Gaussian asymptotics for a non-linear Langevin type equation driven by an α-stable Lévy noise

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Abstract: Consider a one-dimensional process x_t^ε the position of a particle at time t which speed v_t^ε is a solution of a stochastic differential equation driven by a small α-stable Lévy process, $\varepsilon\ell_t$, $\alpha \in (0,2]$, and with a non-linear drift coefficient $-\mathrm{sgn}(v)|v|^\beta$, $\beta > 2 - (\alpha/2)$. The noise could be path continuous (Brownian motion $\alpha = 2$) or pure jump process ($0 < \alpha < 2$). We prove that, as ε goes to 0, the limit in distribution of the process $\{\varepsilon^{(\beta+(\alpha/2)-2)\theta_{\alpha,\beta}}\, x^\varepsilon_{\varepsilon^{-\alpha}t} : t \geq 0\}$ is a Brownian motion with some variance $\kappa_{\alpha,\beta}$, where $\theta_{\alpha,\beta} = \alpha/(\beta+\alpha-1)$. This result is a generalization in some sense of the linear case studied by Hintze and Pavlyukevich [9].

Key words: stable Lévy noise, non-linear Langevin type equation, Lévy driven stochastic differential equation, Brownian motion, exponential ergodic processes, Lyapunov function, convergence in probability, functional central limit theorem for martingales.

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1 Introduction

In many physical, engineering or financial mathematics models based on random perturbations, the usual construction is performed by using the standard white noise and studying the resulting diffusion process. The Gaussian feature and the continuity of paths of the Brownian motion are essential when choosing the tools used for this kind of situation. During the last fifteen years, the study of some particular phenomena, as discontinuous behaviour of paths or self-similarity in time scale, focuses on another type of random perturbation, mainly an α-stable Lévy noise. The resulting processes are called in physical literature anomalous (fractional) diffusions or Lévy flights.

In this paper, we consider the one-dimensional and non-linear Langevin type equation driven by an α-stable Lévy process. Let us denote by x_t^ε the one-dimensional process describing the position of a particle at time $t \geq 0$, having the speed v_t^ε

$$x_t^\varepsilon = x_0 + \int_0^t v_s^\varepsilon \mathrm{d}s,\ t \geq 0, \tag{1.1}$$

and such that v_t^ε is a small α-stable Lévy process in a potential $\mathcal{U}(x) := \frac{2}{\beta+1}|x|^{\beta+1}$,

$$dv_t^\varepsilon = \varepsilon d\ell_t - \frac{1}{2}\mathcal{U}'(v_t^\varepsilon)\mathrm{d}t, \quad v_0^\varepsilon = v_0, \tag{1.2}$$

in other words v_t^ε verifies the following integral equation

$$v_t^\varepsilon = v_0 + \varepsilon\ell_t - \int_0^t \mathrm{sgn}(v_s^\varepsilon)|v_s^\varepsilon|^\beta \mathrm{d}s,\ t \geq 0. \tag{1.3}$$

Here $\beta > -1$ and $\{\ell_t : t \geq 0\}$ is an α-stable Lévy process, $\alpha \in (0,2]$. If $\alpha \in (0,2)$, the Lévy process is a pure jump process with càdlàg paths and the jump measure is given by $\nu(\mathrm{d}z) = |z|^{-1-\alpha}\mathbb{1}_{\mathbb{R}\setminus\{0\}}(z)\mathrm{d}z$. The 2-stable Lévy process is the standard Brownian motion $\{b_t : t \geq 0\}$ which is continuous. In all cases, the process has the property of self-similarity which means that the processes $\{\ell_t : t \geq 0\}$ and $\{c^{-1/\alpha}\ell_{ct} : t \geq 0\}$ have the same law, for any $c > 0$.

The case of a harmonic potential ($\beta = 1$, linear equation), when the speed is a Ornstein-Uhlenbeck process, was already considered by Hintze and Pavlyukevich [9]. The dynamic of the integrated Ornstein-Uhlenbeck process appears in some financial mathematics (volatility) models (see for instance Barndorff-Nielsen and Shephard [3]) or in models in physics of plasma (see for instance Chechkin, Gonchar and Szydlowski [5]). In the paper by Hintze and Pavlyukevich, the authors study the asymptotic behaviour of the integrated Ornstein-Uhlenbeck and prove that this process converges weakly, as $\varepsilon \to 0$, to the underlying α-stable Lévy process. In particular, when the driving process is a Brownian motion ($\alpha = 2$), the asymptotic behaviour is Gaussian. In [9], asymptotics of the first exit time from an interval are deduced. Several physical papers pointed out that new interesting phenomena appear when one considers super-harmonic potentials (see for instance Metzler, Chechkin, Klafter [11]).

Our goal is to answer to the same question in the situation of a super-harmonic potential (non-linear equation): what is the asymptotic behaviour of the position process x_t^ε, as $\varepsilon \to 0$? On the one hand, the non-linear case introduces new technical difficulties, mainly since the solution is no longer explicit. Indeed, this fact was essential to prove weak convergence in the linear case. On the other hand, different conditions on the two parameters α and β will generate different asymptotics for the position process. The intuition suggests that the big jumps should be compensated by a strong negative drift (for instance if $\beta > 1$) and small jumps should have some regularising effect. In the present paper, we answer to the question by showing that for α and β in some unbounded domain, the position process x_t^ε will behave as a Brownian motion when ε goes to 0. In other words, we get Gaussian asymptotic behaviour even if α is smaller than 2, provided that β is not very small, more precisely if $\beta + \frac{\alpha}{2} > 2$. When α and β are in somehow "small" the previous heuristic fails. To get convergence toward a stable process, one needs to change the approach and other technical difficulties appear (this case will be presented in a forthcoming paper, see [8]).

To state the main result of the present paper, we will perform some scaling transformations. Without loss of generality, we can assume that the initial position is the origin $x_0 = 0$. Moreover we will assume that the initial speed vanishes $v_0 = 0$, contrary to the linear case. By using the self-similarity, it is clear that the process $\{L_t^\varepsilon := \varepsilon \ell_{\varepsilon^{-\alpha}t} : t \geq 0\}$ is also an α-stable process. Let us denote, for $t \geq 0$,

$$X_t^\varepsilon := x_{\varepsilon^{-\alpha}t}^\varepsilon \quad \text{and} \quad V_t^\varepsilon := v_{\varepsilon^{-\alpha}t}^\varepsilon \tag{1.4}$$

satisfying, respectively,

$$X_t^\varepsilon = \frac{1}{\varepsilon^\alpha}\int_0^t V_s^\varepsilon \mathrm{d}s \quad \text{and} \quad V_t^\varepsilon = L_t^\varepsilon - \frac{1}{\varepsilon^\alpha}\int_0^t \operatorname{sgn}(V_s^\varepsilon)|V_s^\varepsilon|^\beta \mathrm{d}s. \tag{1.5}$$

Set

$$\check{L}_t^\varepsilon := \frac{L^\varepsilon_{t\,\varepsilon^{\alpha^2/(\alpha+\beta-1)}}}{\varepsilon^{\alpha/(\alpha+\beta-1)}} = \frac{\ell_{t\,\varepsilon^{-\alpha(\beta-1)/(\alpha+\beta-1)}}}{\varepsilon^{(\beta-1)/(\alpha+\beta-1)}} \quad \text{and} \quad \check{V}_t^\varepsilon := \frac{V^\varepsilon_{t\,\varepsilon^{\alpha^2/(\alpha+\beta-1)}}}{\varepsilon^{\alpha/(\alpha+\beta-1)}}, \tag{1.6}$$

provided that $\alpha + \beta - 1 > 0$. Again by self-similarity, $\check{L}^\varepsilon$ is distributed as an α-stable Lévy process and we have

$$X_t^\varepsilon = \varepsilon^{\frac{\alpha(2-\beta)}{\alpha+\beta-1}} \int_0^{t\varepsilon^{-\alpha^2/(\alpha+\beta-1)}} \check{V}_s^\varepsilon \mathrm{d}s \quad \text{and} \quad \check{V}_t^\varepsilon = \check{L}_t^\varepsilon - \int_0^t \operatorname{sgn}(\check{V}_s^\varepsilon)|\check{V}_s^\varepsilon|^\beta \mathrm{d}s. \tag{1.7}$$

Let us note that if $\alpha = 2$, all previous computations hold true with ℓ, L or $\check{L}$ replaced respectively by b, B or $\check{B}$ a standard Brownian motion. Our main result is the following:

Theorem 1.1. *Assume that $0 < \alpha \leq 2$ and $\beta + \frac{\alpha}{2} > 2$. There exists a positive constant $\kappa_{\alpha,\beta}$ such that the process*

$$\left\{\varepsilon^{\frac{\alpha(\beta+\alpha/2-2)}{\alpha+\beta-1}} x_{\varepsilon^{-\alpha}t}^\varepsilon : t \geq 0\right\} = \left\{\varepsilon^{\frac{\alpha(\beta+\alpha/2-2)}{\alpha+\beta-1}} X_t^\varepsilon : t \geq 0\right\} \tag{1.8}$$

converges in distribution toward a Brownian motion process with variance $\kappa_{\alpha,\beta}$, as $\varepsilon \to 0$. Moreover, if $\alpha = 2$, the result is true even for $-1 < \beta \leq 1$.

Remark 1.2. *1. If the driving noise is the Brownian motion $\alpha = 2$, the convergence in the theorem holds in the space of continuous functions $\mathrm{C}([0,\infty))$ endowed by the uniform topology. If the driving* *noise is α-stable with $\alpha \in (0,2)$, the convergence in Theorem* 1.1 *holds in the Skorokhod space of càdlàg functions* $\mathrm{D}([0,\infty))$ *endowed by J_1 (or simple) Skorokhod topology. Our situation is simpler than in* [9] *since the limit is a continuous paths process.*

2. *If the driving noise is the Brownian motion $\alpha = 2$, the normalizing factor is $\varepsilon^{2(\beta-1)/(\beta+1)}$ and it behaves differently following with the position of β with respect to 1 (if $\beta = 1$, the position process X^ε converges in distribution toward a standard Brownian motion, see also* Remark 2.4 *below).*

3. *The case when $\beta + \frac{\alpha}{2} = 2$ should be considered as a critical for some phase transition from Gaussian to stable case. It should be reasonable that there is some continuity but the proof seems more delicate since natural integrability conditions are not fulfilled.*

4. *The constant $\kappa_{\alpha,\beta}$ has an integral representation (see* (2.9) *and* (3.28) *) and it is more explicit when the driving noise is the Brownian motion ($\alpha = 2$).*

5. *Again, as an application, one can find asymptotics of the first exit time from an interval : Corollary* 2.1*, p. 269, in* [9] *applies.*

Let us explain the method of proof and the organisation of the paper. To simplify the notations, all along the paper we will denote

$$\theta = \theta_{\alpha,\beta} := \frac{\alpha}{\alpha+\beta-1} \in (0,1). \tag{1.9}$$

It is a simple observation that

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)} X_t^\varepsilon = \varepsilon^{\frac{\alpha\theta}{2}} \int_0^{t\varepsilon^{-\alpha\theta}} \check{V}_s^\varepsilon \mathrm{d}s\,,$$

hence Theorem 1.1 is in fact a second order type ergodic theorem. By using stochastic calculus, we will show that the latter quantity is a sum of a square integrable martingale and a term which tends in probability toward 0 as $\varepsilon \to 0$. The result is then obtained by using the functional central limit theorem for martingales and the continuous-mapping theorem.

