# Anomalous diffusion behaviour for a time-inhomogeneous Kolmogorov type diffusion

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**Abstract:** We study a kinetic stochastic model with a non-linear time-inhomogeneous drag force and a Brownian-type random force. More precisely, the Kolmogorov type diffusion $(V, X)$ is considered. Here $X$ is the position of the particle and $V$ is its velocity and is solution of a stochastic differential equation driven by a one-dimensional Brownian motion, with the drift of the form $t^{-\beta}F(v)$. The function $F$ satisfies some homogeneity condition and $\beta$ is positive. The behaviour of the process $(V, X)$ in large time is proved and the precise rate of convergence is pointed out by using stochastic analysis tools.
**Key words:** kinetic stochastic equation; time-inhomogeneous diffusions; explosion times; scaling transformations; asymptotic distributions; ergodicity.
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## 1 Introduction

In several domains as fluids dynamics, statistical mechanics, biology, a number of models are based on the Fokker-Planck and Langevin equations driven by Brownian motion or could be non-linear or driven by other random noises. For instance, in [CCM10] the persistent turning walker model was introduced, inspired by the modelling of fish motion. An associated two-component Kolmogorov type diffusion solves a kinetic Fokker-Planck equation based on an Ornstein-Uhlenbeck Gaussian process and the authors studied the large time behaviour of this model by using appropriate tools from stochastic analysis. One of the natural questions is the behaviour in large time of the solution to the corresponding stochastic differential equation. Although the tools of partial differential equations allowed to ask of this kind of questions, since these models are probabilistic, tools based on stochastic processes could be more natural to use.

In the last decade the asymptotic study of solutions of non-linear Langevin's type was the subject of an important number of papers, see for instance [CNP19], [EG15], [FT18]. For instance in [FT18] the following system is studied

$$V_t = v_0 + B_t - \frac{\rho}{2}\int_0^t F(V_s)\,\mathrm{d}s \quad \text{and} \quad X_t = x_0 + \int_0^t V_s\,\mathrm{d}s.$$

In other words one considers a particle moving such that its velocity is a diffusion with an invariant measure behaving like $(1+|v|^2)^{-\rho/2}$, as $|v| \to \infty$. The authors prove that for large time, after a suitable rescaling, the position process behaves as a Brownian motion or other stable processes, following the values of $\rho$. It should be noted that in these cited papers the standard tools associated with time-homogeneous equations are used: invariant measure, scale function, speed measure and so on. Several of these tools will not be available when the drag force is depending explicitly on time. In [GO13], a non-linear SDE driven by a Brownian motion but having time-inhomogeneous drift coefficient was studied and its large time behaviour was described. Moreover, sharp rates of convergence are proved for the 1-dimensional marginal of the solution. In the present paper, we consider the velocity process as satisfying the same kind of SDE.

Let us describe our problem: consider a one-dimensional time-inhomogeneous stochastic kinetic model driven by a Brownian motion. We denote by $(X_t)_{t\geq 0}$ the one-dimensional process describing the position of a particle at time $t$ having the velocity $V_t$. The velocity process $(V_t)_{t\geq 0}$ is supposed to follow a Brownian dynamics in a potential $U(t,v)$, varying in time :

$$\mathrm{d}V_t = \mathrm{d}B_t - \frac{1}{2}\partial_v U(t, V_t)\,\mathrm{d}t \quad \text{and} \quad X_t = X_0 + \int_0^t V_s\,\mathrm{d}s. \tag{1}$$

It can be viewed as the perturbation of the classical two-component Kolmogorov diffusion

$$\mathrm{d}V_t = \mathrm{d}B_t \quad \text{and} \quad X_t = X_0 + \int_0^t V_s\,\mathrm{d}s.$$

In the present paper the potential is supposed to grow slowly to infinity and it will be supposed to be of the form $t^{-\beta}\int_0^v F(u)\,\mathrm{d}u$, with $\beta > 0$ and $F$ satisfying some homogeneity condition. It describes a one dimensional particle evolving in a force field $Ft^{-\beta}$ and undergoing many small random shocks. A natural question is to understand the behaviour of the process $(V,X)$ in large time. More precisely we look for the limit in distribution of $v(\varepsilon)(V_{t/\epsilon}, \epsilon X_{t/\varepsilon})_t$, as $\varepsilon \to 0$, where $v(\varepsilon)$ is some rate of convergence. Our results are proved on the product of path spaces and consequently contain those of [GO13].

When $F = 0$, it is not difficult to see that the rescaled position process $(\epsilon^{1/2}V_{t/\epsilon}, \varepsilon^{3/2}X_{t/\epsilon})_t$ converges in distribution towards the Kolmogorov diffusion $(B_t, \int^t B_s\,\mathrm{d}s)_t$. We prove that this anomalous diffusion behaviour still holds for sufficiently "small at infinity" potential. The strategy to tackle this problem is based on estimates of moments of the velocity process. We can even provide the growth rate of the velocity process, even if we do not use this when proving our convergence results. The main result can be then extended for the case when the potential is equally weighted as the random noise, in some sense (called, in the following critical). The organisation of our paper is as follows: in the next section we introduce notations and we state our main results. Existence and non-explosion of solutions are studied in Section 3. Estimates of the moments of the velocity process are given in Section 4 while the proofs of our main results are presented in Section 5. Section 6 is devoted to the analysis of the growth rates of the velocity and position processes.

## 2 Notations and main results

We consider the following one-dimensional stochastic kinetic model, for $t \geq t_0 > 0$,

$$\mathrm{d}V_t = \mathrm{d}B_t - t^{-\beta}F(V_t)\,\mathrm{d}t,\ V_{t_0} = v_0 > 0, \quad \text{and} \quad \mathrm{d}X_t = V_t\,\mathrm{d}t,\ X_{t_0} = x_0 \in \mathbb{R}. \tag{SKE}$$

Here $(B_t)_{t\geq 0}$ is a standard Brownian motion and $\beta$ is a real number. The function $F$ is supposed to satisfy either

$$\text{for some } \gamma \in \mathbb{R},\ \forall v \in \mathbb{R},\ \lambda > 0,\ F(\lambda v) = \lambda^\gamma F(v), \tag{$H_1^\gamma$}$$

or

$$|F| \leq G \text{ where } G \text{ is a positive function satisfying } (H_1^\gamma). \tag{$H_2^\gamma$}$$

These assumptions implies that there exist a positive constant $K$ such that, for all $v \in \mathbb{R}$, $|F(v)| \le K|v|^\gamma$. Obviously $(H_2^\gamma)$ is a generalization of $(H_1^\gamma)$. In the following, sgn is the sign function with convention $\mathrm{sgn}(0) = 0$. As an example of function satisfying $(H_1^\gamma)$ one can keep in mind $F : v \mapsto \mathrm{sgn}(v)\,|v|^\gamma$ (see also [GO13]), and as an example of function satisfying $(H_2^\gamma)$ (with $\gamma = 0$) $F : v \mapsto v/(1+v^2)$ (see also [FT18]).

**Remark 2.1.** *If a function $\pi$ satisfies $(H_1^\gamma)$, then for all $x \in \mathbb{R}$, $\pi(x) = \pi(\mathrm{sgn}(x))\,|x|^\gamma$.*

In the following, the space of continuous functions $\mathcal{C}((0,\infty),\mathbb{R})$ is endowed by the uniform topology

$$\mathrm{d}_u : f, g \in \mathcal{C}((0,+\infty)) \mapsto \sum_{n=1}^{+\infty} \frac{1}{2^n} \min\Big(1, \sup_{[\frac{1}{n},n]} |f(t) - g(t)|\Big).$$

Let us state our main results.

**Theorem 2.2.** *Consider $\gamma \ge 0$, and $\beta > \frac{\gamma+1}{2}$. Assume that $H_1^\gamma$ either $H_2^\gamma$ are satisfied. Let $(V_t, X_t)_{t\ge t_0}$ be a solution to* (SKE). *When $\gamma \ge 1$, we suppose also that for all $v \in \mathbb{R}$, $vF(v) \ge 0$. Then, as $\varepsilon \to 0$,*

$$(\sqrt{\varepsilon} V_{t/\varepsilon}, \varepsilon^{3/2} X_{t/\varepsilon})_{t\ge \varepsilon t_0} \Longrightarrow (\mathcal{B}_t, \int_0^t \mathcal{B}_s \,\mathrm{d}s)_{t>0}, \tag{2}$$

*in the space of continuous functions $\mathcal{C}((0,\infty))$ endowed by the uniform topology, where $(\mathcal{B}_t)_{t\ge0}$ is a standard Brownian motion.*

**Theorem 2.3.** *Consider $\gamma \ge 0$ and $\beta = \frac{\gamma+1}{2}$. Assume taht $H_1^\gamma$ is satisfied. Let $(V_t, X_t)_{t\ge t_0}$ be a solution to* (SKE). *When $\gamma \ge 1$, we suppose also that for all $v \in \mathbb{R}$, $vF(v) \ge 0$.*
*Call $\tilde{H}$ the ergodic process solution to*

$$\mathrm{d}H_s = \mathrm{d}W_s - \frac{H_s}{2}\,\mathrm{d}s - F\big(H_s\big)\,\mathrm{d}s, \tag{3}$$

*starting at its invariant measure, where $(W_t)_{t\ge0}$ is a standard Brownian motion. Setting $\Lambda_{F,t_1,\cdots,t_d}$ for the f.d.d. of $\tilde{H}$, we call $(\mathcal{V}_t)_{t\ge0}$ the process whose finite dimensional distribution are $T * \Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}$: the pushforward measure of $\Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}$ by the linear mapping $T : u := (u_1,\cdots,u_d) \mapsto (\sqrt{t_1}u_1,\cdots,\sqrt{t_d}u_d)$.*
*Then under $(H_1^\gamma)$ as $\varepsilon \to 0$,*

$$(\sqrt{\varepsilon} V_{t/\varepsilon}, \varepsilon^{3/2} X_{t/\varepsilon})_{t\ge \varepsilon t_0} \Longrightarrow (\mathcal{V}_t, \int_0^t \mathcal{V}_s \,\mathrm{d}s)_{t>0}, \tag{4}$$

*in the space of continuous functions $\mathcal{C}((0,\infty))$ endowed by the uniform topology.*

**Remark 2.4.** *The unidimensional law of $(\mathcal{V}_t)_{t\ge0}$ was explicitely computed (see Theorem 4.1 in [GO13]).*

# 3 Existence and non-explosion of solution

In this section, we will prove the existence of solution to (SKE) up to explosion time and the non explosion of such solution with additional assumption. In the following, we suppose that $\gamma > -1$ and set $\Omega = \overline{\mathcal{C}}([t_0,\infty))$ the set of continuous functions, that equal $\infty$ after their (possibly infinite) explosion time . Following the idea used in [GO13], we first perform a change of time in (SKE) in order to produce at least one time-homogeneous coefficient in the transformed equation.

