leqslant g(\delta u),\, X^{(\sigma)}_{\tau_1 -} \leqslant \eta u\big) \\ &+ \mathbb{P}_x\big(t - \tau_1 \leqslant g(\delta u),\, X^{(\sigma)}_{\tau_1 -} > \eta u,\, \tau_1 - \tau_2 > g(\delta u)\big) =: p_{221}(u) + p_{222}(u). \end{aligned} \tag{2.20}$$

Step 7. We begin with the study of p_{221}. We have

$$\begin{aligned} p_{221}(u) &\leqslant \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) \\ &+ \mathbb{P}_x\big(A^c_{\varepsilon,\gamma,\delta,u,t} \cap \big\{X_t^{(\sigma)} > u,\, t - \tau_1 \leqslant g(\delta u),\, X^{(\sigma)}_{\tau_1 -} \leqslant \eta u\big\}\big) \\ &:= \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) + p_{\mathrm{main}}(u). \end{aligned} \tag{2.21}$$

By using Lemma 3, for all $u \geqslant u_0(\varepsilon, \gamma, \delta)$ and $\sigma \leqslant \sigma_0$,

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) \leqslant C(\varepsilon, \gamma) g(\delta u) (\rho(u))^{\gamma/(4\varepsilon(1 \vee c))}. \tag{2.22}$$

Furthermore, by the definition of g and (2.10), for all $u \geqslant u_0$, on the event

$$A^c_{\varepsilon,\gamma,\delta,u,t} \cap \big\{X^{(\sigma)}_t > u,\, t-\tau_1 \leqslant g(\delta u),\, X^{(\sigma)}_{\tau_1-} \leqslant \eta u\big\},$$

the magnitude $W^{(\sigma)}_{M^{(\sigma)}_1(-\varepsilon u, c\varepsilon u)}$ of the jump at time τ_1 should satisfy

$$t-\tau_1 + g\big(\eta u + W^{(\sigma)}_{M^{(\sigma)}_1(-\varepsilon u, c\varepsilon u)}\big) \leqslant g((1-\gamma)u).$$

Hence, since g is positive and decreasing, we get

$$t-\tau_1 \leqslant g((1-\gamma)u) \text{ and } W^{(\sigma)}_{M^{(\sigma)}_1(-\varepsilon u, c\varepsilon u)} \geqslant g^{-1}(g((1-\gamma)u) - (t-\tau_1) - \eta u).$$

At this level, we need to assume that $(1\vee c)\varepsilon + \gamma + \eta < 1$. For all $s \in (0, g((1-\gamma)u))$,

$$\begin{aligned}
&\mathbb{P}\big(W^{(\sigma)}_{M^{(\sigma)}_1(-\varepsilon u, c\varepsilon u)} \geqslant g^{-1}(g((1-\gamma)u) - s - \eta u)\big) \\
&\quad = \mathbb{P}\big(W^{(\sigma)}_1 \geqslant g^{-1}(g((1-\gamma)u) - s - \eta u) \mid W^{(\sigma)}_1 \notin [-\varepsilon u, c\varepsilon u]\big) \\
&\quad = \frac{\mathbb{P}\big(W^{(\sigma)}_1 \geqslant g^{-1}(g((1-\gamma)u) - s - \eta u)\big)}{\mathbb{P}\big(W^{(\sigma)}_1 \notin [-\varepsilon u, c\varepsilon u]\big)} \\
&\quad = \frac{\rho(g^{-1}(g((1-\gamma)u) - s) - \eta u))}{(c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u)}.
\end{aligned}$$

Since $t-\tau_1$ and $W^{(\sigma)}_{M^{(\sigma)}_1(-\varepsilon u, c\varepsilon u)}$ are independent and recalling that the distribution of $t-\tau_1$ is exponential with parameter $(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)$, we obtain

$$\begin{aligned}
p_{\mathrm{main}}(u) &= \mathbb{P}_x\big(A^c_{\varepsilon,\gamma,\delta,u,t} \cap \big\{X^{(\sigma)}_t > u,\, t-\tau_1 \leqslant g(\delta u),\, X^{(\sigma)}_{\tau_1-} \leqslant \eta u\big\}\big) \\
&\leqslant \int_0^{g((1-\gamma)u)} e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s} \rho(g^{-1}(g((1-\gamma)u) - s) - \eta u))\mathrm{d}s \\
&\leqslant \int_0^{g((1-\gamma)u)} \rho(g^{-1}(g((1-\gamma)u) - s) - \eta u))\mathrm{d}s.
\end{aligned}$$