In the next section, we consider the case when the driving noise is the Brownian motion: in this case computations are performed by using Itô's calculus and are more explicit. For instance, the constant $\kappa_{2,\beta}$ can be written in terms of the scale function and the speed measure. In Section 3, we follow the same structure of the proof for a pure jump driving noise. Computations are more technical and new ideas are needed: for instance, we need to find and use a Lyapunov function which allows to perform the same reasoning by using Lévy-Itô's calculus. We collect in the Appendix the technical proofs.

2 Brownian motion driving noise

Recall that in this case, $\{b_t : t \geq 0\}$ is a standard one-dimensional Brownian motion, $\beta > -1$ and we set

$$\check{B}_t^\varepsilon := \frac{B^\varepsilon_{t\,\varepsilon^{4/(\beta+1)}}}{\varepsilon^{2/(\beta+1)}} = \frac{b_{t\,\varepsilon^{2(1-\beta)/\beta+1}}}{\varepsilon^{(\beta-1)/(\beta+1)}}, \quad \text{and} \quad \check{V}_t^\varepsilon := \frac{V^\varepsilon_{t\,\varepsilon^{4/(\beta+1)}}}{\varepsilon^{2/(\beta+1)}}. \tag{2.1}$$

Recall also that

$$X_t^\varepsilon = \varepsilon^{\frac{2(2-\beta)}{(\beta+1)}} \int_0^{t\varepsilon^{-4/(\beta+1)}} \check{V}_s^\varepsilon \mathrm{d}s \quad \text{and} \quad \check{V}_t^\varepsilon = \check{B}_t^\varepsilon - \int_0^t \operatorname{sgn}(\check{V}_s^\varepsilon)|\check{V}_s^\varepsilon|^\beta \mathrm{d}s\,. \tag{2.2}$$

$\check{B}^\varepsilon$ is distributed as a standard Brownian motion so, to simplify the notation, we will suppress the index ε, as well as for $\check{V}^\varepsilon$.

2.1 The speed process V^ε

2.1.1 Existence and uniqueness

If $\beta \geq 1$, the drift coefficient in (2.2_2) is a locally Lipschitz function hence by well known results (see, for instance, Theorem 12.1, p. 132 in [12]), we get a pathwise unique strong solution $\check{V}$ to equation (2.2_2), whereas if $-1 < \beta < 1$, Girsanov's theorem gives the existence of a weak solution to equation (2.2_2). For both situations, the solution is defined until an explosion time τ_e, but it is no difficult to prove that $\tau_e = \infty$ a.s. by using Theorem 10.2.1, p. 254, in [14] and a convenient Lyapunov function (for instance $h(x) = 1 + x^2$ for all $|x| \geq 1$, $h(x) = 1$ for all $|x| \leq 1/2$ and $h \geq 1$). Introduce the scale function and the speed measure associated to the diffusion

$$s_\beta(x) := \int_0^x \mathrm{e}^{-c_\beta(y)}\mathrm{d}y \quad \text{and} \quad m_\beta(\mathrm{d}x) := 2\mathrm{e}^{c_\beta(x)}\mathrm{d}x, \quad \text{where} \quad c_\beta(x) := -\frac{2}{\beta+1}|x|^{\beta+1}. \tag{2.3}$$

Since $\int_0^\infty m_\beta([0,x])\mathrm{e}^{-c_\beta(x)}\mathrm{d}x = \infty$, by Theorem 52.1, p. 297 in [12], the pathwise uniqueness holds to (2.2_2). Finally, there exists a pathwise unique strong solution $\check{V}$ to the equation (2.2_2).

2.1.2 Convergence in probability

The main result of this section is the following

Proposition 2.1. *As $\varepsilon \to 0$, $\{V_t^\varepsilon : t \geq 0\}$ converges to 0 in probability uniformly on each compact time interval.*

By (2.1_2), the relation between V^ε and $\check{V}$ is $V_t^\varepsilon = \varepsilon^{2/(\beta+1)}\check{V}_{t\,\varepsilon^{-4/(\beta+1)}}$. To prove Proposition 2.1, we need a preliminary result:

Lemma 2.2.

1. *Fix $p \geq 2$. There exists a positive constant $C_{p,\beta}$ such that, for any $t \geq 0$,*

$$\mathbb{E}\Big(|\check{V}_t|^p\Big) \leq C_{p,\beta}\, t. \tag{2.4}$$

2. *Fix $p \geq 4$ and $T > 0$. There exists a positive constant $C'_{p,\beta}$ such that*

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |\check{V}_{t\,\varepsilon^{-4/(\beta+1)}}|\Big)^p \leq C'_{p,\beta}\, T^2\, \varepsilon^{-8/(\beta+1)}. \tag{2.5}$$

Proof of Proposition 2.1. Taking $p > 4$ in Lemma 2.2, we deduce that for any $T > 0$, as $\varepsilon \to 0$, $\sup_{0\leq t\leq T}|V_t^\varepsilon|$ converges to 0 in $\mathrm{L}^p(\Omega)$, and the conclusion follows. □

Proof of Lemma 2.2. By using Itô's formula and the equation (2.2_2), we can write

$$|\check{V}_t|^p = p\int_0^t \mathrm{sgn}(\check{V}_s)|\check{V}_s|^{p-1}\mathrm{d}\check{B}_s + p\int_0^t \Big((1/2)(p-1)|\check{V}_s|^{p-2} - |\check{V}_s|^{p-1+\beta}\Big)\mathrm{d}s$$

Since $\beta > -1$, there exists a constant $C_{p,\beta} > 0$ such that

$$p\Big((1/2)(p-1)|x|^{p-2} - |x|^{p-1+\beta}\Big) \leq C_{p,\beta},\ \forall x \in \mathbb{R}.$$

We deduce that

$$|\check{V}_t|^p \leq C_{p,\beta}t + p\int_0^t \mathrm{sgn}(\check{V}_s)|\check{V}_s|^{p-1}\mathrm{d}\check{B}_s \tag{2.6}$$

We show that $\int_0^t \mathrm{sgn}(\check{V}_s)|\check{V}_s|^{p-1}\mathrm{d}\check{B}_s$ is a martingale. Fix $T > 0$, for all $t \leq T$, since $(a+b)^2 \leq 2(a^2+b^2)$ and $|x|^{2p-2} \leq 1 + |x|^{2p}$, by using the Burkholder-Davis-Gundy inequality, we can see that there exists a positive constant C'_1 such that

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\leq u\leq t}|\check{V}_u|^p\Big)^2 &\leq 2C_{p,\beta}^2\, T^2 + 2p^2\,\mathbb{E}\Big(\sup_{0\leq u\leq t}\int_0^u \mathrm{sgn}(\check{V}_s)|\check{V}_s|^{p-1}\mathrm{d}\check{B}_s\Big)^2 \leq 2C_{p,\beta}^2\, T^2 \\
&+ 2p^2\, C'_1\int_0^t \mathbb{E}\big(|\check{V}_s|^{2p-2}\big)\mathrm{d}s \leq 2p^2\, C'_1\, T + 2C_{p,\beta}^2\, T^2 + 2p^2\, C'_1\int_0^t \mathbb{E}\big(|\check{V}_s|^{2p}\big)\mathrm{d}s \\
&\leq 2p^2\, C'_1\, T + 2C_{p,\beta}^2\, T^2 + 2p^2\, C'_1\int_0^t \mathbb{E}\big(\sup_{0\leq u\leq s}|\check{V}_u|^p\big)^2\mathrm{d}s.
\end{aligned}$$

By Gronwall's lemma, we get, for all $t \leq T$,

$$\mathbb{E}\Big(\sup_{0\leq u\leq t}|\check{V}_u|^p\Big)^2 \leq (2p^2\, C_1'\, T + 2C_{p,\beta}^2\, T^2)\mathrm{e}^{2p^2\, C_2'\, T}\,.$$

Hence $\int_0^t \operatorname{sgn}(\check{V}_s)|\check{V}_s|^{p-1}\mathrm{d}\check{B}_s$ is a martingale and we get (2.4) by taking expectation in (2.6).

It is now possible to improve the inequality (2.4). Indeed, it can be used to see that

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\leq t\leq T}|\check{V}_{t\,\varepsilon^{-4/(\beta+1)}}|\Big)^p = \mathbb{E}\Big(\sup_{0\leq t\leq T}|\check{V}_{t\,\varepsilon^{-4/(\beta+1)}}|^{p/2}\Big)^2 \leq \frac{p^2}{2}\,\mathbb{E}\Big(\sup_{0\leq t\leq T}\int_0^{t\,\varepsilon^{-4/(\beta+1)}}|\check{V}_s|^{p/2-1}\mathrm{d}\check{B}_s\Big)^2 \\ + 2\,C_{p/2,\beta}^2\,T^2\,\varepsilon^{-8/(\beta+1)} \leq \frac{p^2}{2}C_1'\int_0^{T\,\varepsilon^{-4/(\beta+1)}}\mathbb{E}\big(|\check{V}_s|^{p-2}\big)\mathrm{d}s + 2C_{p/2,\beta}^2\,T^2\,\varepsilon^{-8/(\beta+1)} \\ \leq \frac{p^2}{4}C_1'\,C_{p-2,\beta}\,T^2\,\varepsilon^{-8/(\beta+1)} + 2C_{p/2,\beta}^2\,T^2\,\varepsilon^{-8/(\beta+1)}.\end{aligned}$$

Therefore (2.5) follows taking $C'_{p,\beta} := \frac{p^2}{4}C_1'\,C_{p-2,\beta} + 2C_{p/2,\beta}^2\,.$ □

2.1.3 Ergodicity

Recall that we introduced the scale function and the speed measure in (2.3). Since $s_\beta(\infty) = \infty$ and $m_\beta(\mathbb{R}) < \infty$, the diffusion $\check{V}$ is regular (see for instance (45.2) and (46.10) pp. 272-275 in [12]) and is a recurrent and ergodic process with the invariant measure m_β (see for instance Theorem 53.1, p. 300 in [12]). Therefore, for all $f \in \mathrm{L}^1(m_\beta)$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\check{V}_s)\mathrm{d}s = \frac{1}{m_\beta(\mathbb{R})}\int_{\mathbb{R}} f(x)m_\beta(\mathrm{d}x),\ \text{ almost surely.} \tag{2.7}$$

2.2 The position process X^ε

We recall that the infinitesimal generator of $\check{V}$ is given by $\mathcal{L}_{2,\beta} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \operatorname{sgn}(x)|x|^\beta\frac{\mathrm{d}}{\mathrm{d}x}$. Introduce

$$g_\beta(x) := \int_0^x\left(\int_y^{+\infty} -2z\mathrm{e}^{c_\beta(z)}\mathrm{d}z\right)\mathrm{e}^{-c_\beta(y)}\mathrm{d}y,\ x \in \mathbb{R}, \tag{2.8}$$

and note that $(\mathcal{L}_{2,\beta}\, g_\beta)(x) = x$, for all $x \in \mathbb{R}$. Set

$$\kappa_{2,\beta} := \frac{1}{m_\beta(\mathbb{R})}\int_{\mathbb{R}} g_\beta'(x)^2 m_\beta(\mathrm{d}x) = -\frac{2}{m_\beta(\mathbb{R})}\int_{\mathbb{R}} x g_\beta'(x)m_\beta(\mathrm{d}x) \tag{2.9}$$

(the latter equality is obtained by integrating by parts). We can give now the proof of the main result.