For every $\mathcal{C}^2$-diffeomorphism $\varphi : [0, t_1) \to [t_0, \infty)$, let introduce the scaling transformation $\Phi_\varphi$ defined, for $\omega \in \Omega$, by

$$\Phi_\varphi(\omega)(s) := \frac{\omega(\varphi(s))}{\sqrt{\varphi'(s)}}, \text{ with } s \in [0, t_1).$$

The result containing the change of time transformation can be found in [GO13], Proposition 2.1, p. 187:

Let $V$ be a solution to equation (SKE). Thanks to Lévy's characterization theorem of the Brownian motion, $\left(\int_0^t \frac{\mathrm{d}B_{\varphi(s)}}{\sqrt{\varphi'(s)}}\right)_{t\geq 0}$ is a standard Brownian motion. Then, by a change of variable $t = \varphi(s)$, one gets

$$V_{\varphi(t)} - V_{\varphi(0)} = \int_0^t \sqrt{\varphi'(s)}\,\mathrm{d}W_s - \int_0^t \frac{F(V_{\varphi(s)})}{\varphi(s)^\beta}\varphi'(s)\,\mathrm{d}s.$$

The integration by parts formula yields

$$\mathrm{d}\left(\frac{V_{\varphi(s)}}{\sqrt{\varphi'(s)}}\right) = \mathrm{d}W_s - \frac{\sqrt{\varphi'(s)}}{\varphi(s)^\beta}F(V_{\varphi(s)})\,\mathrm{d}s - \frac{\varphi''(s)}{2\varphi'(s)}\frac{V_{\varphi(s)}}{\sqrt{\varphi'(s)}}\,\mathrm{d}s.$$

As a consequence, we can state the following result in our context.

**Proposition 3.1.** *If $V$ is a solution to equation* (SKE)*, then $V^{(\varphi)} := \Phi_\varphi(V)$ is a solution to*

$$\mathrm{d}V_s^{(\varphi)} = \mathrm{d}W_s - \frac{\sqrt{\varphi'(s)}}{\varphi(s)^\beta}F(\sqrt{\varphi'(s)}V_s^{(\varphi)})\,\mathrm{d}s - \frac{\varphi''(s)}{\varphi'(s)}\frac{V_s^{(\varphi)}}{2}\,\mathrm{d}s,\ V_0^{(\varphi)} = \frac{V_{\varphi(0)}}{\sqrt{\varphi'(0)}}, \tag{5}$$

*where $W_t := \int_0^t \frac{\mathrm{d}B_{\varphi(s)}}{\sqrt{\varphi'(s)}}$.*
*If $V^{(\varphi)}$ is a solution to* (5)*, then $\Phi_\varphi^{-1}(V^{(\varphi)})$ is a solution to equation* (SKE)*, where $B_t - B_{t_0} := \int_{t_0}^t \sqrt{(\varphi' \circ \varphi^{-1})(s)}\,\mathrm{d}W_{\varphi^{-1}(s)}$.*
*Furthermore uniqueness in law, pathwise uniqueness or strong existence hold for equation* (SKE) *if and only if they hold for equation* (5)*.*

In the following, we will use two particular changes of time, depending on which term of (5) should become time-homogeneous.

- *The exponential change of time*: denoting $\varphi_e : t \mapsto t_0e^t$, the exponential scaling transformation is defined by $\Phi_e(\omega) : s \in \mathbb{R}^+ \mapsto \frac{\omega_{t_0e^s}}{\sqrt{t_0}e^{s/2}}$, for $\omega \in \Omega$. Thanks to Proposition 3.1, the process $V^{(e)} := \Phi_e(V)$ satisfies the equation

$$\mathrm{d}V_s^{(e)} = \mathrm{d}W_s - \frac{V_s^{(e)}}{2}\,\mathrm{d}s - t_0^{1/2-\beta}e^{(1/2-\beta)s}F\big(\sqrt{t_0}e^{s/2}V_s^{(e)}\big)\,\mathrm{d}s, \tag{6}$$

  where $(W_t)_{t\geq 0}$ is a standard Brownian motion.
- *The power change of time*: for $q := \frac{2\beta}{\gamma+1} \neq 1$, consider $\varphi_q \in \mathcal{C}^2([0, t_1))$ the solution to the Cauchy problem

$$\varphi_q' = \varphi_q^q,\ \varphi_q(0) = t_0.$$

Clearly, $\varphi_q(t) = \left(t_0^{1-q} + (1-q)t\right)^{1/(1-q)}$, when $q \neq 1$, and $\varphi_q = \varphi_e$, when $q = 1$.
The time $t_1$ satisfies $t_1 = \infty$, when $q \leq 1$, and $t_1 = t_0^{1-q}(q-1)^{-1}$, when $q > 1$. The power scaling transformation is defined by $\Phi_q(\omega) : s \in \mathbb{R}^+ \mapsto \dfrac{\omega(\varphi_q(s))}{\varphi_q(s)^{q/2}}$. The process $V^{(q)} := V^{(\varphi_q)}$ satisfies the equation

$$\mathrm{d}V_s^{(q)} = \mathrm{d}W_s - \rho\varphi_q^{-\gamma\beta/(\gamma+1)}(s)F\left(\sqrt{\varphi_q'(s)}V_s^{(q)}\right)\mathrm{d}s - q\varphi_q^{q-1}(s)\frac{V_s^{(q)}}{2}\,\mathrm{d}s, \qquad (7)$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion.

We now study the existence and the explosion of the solution to (SKE), first under the homogeneity assumption $(H_1^\gamma)$ and then under the domination assumption $(H_2^\gamma)$.

**Until further notice, $(H_1^\gamma)$ is supposed to be satisfied.** As a consequence, the power scaling process $V^{(q)}$ satisfies the equation

$$\mathrm{d}V_s^{(q)} = \mathrm{d}W_s - F(V_s^{(q)})\,\mathrm{d}s - q\varphi_q^{q-1}(s)\frac{V_s^{(q)}}{2}\,\mathrm{d}s, \;\; s \in [0, t_1), \qquad (8)$$

which can be written, when $q > 1$, as

$$\mathrm{d}V_s^{(q)} = \mathrm{d}W_s - F(V_s^{(q)})\,\mathrm{d}s - \delta\frac{V_s^{(q)}}{t_1 - s}\,\mathrm{d}s, \;\; s \in [0, t_1), \qquad (9)$$

where $\delta = \dfrac{q}{2(q-1)}$. Proposition 3.2, p. 188, in [GO13] can be stated in the present situation.

**Proposition 3.2.** *For $\gamma > -1$, there exists a pathwise unique strong solution to* (SKE)*, defined up to the explosion time.*

**Remark 3.3.** *In the linear case ($\gamma = 1$), drift and diffusion are Lipschitz and satisfy locally linear growth. The existence and non-explosion of $V$ follow from Theorem 2.9, p. 289, in [KS98].*

*Proof.* We sketch the proof in our context. Remark first that, since $\gamma > -1$, $x \mapsto |x|^\gamma$ is locally integrable. Leaving out the third term on the right-hand side of (8), one gets a time-homogeneous equation:

$$\mathrm{d}H_s = \mathrm{d}W_s - F(H_s)\,\mathrm{d}s, \;\; s \in [0, t_1). \qquad (10)$$

By using Proposition 2.2, p. 28, in [CE05], there exists a unique weak solution $H$ to this time-homogeneous equation (10) defined up to the explosion time. Moreover, the Girsanov transformation induces a linear bijection between weak solutions defined up to the explosion time to equations (8) and (10). It follows that there exists a unique weak solution $V^{(q)}$ to equation (8). Therefore, by using the bijection induces by the change of time (Proposition 3.1), there exists a unique weak solution $V$ to equation (SKE). Besides, by using Corollary 3.4 and Proposition 3.2, pp. 389-390, in [RY05], pathwise uniqueness holds for the equation (SKE). The conclusion follows (Theorem 1.7, p. 368, in [RY05]). □

**Proposition 3.4.**

- *When $\gamma \leq 1$ or for all $v \in \mathbb{R}$, $vF(v) \geq 0$, the explosion time of $V$ is a.s. infinite.*
- *When $2\beta > \gamma + 1$, then $\mathbb{P}(\tau_\infty = \infty) > 0$.*

- *If $\gamma > 1$ and $(F(-1), F(1)) \in ((0,\infty)) \times [0,\infty)) \cup (\mathbb{R} \times (-\infty, 0))$, $\mathbb{P}(\tau_\infty = \infty) < 1$, where $\tau_\infty$ denotes the explosion time of $V$.*

**Remark 3.5.** *If $2\beta < \gamma + 1$ and $F(1) = -F(-1) < 0$, it follows from Proposition 3.6 p.9 in [GO13] that the explosion time of $V$ is finite a.s.*

*Proof.* This proof is inspired by those of Propositions 3.6 and 3.7 in [GO13]. We split the proof into several steps.