We perform the change of variable $y = g^{-1}(g((1-\gamma)u) - s)$ and we get

$$\begin{aligned}
\text{(2.23)}\quad p_{\mathrm{main}}(u) &\leqslant \int_{(1-\gamma)u}^{+\infty} \frac{\rho(y-\eta u)}{f(y)}\mathrm{d}y \leqslant \int_{(1-\gamma)u}^{+\infty} \frac{\rho(y(1-\eta/(1-\gamma)))}{f(y)}\mathrm{d}y \\
&= \left(1 - \frac{\eta}{1-\gamma}\right)^{-\alpha} \int_{(1-\gamma)u}^{+\infty} \frac{\rho(y)}{f(y)}\mathrm{d}y = \left(1 - \frac{\eta}{1-\gamma}\right)^{-\alpha} h((1-\gamma)u).
\end{aligned}$$

Putting together (2.21), (2.22) and (2.23), we deduce, for all $u \geqslant u_0(\varepsilon,\gamma,\delta)$ and $\sigma \leqslant \sigma_0$,

(2.24)
$$p_{221}(u) \leqslant \left(1 - \frac{\eta}{1-\gamma}\right)^{-\alpha} h((1-\gamma)u) + C(\varepsilon,\gamma) g(\delta u)(\rho(u))^{\gamma/(4\varepsilon(1\vee c))}.$$

It remains to estimate p_{222}. Since $\tau_1 - \tau_2$ and $t - \tau_1$ are independent, we can split

$$p_{222}(u) = \mathbb{P}(t - \tau_1 \leqslant g(\delta u)) \cdot \mathbb{P}_x\big(X^{(\sigma)}_{\tau_1-} > \eta u,\, \tau_1 - \tau_2 > g(\delta u)\big).$$

We can write

$$\mathbb{P}_x\big(X^{(\sigma)}_{\tau_1-} > \eta u,\, \tau_1 - \tau_2 > g(\delta u)\big) \leqslant \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,\tau_1}) + \mathbb{P}_x\big(A^c_{\varepsilon,\gamma,\delta,u,\tau_1} \cap \big\{X^{(\sigma)}_{\tau_1-} > \eta u,\, \tau_1 - \tau_2 > g(\delta u)\big\}\big).$$

By choosing γ, δ and ε small enough, we can assume that $\delta + \gamma < \eta$. By employing the same argument used to estimate p_1, we deduce

$$\mathbb{P}_x\big(A^c_{\varepsilon,\gamma,\delta,u,\tau_1} \cap \big\{X^{(\sigma)}_{\tau_1-} > \eta u,\, \tau_1 - \tau_2 > g(\delta u)\big\}\big) = 0.$$

We use again Lemma 3 and the exponential distribution of $t - \tau_1$ with parameter $(c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u)$ to obtain that, for all $u \geqslant u_0(\varepsilon,\delta,\gamma)$ and $\sigma \leqslant \sigma_0$,

$$p_{222}(u) \leqslant C(\varepsilon,\delta,\gamma,\eta)\rho(u)^{(1+\gamma/(4(1\vee c)\varepsilon))} g(u)^2. \tag{2.25}$$

Step 8. Finally, summarizing the inequalities (2.16), (2.19), (2.24) and (2.25), for ε, γ, δ and η such that $\delta + \gamma < \eta < 1$, $(1 \vee c)\varepsilon < \gamma/4$ and $(1 \vee c)\varepsilon + \gamma + \eta < 1$, there exist $u_0(\varepsilon,\gamma,\delta,\eta)$ and σ_0 such that, for all $u \geqslant u_0(\varepsilon,\gamma,\delta,\eta)$ and $\sigma \leqslant \sigma_0$,

$$\mathbb{P}_x(X^{(\sigma)}_t > u) \leqslant \left(1 - \frac{\eta}{1-\gamma}\right)^{-\alpha} h((1-\gamma)u) + (c^{1-\alpha} + c^{-\alpha})^2 \rho(\varepsilon u)^2 g(\delta u)^2 + C(\varepsilon,\gamma,\delta,\eta) g(u)\rho(u)^{\gamma/(4(1\vee c)\varepsilon)}.$$

Since h is regularly varying at infinity with exponent $1 - \alpha - \beta$, g is regularly varying at infinity with exponent $1 - \beta$ and $\rho(u)$ is regularly varying at infinity with exponent $-\alpha$, choosing ε, γ, δ and η small enough, we get that for all $\xi > 0$, there exists $u_0(\xi)$ such that, for all $u \geqslant u_0(\xi)$, all $x \in \mathbb{R}$ and all $t \geqslant 1$,

$$\frac{\mathbb{P}_x(X_t^{(\sigma)} > u)}{h(u)} \leqslant 1 + \xi,$$

hence we have established the upper bound of the main result.