Proof of Theorem 1.1 for the case $\alpha = 2$. By applying Itô's formula, we can see that

$$g_\beta(\check{V}_t) = \int_0^t g_\beta'(\check{V}_s)\mathrm{d}\check{B}_s + \int_0^t(\mathcal{L}_{2,\beta}\,g_\beta)(\check{V}_s)\mathrm{d}s = \int_0^t g_\beta'(\check{V}_s)\mathrm{d}\check{B}_s + \int_0^t \check{V}_s\mathrm{d}s,$$

and therefore

$$\varepsilon^{2(\beta-1)/(\beta+1)}X_t^\varepsilon = -\varepsilon^{2/(\beta+1)}\int_0^{t\,\varepsilon^{-4/(\beta+1)}} g_\beta'(\check{V}_s)\mathrm{d}\check{B}_s + \varepsilon^{2/(\beta+1)}g_\beta(\check{V}_{t\,\varepsilon^{-4/(\beta+1)}})\,.$$

The continuous local martingale

$$M_t^\varepsilon := -\varepsilon^{2/(\beta+1)}\int_0^{t\,\varepsilon^{-4/(\beta+1)}} g_\beta'(\check{V}_s)\mathrm{d}\check{B}_s$$

has the quadratic variation

$$\langle M^\varepsilon\rangle_t = \varepsilon^{4/(\beta+1)}\int_0^{t\,\varepsilon^{-4/(\beta+1)}} g_\beta'(\check V_s)^2 \mathrm{d}s\,.$$

As a consequence of (2.7), for all t, $\langle M^\varepsilon\rangle_t \to \kappa_{2,\beta} t$ a.s., as $\varepsilon \to 0$, where $\kappa_{2,\beta}$ is given by (2.9), and it is the constant in the statement of Theorem 1.1. Indeed, using Whitt's theorem (see Theorem 2.1(ii), p. 270 in [15]), we deduce that M^ε converges in distribution (as a process) toward $\kappa_{2,\beta}^{1/2}\check B$.

We will prove that the second term in the right hand side converges in probability uniformly on compact sets to 0. At this level, we need a technical result:

Lemma 2.3. *There exist two positive constants μ_β, ν_β such that for all $x \in \mathbb{R}$,*

$$|g_\beta(x)| \le \mu_\beta |x|^{(2-\beta)\vee 1} + \nu_\beta. \tag{2.10}$$

We postpone the proof of the lemma to the Appendix and finish the proof of Theorem 1.1 in the case $\alpha = 2$. By using the classical inequality $(a+b)^{2m} \le 2^{2m-1}(a^{2m}+b^{2m})$, ($m \ge 1$ integer), we obtain

$$|\varepsilon^{2/(\beta+1)} g_\beta(\check V_{t\,\varepsilon^{-4/(\beta+1)}})|^{2m} \le 2^{2m-1}\mu_\beta^{2m}\varepsilon^{(4m)/(\beta+1)}|\check V_{t\,\varepsilon^{-4/(\beta+1)}}|^{2m((2-\beta)\vee 1)} + 2^{2m-1}\nu_\beta^{2m}\varepsilon^{(4m)/(\beta+1)}\,.$$

By choosing the integer $m \ge 1$ such that $p := 2m((2-\beta)\vee 1) > 4$, we can use Lemma 2.2 and we get for all $T > 0$,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\sup_{0\le t\le T} \varepsilon^{4m/(\beta+1)} g_\beta^{2m}(\check V_{t\,\varepsilon^{-4/(\beta+1)}})\Big] = 0.$$

We finish the proof of the theorem by employing the joint convergence theorem and the simple continuous-mapping theorem (Theorem 11.4.5 p. 379 and Theorem 3.4.1, p. 85 in [16]) on the space of continuous functions $\mathrm{C}([0,\infty))$ endowed with the uniform topology. □

Remark 2.4. *Let us note that if $\beta = 1$ (Ornstein-Uhlenbeck case), $g_\beta(x) = -x$, $\kappa_{2,\beta} = 1$ and the result of Theorem* 1.1 *coincides with the result of Proposition* 2.1*, p. 268, in* [9].

3 α-stable driving noise

Recall that $\check L^\varepsilon$ is distributed as a α-stable Lévy process (see (1.6_1)) so, to simplify the notation, we will suppress the index ε, as well as for $\check V^\varepsilon$ (see (1.7_2)).

3.1 The speed process V^ε

3.1.1 Existence and uniqueness

If $\beta > 1$, the drift coefficient in (1.7_2) is a locally Lipschitz function and it is well known (see, for instance, Theorem 6.2.11, p. 376 in [1]) that there exists a locally pathwise unique strong solution $\check V$ for equation (1.7_2) defined up to an explosion random time τ. Moreover it can be proved that $\tau = \infty$ a.s. hence $\check V$ is a global solution. For the sake of completeness, we give the proof of the latter statement (see also [13], p. 73) by following some ideas in [6], pp. 156-157.

Lemma 3.1. *For any $\alpha \in (0,2)$, any $\delta \in (0,\alpha)$ and any $T > 0$, $\mathbb{E}\Big[\sup_{t\in[0,T]} |\check V_t|^\delta\Big] < \infty$.*

Proof. By Itô-Lévy's decomposition, there exists a Poisson process N and its compensated $\tilde N$ such that

$$\check L_t = \int_0^t\int_{|z|\le 1} z\tilde N(\mathrm{d}s,\mathrm{d}z) + \int_0^t\int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z)$$

and so the equation satisfied by $\check V$, starting from any $x \in \mathbb{R}$, is

$$\check V_t = x + \int_0^t\int_{|z|\le 1} z\tilde N(\mathrm{d}s,\mathrm{d}z) + \int_0^t\int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z) - \int_0^t \mathrm{sgn}(\check V_s)|\check V_s|^\beta \mathrm{d}s. \tag{3.1}$$

Consider another equation where we skip the (third) big jumps term

$$Y_t = x + \int_0^t \int_{|z|\le 1} z\tilde{N}(\mathrm{d}s,\mathrm{d}z) - \int_0^t \mathrm{sgn}(Y_s)|Y_s|^\beta \mathrm{d}s, \tag{3.2}$$

and apply Itô-Lévy's formula. We obtain

$$\begin{aligned} Y_t^2 &= x^2 + M_t + \int_0^t \int_{|z|\le 1} [(Y_s+z)^2 - Y_s^2 - 2zY_s]\nu(\mathrm{d}z)\mathrm{d}s - 2\int_0^t |Y_s|^{\beta+1}\mathrm{d}s \\ &= x^2 + \tilde{M}_t + t\int_{|z|\le 1} z^2\nu(\mathrm{d}z) - 2\int_0^t |Y_s|^{\beta+1}\mathrm{d}s, \end{aligned} \tag{3.3}$$

where the local martingale term is given by

$$\tilde{M}_t := \int_0^t \int_{|z|\le 1} [(Y_s+z)^2 - Y_s^2]\tilde{N}(\mathrm{d}s,\mathrm{d}z).$$

The constants depending only on α and β will be denoted c_α or $k_{\alpha,\beta}$ and could change from line to line in this proof. Let us write the third term in (3.3) as $c_\alpha t$ and note that $\lim_{|y|\to\infty}(c_\alpha - 2|y|^{\beta+1}) = -\infty$. We deduce that there exists a positive constant $k_{\alpha,\beta}$ such that, for all $t \ge 0$,

$$Y_t^2 \le x^2 + k_{\alpha,\beta}t + \tilde{M}_t. \tag{3.4}$$

By Kunita's inequality (see for instance [1], p. 265) and by our convention on constants,

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le s\le t} Y_s^2\right] &\le x^2 + k_{\alpha,\beta}t + c_\alpha \int_0^t \int_{|z|\le 1} \mathbb{E}\left[(Y_s+z)^2 - Y_s^2\right]^2 \nu(\mathrm{d}z)\mathrm{d}s \\ &\le x^2 + k_{\alpha,\beta}t + c_\alpha \int_0^t \mathbb{E}[Y_s^2]\mathrm{d}s \le x^2 + k_{\alpha,\beta}t + c_\alpha \int_0^t \mathbb{E}\left[\sup_{0\le u\le s} Y_u^2\right]\mathrm{d}s. \end{aligned} \tag{3.5}$$

Applying Gronwall's inequality, we get

$$\mathbb{E}\left[\sup_{0\le u\le t} Y_u^2\right] \le (x^2 + k_{\alpha,\beta}t)e^{c_\alpha t}. \tag{3.6}$$

Hence M is a (true) square integrable martingale and, taking expectation in (3.4), we obtain

$$\mathbb{E}[Y_t^2] \le x^2 + k_{\alpha,\beta}t. \tag{3.7}$$

Re-injecting this in (3.5), we get that, for any $T > 0$, there exists a positive constant $C_{\alpha,\beta,T}$ depending also on T, such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} Y_t^2\Big] \le C_{\alpha,\beta,T}(1+x^2). \tag{3.8}$$

We proceed with the study of (3.1). Denote by $0 < T_1 < T_2 < \dots$ the jumping times of N restricted to $\{|z| > 1\}$, and by (Z_n) the jumps which are i.i.d. random variables with distribution $\lambda^{-1}\mathbb{1}_{\{|z|>1\}}\nu(\mathrm{d}z)$, where $\lambda := \int_{\{|z|>1\}}\nu(\mathrm{d}z)$. Therefore $\int_0^t\int_{|z|>1} zN(\mathrm{d}s,\mathrm{d}z) = \sum_{n\in\mathbb{N}} Z_n\mathbb{1}_{\{T_n\le t\}}$ and (3.1) coincides with (3.2) on each time interval (T_n, T_{n+1}). Since $\check{V}$ is a solution of (3.2) on $[0, T_1)$, by using (3.8),

$$\mathbb{E}\Big[\sup_{t\in[0,T_1\wedge T)} \check{V}_t^2\Big|\mathcal{G}\Big] \le C_{\alpha,\beta,T}(1+x^2), \qquad \text{with } \mathcal{G} := \sigma(T_1, T_2, \dots).$$

By using the Jensen inequality and the classical inequality $(a+b)^\delta \le c_\delta(a^\delta + b^\delta)$, we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,T_1\wedge T)} \check{V}_t^\delta\Big|\mathcal{G}\Big] \le C_{\alpha,\beta,\delta,T}(1+|x|^\delta).$$

Furthermore, $\check{V}_{T_1} = \check{V}_{T_1-} + Z_1$, hence $|\check{V}_{T_1}|^\delta \le c_\delta\big(|\check{V}_{T_1-}|^\delta + |Z_1|^\delta\big)$. Since $\delta < \alpha$, $\mathbb{E}(|Z_1|^\delta) < \infty$. Consequently, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T_1\wedge T]} \check{V}_t^\delta \Big| \mathcal{G}\Big] \le C_{\alpha,\beta,\delta,T}(1+|x|^\delta).$$

Using the same inequality on (T_n, T_{n+1}), but starting from $\check{V}_{T_n}$, we can show that, for any $n \ge 0$,

$$u_n := \mathbb{E}\Big[\sup_{t\in[T_n\wedge T, T_{n+1}\wedge T]} \check{V}_t^\delta \Big| \mathcal{G}\Big] \le C'_{T,\delta}(1+\mathbb{E}[|\check{V}_{T_n}|^\delta|\mathcal{G}]) \quad (\text{with } T_0 = 0).$$