STEP 1. Assume first that $\gamma \leq 1$ or $vF(v) \geq 0$. We will use a criterion of non explosion stated in [SV06]. Call $\mathcal{L}_t$ the time-inhomogeneous infinitesimal generator of $V$, then

$$\mathcal{L}_t := \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{F(x)}{t^\beta}\frac{\partial}{\partial x}. \tag{11}$$

Let $\psi$ be a twice continuous differentiable positive function such that

$$\text{for all } |x| \geq 1,\ \psi(x) = 1 + x^2,\ \text{for all } |x| \leq \frac{1}{2},\ \psi(x) = 1,\ \text{and } \psi \geq 1, \text{on } \mathbb{R}.$$

Note that $\psi$ does not depend on time. Hence $\big(\partial_t + \mathcal{L}_t\big)\psi = \mathcal{L}_t\psi$.
Fix $T \geq t_0$ and call $c_T$ the supremum of the continuous function $\mathcal{L}_t\psi$ on $[t_0, T] \times [-1, 1]$. Then, for all $|x| \leq 1$ and $t \in [t_0, T]$,

$$\mathcal{L}_t\psi(x) \leq c_T \leq c_T\psi(x).$$

Moreover, for all $|x| > 1$ and $t \in [t_0, T]$, for $C$ a positive constant,

$$\mathcal{L}_t\psi(x) = -2x\frac{F(x)}{t^\beta} + 1 \leq \begin{cases} 1 \leq \psi(x), & \text{if for all } v \in \mathbb{R},\ vF(v) \geq 0, \\ 2\max(|F(1)|, |F(-1)|)x^2 + 1 \leq C\psi(x), & \text{if } \gamma \leq 1. \end{cases}$$

So, by using Theorem 10.2.1, p. 254, in [SV06], we deduce that $\tau_\infty$ is infinite a.s.

STEP 2. In this step, we suppose that $2\beta > \gamma + 1$. We follow the ideas of the proof of Proposition 3.7, pp. 191-192, in [GO13]. We first show that $\mathbb{P}(\tau_\infty = \infty) > 0$. Let $V^{(q)}$ be the pathwise unique strong solution to equation (9). Also denote by $b$, the $\delta$-Brownian bridge, the pathwise unique strong solution to equation

$$\mathrm{d}b_s = \mathrm{d}W_s - \delta\frac{b_s}{t_1 - s}\,\mathrm{d}s,\ b_0 = x_0,\ s \in [0, t_1). \tag{12}$$

Note that the equation (12) is obtained from (9) by omitting the second term on the right-hand side. Setting $\tau_\infty^{(q)}$ for the explosion time of $V^{(q)}$, then $\tau_\infty^{(q)} \in [0, t_1] \cup \{\infty\}$ a.s. and $\{\tau_\infty^{(q)} \geq t_1\} = \{\tau_\infty = \infty\}$. Note that $b$ becomes continuous on $[0, t_1]$ by setting $b_{t_1} = 0$ a.s.

Fix $n \geq 1$ and for all $s \in [0, t_1]$, define

$$T_n := \inf\left\{s \in [0, t_1), \left|V_s^{(q)}\right| \geq n\right\},\ \sigma_n := \inf\{s \in [0, t_1], |b_s| \geq n\},$$

and

$$\mathcal{E}(s) := \exp\left(\int_0^s -F(b_u)\,\mathrm{d}W_u - \frac{1}{2}\int_0^s F(b_u)^2\,\mathrm{d}u\right).$$

Since $\gamma > 1 \geq 0$, we have

$$\begin{aligned}\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^{s\wedge\sigma_n} F(b_u)^2\,\mathrm{d}u\right)\right] &\leq \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^{s\wedge\sigma_n} n^{2\gamma}\max(F(1)^2, F(-1)^2)\,\mathrm{d}u\right)\right] \\ &\leq \exp\left(\frac{t_1}{2}n^{2\gamma}\max(F(1)^2, F(-1)^2)\right).\end{aligned}$$

We observe that Novikov's condition applies to $(\mathcal{E}_{s\wedge\sigma_n})_{s\geq 0}$. Therefore, by using the Girsanov transformation between $b$ and $V^{(q)}$, we can write for every integer $n \geq 1$, $s \in [0, t_1]$ and $A \in \mathcal{F}_s$,

$$\mathbb{E}\left[\mathbb{1}_A\left(V^{(q)}_{\bullet\wedge T_n}\right)\mathbb{1}_{T_n>s}\right] = \mathbb{E}\big[\mathbb{1}_A\left(b_{\bullet\wedge\sigma_n}\right)\mathcal{E}(s\wedge\sigma_n)\mathbb{1}_{\sigma_n>s}\big].$$

Letting $n \to \infty$, by dominated convergence theorem and Fatou's lemma, we obtain

$$\mathbb{E}\left[\mathbb{1}_A\left(V^{(q)}\right)\mathbb{1}_{\tau^{(q)}_\infty>s}\right] \geq \mathbb{E}\left[\mathbb{1}_A\left(b\right)\mathcal{E}(s)\right].$$

Hence, $\mathbb{P}(\tau_\infty = \infty) = \mathbb{P}(\tau^{(q)}_\infty \geq t_1) \geq \mathbb{E}[\mathcal{E}(t_1)] > 0$.

Step 3.Assume now that $\gamma > 1$ and $(F(-1), F(1)) \in ((0,\infty)) \times [0,\infty)) \cup (\mathbb{R} \times (-\infty, 0))$. We will show that $\mathbb{P}(\tau_\infty = \infty) < 1$ when $F(1) > 0$ and $F(-1) > 0$. Our strategy is to apply the criterion for explosion stated in [SV06], Theorem 10.2.1, p. 254. Let $T > t_0$ and choose $a \in (1, \gamma)$. Also, one can choose $k \geq 1$ such that $a(a-1)^{-1} < k(T - t_0)$. Introduce the continuous differentiable negative function $f_1 : x \mapsto \dfrac{-1/2}{1+|x|^a}$, and, for $\mu > 0$, the bounded twice continuous differentiable function

$$G_{1,\mu}(x) = \exp\left(\mu\int_{-\infty}^{x} f_1(y)\,\mathrm{d}y\right),\ x \in \mathbb{R}.$$

For all $t \in [t_0, T]$ and all $x \in \mathbb{R}$,

$$\begin{aligned}\big(\partial_t + \mathcal{L}_t\big)G_{1,\mu}(x) = \mathcal{L}_t G_{1,\mu}(x) &= \mu G_{1,\mu}(x)\left[F(x)t^{-\beta}\left|f_1(x)\right| + \frac{1}{2}f_1'(x) + \frac{\mu}{2}f_1^2(x)\right]\\ &\geq \mu G_{1,\mu}(x)\left[F(x)T^{-\beta}\left|f_1(x)\right| + \frac{1}{2}f_1'(x) + \frac{\mu}{2}f_1^2(x)\right].\end{aligned}$$

Since $|f_1(x)| \underset{|x|\to\infty}{\sim} \frac{1}{2}|x|^{-a}$, $\lim_{|x|\to\infty} F(x)\left|f_1(x)\right| = +\infty$, and using that $\lim_{|x|\to\infty} f_1'(x) = 0$, there exists $r \geq 1$ such that, for all $\mu > 0$,

$$\big(\partial_t + \mathcal{L}_t\big)G_{1,\mu}(x) \geq \mu G_{1,\mu}(x)\left[F(x)T^{-\beta}\left|f_1(x)\right| + \frac{1}{2}f_1'(x)\right] \geq k\mu G_{1,\mu}(x) \text{ on } [t_0, T] \times [-r, r]^c.$$

Moreover, since $f_1^2$ is bounded away from zero, while $|f_1'|$ is bounded on $[-r, r]$ and since $F$ is non-negative, there exists $\mu_0$, such that,

$$\big(\partial_t + \mathcal{L}_t\big)G_{1,\mu_0}(x) \geq \mu_0 G_{1,\mu_0}(x)\left[\frac{1}{2}f_1'(x) + \frac{\mu_0}{2}f_1^2(x)\right] \geq k\mu_0 G_{1,\mu_0}(x) \text{ on } [t_0, T] \times [-r, r].$$

Hence, for all $t \in [t_0, T]$ and all $x \in \mathbb{R}$, $\big(\partial_t + \mathcal{L}_t\big)G_{1,\mu_0}(x) \geq k\mu_0 G_{1,\mu_0}(x)$. Besides, since $|f_1(x)| \leq 1 \wedge |x|^{-a}$,

$$\int_{-\infty}^{x_0}\big(-f_1(x)\big)\,\mathrm{d}x \leq \int_{\mathbb{R}}\big(1 \wedge |x|^{-a}\big)\,\mathrm{d}x = a(a-1)^{-1} < k(T - t_0).$$

Thus, we get a lower bound

$$G_{1,\mu_0}(x_0) > e^{-k\mu_0(T-t_0)} \geq e^{-k\mu_0(T-t_0)}\sup_{x\in\mathbb{R}} G_{1,\mu_0}(x).$$

Therefore, Theorem 10.2.1, p. 254, in [SV06] applies and $V$ explodes in finite time with positive probability.

When $F(-1) < 0$ and $F(1) < 0$, one can proceed in the same way, using the function $x \mapsto \exp\left(\mu \int_x^{+\infty} f_1(y)\,\mathrm{d}y\right)$ instead of $G_{1,\mu}$, in order to get that $\mathbb{P}(\tau_\infty = \infty) < 1$.