REMARK 5. At this level we note that, if instead of the regular variation at infinity of the function f, we made only the assumption $f(x) \geqslant \hat{f}(x)$ for all $x \geqslant A$ for some function $\hat{f}$ which is regularly varying at infinity with exponent greater than one, we would still have the upper bound, for all $\xi > 0$, there exists $u_0(\xi)$ such that, for all $u \geqslant u_0(\xi)$, all $x \in \mathbb{R}$ and all $t \geqslant 1$,

$$\frac{\mathbb{P}_x(X_t^{(\sigma)} > u)}{\hat{h}(u)} \leqslant 1 + \xi \quad \text{with} \quad \hat{h}(u) = \int_u^{+\infty} \frac{\nu((y, +\infty))}{\hat{f}(y)} \mathrm{d}y.$$

Step 9. We proceed with the proof of the lower bound. For all $\varepsilon < 1$, $\delta < 1$ and $\eta < 1$, we get, by the strong Markov property and (2.10)

$$\begin{aligned} \mathbb{P}_x(X_t^{(\sigma)} > u) &\geqslant \mathbb{P}_x\big(X_t^{(\sigma)} > u,\, \tau_1 \geqslant t - g(u(1+\delta)),\, X_{\tau_1-}^{(\sigma)} \geqslant -\eta u\big) \\ &\geqslant \int_0^{g(u(1+\delta))} (c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u) e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s}\, \mathbb{P}_x(X_{(t-s)-}^{(\sigma)} \geqslant -\eta u) \\ &\qquad \times \int_{c\varepsilon u}^{+\infty} \mathbb{P}_{y-\eta u}(X_s^{(\varepsilon u)} > u) \frac{\nu(\mathrm{d}y)}{(c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u)} \mathrm{d}s. \end{aligned}$$

Let us observe that $X^{(\sigma)}$ has, under $\mathbb{P}_x$, the same distribution as $-X^{(\sigma)}$ under the distribution $\mathbb{P}_{-x}$, but with a drift $\hat{f}(x) = -f(-x)$ and an asymmetric driving noise where the coefficients a_+, a_- in the expressions of its Lévy measure are inverted. By using the hypothesis on f and Remark 5, we obtain that for all $u \geqslant u_0$, for all $\sigma \leqslant \sigma_0$, all $x \in \mathbb{R}$ and all $s < g(u(1+\delta))$,

$$\mathbb{P}_x(X_{(t-s)-}^{(\sigma)} \geqslant -\eta u) \geqslant 1 - r(u),$$

where r is a function converging to zero. In the sequel, the function r can change from line to line. Observe that, according to (2.10), in a similar manner as we studied p_1, if

$$y \geqslant \eta u + g^{-1}(g(u(1+\delta)) - s) \tag{2.26}$$

then, under the distribution $\mathbb{P}_{y-\eta u}$, the event $\left\{ X_s^{(\varepsilon u)} > u \right\}$ contains, up to an event of probability zero, the event $A^c_{\varepsilon,\delta,1+\delta,u,t}$. Hence, for all s and y satisfying (2.26), we get

$$\mathbb{P}_{y-\eta u}(X_s^{(\varepsilon u)} > u) \geqslant 1 - \mathbb{P}_x(A_{\varepsilon,\delta,1+\delta,u,t}).$$