Then the sequence $(u_n)_{n\ge 0}$ satisfies $u_0 \le C'_{T,\delta}(1+|x|^\delta)$ and $u_{n+1} \le C'_{T,\delta}(1+u_n)$, implying that there exists $C_{T,\delta,x} > 1$ such that $u_n \le C_{T,\delta,x}^{n+1}$. We deduce that

$$\mathbb{E}\Big[\sup_{t\in[0,T_n\wedge T]} \check{V}_t^\delta \Big| \mathcal{G}\Big] \le u_0 + \cdots + u_{n-1} \le \frac{C_{T,\delta,x}^{n+1}}{C_{T,\delta,x}-1}.$$

Finally,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \check{V}_t^\delta\Big] \le \sum_{n\ge 0} \mathbb{E}\Big[\mathbb{1}_{T_n<T<T_{n+1}} \mathbb{E}\big(\sup_{t\in[0,T_n\wedge T]} \check{V}_t^\delta |\mathcal{G}\big)\Big] \le \frac{1}{C_{T,\delta,x}-1}\sum_{n\ge 0} C_{T,\delta,x}^{n+2}\frac{(\lambda T)^n}{n!}e^{-\lambda T} < \infty.$$

□

3.1.2 Ergodicity

The ergodic feature of the process $\check{V}$ is a consequence of Proposition 0.1, p. 604 in [10]. Indeed, provided that $\beta > 1$, the drift coefficient $b(x) = -\mathrm{sgn}(x)|x|^\beta$ and the jump measure $\nu(\mathrm{d}z) = |z|^{-1-\alpha}\mathbb{1}_{\mathbb{R}\setminus\{0\}}\mathrm{d}z$ clearly satisfy the conditions in the cited result. Hence $\check{V}$ is an exponential ergodic (and Harris recurrent) process having an unique invariant distribution, denoted by $m_{\alpha,\beta}$, which satisfies

$$m_{\alpha,\beta}([x,+\infty)) \underset{|x|\to\infty}{\sim} \int_{|x|}^{+\infty} \frac{\nu([u,+\infty))}{-b(x)}\mathrm{d}u = \frac{C}{|x|^{\alpha+\beta-1}} \tag{3.9}$$

as follows from Theorem 4.1, p. 92 in [13]. Clearly, the identity function, id $\in \mathrm{L}^1(m_{\alpha,\beta})$ under the hypothesis of Theorem 1.1, $\beta + \frac{\alpha}{2} - 2 > 0$. By the classical ergodic theorem, for all $f \in \mathrm{L}^1(m_{\alpha,\beta})$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\check{V}_s)\mathrm{d}s = \int_{\mathbb{R}} f(x) m_{\alpha,\beta}(\mathrm{d}x), \text{ a.s.} \tag{3.10}$$

Recall that we are interested on the behaviour as $\varepsilon \to 0$ of

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)} x^\varepsilon_{\varepsilon^{-\alpha}t} = \varepsilon^{\frac{\alpha\theta}{2}} \int_0^{t\varepsilon^{-\alpha\theta}} \check{V}_s \mathrm{d}s, \tag{3.11}$$

where θ is given by (1.9). In other words, we are studying a large time behaviour of a functional of $\check{V}$, hence it is quite natural to perform the study in steady state. In fact, we can prove the following lemma (see also [2], Theorem 2.6, p. 194):

Lemma 3.2. *Suppose that $\beta + \frac{\alpha}{2} - 2 > 0$. Assume that the process $\big\{\varepsilon^{\alpha\theta/2}\int_0^{t\varepsilon^{-\alpha\theta}} \check{V}_s\mathrm{d}s : t \ge 0\big\}$ converges, as $\varepsilon \to 0$, in distribution toward a Brownian motion, provided that $\check{V}$ is starting with $m_{\alpha,\beta}$ as an initial distribution. Then the same process converges in distribution toward a Brownian motion when $\check{V}_0 = 0$.*

Proof. In this proof we will denote the process in (3.11) by $Z_{\varepsilon,0}(t)$, and for $\Delta \ge 0$,

$$Z_{\varepsilon,\Delta}(t) := \varepsilon^{\frac{\alpha\theta}{2}} \int_\Delta^{t\varepsilon^{-\alpha\theta}+\Delta} \check{V}_s \mathrm{d}s.$$

First, let us prove that $Z_{\varepsilon,\Delta}(\cdot)$ converges in distribution, as $\Delta \to \infty$ and $\varepsilon \to 0$, toward a Brownian motion, when $\check{V}_0 = 0$. Denoting by μ_Δ the distribution of $\check{V}_\Delta$, for each bounded continuous real function ψ on $\mathrm{C}([0,+\infty))$, by the Markov property, we have

$$\mathbb{E}\big[\psi(Z_{\varepsilon,\Delta}(\cdot)) \mid \check{V}_0 = 0\big] = \mathbb{E}\big[\psi(Z_{\varepsilon,0}(\cdot)) \mid \check{V}_0 \sim \mu_\Delta\big].$$

We can write, for all $\varepsilon > 0$,

$$\begin{aligned}\big|\mathbb{E}\big[\psi(Z_{\varepsilon,0}(\cdot)) \mid \check{V}_0 \sim \mu_\Delta\big]-\mathbb{E}\big[\psi(Z_{\varepsilon,0}(\cdot)) \mid \check{V}_0 \sim m_{\alpha,\beta}\big]\big| &= \Big|\int_{\mathbb{R}} \mathbb{E}\big[\psi(Z_{\varepsilon,0}(\cdot)) \mid \check{V}_0 = y\big]\big(\mu_\Delta(\mathrm{d}y)-m_{\alpha,\beta}(\mathrm{d}y)\big)\Big| \\ &\leq \|\psi\|_\infty \int_{\mathbb{R}} \big|p(\Delta,0,\mathrm{d}y) - m_{\alpha,\beta}(\mathrm{d}y)\big| \leq \|\psi\|_\infty \|p(\Delta,0,\mathrm{d}y) - m_{\alpha,\beta}(\mathrm{d}y)\|_{\mathrm{TV}}\,,\end{aligned}$$

where $p(t,x,\mathrm{d}y) = \mathbb{P}_x(\check{V}_t \in \mathrm{d}y)$ is the transition kernel of $\check{V}$ (and therefore $p(\Delta,0,\mathrm{d}y) = \mu_\Delta(\mathrm{d}y)$) , and $\|\cdot\|_{\mathrm{TV}}$ is the norm in total variation. Since $\check{V}$ is (exponentially) ergodic, we get that

$$\lim_{\Delta\to\infty} \big|\mathbb{E}\big[\psi(Z_{\varepsilon,0}(\cdot)) \mid \check{V}_0 \sim \mu_\Delta\big] - \mathbb{E}\big[\psi(Z_{\varepsilon,0}(\cdot)) \mid \check{V}_0 \sim m_{\alpha,\beta}\big]\big| = 0, \quad \text{uniformly in } \varepsilon.$$

Second, by choosing $\Delta = \Delta(\varepsilon) = \varepsilon^{-\alpha\theta/4}$ we obtain

$$\sup_{t\geq 0}\Big\{\Big|Z_{\varepsilon,\Delta(\varepsilon)}(t) - \varepsilon^{\frac{\alpha\theta}{2}}\int_0^{t\varepsilon^{-\alpha\theta}+\Delta(\varepsilon)} \check{V}_s\mathrm{d}s\Big|\Big\} \leq \varepsilon^{\frac{\alpha\theta}{2}}\int_0^{\Delta(\varepsilon)} |\check{V}_s|\mathrm{d}s = \varepsilon^{\frac{\alpha\theta}{4}}\frac{1}{\Delta(\varepsilon)}\int_0^{\Delta(\varepsilon)} |\check{V}_s|\mathrm{d}s.$$

The right hand side term of the latter inequality tends to 0 almost surely, by using the ergodicity (3.10). Therefore $\varepsilon^{\alpha\theta/2}\int_0^{\bullet\varepsilon^{-\alpha\theta}+\Delta(\varepsilon)} \check{V}_s\mathrm{d}s$ converges in distribution, as $\varepsilon \to 0$, toward a Brownian motion when $\check{V}_0 = 0$. Clearly, $\lim_{\varepsilon\to 0}(t - \Delta(\varepsilon)\varepsilon^{\alpha\theta}) = t$, and applying Lemma p. 151 in [4] (a consequence of the continuous mapping theorem for the composition function), we can conclude. □

In the sequel, we will always assume that $\check{V}$ is starting with $m_{\alpha,\beta}$ as an initial distribution. Let us recall that the infinitesimal generator of $\check{V}$ is given by

$$(\mathcal{L}_{\alpha,\beta}\, g)(x) = -\mathrm{sgn}(x)|x|^\beta g'(x) + \int_{\mathbb{R}} \Big[g(x+y) - g(x) - yg'(x)\mathbb{1}_{|y|\leq 1}\Big]\nu(\mathrm{d}y)\,, \tag{3.12}$$

with the domain $D_{\mathcal{L}_{\alpha,\beta}}$. Also denote $(\mathcal{T}_t)_{t\geq 0}$ the semi-group associated to the operator $\mathcal{L}_{\alpha,\beta}$ (or to the process $\check{V}$). We collect in the following lemma some useful properties of the process $\check{V}$.

Lemma 3.3.

1. *The domain $D_{\mathcal{L}_{\alpha,\beta}}$ contains the space of bounded twice differentiable functions $\mathrm{C}_b^2(\mathbb{R})$.*
2. *For all $p \geq 1$, $\mathcal{T}_t$ is a contraction semi-group on $\mathrm{L}^p(m_{\alpha,\beta})$ and for each $f \in \mathrm{L}^p(m_{\alpha,\beta})$,*

$$\lim_{t\to 0} \|\mathcal{T}_t f - f\|_{\mathrm{L}^p(m_{\alpha,\beta})} = 0. \tag{3.13}$$

Proof. To prove the first point, we fix $f \in \mathrm{C}_b^2(\mathbb{R})$ and we show that $(\mathcal{L}_{\alpha,\beta}\, f)(x) < \infty$. First, $-\mathrm{sgn}(x)|x|^\beta f'(x)$ is well defined for all $x \in \mathbb{R}$. Since $f \in \mathrm{C}_b^2(\mathbb{R})$, $\forall y \in [-1,1]$,

$$\Big|f(x+y) - f(x) - yf'(x)\Big| \leq y^2 \sup_{z\in[x-1,x+1]} |f''(z)| < \infty,$$

and we find

$$\int_{|y|\leq 1} \Big[f(x+y) - f(x) - yf'(x)\Big]\nu(\mathrm{d}y) \leq \Big[\sup_{z\in[x-1,x+1]} |f''(z)|\Big]\int_{|y|\leq 1} y^2\nu(\mathrm{d}y) < \infty.$$

Since f is bounded, we have

$$\int_{|y|>1} \Big[f(x+y) - f(x)\Big]\nu(\mathrm{d}y) \leq 2\|f\|_\infty \int_{|y|>1} \nu(\mathrm{d}y) < \infty\,,$$

hence $f \in \mathcal{D}_{\mathcal{L}_{\alpha,\beta}}$.