Step 4. It remains to show that $\mathbb{P}(\tau_\infty = \infty) < 1$ when $F(1) < 0$ and $F(-1) > 0$. As in the previous step, we choose $a \in (1, \gamma)$ and for every $T > t_0$, we choose again $k \geq 1$ such that $a(a-1)^{-1} < k(T - t_0)$. Moreover, it can be noted that there exists a continuous differentiable odd function $f_2$, defined on $\mathbb{R}$, vanishing only at $x = 0$, such that $|f_2(x)| \leq 1 \wedge |x|^{-a}$, and satisfying

$$f_2(x) := kx,\ x \in \left[-\frac{1}{2k}, \frac{1}{2k}\right],\ \lim_{|x|\to\infty} |x|^\gamma |f_2(x)| = \infty \text{ and } \lim_{|x|\to\infty} f_2'(x) = 0.$$

For $\mu > 0$, we introduce the bounded twice continuous differentiable function

$$G_{2,\mu}(x) := \exp\left(\mu \int_0^x f_2(y)\,\mathrm{d}y\right),\ x \in \mathbb{R}.$$

Then for all $t \in [t_0, T]$ and all $x \in \mathbb{R}$,

$$\begin{aligned}\big(\partial_t + \mathcal{L}_t\big) G_{2,\mu}(x) = \mathcal{L}_t G_{2,\mu}(x) &= \mu G_{2,\mu}(x) \left[\frac{|F(x) f_2(x)|}{t^\beta} + \frac{1}{2} f_2'(x) + \frac{\mu}{2} f_2^2(x)\right] \\ &\geq \mu G_{2,\mu}(x) \left[\rho \frac{|x|^\gamma |f_2(x)|}{t^\beta} + \frac{1}{2} f_2'(x) + \frac{\mu}{2} f_2^2(x)\right],\end{aligned}$$

where $\rho = \min\left\{ |F(1)|, |F(-1)| \right\} > 0$. One can conclude, using the same argument as in the proof of Proposition 3.7, p. 13, in [GO13]. □

**Assume now that $(H_2^\gamma)$ is satisfied.** Since the equation satisfied by the power scaling process (7) doesn't have any time-homogeneous term, the previous method cannot be used to conclude to the existence up to the explosion. Instead, one get the following results in the same way as previously, by using the exponential change of time process or by considering $G$ instead of $|F|$

**Proposition 3.6.** *If $\gamma \geq 0$, there exists a pathwise unique strong solution to* (SKE)*, defined up to the explosion time.*

**Proposition 3.7.**

- *When $\gamma \leq 1$ or for all $v \in \mathbb{R}$, $vF(v) \geq 0$, the explosion time of $V$ is a.s. infinite.*
- *Else, when $2\beta > \gamma + 1$, $\mathbb{P}(\tau_\infty = \infty) > 0$, where $\tau_\infty$ denotes the explosion time of $V$.*

# 4 Moments estimates of the velocity process

In this section, we give estimates for the moment of the velocity process. It will be useful to control some stochastic terms appearing later.

**Proposition 4.1.** *Assume that $\gamma \geq 0$. The inequality*

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0} t^{\frac{\kappa}{2}}$$

*yields for*

- $\kappa \in [0,1]$ *when* $\gamma < 1$,
- $\kappa \geq 0$, *when for all* $v \in \mathbb{R}$, $vF(v) \geq 0$.

**Remark 4.2.** *It can proved, when* $-1 < \gamma < 0$, *that for all* $t \geq t_0$, $\mathbb{E}[|V_t|] \leq C_{\gamma,\beta,t_0}\sqrt{t}$, *without hypothesis of the positivity of the function* $v \mapsto vF(v)$.

*Proof.* Step 1. Assume that $\gamma \geq 1$ and that for all $v \in \mathbb{R}$, $vF(v) \geq 0$.
Define, for all $n \geq 0$, the stopping times $T_n := \inf\{t \geq t_0,\ |V_t| \geq n\}$. By Itô's formula, for all $t \geq t_0$,

$$\begin{aligned} V_{t\wedge T_n}^2 &= v_0 + \int_{t_0}^{t\wedge T_n} 2V_s \, \mathrm{d}B_s - \int_{t_0}^{t\wedge T_n} 2s^{-\beta} V_s F(V_s) \, \mathrm{d}s + (t \wedge T_n - t_0) \\ &= v_0^2 + \int_{t_0}^{t} \mathbb{1}_{s\leq T_n} 2V_s \, \mathrm{d}B_s - \int_{t_0}^{t\wedge T_n} 2s^{-\beta} V_s F(V_s) \, \mathrm{d}s + (t - t_0) \\ &\leq v_0^2 + \int_{t_0}^{t} \mathbb{1}_{s\leq T_n} 2V_s \, \mathrm{d}B_s + (t - t_0). \end{aligned}$$

Since $\int_{t_0}^t 4\mathbb{1}_{s\leq T_n} V_s^2 \, \mathrm{d}s \leq 4n^2(t-t_0) < \infty$, taking expectation yields

$$\mathbb{E}\left[V_{t\wedge T_n}^2\right] \leq v_0^2 + (t - t_0) \leq C_{t_0} t.$$

Set $\kappa \in [0,2]$, then, by Jensen's inequality,

$$\mathbb{E}\left[|V_t|^\kappa\right] \leq \mathbb{E}\left[|V_t|^2\right]^{\frac{\kappa}{2}} \leq \left(\liminf_{n\to\infty} \mathbb{E}\left[V_{t\wedge T_n}^2\right]\right)^{\frac{\kappa}{2}} \leq C_{\kappa,t_0} t^{\frac{\kappa}{2}}. \tag{13}$$

When $\kappa > 2$, $v \mapsto |v|^\kappa$ is $\mathcal{C}^2$, so by Itô's formula, for all $t \geq t_0$,

$$\begin{aligned} |V_{t\wedge T_n}|^\kappa = |v_0|^\kappa + \int_{t_0}^{t\wedge T_n} \kappa \operatorname{sgn}(V_s) |V_s|^{\kappa-1} \mathrm{d}B_s - \int_{t_0}^{t\wedge T_n} \kappa s^{-\beta} |V_s|^{\kappa-1} \operatorname{sgn}(V_s) F(V_s) \, \mathrm{d}s \\ + \int_{t_0}^{t\wedge T_n} \frac{\kappa(\kappa-1)}{2} |V_s|^{\kappa-2} \mathrm{d}s. \end{aligned} \tag{14}$$

In addition, using the hypothesis on the sign of $F$,

$$|V_{t\wedge T_n}|^\kappa \leq |v_0|^\kappa + \int_{t_0}^{t} \mathbb{1}_{s\leq T_n} \kappa \operatorname{sgn}(V_s) |V_s|^{\kappa-1} \mathrm{d}B_s + \int_{t_0}^{t\wedge T_n} \frac{\kappa(\kappa-1)}{2} |V_s|^{\kappa-2} \mathrm{d}s. \tag{15}$$

We observe that $\int_{t_0}^t \kappa^2 V_s^{2\kappa-2} \mathbb{1}_{s\leq T_n} \, \mathrm{d}s \leq \kappa^2 n^{2\kappa-2}(t-t_0) < \infty$. Taking expectation in 15, we have

$$\mathbb{E}\left[|V_t|^\kappa\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[|V_{t\wedge T_n}|^\kappa\right] \leq |v_0|^\kappa + \int_{t_0}^t \frac{\kappa(\kappa-1)}{2} \mathbb{E}\left[|V_s|^{\kappa-2}\right] \mathrm{d}s.$$

When $0 \leq \kappa - 2 \leq 2$, we can upper bound $\mathbb{E}\left[|V_s|^{\kappa-2}\right]$ by injecting 13 and get

$$\mathbb{E}\left[|V_t|^\kappa\right] \leq |v_0|^\kappa + \int_{t_0}^t \frac{\kappa(\kappa-1)}{2} C_{\kappa,t_0} s^{\frac{\kappa-2}{2}} \, \mathrm{d}s \leq C_{\kappa,t_0} s^{\frac{\kappa}{2}}.$$

The method is then applied step by step to prove the inequality for all $\kappa > 2$.
Step 2. Assume now that $\gamma \in [0,1[$. Fix $\kappa \in [0,1]$. Then Jensen's inequality yields, for all

$t \geq t_0$, $\mathbb{E}[|V_t|^\kappa] \leq \mathbb{E}[|V_t|]^\kappa$, hence it suffices to verify only the inequality for $\kappa = 1$.
Define, for all $n \geq 0$, the stopping times $T_n := \inf\{t \geq t_0,\ |V_t| \geq n\}$ and let us recall that under both hypotheses ($(H_1^\gamma)$ or $(H_2^\gamma)$), there exists a positive constant $K$, such that $|F(v)| \leq K|v|^\gamma$. One can write, for $t \geq t_0$ and $n \geq 0$,

$$\begin{aligned}|V_{t\wedge T_n}| &\leq |v_0 - B_{t_0}| + |B_{t\wedge T_n}| + \int_{t_0}^{t\wedge T_n} s^{-\beta} |F(V_{s\wedge T_n})| \,\mathrm{d}s \\ &\leq |v_0 - B_{t_0}| + |B_{t\wedge T_n}| + \int_{t_0}^{t\wedge T_n} s^{-\beta} K |V_{s\wedge T_n}|^\gamma \,\mathrm{d}s.\end{aligned}$$

By noting that $\gamma \in [0,1[$ and $(B_t^2 - t)_{t\geq 0}$ is a martingale, taking expectation we get

$$\begin{aligned}\mathbb{E}[|V_{t\wedge T_n}|] &\leq \mathbb{E}[|v_0 - B_{t_0}|] + \mathbb{E}[|B_{t\wedge T_n}|] + \int_{t_0}^{t} s^{-\beta} K \mathbb{E}[|V_{s\wedge T_n}|^\gamma] \,\mathrm{d}s \\ &\leq \mathbb{E}[|v_0 - B_{t_0}|] + \sqrt{\mathbb{E}\left[B_{t\wedge T_n}^2\right]} + \int_{t_0}^{t} s^{-\beta} K \mathbb{E}[|V_{s\wedge T_n}|]^\gamma \,\mathrm{d}s \\ &\leq \mathbb{E}[|v_0 - B_{t_0}|] + \sqrt{\mathbb{E}[t \wedge T_n]} + \int_{t_0}^{t} s^{-\beta} K \mathbb{E}[|V_{s\wedge T_n}|]^\gamma \,\mathrm{d}s \\ &\leq C_{t_0}\sqrt{t} + \int_{t_0}^{t} s^{-\beta} K \mathbb{E}[|V_{s\wedge T_n}|]^\gamma \,\mathrm{d}s.\end{aligned}$$