Therefore, by using Lemma 3, for all $\sigma \leqslant \sigma_0$ and $u \geqslant u_0(\varepsilon,\delta)$,

$$\mathbb{P}_{y-\eta u}(X_s^{(\varepsilon u)} > u) \geqslant 1 - r(u),$$

for all s and y satisfying (2.26), as long as ε is small relatively to δ. So, for all $\varepsilon < 1$, $\delta < 1$ and $\eta < 1$ such that ε is small relatively to δ, for all $\sigma \leqslant \sigma_0$ and all $u \geqslant u_0(\varepsilon,\delta)$,

$$\begin{aligned}
\mathbb{P}_x(X_t^{(\sigma)} > u) &\geqslant \int_0^{g(u(1+\delta))} e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s} \mathbb{P}_x(X^{(\sigma)}_{(t-s)-} \geqslant -\eta u) \\
&\quad \times \int_{\eta u + g^{-1}(g(u(1+\delta))-s)}^{+\infty} \mathbb{P}_{y-\eta u}(X_s^{(\varepsilon u)} > u)\nu(\mathrm{d}y)\mathrm{d}s \\
&\geqslant (1-r(u))^2 \int_0^{g(u(1+\delta))} e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s} \rho(\eta u + g^{-1}(g(u(1+\delta)) - s))\mathrm{d}s \\
&\geqslant (1-r(u))^2 e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u) g(u(1+\delta))} \int_{u(1+\delta)}^{+\infty} \frac{\rho(\eta u + y)}{f(y)}\mathrm{d}y \\
&\geqslant (1-r(u))^3 \int_{u(1+\delta)}^{+\infty} \frac{\rho(y(1+\eta/(1+\delta)))}{f(y)}\mathrm{d}y \\
&= (1-r(u))^3 \left(1 + \frac{\eta}{1+\delta}\right)^{-\alpha} h(u(1+\delta)).
\end{aligned}$$

We conclude that, for all $\xi > 0$, choosing η, ε and δ small enough, there exist $u_0(\xi)$ and $\sigma_0(\xi)$ such that

$$\frac{\mathbb{P}_x(X_t^{(\sigma)} > u)}{\hat{h}(u)} \geqslant 1 - \xi,$$

for all $u \geqslant u_0(\xi)$, all $\sigma \leqslant \sigma_0(\xi)$, all $x \in \mathbb{R}$ and $t \geqslant 1$. □

Proof of Lemma 3. Recall that we denoted $\rho(u) = \nu\big((u, +\infty)\big)$ and

$$\lambda_\sigma = \frac{\sigma^{-\alpha}}{\alpha}(a_- + a_+ c^{-\alpha}).$$

Set $q := \frac{a_-}{a_- + a_+ c^{-\alpha}}$. For all ε, u and σ, 0 is a quantile of order q for the random variable $W_1^{(\sigma)} \mathbb{1}_{\{W_1^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}}$ since, by using (2.2),

$$\begin{aligned}\mathbb{P}(W_1^{(\sigma)} \mathbb{1}_{\{W_1^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} < 0) &= \mathbb{P}(W_1^{(\sigma)} \in [-\varepsilon u, -\sigma]) \\ &= \frac{1}{\lambda_\sigma \alpha}(a_- \sigma^{-\alpha} - a_-(\varepsilon u)^{-\alpha}) = \frac{q}{\sigma^{-\alpha}}(\sigma^{-\alpha} - (\varepsilon u)^{-\alpha}) \leqslant q,\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(W_1^{(\sigma)} \mathbb{1}_{\{W_1^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} \leqslant 0) &= \mathbb{P}(W_1^{(\sigma)} \leqslant -\sigma) + \mathbb{P}(W_1^{(\sigma)} > c\varepsilon u) \\ &= \frac{1}{\lambda_\sigma \alpha}(a_- \sigma^{-\alpha} + a_+ c^{-\alpha}(\varepsilon u)^{-\alpha}) \geqslant \frac{a_- \sigma^{-\alpha}}{\lambda_\sigma \alpha} = q.\end{aligned}$$

Recall that $N_{g(\delta u)}^{(\sigma)}$ has the same distribution as the number of jumps of $\ell^{(\sigma)}$ in $[s - g(\delta u), s]$. By using Theorem 2.1 p. 50 in [6], we get

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s}) \leqslant \frac{1}{q} \mathbb{P}\Big(\sum_{i=1}^{N_{g(\delta u)}^{(\sigma)}} W_i^{(\sigma)} \mathbb{1}_{\{W_i^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} \geqslant \gamma u \Big).$$