We proceed with the proof of the second point. Fix $f \in \mathrm{L}^p(m_{\alpha,\beta})$ and we show first that

$$\|\mathcal{T}_t f\|_{\mathrm{L}^p(m_{\alpha,\beta})} \leq \|f\|_{\mathrm{L}^p(m_{\alpha,\beta})}.$$

Since

$$\|\mathcal{T}_t f\|^p_{\mathrm{L}^p(m_{\alpha,\beta})} = \int_{\mathbb{R}} |\mathcal{T}_t f(x)|^p m_{\alpha,\beta}(\mathrm{d}x) = \int_{\mathbb{R}} |\mathbb{E}_x(f(\check{V}_t))|^p m_{\alpha,\beta}(\mathrm{d}x),$$

by the Jensen inequality ($p \geq 1$), we get

$$\|\mathcal{T}_t f\|_p^p \leq \int_{\mathbb{R}} \mathbb{E}_x(|f(\check{V}_t)|^p) m_{\alpha,\beta}(\mathrm{d}x) = \mathbb{E}_{m_{\alpha,\beta}}(|f(\check{V}_t)|^p) = ||f||^p_{\mathrm{L}^p(m_{\alpha,\beta})}.$$

Finally, we prove (3.13). Since $\mathrm{C}^2_b(\mathbb{R})$ is dense in $\mathrm{L}^p(m_{\alpha,\beta})$, there exists $f_\eta \in \mathrm{C}^2_b(\mathbb{R})$ such that $||f - f_\eta||_{\mathrm{L}^p(m_{\alpha,\beta})} \leq \eta/3$. Since $\mathcal{T}_t$ is a contraction semi-group and $m_{\alpha,\beta}$ is a probability measure, we get

$$\|\mathcal{T}_t f - f\|_{\mathrm{L}^p(m_{\alpha,\beta})} \leq 2\|f - f_\eta\|_{\mathrm{L}^p(m_{\alpha,\beta})} + \|\mathcal{T}_t f_\eta - f_\eta\|_\infty \leq (2\eta)/3 + \|\mathcal{T}_t f_\eta - f_\eta\|_\infty.$$

Since $\mathcal{T}_t$ is a Feller semi-group (see for instance, [1], p. 151), for t small enough, we have $\|\mathcal{T}_t f_\eta - f_\eta\|_\infty \leq \eta/3$ and we deduce (3.13). The proof is complete. □

3.1.3 Convergence in probability

The main result of this section concerns the behaviour of the speed process which is described by using a Lyapunov function.

Proposition 3.4. *Suppose that $\beta + \frac{\alpha}{2} > 2$ and let p and γ such that*

$$p > 1, \;\; p\gamma > 2, \;\; 2 - \beta < \gamma < \frac{\alpha}{2}. \tag{3.14}$$

Introduce the Lyapunov function

$$h_{p,\gamma}(x) := (1 + |x|^{p\gamma})^{1/p}. \tag{3.15}$$

Then, as $\varepsilon \to 0$, $\{\varepsilon^{\alpha\theta/2} h_{p,\gamma}(\varepsilon^{-\theta} V_t^\varepsilon) : t \geq 0\}$ converges to 0 in probability uniformly on each compact time interval. More precisely, there exists $q > 2$ such that, for any fixed $T > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{E}\Big[\Big(\sup_{t \in [0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\Big(\varepsilon^{-\theta} V_t^\varepsilon\Big)\Big)^q\Big] = \mathbb{E}\Big[\Big(\sup_{t \in [0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\big(\check{V}_{t\,\varepsilon^{-\alpha\theta}}\big)\Big)^q\Big] = 0. \tag{3.16}$$

In order to prove this result, we need the following lemma whose proof is postponed to the Appendix.

Lemma 3.5.

1. *If $p\gamma > 2$, $h_{p,\gamma}$ is a twice differentiable function and there exists a positive constant k such that for all $(x, y) \in \mathbb{R}^2$,*

   - *if $|x| < 1$ then*
   $$|h_{p,\gamma}(x + y) - h_{p,\gamma}(x)| \leq k(|y|\mathbb{1}_{\{|y|\leq 1\}} + |y|^\gamma \mathbb{1}_{\{|y|>1\}});$$
   - *if $|x| \geq 1$ then*
   $$|h_{p,\gamma}(x + y) - h_{p,\gamma}(x)| \leq k(|y||x|^{\gamma-1}\mathbb{1}_{\{|y|\leq i(x)\}} + |y|^\gamma \mathbb{1}_{\{i(x)<|y|\}}),$$

   where $i(x) := (2|x|^{p\gamma} + 1)^{1/p\gamma} - |x|$.

2. *Assume that $p\gamma > 2$ and $2 - \beta < \gamma < \alpha$. There exist a continuous function $f_{p,\alpha,\beta,\gamma}$, a compact set K and a constant d (depending only on p, α, β, γ) such that*

$$\forall x \in \mathbb{R}, \;\; f_{p,\alpha,\beta,\gamma}(x) \geq 1 + |x|, \quad f_{p,\alpha,\beta,\gamma}(x) \underset{|x|\to\infty}{\sim} \gamma|x|^{\gamma+\beta-1}, \tag{3.17}$$

   and

$$(\mathcal{L}_{\alpha,\beta}\, h_{p,\gamma})(x) \leq -f_{p,\alpha,\beta,\gamma}(x) + d\mathbb{1}_K. \tag{3.18}$$

Proof of Proposition 3.4. By (1.6_2), we can write

$$\varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\Big(\frac{V_t^\varepsilon}{\varepsilon^\theta}\Big) = \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\big(\check{V}_{t\,\varepsilon^{-\alpha\theta}}\big) \tag{3.19}$$

and the first equality in (3.16) is clear. Since $2-\beta < \frac{\alpha}{2}$ and $\beta > 1$, we can fix q such that $\frac{2}{p} \vee (2-\beta) < \gamma < 2\gamma < q\gamma < \alpha$ and $2 < q < \frac{\beta-1}{\alpha} + 2$. By noting that $h_{p,\gamma}(x)^q = h_{\frac{p}{q},q\gamma}(x)$, we can write

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\big(\check{V}_{t\,\varepsilon^{-\alpha\theta}}\big)\Big)^q\Big] = \varepsilon^{q\frac{\alpha\theta}{2}}\, \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} h_{\frac{p}{q},q\gamma}\Big(\check{V}_{t\,\varepsilon^{-\alpha\theta}}\Big)\Big)\Big].$$

Employing Itô's formula with $h_{\frac{p}{q},q\gamma}$, we get

$$h_{\frac{p}{q},q\gamma}(\check{V}_t) - h_{\frac{p}{q},q\gamma}(\check{V}_0) = R_t + \int_0^t (\mathcal{L}_{\alpha,\beta}\, h_{\frac{p}{q},q\gamma})(\check{V}_s)\mathrm{d}s, \tag{3.20}$$

where

$$R_t := \int_0^t \int_{\mathbb{R}} \Big(h_{\frac{p}{q},q\gamma}(\check{V}_s + y) - h_{\frac{p}{q},q\gamma}(\check{V}_s)\Big)\tilde{N}(\mathrm{d}y, \mathrm{d}s).$$

By Lemma 3.5 applied to the function $h_{\frac{p}{q},q\gamma}$, we see that there exists $c > 0$ such that, for all $t \in [0, T]$,

$$\int_0^t (\mathcal{L}_{\alpha,\beta}\, h_{\frac{p}{q},q\gamma})(\check{V}_s)\mathrm{d}s \le ct.$$

Moreover, let us note that $h_{\frac{p}{q},q\gamma}$ is continuous and that $h_{\frac{p}{q},q\gamma}(x) \sim |x|^{q\gamma}$, as $|x| \to \infty$. Hence, by the choice of q, we have $h_{\frac{p}{q},q\gamma} \in \mathrm{L}^1(m_{\alpha,\beta})$. Replacing in (3.20), we obtain

$$\varepsilon^{q\frac{\alpha\theta}{2}}\, \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} h_{\frac{p}{q},q\gamma}\Big(\check{V}_{t\,\varepsilon^{-\alpha\theta}}\Big)\Big)\Big] \le \varepsilon^{q\frac{\alpha\theta}{2}}\, \|h_{\frac{p}{q},q\gamma}\|_{\mathrm{L}^1(m_{\alpha,\beta})} + \varepsilon^{(q-2)\frac{\alpha\theta}{2}}\, cT + \varepsilon^{q\frac{\alpha\theta}{2}}\, \mathbb{E}\Big(\sup_{t\in[0,T]} R_{t\,\varepsilon^{-\alpha\theta}}\Big).$$

Since $q > 2$, the first and the second term converge toward 0. For the last term, we use Kunita's first inequality (see for instance [1], p. 265): since $\check{V}_0 \sim m_{\alpha,\beta}$, then for all t, $\check{V}_t \sim m_{\alpha,\beta}$ and there exists a positive constant C such that

$$\mathbb{E}\Big(\sup_{t\in[0,T]} R_{t\,\varepsilon^{-\alpha\theta}}\Big) \le \mathbb{E}\Big(\sup_{t\in[0,T]} R^2_{t\,\varepsilon^{-\alpha\theta}}\Big)^{1/2} \le C\,\sqrt{T}\,\varepsilon^{-\frac{\alpha\theta}{2}} \iint_{\mathbb{R}^2} \big(h_{\frac{p}{q},q\gamma}(x+y) - h_{\frac{p}{q},q\gamma}(x)\big)^2 \nu(\mathrm{d}y) m_{\alpha,\beta}(\mathrm{d}x).$$

It is sufficient to show that

$$\iint_{\mathbb{R}^2} \big(h_{\frac{p}{q},q\gamma}(x+y) - h_{\frac{p}{q},q\gamma}(x)\big)^2 \nu(\mathrm{d}y) m_{\alpha,\beta}(\mathrm{d}x) < \infty. \tag{3.21}$$

This fact is obtained by using Lemma 3.5. If $|x| \ge 1$,

$$\big(h_{\frac{p}{q},q\gamma}(x+y) - h_{\frac{p}{q},q\gamma}(x)\big)^2 \le k^2(|y|^2|x|^{2q\gamma-2}\mathbb{1}_{\{|y|\le i(x)\}} + |y|^{2q\gamma}\mathbb{1}_{\{i(x)<|y|\}}),$$

hence

$$\int_{\mathbb{R}} \Big(h_{\frac{p}{q},q\gamma}(x+y) - h_{\frac{p}{q},q\gamma}(x)\Big)^2 \nu(\mathrm{d}y) = O(|x|^{2q\gamma-\alpha}), \text{ as } |x| \to +\infty,$$

and, since $q < \frac{\beta-1}{\alpha} + 2$, we get (3.21). If $|x| < 1$,

$$\big(h_{\frac{p}{q},q\gamma}(x+y) - h_{\frac{p}{q},q\gamma}(x)\big)^2 \le k^2(|y|^2\mathbb{1}_{\{|y|\le 1\}} + |y|^{2q\gamma}\mathbb{1}_{\{|y|>1\}})$$

and $\int_{\mathbb{R}^2} \big(h_{\frac{p}{q},q\gamma}(x+y) - h_{\frac{p}{q},q\gamma}(x)\big)^2 \nu(\mathrm{d}y)$ is finite independently of x. Since $m_{\alpha,\beta}$ is a probability measure, (3.21) is verified again. The proof is complete except for Lemma 3.5. □

3.2 The position process X^ε

We are ready to prove our main result concerning the behaviour of the position process. Recall that, thanks to Lemma 3.2, we assume that $\check{V}$ is starting with $m_{\alpha,\beta}$ as an initial distribution.