The function $g_n : t \mapsto \mathbb{E}[|V_{t\wedge T_n}|]$, which is bounded by $n$. Applying a Gronwall-type lemma, stated below (Lemma 4.3) and Fatou's lemma, we end up, for all $t \geq t_0$, with

$$\begin{aligned}\mathbb{E}[|V_t|] \leq \liminf_{n\to\infty} \mathbb{E}[|V_{t\wedge T_n}|] &\leq C_\gamma \left[ C_{t_0}\sqrt{t} + \left( \frac{1-\gamma}{1-\beta} K (t^{1-\beta} - t_0^{1-\beta}) \right)^{\frac{1}{1-\gamma}} \right] \\ &\leq C_{\gamma,\beta,t_0}\sqrt{t}.\end{aligned}$$

□

**Lemma 4.3** (Gronwall-type lemma). *Fix $r \in [0,1)$ and $t_0 \in \mathbb{R}$. Assume that $g$ is a non-negative real-valued function, $b$ is a positive function and $a$ is a differentiable real-valued function. Moreover, suppose that the function $bg^r$ is a continuous function. If*

$$\forall t \geq t_0,\ g(t) \leq a(t) + \int_{t_0}^{t} b(s) g(s)^r \,\mathrm{d}s \tag{16}$$

*Then,*

$$\forall t \geq t_0,\ g(t) \leq C_r \left[ a(t) + \left( (1-r) \int_{t_0}^{t} b(s) \,\mathrm{d}s \right)^{\frac{1}{1-r}} \right],$$

*where $C_r := 2^{\frac{1}{1-r}}$.*

*Proof.* For $t \geq t_0$, since $r \geq 0$,

$$g(t)^r \leq \left( a(t) + \int_{t_0}^{t} b(s) g(s)^r \,\mathrm{d}s \right)^r,$$

then, multiplying by $b(t) > 0$,

$$b(t)g(t)^r \leq b(t)\left(a(t) + \int_{t_0}^{t} b(s)g(s)^r \, \mathrm{d}s\right)^r.$$

Now, let make us appearing the derivative of $H$

$$a'(t) + b(t)g(t)^r \leq a'(t) + b(t)\left(a(t) + \int_{t_0}^{t} b(s)g(s)^r \, \mathrm{d}s\right)^r.$$

That is

$$\frac{a'(t) + b(t)g(t)^r}{\left(a(t) + \int_{t_0}^{t} b(s)g(s)^r \, \mathrm{d}s\right)^r} \leq \frac{a'(t)}{\left(a(t) + \underbrace{\int_{t_0}^{t} b(s)g(s)^r \, \mathrm{d}s}_{\geq 0}\right)^r} + b(t) \leq b(t) + \frac{a'(t)}{a(t)^r}.$$

Integrating, since $r \neq 1$,

$$(1-r)^{-1}\left[\left(a(t) + \int_{t_0}^{t} b(s)g(s)^r \, \mathrm{d}s\right)^{1-r} - a(t_0)^{1-r}\right] \leq (1-r)^{-1}\left[a(t)^{1-r} - a(t_0)^{1-r}\right] + \int_{t_0}^{t} b(s)\, \mathrm{d}s$$

or equivalently, using that $r < 1$,

$$H(t)^{1-r} \leq a(t)^{1-r} + (1-r)\int_{t_0}^{t} b(s)\, \mathrm{d}s.$$

Since $\frac{1}{1-r} > 0$ and using (16)

$$g(t) \leq \left(a(t)^{1-r} + (1-r)\int_{t_0}^{t} b(s)\, \mathrm{d}s\right)^{\frac{1}{1-r}} \leq C_r\left[a(t) + \left((1-r)\int_{t_0}^{t} b(s)\, \mathrm{d}s\right)^{\frac{1}{1-r}}\right].$$

This concludes the proof of the lemma. □

**Remark 4.4.** *Call $H(t)$ the right-hand side of* (16). *If $g$ is not continuous, note that the function $H$ is continuous and satisfies* (16) *(since $b$ is positive and $g \leq H$). So, one can apply the lemma to $H$ and then use the inequality $g \leq H$.* [1]

## 5 Proof of the asymptotic behaviour of the solution

This section is devoted to the proofs of our main results.

[1] We are thankful to Thomas Cavallazzi for this remark.

## 5.1 Asymptotic behaviour above the critical line under both assumptions

In this section, we assume that $\gamma \geq 0$ and $\beta > \frac{\gamma+1}{2}$.

*Proof of Theorem 2.2.* We split the proof into three steps.

Step 1. We note that it is enough to prove that $(V_t^{(\varepsilon)})_{t>0} := (\sqrt{\varepsilon} V_{t/\varepsilon})_{t>0}$ converges in distribution to a Brownian motion in the space of continuous functions $\mathcal{C}((0,\infty))$ endowed by the uniform topology. Indeed, assume that the convergence of the rescaled velocity process is proved. For $\epsilon \in (0,1]$ and $t \geq \epsilon t_0$ we can write

$$\varepsilon^{3/2} X_{t/\varepsilon} = \varepsilon^{3/2} x_0 + \int_{\varepsilon t_0}^{t} V_s^{(\varepsilon)} \, \mathrm{d}s.$$

Clearly, the theorem will be proved once we show that $(V_\bullet^{(\varepsilon)}, \int_{\varepsilon t_0}^{\bullet} V_s^{(\varepsilon)} \, \mathrm{d}s) =: g_\varepsilon(V_\bullet^{(\varepsilon)})$ converges weakly in $\mathcal{C}((0,\infty))$ endowed by the uniform topology. Here the mapping $g_\varepsilon : v \mapsto \left(v_t, \int_{\varepsilon t_0}^t v_s \, \mathrm{d}s\right)_{t>0}$ is defined and valued on $\mathcal{C}((0,\infty))$. This mapping is converging, as $\varepsilon \to 0$, to the continuous mapping $g : v \mapsto \left(v_t, \int_0^t v_s \, \mathrm{d}s\right)_{t>0}$.

Let $h : \mathcal{C}((0,\infty)) \times \mathcal{C}((0,\infty)) \to \mathbb{R}$ be a bounded and uniformly continuous function. We will show that $\mathbb{E}[(h \circ g_\epsilon)(V^{(\epsilon)})] \underset{\epsilon \to 0}{\longrightarrow} \mathbb{E}[(h \circ g)(\mathcal{B})]$, where $\mathcal{B}$ is a standard Brownian motion. Using the continuous mapping theorem (see Theorem 2.7 p. 21, in [Bil99]), $\mathbb{E}\left[(h \circ g)(V^{(\epsilon)})\right]$ converges to $\mathbb{E}[(h \circ g)(\mathcal{B})]$. Since $h$ is bounded and uniformly continuous, for all $\eta > 0$, there exists $\delta > 0$ such that,

$$\begin{aligned}
\limsup_{\varepsilon \to 0} \mathbb{E}\left[\left|h \circ g_\epsilon(V^{(\epsilon)}) - h \circ g(V^{(\epsilon)})\right|\right] &\leq \eta + 2\|h\|_\infty \limsup_{\varepsilon \to 0} \mathbb{P}\left(\mathrm{d}_u\left(g_\varepsilon(V^\varepsilon), g(V^{(\varepsilon)})\right) > \delta\right) \\
&\leq \eta + 2\|h\|_\infty \limsup_{\varepsilon \to 0} \mathbb{P}\left(C \int_0^{\varepsilon t_0} \left|V_s^{(\varepsilon)}\right| \mathrm{d}s > \delta\right) \\
&\leq \eta.
\end{aligned}$$

One can prove the almost sure convergence of $\int_0^{\varepsilon t_0} \left|V_s^{(\varepsilon)}\right| \mathrm{d}s$ towards 0 using moment estimates (Proposition 4.1). One concludes, letting $\eta \to 0$ and using the Portmanteau theorem (see Theorem 2.1. p. 16, in [Bil99]).

Step 2. Let us prove now the convergence of the rescaled velocity process.
Let $\epsilon \in (0,1]$ and $t \geq \epsilon t_0$. One can write

$$\begin{aligned}
V_t^{(\varepsilon)} = \sqrt{\varepsilon} V_{t/\varepsilon} &= \sqrt{\varepsilon}(v_0 - B_{t_0}) + \sqrt{\varepsilon} B_{t/\varepsilon} - \sqrt{\varepsilon} \int_{t_0}^{t/\varepsilon} F(V_s) s^{-\beta} \, \mathrm{d}s \\
&= \sqrt{\varepsilon}(v_0 - B_{t_0}) + B_t^{(\varepsilon)} - \varepsilon^{\beta - 1/2} \int_{\varepsilon t_0}^{t} F(V_{u/\varepsilon}) u^{-\beta} \, \mathrm{d}u,
\end{aligned}$$

by self-similarity, $B^{(\varepsilon)} := (\sqrt{\varepsilon} B_{t/\varepsilon})_{t \geq 0}$ has the same distribution as a standard Brownian motion. The proof will be complete once we prove that

$$\forall T > 0 \sup_{\varepsilon t_0 \leq t \leq T} \left|V_t^{(\varepsilon)} - B_t^{(\varepsilon)}\right| \xrightarrow{\mathbb{P}} 0, \quad \text{as } \varepsilon \to 0. \tag{17}$$

Indeed, fix $a > 0$ and choose $N > 0$ such that $\sum_{n=N+1}^{+\infty} \frac{1}{2^n} \leq \frac{a}{2}$. Then,

$$\mathrm{d}_u\left(V^{(\varepsilon)}, B^{(\varepsilon)}\right) \leq \frac{a}{2} + \sum_{n=1}^{N} \frac{1}{2^n} \sup_{[\frac{1}{n}, n]} \left|V_t^{(\varepsilon)} - B_t^{(\varepsilon)}\right|.$$

It follows that

$$\mathbb{P}\left(\mathrm{d}_u\left(V^{(\varepsilon)}, B^{(\varepsilon)}\right) > a\right) \leq \sum_{n=1}^{N} \mathbb{P}\left(\sup_{[\frac{1}{n}, n]} \left|V_t^{(\varepsilon)} - B_t^{(\varepsilon)}\right| > a'\right) \underset{\epsilon\to 0}{\longrightarrow} 0,$$

where $a' = a(\sum_{n\geq 1}^{+\infty} 1/2^n)^{-1}$. It remains to apply Theorem 3.1, p. 27, in [Bil99] to conclude.