Again we use the fact that $N_{g(\delta u)}^{(\sigma)}$ is a Poisson distributed random variable of parameter $\lambda_\sigma g(\delta u)$ and is independent of the $W_i^{(\sigma)}$. By conditioning, we obtain

$$\begin{aligned}\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s}) &\leqslant \frac{1}{q} \exp(-\lambda_\sigma g(\delta u)) \\ &\quad \times \sum_{n \geqslant 1} \frac{\big(\lambda_\sigma g(\delta u)\big)^n}{n!} \mathbb{P}\Big(\sum_{i=1}^{n} W_i^{(\sigma)} \mathbb{1}_{\{W_i^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} \geqslant \gamma u \Big).\end{aligned} \tag{2.27}$$

Recall that $W_i^{(\sigma)} \mathbb{1}_{\{W_i^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}}$ are i.i.d. random variables with expectation 0, bounded by $(1 \vee c)\varepsilon u$, we can use Theorem 1 in [7], p. 201. We get

$$\mathbb{P}\big(\sum_{i=1}^{n} W_i^{(\sigma)} \mathbb{1}_{\{W_i^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} \geqslant \gamma u\big) \\ \leqslant \exp\left[-\frac{\gamma}{2\varepsilon(1\vee c)} \operatorname{arcsinh}\big(\frac{\gamma u^2 \varepsilon(1\vee c)}{n\mathbb{V}\mathrm{ar}\big(\mathrm{W}_1^{(\sigma)}\mathbb{1}_{\{\mathrm{W}_1^{(\sigma)}\in[-\varepsilon \mathrm{u}, c\varepsilon \mathrm{u}]\}}\big)}\big)\right].$$

Furthermore, we can estimate

$$\mathbb{V}\mathrm{ar}\big(\mathrm{W}_1^{(\sigma)}\mathbb{1}_{\{\mathrm{W}_1^{(\sigma)}\in[-\varepsilon \mathrm{u}, c\varepsilon \mathrm{u}]\}}\big) = \mathbb{E}\big((\mathrm{W}_1^{(\sigma)})^2\mathbb{1}_{\{\mathrm{W}_1^{(\sigma)}\in[-\varepsilon \mathrm{u}, c\varepsilon \mathrm{u}]\}}\big) \\ = \frac{1}{\lambda_\sigma}\left(\int_{-\varepsilon u}^{-\sigma} a_-|z|^{1-\alpha}\mathrm{d}z + \int_{c\sigma}^{c\varepsilon u} a_+ z^{1-\alpha}\mathrm{d}z\right) \leqslant \frac{\alpha(c^{1-\alpha}+c^{2-\alpha})}{\lambda_\sigma(2-\alpha)}\varepsilon^{2-\alpha}u^2\rho(u).$$

Setting $\hat{C} := \frac{(1\vee c)(2-\alpha)}{\alpha(c^{1-\alpha}+c^{2-\alpha})}$, we can write

$$\mathbb{P}\big(\sum_{i=1}^{n} W_i^{(\sigma)} \mathbb{1}_{\{W_i^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} \geqslant \gamma u\big) \\ \leqslant \exp\left[-\frac{\gamma}{2\varepsilon(1\vee c)} \operatorname{arcsinh}\big(\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{n\rho(u)}\big)\right].$$

Since $\operatorname{arcsinh}(x) \sim \log(x)$ when $x \to +\infty$, there exists $a > 0$ such that for all $x \geqslant a$, $\operatorname{arcsinh}(x) \geqslant \frac{1}{2}\log(x)$. Therefore, if $n \leqslant \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\,\rho(u)}$, we get

$$\mathbb{P}_x\big(\sum_{i=1}^{n} W_i^{(\sigma)} \mathbb{1}_{\{W_i^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} \geqslant \gamma u\big) \\ \leqslant \exp\big[-\frac{\gamma}{4\varepsilon(1\vee c)} \log\big(\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{n\rho(u)}\big)\big] = \big(\frac{n\rho(u)}{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}\big)^{\gamma/(4\varepsilon(1\vee c))}.$$

By injecting this result in (2.27), we obtain

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s}) \leqslant \frac{1}{q}\big(\frac{\rho(u)}{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}\big)^{\gamma/(4\varepsilon(1\vee c))}\mathbb{E}\big((N_{g(\delta u)}^{(\sigma)})^{\gamma/(4\varepsilon(1\vee c))}\big) \\ + \frac{1}{q}\mathbb{P}\big(N_{g(\delta u)}^{(\sigma)} > \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\big). \tag{2.28}$$