Proof of Theorem 1.1 for the case $\alpha \in (0,2)$. Thanks to (3.17), Theorem 3.2, p. 924 in [7] applies and we deduce that the Poisson equation $\mathcal{L}g = \mathrm{id}$ admits a solution $\hat{g}$ satisfying $|\hat{g}| \leq c(h_{p,\gamma}+1)$, with c a positive constant. Applying Itô-Levy's formula with $\hat{g}$, we get

$$\hat{g}(\check{V}_t) - \hat{g}(\check{V}_0) = \int_0^t \check{V}_s \mathrm{d}s + M_t, \tag{3.22}$$

where

$$M_t := \int_0^t \int_{\mathbb{R}} [\hat{g}(z+\check{V}_s) - \hat{g}(\check{V}_s)] \tilde{N}(\mathrm{d}s, \mathrm{d}z). \tag{3.23}$$

Step 1) We prove that M given by the latter formula is a square integrable true martingale. On one hand we have

$$\mathbb{E}[\hat{g}(\check{V}_t)^2] = \mathbb{E}[\hat{g}(V_0)^2] = \int_{\mathbb{R}} \hat{g}(x)^2 m_{\alpha,\beta}(\mathrm{d}x) < \infty.$$

Indeed, recall that $h_{p,\gamma}^2$ is continuous and it behaves as $|x|^{2\gamma}$ in the neighbourhood of the infinity. Recalling that γ was chosen such that $\frac{4}{p} \vee (4-2\beta) < 2\gamma < \alpha$, by using (3.9), we see that

$$\int_{\mathbb{R}} h_{p,\gamma}(x)^2 m_{\alpha,\beta}(\mathrm{d}x) < \infty.$$

On the other hand, we can write

$$\mathbb{E}\left[\Big(\int_0^t \check{V}_s \mathrm{d}s\Big)^2\right] = \mathbb{E}\int_0^t\int_0^t \check{V}_u\check{V}_s \,\mathrm{d}u\mathrm{d}s = 2\mathbb{E}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}u\, \check{V}_u\check{V}_s \leq 2\mathbb{E}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}u\,|\check{V}_u||\check{V}_s|.$$

Using Markov's property and that $\check{V}_u$ and $\check{V}_0$ follow the invariant law, we get, for $u<s$, $\mathbb{E}\big(|\check{V}_s||\check{V}_u|\big) = \mathbb{E}\big(|\check{V}_{s-u}||\check{V}_0|\big)$. Therefore

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^t \check{V}_s \mathrm{d}s\Big)^2\Big] &\leq 2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}u\,\mathbb{E}\big(|\check{V}_{s-u}||\check{V}_0|\big) = 2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}u\,\mathbb{E}\big(|\check{V}_u||\check{V}_0|\big)\\ &= 2\int_0^t \mathrm{d}s\,\mathbb{E}\Big(|\check{V}_0|\int_0^s \mathcal{T}_u|\mathrm{id}|(\check{V}_0)\mathrm{d}u\Big).\end{aligned}$$

Applying again Theorem 3.2, p. 924 in [7], we deduce that the Poisson equation $\mathcal{L}_{\alpha,\beta}\, g = |\mathrm{id}|$ admits a solution $\tilde{g}$ satisfying $|\tilde{g}| \leq c'(h_{p,\gamma}+1)$ with c' a positive constant. Moreover

$$\int_0^s \mathcal{T}_u|\mathrm{id}|(\check{V}_0)\mathrm{d}u = \mathcal{T}_s\tilde{g}(\check{V}_0) - \tilde{g}(\check{V}_0).$$

Replacing in the latter inequality

$$\mathbb{E}\Big[\Big(\int_0^t \check{V}_s \mathrm{d}s\Big)^2\Big] \leq 2\int_0^t \mathbb{E}\left(|\check{V}_0||\mathcal{T}_s\tilde{g}(\check{V}_0) - \tilde{g}(\check{V}_0)|\right)\mathrm{d}s = 2\int_0^t \mathrm{d}s\int_{\mathbb{R}} |x||\mathcal{T}_s\tilde{g}(x) - \tilde{g}(x)| m_{\alpha,\beta}(\mathrm{d}x).$$

At this level, we need to apply the Hölder inequality to conclude that

$$\mathbb{E}\Big[\int_0^t \check{V}_s \mathrm{d}s\Big]^2 < \infty. \tag{3.24}$$

First, if $\beta < 2$ then we choose γ close enough to $2-\beta$ such that $\tilde{g} \in \mathrm{L}^{(3-\beta)/(2-\beta)}(m_{\alpha,\beta})$. Since $\frac{3-\beta}{2-\beta} > 1$, using the second part of Lemma 3.3, we get

$$\|\mathcal{T}_s\tilde{g} - \tilde{g}\|_{\mathrm{L}^{(3-\beta)/(2-\beta)}(m_{\alpha,\beta})} \leq 2\|\tilde{g}\|_{\mathrm{L}^{(3-\beta)/(2-\beta)}(m_{\alpha,\beta})}.$$

By the Hölder inequality and the fact that $|\mathrm{id}| \in \mathrm{L}^{3-\beta}(m_{\alpha,\beta})$, we get (3.24). Second, if $\beta \geq 2$, we choose $\gamma < 1$ close enough to 0 such that $|\mathrm{id}| \in \mathrm{L}^{1/(1-\gamma)}(m_{\alpha,\beta})$. Since $\tilde{g} \in \mathrm{L}^{1/\gamma}(m_{\alpha,\beta})$, using again Lemma 3.3, we get

$$\|\mathcal{T}_t\tilde{g} - \tilde{g}\|_{\mathrm{L}^{1/\gamma}(m_{\alpha,\beta})} \leq 2\|\tilde{g}\|_{\mathrm{L}^{1/\gamma}(m_{\alpha,\beta})}.$$

Since $|\mathrm{id}| \in \mathrm{L}^{1/(1-\gamma)}(m_{\alpha,\beta})$, we can apply the Hölder inequality and get (3.24) again.

We conclude that M given by (3.23) is a square integrable true martingale. Moreover, we can compute its quadratic variation

$$\langle M\rangle_t = \int_0^t \int_{\mathbb{R}} [\hat{g}(y + \check{V}_s) - \hat{g}(\check{V}_s)]^2 \nu(\mathrm{d}y)\mathrm{d}s, \tag{3.25}$$

hence

$$\mathbb{E}[\langle M\rangle_t] = t \iint_{\mathbb{R}^2} [\hat{g}(x + y) - \hat{g}(x)]^2 \nu(\mathrm{d}y) m_{\alpha,\beta}(\mathrm{d}x) < \infty. \tag{3.26}$$

Step 2) Performing a simple time change in (3.22), we see that the process in (1.8) can be written

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)} X_t^\varepsilon = \varepsilon^{\frac{\alpha\theta}{2}} \Big[\hat{g}(\check{V}_{t\,\varepsilon^{-\alpha\theta}}) - \hat{g}(\check{V}_0)\Big] - \varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}. \tag{3.27}$$

In this step, we show that the martingale term on the right hand side of the latter equality converges to a Brownian motion by using Whitt's theorem (see Theorem 2.1 (ii) in [15], pp. 270-271). We need to verify the hypotheses of this result. In order, since the function

$$x \mapsto \int_{\mathbb{R}} [\hat{g}(x + y) - \hat{g}(x)]^2 \nu(\mathrm{d}y) \in \mathrm{L}^1(m_{\alpha,\beta}),$$

by using (3.25) and the ergodic theorem (3.10), we deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0} \langle \varepsilon^{\frac{\alpha\theta}{2}} M_{\bullet\,\varepsilon^{-\alpha\theta}}\rangle_t = \lim_{\varepsilon\to 0} \varepsilon^{\frac{\alpha\theta}{2}} \int_0^{t\varepsilon^{-\alpha\theta}} \int_{\mathbb{R}} [\hat{g}(y + \check{V}_s) - \hat{g}(\check{V}_s)]^2 \nu(\mathrm{d}y)\mathrm{d}s \\ = t \iint_{\mathbb{R}^2} [\hat{g}(x + y) - \hat{g}(x)]^2 \nu(\mathrm{d}y) m_{\alpha,\beta}(\mathrm{d}x).\end{aligned}$$

The condition (6) in [15], p. 271 is fulfilled. Again by (3.25), we see that $\langle M\rangle$ has no jump, hence the condition (4) in [15], p. 270 is trivial. Let us note also that, by (3.22), the jumps of the martingale M_t are $J(M_t) := \hat{g}(\check{V}_t) - \hat{g}(\check{V}_{t-})$. Therefore we deduce that the jumps of the martingale term on the right hand side of (3.27) are

$$\begin{aligned}J\Big(\varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}\Big) := \varepsilon^{\frac{\alpha\theta}{2}} \Big[\hat{g}\big(\check{V}_{\varepsilon^{-\alpha\theta}t}\big) - \hat{g}\big(\check{V}_{\varepsilon^{-\alpha\theta}t-}\big)\Big] &\leq c\,\varepsilon^{\frac{\alpha\theta}{2}} \Big[\big|h_{p,\gamma}\big(\check{V}_{\varepsilon^{-\alpha\theta}t}\big)\big| + \big|h_{p,\gamma}\big(\check{V}_{\varepsilon^{-\alpha\theta}t-}\big)\big| + 2\Big] \\ &\leq 2c\,\varepsilon^{\frac{\alpha\theta}{2}} \Big[\sup_{t\in[0,T]} \Big|h_{p,\gamma}\Big(\varepsilon^{-\theta} V_t^\varepsilon\Big)\Big| + 1\Big],\end{aligned}$$

by using the fact that $|\hat{g}| \leq c(h_{p,\gamma} + 1)$ and (3.19). By Proposition 3.4,

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[\sup_{t\in[0,T]} J\Big(\varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}\Big)^2\Big] = 0.$$

Therefore we can apply Whitt's theorem to deduce that $\big\{\varepsilon^{(\alpha\theta)/2} M_{t\,\varepsilon^{-\alpha\theta}} : t \geq 0\big\}$ converges in distribution (as a process) toward $\kappa_{\alpha,\beta}^{1/2} \check{B}$, where $\check{B}$ is a standard Brownian motion and

$$\kappa_{\alpha,\beta} := \iint_{\mathbb{R}^2} [\hat{g}(x + y) - \hat{g}(x)]^2 \nu(\mathrm{d}y) m_{\alpha,\beta}(\mathrm{d}x) > 0. \tag{3.28}$$

The constant $\kappa_{\alpha,\beta}$ is positive by noting that ν is absolutely continuous with respect to the Lebesgue measure, that $m_{\alpha,\beta}$ has a non-empty support, and that $\hat{g}$ could not be a constant function, since $\mathcal{L}\hat{g} = \mathrm{id}$.

Step 3) By using that $|\hat{g}| \le c(h_{p,\gamma}+1)$, we get

$$\Big|\hat{g}\big(\check{V}_{t\,\varepsilon^{-\alpha\theta}}\big)-\hat{g}(\check{V}_0)\Big|^2 \le 4c^2\Big(\Big|h_{p,\gamma}\big(\check{V}_{t\,\varepsilon^{-\alpha\theta}}\big)\Big|^2+\Big|h_{p,\gamma}(\check{V}_0)\Big|^2+2\Big)$$

hence, using Proposition 3.4,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\varepsilon^{\alpha\theta}\sup_{t\in[0,T]}\Big|\hat{g}\big(\check{V}_{t\,\varepsilon^{-\alpha\theta}}\big)-\hat{g}(\check{V}_0)\Big|^2\Big]=0$$

hence $\big\{\varepsilon^{(\alpha\theta)/2}\big[\hat{g}\big(\check{V}_{t\,\varepsilon^{-\alpha\theta}}\big)-\hat{g}(\check{V}_0)\big] : t\ge 0\big\}$ converges in probability toward 0, uniformly on compact sets.