Step 3. Let us prove now (17). Recall that under both hypothesis ($(H_1^\gamma)$ and $(H_2^\gamma)$), there exists a positive constant $K$, such that $(\sqrt{\varepsilon})^\gamma \left|F\left(\frac{V_u^{(\varepsilon)}}{\sqrt{\varepsilon}}\right)\right| \leq K \left|V_u^{(\varepsilon)}\right|^\gamma$. Modifying the factor in front of the integral part, we get

$$V_t^{(\varepsilon)} = \sqrt{\varepsilon}(v_0 - B_{t_0}) + \sqrt{\varepsilon} B_{t/\varepsilon} - \varepsilon^{\beta-(\gamma+1)/2} \int_{\varepsilon t_0}^{t} (\sqrt{\varepsilon})^\gamma F\left(\frac{V_u^{(\varepsilon)}}{\sqrt{\varepsilon}}\right) u^{-\beta}\,\mathrm{d}u. \tag{18}$$

It follows that, for all $\varepsilon t_0 \leq T$,

$$\begin{aligned}\sup_{\varepsilon t_0\leq t\leq T}\left|V_t^{(\varepsilon)} - B_t^{(\varepsilon)}\right| &\leq \sqrt{\varepsilon}\,|v_0 - B_{t_0}| + \varepsilon^{\beta-(\gamma+1)/2} \sup_{\varepsilon t_0\leq t\leq T}\left|\int_{\varepsilon t_0}^{t} (\sqrt{\varepsilon})^\gamma F\left(\frac{V_u^{(\varepsilon)}}{\sqrt{\varepsilon}}\right) u^{-\beta}\,\mathrm{d}u\right| \\ &\leq \sqrt{\varepsilon}\,|v_0 - B_{t_0}| + \varepsilon^{\beta-(\gamma+1)/2} \int_{\varepsilon t_0}^{T} K\left|V_u^{(\varepsilon)}\right|^\gamma u^{-\beta}\,\mathrm{d}u.\end{aligned}$$

Taking the expectation and using moment estimates (Proposition 4.1), we obtain

$$\begin{aligned}\varepsilon^{\beta-(\gamma+1)/2}\mathbb{E}\left[\int_{\varepsilon t_0}^{T} K\left|V_u^{(\varepsilon)}\right|^\gamma u^{-\beta}\,\mathrm{d}u\right] &= \varepsilon^{\beta-(\gamma+1)/2}\int_{\varepsilon t_0}^{T} K\mathbb{E}\left[\left|V_u^{(\varepsilon)}\right|^\gamma\right] u^{-\beta}\,\mathrm{d}u \\ &\leq \varepsilon^{\beta-(\gamma+1)/2}\int_{\varepsilon t_0}^{T} KC_{\gamma,\beta,t_0} u^{\frac{\gamma}{2}-\beta}\,\mathrm{d}u \\ &\leq C\left(\varepsilon^{\beta-(\gamma+1)/2} T^{\frac{\gamma}{2}-\beta+1} - t_0^{\frac{\gamma}{2}-\beta+1}\sqrt{\varepsilon}\right) \underset{\varepsilon\to 0}{\longrightarrow} 0,\end{aligned}$$

since $\beta > \frac{\gamma+1}{2}$. Hence

$$\mathbb{E}\left[\sup_{\varepsilon t_0\leq t\leq T}\left|V_t^{(\varepsilon)} - B_t^{(\varepsilon)}\right|\right] = O(\varepsilon^q),$$

where $q = \min(\frac{1}{2}, \beta - (\gamma+1)/2) > 0$. This concludes the proof. □

**Remark 5.1.** *One can observe that the only moment in this proof, when we need the condition "$\gamma < 1$ or for all $v\mathbb{R}$, $vF(v)$" is to get the moment estimates.*

## 5.2 Asymptotic behaviour on the critical line under $(H_1^\gamma)$

Assume in this section that $\beta = \frac{\gamma+1}{2}$ and $(H_1^\gamma)$ is satisfied. As in the first step of the previous section, it suffices to prove the convergence of the rescaled velocity process $(\sqrt{\varepsilon}V_{t/\varepsilon})_t$.

Step 1. We first prove the finite dimensional convergence. The exponential scaling process $V^{(e)}$ satisfies the time-homogeneous equation

$$\mathrm{d}V_s^{(e)} = \mathrm{d}W_s - \frac{V_s^{(e)}}{2}\,\mathrm{d}s - F\big(V_s^{(e)}\big)\,\mathrm{d}s, \tag{19}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion.
By using Proposition 2.2, p. 28, in [CE05], there exists a unique weak solution $H$ to the time-homogeneous equation (19) defined up to the explosion time. Using the bijection induced by the exponential change of time (Proposition 3.1) and the non explosion of the velocity process $V$ (Proposition 3.4), we can conclude that the explosion time of $H$ is almost surely infinite. Hence,

$$\left(\frac{V_{t_0 e^t}}{\sqrt{t_0}e^{t/2}}\right)_{t\geq 0} = (H_t)_{t\geq 0},$$

as solutions of the same SDE, starting at the same point. This can also be written as

$$\left(\frac{V_t}{\sqrt{t}}\right)_{t\geq t_0} = (H_{\log(t/t_0)})_{t\geq t_0}. \tag{20}$$

So, we have, for all $\epsilon > 0$, and $t_1, \cdots, t_d \geq t_0$

$$\left(\frac{V_{\epsilon^{-1}t_1}}{\sqrt{\epsilon^{-1}t_1}}, \cdots, \frac{V_{\epsilon^{-1}t_d}}{\sqrt{\epsilon^{-1}t_d}}\right) = \big(H_{\log(t_1/t_0)+\log(\epsilon^{-1})}, \cdots, H_{\log(t_d/t_0)+\log(\epsilon^{-1})}\big).$$

As in [GO13], the scale function and the speed measure of $H$ are respectively

$$\mathtt{p}(x) := \int_0^x \exp\left(\frac{y^2}{2} + \frac{2}{\gamma+1}\operatorname{sgn}(y)F(\operatorname{sgn}(y))\,|y|^{\gamma+1}\right)\mathrm{d}y$$

and

$$\nu_F(\mathrm{d}x) := \exp\left(-\frac{x^2}{2} - \frac{2}{\gamma+1}\operatorname{sgn}(x)F(\operatorname{sgn}(x))\,|x|^{\gamma+1}\right)\mathrm{d}x.$$

By the ergodic theorem (Theorem 23.15 p. 465 in [Kal02]), $H$ is $\Lambda_F$-ergodic, where $\Lambda_F$ is the probability measure associated to $\nu_F$. Call $\tilde{H}$ the solution of the time homogeneous equation (19) starting from $\Lambda_F$.
For $t_1, \cdots, t_d \in \mathbb{R}^d$, let $\Lambda_{F,t_1,\cdots,t_d} := \mathcal{L}(\tilde{H}_{t_1}, \cdots, \tilde{H}_{t_d})$ be the law of $(\tilde{H}_{t_1}, \cdots, \tilde{H}_{t_d})$. Then, for all $s \geq 0$, $\Lambda_{F,t_1,\cdots,t_d} = \Lambda_{F,t_1+s,\cdots,t_d+s}$. Indeed, thanks to the invariance property of $\Lambda_F$, $(\tilde{H})$ and $(\tilde{H}_{\cdot+s})$ satisfy the same SDE, starting at the same point. As a consequence, for all $\epsilon \to 0$,

$$\mathcal{L}\left(\tilde{H}_{\log(t_1/t_0)+\log(\epsilon^{-1})}, \cdots, \tilde{H}_{\log(t_d/t_0)+\log(\epsilon^{-1})}\right) = \Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}.$$

Moreover, thanks to Theorem 23.17 p.466 in [Kal02]

$$\|\mathcal{L}\left(H_{\log(t_1/t_0\epsilon)}, \cdots, H_{\log(t_d/t_0\epsilon)}\right) - \mathcal{L}\left(\tilde{H}_{\log(t_1/t_0\epsilon)}, \cdots, \tilde{H}_{\log(t_d/t_0\epsilon)}\right)\|_{TV} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Hence,

$$\left(\frac{V_{\epsilon^{-1}t_1}}{\sqrt{\epsilon^{-1}t_1}}, \cdots, \frac{V_{\epsilon^{-1}t_d}}{\sqrt{\epsilon^{-1}t_d}}\right) \underset{\varepsilon\to 0}{\Longrightarrow} \Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}.$$

Or,

$$\big(\sqrt{\varepsilon}V_{t_1/\varepsilon}, \cdots, \sqrt{\varepsilon}V_{t_d/\varepsilon}\big) \underset{\varepsilon\to 0}{\Longrightarrow} T * \Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)},$$

where $T * \Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}$ is the pushforward of the measure $\Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}$ by the linear mapping $T : u := (u_1, \cdots, u_d) \mapsto (\sqrt{t_1}u_1, \cdots, \sqrt{t_d}u_d)$.
STEP 2. Let us prove now the tightness of the family of laws of continuous process $\left(V^{(\varepsilon)}\right)_{t\geq t_0} = \left(\sqrt{\varepsilon}V_{t/\varepsilon}\right)_{t\geq t_0}$ on every compact interval $[t_0, M]$.
Fix $\eta > 0$. Using Markov inequality and moment estimates (Proposition 4.1), for $R$ big enough, and $t_0 \leq t \leq M$,

$$\sup_{\varepsilon>0} \mathbb{P}\left(\left|V_t^{(\varepsilon)}\right| \leq R\right) \leq C_{\gamma,\beta,t_0}\frac{\sqrt{t}}{R} \leq C_{\gamma,\beta,t_0}\frac{\sqrt{M}}{R} \leq \eta.$$

By Definition 30.3 p. 239 in [Bil99], we deduce that $(V_t^{(\varepsilon)})_{t_0\leq t\leq M}$ is tight.
STEP 3. The two first steps yields weak convergence on every compact set (Theorem 13.1 p.139 in [Bil99]). The conclusion follows from Theorem 16.7 p.174 in [Bil99], since all processes considered are continuous.