It is no difficult to see that, if ξ is a Poisson distributed random variable, for all $p \geqslant 1$, there exists C_p such that

$$\mathbb{E}\xi^p \leqslant C_p\big(\mathbb{E}\xi + (\mathbb{E}\xi)^p\big).$$

Since $(1 \vee c)\varepsilon \leqslant \gamma/4$, we can apply this result to $N^{(\sigma)}_{g(\delta u)}$ and we deduce

$$\mathbb{E}\big((N^{(\sigma)}_{g(\delta u)})^{\gamma/(4\varepsilon(1\vee c))}\big) \leqslant C'_{\varepsilon,\gamma}\big(\lambda_\sigma g(\delta u) + (\lambda_\sigma g(\delta u))^{\gamma/(4\varepsilon(1\vee c))}\big).$$

We obtain an estimate for the first term on the right hand side of (2.28): there exists $C(\varepsilon,\gamma)$ such that

(2.29)
$$\frac{1}{q}\big(\frac{\rho(u)}{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}\big)^{\gamma/(4\varepsilon(1\vee c))}\mathbb{E}\big((N^{(\sigma)}_{g(\delta u)})^{\gamma/(4\varepsilon(1\vee c))}\big) \leqslant C(\varepsilon,\gamma) g(\delta u)\rho(u)^{\gamma/(4\varepsilon(1\vee c))}.$$

To study the second term on the right hand side of (2.28), we set

$$\vartheta := \log\big(\frac{\varepsilon^{\alpha-1}\gamma}{g(\delta u)\rho(u)}\big).$$

There exists $u_0(\varepsilon,\gamma,\delta)$ such that for all $u \geqslant u_0(\varepsilon,\gamma,\delta)$, ϑ is strictly positive. We get, for all $u \geqslant u_0(\varepsilon,\gamma,\delta)$,

$$\begin{aligned}\mathbb{P}\big(N^{(\sigma)}_{g(\delta u)} > \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\big) &= \mathbb{P}\big(e^{\vartheta N^{(\sigma)}_{g(\delta u)}} > \exp\big(\vartheta\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\big)\big)\\ &\leqslant \exp\big((e^{\vartheta}-1)\lambda_\sigma g(\delta u) - \vartheta\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\big),\end{aligned}$$

by using Markov's inequality. By choosing $C(\varepsilon,\gamma)$ and $u_0(\varepsilon,\gamma,\delta)$ large enough, we obtain, using the expression of ϑ,

$$\mathbb{P}\big(N^{(\sigma)}_{g(\delta u)} > \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\big) \leqslant C(\varepsilon,\gamma)\,(g(\delta u)\rho(u))^{C(\varepsilon,\gamma)\lambda_\sigma/\rho(u)}. \tag{2.30}$$

Replacing (2.29) and (2.30) in (2.28), we get (2.14). □

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REFERENCES

[1] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Second Edition, Cambridge University Press, 2011.

[2] R. Eon, *Asymptotique des solutions d'équations différentielles de type frottement perturbées par des bruits de Lévy stables*, Thèse de doctorat, Université de Rennes 1, 2016. https://tel.archives-ouvertes.fr/tel-01388319v1

[3] R. Eon and M. Gradinaru, *Gaussian asymptotics for a non-linear Langevin type equation driven by an α-stable Lévy noise*, Elec. J. Probab. **20** (2015), Paper 100, pp. 1–19.

[4] O. Kallenberg, *Foundations of Modern Probability*, Springer, 2000.

[5] A.M. Kulik, *Exponential ergodicity of the solutions to SDE's with a jump noise*, Stochastic Proc. Appl. **119** (2009), pp. 602—632.

[6] V. Petrov, *Limit theorems of probability theory. Sequences of independent random variables*, Oxford Studies in Probability, 1995.

[7] Yu.V. Prokhorov, *An Extremal Problem in Probability Theory*, Theory Probab. Its Appl. **4** (1959), pp. 201–203.

[8] G. Samorodnitsky and M. Grigoriu, *Tails of solutions of certain nonlinear stochastic differential equations driven by heavy tailed Levy motions*, Stochastic Proc. Appl. **105** (2003), pp. 69–97.

[9] T. Srokowski, *Asymmetric Lévy flights in nonhomogeneous environments*, J. Stat. Mech. Theory Exp. **5** (2014), pp. 5–24.

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