Step 4) Our processes are valued in the Skorokhod space of càdlàg functions $\mathrm{D}([0,\infty))$ endowed with J_1 (or simple) Skorokhod topology (see [16], §3.3). It is not difficult to see that a sequence which converges in probability toward 0, uniformly on compact sets, is also convergent in probability for J_1 metric, hence in distribution in J_1 topology. Recall that in the Skorokhod space, the addition is not a continuous map (see for instance [16], p. 84). In our case, the limits of the terms on the right hand side of equality (3.27) are, respectively 0 and a Brownian motion which have continuous paths. By using the joint convergence theorem (Theorem 11.4.5, p. 379 in [16]) and the continuous-mapping theorem (Theorem 3.4.3, p. 86 in [16]), we obtain the conclusion of Theorem 1.1. □

Proposition 3.6. *The constant* $\kappa_{\alpha,\beta}$ *in Theorem* 1.1 *given in* (3.28) *satisfies*

$$\kappa_{\alpha,\beta}=-2\int_{\mathbb{R}}x\hat{g}(x)m_{\alpha,\beta}(\mathrm{d}x)>0. \tag{3.29}$$

Proof. Since, by (3.26) and (3.28), $\kappa_{\alpha,\beta}=\frac{1}{t}\mathbb{E}\big[M_t^2\big]$, for all $t>0$, by taking $t=\varepsilon^{\alpha\theta}$ and using Itô's formula, we get

$$\begin{aligned}\kappa_{\alpha,\beta}=\varepsilon^{-\alpha\theta}\mathbb{E}\Big[\Big(\hat{g}\big(\check{V}_{\varepsilon^{\alpha\theta}}\big)-\hat{g}(\check{V}_0)-\int_0^{\varepsilon^{\alpha\theta}}\check{V}_s\mathrm{d}s\Big)^2\Big]&=\varepsilon^{-\alpha\theta}\Big\{\mathbb{E}\Big[\Big(\hat{g}\big(\check{V}_{\varepsilon^{\alpha\theta}}\big)-\hat{g}(\check{V}_0)\Big)^2\Big]\\&+\mathbb{E}\Big[\Big(\int_0^{\varepsilon^{\alpha\theta}}\check{V}_s\mathrm{d}s\Big)^2\Big]-2\mathbb{E}\Big[\Big(\hat{g}\big(\check{V}_{\varepsilon^{\alpha\theta}}\big)-\hat{g}(\check{V}_0)\Big)\int_0^{\varepsilon^{\alpha\theta}}\check{V}_s\mathrm{d}s\Big]\Big\}.\end{aligned} \tag{3.30}$$

The first term on the right hand side of (3.30) can be written :

$$\begin{aligned}\mathbb{E}\Big[\Big(\hat{g}\big(\check{V}_{\varepsilon^{\alpha\theta}}\big)-\hat{g}(\check{V}_0)\Big)^2\Big]&=2\int\hat{g}(x)^2m_{\alpha,\beta}(\mathrm{d}x)-2\mathbb{E}\Big[\hat{g}(\check{V}_0)\hat{g}\big(\check{V}_{\varepsilon^{\alpha\theta}}\big)\Big]=2\int\hat{g}(x)^2m_{\alpha,\beta}(\mathrm{d}x)\\&-2\mathbb{E}\Big[\hat{g}(\check{V}_0)\mathbb{E}\Big(\hat{g}\big(\check{V}_{\varepsilon^{\alpha\theta}}\big)\mid\check{V}_0\Big)\Big]=2\int\hat{g}(x)^2m_{\alpha,\beta}(\mathrm{d}x)-2\mathbb{E}\Big[\hat{g}(\check{V}_0)\big(\mathcal{T}_{\varepsilon^{\alpha\theta}}\hat{g}\big)(\check{V}_0)\Big]\\&=2\int\hat{g}(x)^2m_{\alpha,\beta}(\mathrm{d}x)-2\mathbb{E}\Big[\hat{g}(\check{V}_0)\Big(\hat{g}(\check{V}_0)+\int_0^{\varepsilon^{\alpha\theta}}(\mathcal{T}_s\mathrm{id})(\check{V}_0)\mathrm{d}s\Big)\Big]\\&=-2\mathbb{E}\Big[\hat{g}(\check{V}_0)\int_0^{\varepsilon^{\alpha\theta}}(\mathcal{T}_s\mathrm{id})(\check{V}_0)\mathrm{d}s\Big]=-2\int\hat{g}(x)\,m_{\alpha,\beta}(\mathrm{d}x)\int_0^{\varepsilon^{\alpha\theta}}(\mathcal{T}_s\mathrm{id})(x)\mathrm{d}s\\&=-2\,\varepsilon^{\alpha\theta}\int x\hat{g}(x)m_{\alpha,\beta}(\mathrm{d}x)-2\int\hat{g}(x)\,m_{\alpha,\beta}(\mathrm{d}x)\int_0^{\varepsilon^{\alpha\theta}}\big((\mathcal{T}_s\mathrm{id})-\mathrm{id}\big)(x)\mathrm{d}s.\end{aligned}$$

By using the Hölder inequality, we prove that,

$$\mathbb{E}\left[\Big(\hat{g}\big(\check{V}_{\varepsilon^{\alpha\theta}}\big)-\hat{g}(\check{V}_0)\Big)^2\right]\sim-2\,\varepsilon^{\alpha\theta}\int x\hat{g}(x)m_{\alpha,\beta}(\mathrm{d}x),\ \text{ as }\varepsilon\to 0. \tag{3.31}$$

Indeed, if $2-\frac{\alpha}{2}<\beta<2$, $\hat{g}\in\mathrm{L}^{3-\beta/2-\beta}(m_{\alpha,\beta})$ and $\lim_{s\to 0}\|(\mathcal{T}_s\mathrm{id})-\mathrm{id}\|_{\mathrm{L}^{3-\beta}(m_{\alpha,\beta})}=0$, and if $\beta\ge 2$, $\hat{g}\in\mathrm{L}^{\frac{1}{\gamma}}(m_{\alpha,\beta})$ and $\lim_{s\to 0}\|(\mathcal{T}_s\mathrm{id})-\mathrm{id}\|_{\mathrm{L}^{1/(1-\gamma)}(m_{\alpha,\beta})}=0$.

By using (3.24) and Fubini's theorem, the second term on the right hand side of (3.30) can be written

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^{\varepsilon^{\alpha\theta}}\check{V}_s\mathrm{d}s\Big)^2\Big] &= \int_0^{\varepsilon^{\alpha\theta}}\mathrm{d}s\int_0^s\mathbb{E}\Big(\check{V}_s\check{V}_u\Big)\mathrm{d}u = \int_0^{\varepsilon^{\alpha\theta}}\mathrm{d}s\int_0^s\mathbb{E}\Big(\check{V}_{s-u}\check{V}_0\Big)\mathrm{d}u\\ &= \int_0^{\varepsilon^{\alpha\theta}}\mathrm{d}s\int_0^s\mathbb{E}\Big(\check{V}_0\,(\mathcal{T}_{s-u}\mathrm{id})(\check{V}_0)\Big)\mathrm{d}u = \int_0^{\varepsilon^{\alpha\theta}}\mathrm{d}u\,\mathbb{E}\Big(\check{V}_0\int_u^{\varepsilon^{\alpha\theta}}(\mathcal{T}_{s-u}\mathrm{id})(\check{V}_0)\,\mathrm{d}s\Big)\\ &= \int_0^{\varepsilon^{\alpha\theta}}\mathrm{d}u\,\mathbb{E}\Big[\check{V}_0\Big((\mathcal{T}_{\varepsilon^{\alpha\theta}-u}\hat{g})(\check{V}_0)-\hat{g}(\check{V}_0)\Big)\Big] = \int_0^{\varepsilon^{\alpha\theta}}\mathrm{d}u\int x\Big((\mathcal{T}_{\varepsilon^{\alpha\theta}-u}\hat{g})-\hat{g}\Big)(x)m_{\alpha,\beta}(\mathrm{d}x).\end{aligned}$$

Once again by the Hölder inequality, we prove that

$$\mathbb{E}\left[\Big(\int_0^{\varepsilon^{\alpha\theta}}\check{V}_s\mathrm{d}s\Big)^2\right] = o(\varepsilon^{\alpha\theta}),\ \text{ as } \varepsilon\to 0. \tag{3.32}$$

Indeed, if $2-\frac{\alpha}{2}<\beta<2$ then $\mathrm{id}\in \mathrm{L}^{3-\beta}(m_{\alpha,\beta})$, we can see that

$$\lim_{\varepsilon\to 0}\sup_{0\le u\le\varepsilon^{\alpha\theta}}\|(\mathcal{T}_{\varepsilon^{\alpha\theta}-u}\hat{g})-\hat{g}\|_{\mathrm{L}^{3-\beta/2-\beta}(m_{\alpha,\beta})}=0.$$

Similarly, if $\beta\ge 2$ then $\mathrm{id}\in\mathrm{L}^{1/(1-\gamma)}(m_{\alpha,\beta})$, we see that

$$\lim_{\varepsilon\to 0}\sup_{0\le u\le\varepsilon^{\alpha\theta}}\|\mathcal{T}_{\varepsilon^{\alpha\theta}-u}\hat{g})-\hat{g}\|_{\mathrm{L}^{\frac{1}{\gamma}}(m_{\alpha,\beta})}=0.$$

Finally, the third term in (3.30) is analysed by using the Cauchy-Schwartz inequality and the behaviour of the other terms. We get that

$$-2\,\mathbb{E}\Big[\Big(\hat{g}(\check{V}_{\varepsilon^{\alpha\theta}})-\hat{g}(\check{V}_0)\Big)\int_0^{\varepsilon^{\alpha\theta}}\check{V}_s\mathrm{d}s\Big] = o(\varepsilon^{\alpha\theta}),\ \text{ as }\ \varepsilon\to 0. \tag{3.33}$$

Putting together (3.30)-(3.32), we obtain that

$$\kappa_{\alpha,\beta} = -2\int x\hat{g}(x)m_{\alpha,\beta}(\mathrm{d}x)+o(1),\ \text{ as }\ \varepsilon\to 0.$$

and the result is proved. □

3.3 Appendix

Proof of Lemma 2.3. Note that g_β is an odd function. Introduce $\varphi_\beta(x)=-\int_x^{+\infty}2y\mathrm{e}^{c_\beta(y)}\mathrm{d}y$. By the continuity of g_β on $[0,1]$, it is sufficient to prove (2.3) for $x>1$. Assume $\beta\in[1,\infty)$, then, since $x>1$,

$$\varphi_\beta(x)=\int_x^{+\infty}z^{1-\beta}\left(-2z^\beta\mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}}\right)\mathrm{d}z\ge\int_x^{+\infty}-2z^\beta\mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}}\mathrm{d}z=-\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}},$$

hence

$$\int_1^x\mathrm{e}^{\frac{2}{\beta+1}y^{\beta+1}}\varphi_\beta(y)\mathrm{d}y\ge 1-x,$$

and (2.3) is true in this case. If $\beta\in[0,1)$, by integration by parts,

$$\begin{aligned}\varphi_\beta(x)=\int_x^{+\infty}z^{1-\beta}\big(-2z^\beta\mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}}\big)\mathrm{d}z &= -x^{1-\beta}\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}}+\frac{1-\beta}{2}\int_x^{+\infty}z^{-2\beta}\big(-2z^\beta\mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}}\big)\mathrm{d}z\\ &\ge -x^{1-\beta}\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}}-\frac{1-\beta}{2}x^{-2\beta}\mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}},\end{aligned}$$

hence,

$$\int_1^x\mathrm{e}^{\frac{2}{\beta+1}y^{\beta+1}}\varphi_\beta(y)\mathrm{d}y\ge\int_1^x\left(-y^{1-\beta}-\frac{1-\beta}{2}y^{-2\beta}\right)\mathrm{d}y,$$

and (2.3) follows. More generally, assume $\beta \in [-\frac{n}{n+2}, \frac{1-n}{n+1})$, for an integer $n \geq 0$. Set $d_0 = 1$ and $d_k := 2^{-k} \prod_{j=0}^{k-1} ((1-\beta) - j(1+\beta))$, for $k \geq 1$ integer. By the choice of n, we can see that $d_n > 0$. If we iterate n times the integration by parts, we get:

$$\varphi_\beta(x) = -\sum_{k=0}^{n} d_k x^{(1-\beta)-k(1+\beta)} \mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}} + d_n \int_x^{+\infty} z^{(1-\beta)-(n+1)(\beta+1)} (-2z^\beta \mathrm{e}^{-\frac{2}{\beta+1}z^{\beta+1}}) \mathrm{d}z.$$