**Example 5.1.** We will see that the limiting process $\mathcal{V}$ is more explicit in the linear case ($\gamma = 1$). Choose $F(1) = 1$, $F(-1) = -1$, the process $\tilde{H}$ solution of (19) is in fact an Ornstein Uhlenbeck process with speed measure $\Lambda_F(dx) := \exp\left(-\frac{3x^2}{2}\right) \mathrm{d}x$. It is a Gaussian process, hence for all $s_1, \cdots, s_d$, its f.d.d. $\Lambda_{F,s_1,\cdots,s_d}$ are Gaussian. As a consequence, knowing the expectaction function $m$ and the covariance function $K$ is enough to provide the law of the process. Since, $\tilde{H}$ is a stationnary Ornstein-Uhlenbeck process, one has $m \equiv 0$ and $K : s,t \mapsto \frac{1}{3}e^{-\frac{3}{2}|t-s|}$. Hence, the limiting process $\mathcal{V}$ having f.d.d $T * \Lambda_{F,\log(t_1/t_0),\cdots,\log(t_d/t_0)}$ is a centered Gaussian process with covariance function $s,t \mapsto \frac{1}{3}\frac{(s\wedge t)^2}{s\vee t}$.

# 6 Growth rate of velocity and position processes

We turn now to the study of the growth rate of the velocity process $V$. In this section, we assume that $(\gamma+1)/2 - \beta \leq 0$ and

$$\forall x \in \mathbb{R},\ \forall \lambda > 0,\ \mathrm{sgn}(F(\lambda x)) = \mathrm{sgn}(F(x)). \tag{$H_3$}$$

**Proposition 6.1.** *When $\gamma \geq 1$, assume also that for all $v \in \mathbb{R}$, $vF(v) \geq 0$,*

$$\limsup_{t\to\infty} \frac{|V_t|}{\sqrt{2\ln(\ln(t))}} \leq 1 \ \textit{a.s. and} \ \limsup_{t\to\infty} \frac{V_t}{\sqrt{2\ln(\ln(t))}} = 1 \ \textit{a.s.} \tag{21}$$

*Besides,*

$$\limsup_{t\to\infty} \frac{|X_t|}{\int_1^t \sqrt{2\ln(\ln(s))}\,\mathrm{d}s} \leq 1 \ \textit{a.s.} \tag{22}$$

First, we discuss results of existence and behaviour of some time-homogeneous processes $V^{\pm}$ such that $V^- \leq V^{(e)} \leq V^+$ almost surely. For $\gamma > -1$, let $\pi$ be a non-negative function satisfying

$$\forall x \in \mathbb{R}, \lambda > 0,\ \pi(\lambda x) = \lambda^\gamma \pi(x).$$

Recall that $\pi$ satisfies Eq. $(H_1^\gamma)$. Under $(H_1^\gamma)$ we take $\pi = |F|$ and under $(H_2^\gamma)$, we take $\pi = G$.

Define the pathwise unique strong solution (up to the explosion time) to the time-homogeneous equation

$$\mathrm{d}V_s^{\pm} = \mathrm{d}W_s - \frac{V_s^{\pm}}{2}\,\mathrm{d}s \pm t_0^{\frac{\gamma+1}{2}-\beta}\pi(V_s^{\pm})\mathbb{1}_{\{\pm F(V_s^{\pm})<0\}}\,\mathrm{d}s. \tag{23}$$

**Lemma 6.2.** *Set $\tau_\infty^\pm$, $\tau_\infty$, respectively the explosion time of $V^\pm$ and $V$.*

*(i) If $\gamma \le 1$ or $F(1) \ge 0$, then $\tau_\infty^+ = \infty$ a.s.*

*(ii) If $\gamma \le 1$ or $F(-1) \le 0$, then $\tau_\infty^- = \infty$ a.s.*

*(iii) If $\gamma > 1$ and $F(1) < 0$, then $\mathbb{P}(\tau_\infty^+ = \infty) = 0$.*

*(iv) If $\gamma > 1$ and $F(-1) > 0$, then $\mathbb{P}(\tau_\infty^- = \infty) = 0$.*

*Proof.* Step 1. Firstly, let us prove that $V^- \le V^{(e)} \le V^+$ almost surely. Indeed, if we denote

$$\mathtt{b}(t,x) = -\frac{x}{2} - t_0^{1/2-\beta} e^{(1/2-\beta)t} F(\sqrt{t_0} e^{t/2} x) \quad \text{and} \quad \mathtt{b}^+(x) = -\frac{x}{2} + t_0^{(\gamma+1)/2-\beta} \pi(x) \mathbb{1}_{\{F(x)\le 0\}},$$

we can write, for all $t \ge 0$ and all $x \in \mathbb{R}$,

$$\mathtt{b}(t,x) \le \mathtt{b}^+(x) \Longleftrightarrow -e^{(1/2-\beta)t} F(\sqrt{t_0} e^{t/2} x) \le t_0^{\frac{\gamma}{2}} \pi(x) \mathbb{1}_{\{F(x)\le 0\}}.$$

This inequality holds by the choice of $\pi$, $(H_3)$, and the assumption $(\gamma+1)/2 - \beta \le 0$. By using the comparison theorem (see Theorem 1.3 in [Yam73]) we get, $V^{(e)} \le V^+$, almost surely. The other inequality can be handled in the same way.
Step 2. Call $\tau_\infty^{(e)}$ the explosion time of $V^{(e)}$. Then $\{\tau_\infty^{(e)} = \infty\} = \{\tau_\infty = \infty\}$, so

$$\{\tau_\infty^- = \infty\} \cap \{\tau_\infty^+ = \infty\} \subset \{\tau_\infty = \infty\}.$$

We give the detailed proof for (i) and (iii), the other parts could be obtained by changing "+" and "−" in the reasoning. First, we prove (i). The scale function of $V^+$ is defined, for $x \in \mathbb{R}$, by

$$\mathtt{p}^+(x) := \int_0^x \exp\left(\frac{y^2}{2} - 2t_0^{\frac{(\gamma+1)}{2}-\beta} \int_0^y \pi(z) \mathbb{1}_{\{F(z)\le 0\}} \,\mathrm{d}z\right) \mathrm{d}y.$$

Note that, if $x < 0$,

$$-\mathtt{p}^+(x) \ge \int_x^0 e^{y^2/2} \,\mathrm{d}y.$$

Thus $\mathtt{p}^+(-\infty) = -\infty$. Suppose that $F(1) \ge 0$, then for $x \ge 0$, $\mathtt{p}^+(x) = \int_0^x e^{y^2/2} \,\mathrm{d}y$, so $\mathtt{p}^+(\infty) = \infty$. By Proposition 5.22, p. 345, in [KS98], the conclusion follows. Assume now that $\gamma < 1$ and $F(1) < 0$. Then, for $x \ge 0$,

$$\mathtt{p}^+(x) = \int_0^x \exp\left(\frac{y^2}{2} - 2t_0^{\frac{\gamma+1}{2}-\beta} \pi(1) \frac{y^{\gamma+1}}{\gamma+1}\right) \mathrm{d}y,$$

so $\mathtt{p}^+(\infty) = \infty$. Using the same result in [KS98], the conclusion follows. If $\gamma = 1$, the drift has linear growth and the conclusion is clear.

Step 3. We proceed with the proof of (iii). Assume that $\gamma > 1$ and $F(1) < 0$. As previously, $\mathtt{p}^+(-\infty) = -\infty$. Besides, $\mathtt{p}^+(\infty) < \infty$. Denote $\mathtt{m}^+ : y \mapsto 2/(\mathtt{p}^+)'(y)$ the speed measure of $V^+$. Fix $y > 0$, then, setting $\mathtt{c} = 2t_0^{\frac{\gamma+1}{2}-\beta} \pi(1) > 0$, one can apply integration by parts to get:

$$\begin{aligned}\big(\mathtt{p}^+(\infty) - \mathtt{p}^+(\mathtt{y})\big)\, \mathtt{m}^+(y) =& 2\exp\left(-\frac{y^2}{2} + \mathtt{c}\frac{y^{\gamma+1}}{\gamma+1}\right) \int_y^{+\infty} \exp\left(\frac{z^2}{2} - \mathtt{c}\frac{z^{\gamma+1}}{\gamma+1}\right) \mathrm{d}z \\ =& \frac{2}{\mathtt{c}y^\gamma - y} + 2\exp\left(-\frac{y^2}{2} + \mathtt{c}\frac{y^{\gamma+1}}{\gamma+1}\right) \int_y^\infty e^{\frac{z^2}{2} - \mathtt{c}\frac{z^{\gamma+1}}{\gamma+1}} \frac{1 - \mathtt{c}\gamma z^{\gamma-1}}{(z - \mathtt{c}z^\gamma)^2} \,\mathrm{d}z.\end{aligned}$$