Since $(1-\beta) - (n+1)(\beta+1) \leq 0$ we can write

$$\varphi_\beta(x) \geq -\left(\sum_{k=0}^{n} d_k x^{(1-\beta)-k(1+\beta)} + d_n x^{(1-\beta)-(n+1)(\beta+1)} \right) \mathrm{e}^{-\frac{2}{\beta+1}x^{\beta+1}}.$$

By integrating, we have

$$\int_1^x \mathrm{e}^{\frac{2}{\beta+1}y^{\beta+1}} \varphi_\beta(y) \mathrm{d}y \geq \int_1^x \left(\sum_{k=0}^{n} d_k y^{(1-\beta)-k(1+\beta)} + d_n y^{(1-\beta)-(n+1)(\beta+1)} \right) \mathrm{d}y,$$

and we easily deduce (2.3). The proof of (2.3) is complete for all $\beta \in (-1, \infty)$. □

Proof of Lemma 3.5. Recall that $h_{p,\gamma}(x) = (1 + |x|^{p\gamma})^{1/p}$ and assume firstly that $|x| < 1$. Since $h_{p,\gamma}$ is continuously differentiable and equivalent to $|x|^\gamma$ at infinity, there exists $k > 0$ such that

$$|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| \leq |y| \sup_{z \in [-2,2]} |h'_{p,\gamma}(z)| \mathbb{1}_{\{|y| \leq 1\}} + k|y|^\gamma \mathbb{1}_{\{|y|>1\}}.$$

The desired inequality is then clear. Secondly, assume that $|x| \geq 1$. It is a simple computation to see that for all $z \geq 0$ and $r > 0$, there exists $c_r > 0$, such that

$$(1+z)^r - 1 \leq c_r \big(z \mathbb{1}_{\{z \leq 1\}} + z^r \mathbb{1}_{\{z>1\}} \big).$$

We deduce that, for all $(u,v) \in [0,\infty) \times [0,\infty)$, there exist $k_r > 0$ such that

$$(u+v)^r - u^r = u^r \left[\Big(1 + \frac{v}{u} \Big)^r - 1 \right] \leq k_r \big(v u^{r-1} \mathbb{1}_{\{v \leq u\}} + v^r \mathbb{1}_{\{u<v\}} \big). \tag{3.34}$$

Since $x \neq 0$,

$$\begin{aligned} |h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| &= |x|^\gamma \left| \Big(\frac{1}{|x|^{p\gamma}} + \Big| 1 + \frac{y}{x} \Big|^{p\gamma} \Big)^{1/p} - \Big(\frac{1}{|x|^{p\gamma}} + 1 \Big)^{1/p} \right| \\ &\leq |x|^\gamma \left[\Big(\frac{1}{|x|^{p\gamma}} + \Big(1 + |\frac{y}{x}| \Big)^{p\gamma} \Big)^{1/p} - \Big(\frac{1}{|x|^{p\gamma}} + 1 \Big)^{1/p} \right]. \end{aligned}$$

Applying (3.34) with $u = \frac{1}{|x|^{p\gamma}} + 1$, $v = \Big(1 + \Big|\frac{y}{x}\Big| \Big)^{p\gamma} - 1$ and $r = \frac{1}{p}$, we obtain

$$\begin{aligned} |h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| \leq k_{1/p} |x|^\gamma \Bigg[&\Big((1 + |\frac{y}{x}|)^{p\gamma} - 1 \Big)^{1/p} \mathbb{1}_{\{i(x) \leq |y|\}} \\ &+ \Big((1 + |\frac{y}{x}|)^{p\gamma} - 1 \Big) \Big(\frac{1}{|x|^{p\gamma}} + 1 \Big)^{\frac{1-p}{p}} \mathbb{1}_{\{|y| < i(x)\}} \Bigg]. \end{aligned}$$

Since $i(x) > |x|$, we can use again (3.34) to estimate the first term in the bracket on the right hand of the latter inequality. We let $u = 1$, $v = \big|\frac{y}{x}\big|$ and $r = p\gamma$ and we get

$$|h_{p,\gamma}(x+y) - h_{p,\gamma}(x)| \leq k_{1/p} k_{p\gamma} |y|^\gamma \mathbb{1}_{\{i(x) \leq |y|\}} + k_{1/p} |x|^\gamma \Big((1 + |\frac{y}{x}|)^{p\gamma} - 1 \Big) \Big(\frac{1}{|x|^{p\gamma}} + 1 \Big)^{\frac{1-p}{p}} \mathbb{1}_{\{|y| < i(x)\}}.$$

Since $|x|\geq 1$, ${}^{i(x)}/_{|x|}$ is bounded, and since $p>1$, $({}^{1}/_{|x|^{p\gamma}}+1)^{(1-p)/p}\leq 1$. Using that $p\gamma>2$ and the fact that ${}^{|y|}/_{|x|}$ is bounded, we have the existence of a $k'>0$ such that

$$\Big((1+\frac{|y|}{|x|})^{p\gamma}-1\Big)\leq k'\frac{|y|}{|x|}.$$

Taking $k=\max(k_{1/p}k_{p\gamma},k_{1/p}k')$, we get the second inequality in the first part of Lemma 3.5.

We proceed with the second part and we note that, since $p\gamma>2$, $h_{p,\gamma}$ is twice differentiable with

$$h''_{p,\gamma}(x)=\gamma\,|x|^{p\gamma-2}\big[(\gamma-1)|x|^{p\gamma}+p\gamma-1\big](1+|x|^{p\gamma})^{1/p-2}.$$

Moreover, since $\gamma<\alpha<2$, $h''_{p,\gamma}\in\mathrm{L}^\infty$. We split $(\mathcal{L}_{\alpha,\beta}\,h_{p,\gamma})(x)$ into three terms

$$\mathcal{L}_{\alpha,\beta}h_{p,\gamma}(x)=-\gamma\frac{|x|^{p\gamma+\beta-1}}{(1+|x|^{p\gamma})^{1-1/p}}+\int_{|y|\leq 1}\Big[h_{p,\gamma}(x+y)-h_{p,\gamma}(x)-yh'_{p,\gamma}(x)\Big]\nu(\mathrm{d}y)$$
$$+\int_{|y|>1}\Big[h_{p,\gamma}(x+y)-h_{p,\gamma}(x)\Big]\nu(\mathrm{d}y).$$

The first term on the right hand side is equivalent to $-\gamma|x|^{\gamma+\beta-1}$ at infinity, while for the second term, since $|y|\leq 1$, we have

$$\Big|h_{p,\gamma}(x+y)-h_{p,\gamma}(x)-yh'_{p,\gamma}(x)\Big|\leq y^2\sup_{|z|\leq 1}|h''_{p,\gamma}(x+z)|\leq y^2\|h''_{p,\gamma}\|_\infty.$$

Hence

$$\Big|\int_{|y|\leq 1}\Big[h_{p,\gamma}(x+y)-h_{p,\gamma}(x)-yh'_{p,\gamma}(x)\Big]\nu(\mathrm{d}y)\Big|\leq c_\alpha\|h''_{p,\gamma}\|_\infty,$$

where $c_\alpha:=\int_{|y|\leq 1}y^2\nu(\mathrm{d}y)$. We use the first part of the lemma to estimate the third term on the right hand side. There are two situations : if $|x|\geq 1$, we get

$$\Big|h_{p,\gamma}(x+y)-h_{p,\gamma}(x)\Big|\leq k(|y||x|^{\gamma-1}\mathbb{1}_{\{|y|\leq i(x)\}}+|y|^\gamma\mathbb{1}_{\{i(x)<|y|\}}).$$

Hence

$$\Big|\int_{|y|>1}\Big[h_{p,\gamma}(x+y)-h_{p,\gamma}(x)\Big]\nu(\mathrm{d}y)\Big|\leq k|x|^{\gamma-1}\int_{\{i(x)\geq|y|>1\}}|y|\nu(\mathrm{d}y)+k\int_{\{\max(1,i(x))\leq|y|\}}|y|^\gamma\nu(\mathrm{d}y)$$
$$\leq k|x|^{\gamma-1}\int_{\{i(x)\geq|y|>1\}}|y|\nu(\mathrm{d}y)+kc'_{\alpha,\gamma}\,,$$

where $c'_{\alpha,\gamma}:=\int_{\{|y|>1\}}|y|^\gamma\nu(\mathrm{d}y)$. Since $i(x)=O(|x|)$, as $|x|\to\infty$,

$$k|x|^{\gamma-1}\int_{\{i(x)\geq|y|>1\}}|y|\nu(\mathrm{d}y)=O(|x|^{\gamma-1})+O(|x|^{\gamma-\alpha}),\qquad\text{as }|x|\to\infty.$$

If $|x|<1$, since $|\mathrm{y}|>1$,

$$|h_{p,\gamma}(x+y)-h_{p,\gamma}(x)|\leq k|y|^\gamma,$$

so

$$\Big|\int_{|y|>1}\Big[h_{p,\gamma}(x+y)-h_{p,\gamma}(x)\Big]\nu(\mathrm{d}y)\Big|\leq\int_{|y|>1}|y|^\gamma\nu(\mathrm{d}y)<+\infty.$$

Denote by u the continuous function $-\mathcal{L}_{\alpha,\beta}h_{p,\gamma}$. Putting together the previous estimates, since $\beta>1$ and $\frac{2}{p}<\gamma<\alpha$, we obtain that

$$u(x)\sim|x|^{\gamma+\beta-1},\qquad\text{as }|x|\to\infty,$$

and since $\gamma>2-\beta$,

$$1+|x|=o(u(x)),\qquad\text{as }|x|\to\infty.$$

Set $K = [k^-, k^+]$ with

$$k^+ := \inf\{x > 0 : y \geq x \Rightarrow u(y) > y + 1\}, \quad k^- := \sup\{x < 0 : y \leq x \Rightarrow u(y) > -y + 1\},$$

and

$$d := -\inf_{\{x \in K\}} (u(x) - 1 - |x|), \quad f_{p,\alpha,\beta,\gamma}(x) := u(x) \mathbb{1}_{K^c} + (1 + |x|) \mathbb{1}_K .$$

Then relations (3.17)-(3.18) hold true and the proof is complete. □

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