One can deduce, by integrating small $o$, that

$$\big(\mathtt{p}^+(\infty) - \mathtt{p}^+(\mathtt{y})\big)\, \mathtt{m}^+(y) \underset{y\to\infty}{\sim} \frac{2}{\mathtt{c}y^\gamma - y},$$

which is an integrable function at $\infty$. The conclusion follows from Theorem 5.29, p. 348, in [KS98]. □

*Proof of Proposition 6.1.*

STEP 1. Remark first that the first inequality of (21) is equivalent to

$$\limsup_{t\to\infty} \frac{\left|V_t^{(e)}\right|}{\sqrt{2\ln(t)}} \le 1.$$

Assuming

$$\limsup_{t\to\infty} \frac{V_t^+}{\sqrt{2\ln(t)}} \le 1 \text{ a.s.,} \quad \text{and} \quad \limsup_{t\to\infty} \frac{-V_t^-}{\sqrt{2\ln(t)}} \le 1 \text{ a.s,} \tag{24}$$

one gets the first inequality of (21) writting

$$\limsup_{t\to\infty} \frac{V_t^{(e)}}{\sqrt{2\ln(t)}} \le \limsup_{t\to\infty} \frac{V_t^+}{\sqrt{2\ln(t)}} \le 1 \text{ a.s. and } \limsup_{t\to\infty} \frac{-V_t^{(e)}}{\sqrt{2\ln(t)}} \le \limsup_{t\to\infty} \frac{-V_t^-}{\sqrt{2\ln(t)}} \le 1 \text{ a.s.}$$

To prove (24), we use Motoo's theorem, borrowed from [Mot59]. This lemma is recalled here for the sake of completeness.

**Theorem 6.3** (Motoo)**.** *Let $Z$ be a regular continuous strong Markov process in $(a,\infty)$, $a \in [-\infty,\infty)$. Assume also that $Z$ is time-homogeneous, with scale function $\mathtt{s}$. For every real positive increasing function $h$,*

$$\mathbb{P}\left(\limsup_{t\to\infty} \frac{Z_t}{h(t)} \ge 1\right) = 0 \text{ or } 1 \text{ according to whether } \int^\infty \frac{\mathrm{d}t}{\mathtt{s}(h(t))} < \infty \text{ or } = \infty.$$

Motoo's theorem yields for all $\varepsilon > 0$,

$$\mathbb{P}\Big(\limsup_{t\to\infty} \frac{V_t^+}{\sqrt{2\ln(t)}} \ge 1+\varepsilon\Big) = 0 \text{ and } \mathbb{P}\Big(\limsup_{t\to\infty} \frac{-V_t^-}{\sqrt{2\ln(t)}} \ge 1+\varepsilon\Big) = 0.$$

Indeed, define $\tilde{V}^- := -V^-$, then

$$\mathrm{d}\tilde{V_s}^- = -\,\mathrm{d}W_s - \frac{\tilde{V_s}^-}{2}\,\mathrm{d}s + t_0^{(\gamma+1)/2-\beta}\pi(-\tilde{V_s}^-)\mathbb{1}_{\{F(-\tilde{V_s}^-)>0\}}\,\mathrm{d}s.$$

Fix $y_0 > 0$. The scale function of $V^+$ and $\tilde{V}^-$ is defined, for $y \ge y_0$, by

$$\mathtt{s}^\pm(y) := \kappa \int_{y_0}^y \exp\Big(\frac{z^2}{2} - 2C^\pm \frac{z^{\gamma+1}}{\gamma+1}\Big)\,\mathrm{d}z.$$

Here and elsewhere $\kappa$ denotes positive constants that can change of value from line to line, and

$$C^+ := t_0^{\frac{\gamma+1}{2}-\beta}\pi(1)\mathbb{1}_{\{F(1)<0\}} \quad \text{for } V^+ \quad \text{and} \quad C^- := t_0^{\frac{\gamma+1}{2}-\beta}\pi(-1)\mathbb{1}_{\{F(-1)>0\}} \quad \text{for } \tilde{V}^-.$$

Note that under our hypothesis $C^{\pm} = 0$ when $\gamma \geq 1$.
Let $\epsilon > 0$. Define the positive increasing function $h : t \mapsto (1+\epsilon)\sqrt{2\ln(t)}$. We will show that $1/\mathtt{s}(h)$ is integrable at infinity. Firstly, remark that

$$\int_{y_0}^{+\infty} \frac{1}{\mathtt{s}(h(t))}\,\mathrm{d}t = \int_{h(y_0)}^{+\infty} \frac{1}{\mathtt{s}(y)} \frac{\mathrm{d}y}{h'(h^{-1}(y))} = \int_{h(y_0)}^{+\infty} \frac{1}{\mathtt{s}(y)} \frac{y\exp\left(y^2/\left(2(1+\epsilon)^2\right)\right)}{(1+\varepsilon)^2}\,\mathrm{d}y.$$

It remains to find an equivalent of $\mathtt{s}$ at infinity. In the following "$\asymp$" means equality up to a multiplicative positive constant. Fix $y > y_0$, integrating by parts, we get,

$$\begin{aligned}
\mathtt{s}(y) &\asymp \int_{y_0}^{y} \exp\left(\frac{z^2}{2} - 2C^{\pm}\frac{z^{\gamma+1}}{\gamma+1}\right)\left(z - 2C^{\pm} z^{\gamma}\right)\cdot \frac{1}{z - 2C^{\pm} z^{\gamma}}\,\mathrm{d}z \\
&\asymp \frac{\exp\left(\frac{y^2}{2} - 2C^{\pm}\frac{y^{\gamma+1}}{\gamma+1}\right)}{y - 2C^{\pm} y^{\gamma}} - \kappa + \int_{y_0}^{y} \frac{1 - 2\gamma C^{\pm} z^{\gamma-1}}{(z - 2C^{\pm} z^{\gamma})^2} \exp\left(\frac{z^2}{2} - 2C^{\pm}\frac{z^{\gamma+1}}{\gamma+1}\right)\mathrm{d}z.
\end{aligned}$$

Since $\gamma > -1$, $\lim_{y\to\infty} \dfrac{1 - 2\gamma C^{\pm} y^{\gamma-1}}{(y - 2C^{\pm} y^{\gamma})^2} = 0$, Moreover, the function $y \mapsto \exp\left(\frac{y^2}{2} - 2C^{\pm}\frac{y^{\gamma+1}}{\gamma+1}\right)$ is not integrable at infinity, since $C^{\pm} = 0$, or $\gamma < 1$. Hence we get, by integration,

$$\mathtt{s}(y) \underset{y\to\infty}{\sim} \kappa \frac{\exp\left(\frac{y^2}{2} - 2C^{\pm}\frac{y^{\gamma+1}}{\gamma+1}\right)}{y - 2C^{\pm} y^{\gamma}}.$$

As a consequence,

$$\frac{1}{\mathtt{s}(y)} \frac{y\exp\left(\frac{y^2}{2(1+\varepsilon)^2}\right)}{(1+\varepsilon)^2} \underset{y\to\infty}{\sim} \kappa\left(y^2 - 2C^{\pm} y^{\gamma+1}\right)\exp\left(-\frac{y^2}{2}\left(1 - \frac{1}{(1+\varepsilon^2)}\right) + 2C^{\pm}\frac{y^{\gamma+1}}{\gamma+1}\right).$$

which is integrable. We can conclude using Motoo's theorem.

Step 2. The second inequality of (21) is equivalent to

$$\limsup_{t\to\infty} \frac{V_t^{(e)}}{\sqrt{2\ln(t)}} = 1 \text{ a.s.}$$

From the first step, we have

$$\limsup_{t\to\infty} \frac{V_t^{(e)}}{\sqrt{2\ln(t)}} \leq 1.$$

We need to prove the opposite inequality. Note first that the first inequality of (21) implies

$$\lim_{t\to\infty} t_0^{1/2-\beta} e^{(1/2-\beta)s} F\left(\sqrt{t_0}e^{s/2}V_s^{(e)}\right) = 0 \text{ a.s.} \tag{25}$$

For $u \geq 0$, introduce the pathwise unique strong solution of

$$\mathrm{d}V_s(u) = \mathrm{d}W_s - \left(\frac{V_s(u)}{2} + 1\right)\mathrm{d}s,\ V_u(u) = V_u^{(e)},$$

and define the stopping time

$$\tau_u := \inf\left\{t \geq u,\ t_0^{1/2-\beta} e^{(1/2-\beta)t}\left|F\left(\sqrt{t_0}e^{t/2}V_t^{(e)}\right)\right| > 1\right\}.$$

By using the comparison theorem (see Theorem 3.7 p 394, in [RY05]) we get, $V_{\cdot\wedge\tau_u}(u) \leq V^{(e)}_{\cdot\wedge\tau_u}$, almost surely. Hence

$$\forall t \geq u,\ V_t(u) \leq V_t^{(e)} \text{ a.s. on } \{\tau_u = \infty\} = \left\{\sup_{t\geq u} t_0^{1/2-\beta} e^{(1/2-\beta)t} \left|F\left(\sqrt{t_0} e^{t/2} V_t^{(e)}\right)\right| \leq 1\right\} := \Omega_u. \tag{26}$$

The scale function of $V(u)$ is defined, for $y \geq 1$, by

$$\mathtt{s_u}(y) := \kappa \int_1^y \exp\left(\frac{z^2}{2} + 2z\right) \mathrm{d}z.$$

Hence, with $g : t \mapsto (1-\varepsilon)\sqrt{2\ln(t)}$, for $t$ big enough,

$$\begin{aligned}\frac{1}{\mathtt{s_u}(g(t))} &= \left(\kappa \int_1^{g(t)} \exp\left(\frac{z^2}{2} + 2z\right) \mathrm{d}z\right)^{-1} \geq \left(\kappa g(t) \exp\left(\frac{g(t)^2}{2} + 2g(t)\right)\right)^{-1} \\ &\geq \frac{\exp\left(-2\sqrt{2\ln(t)}(1-\varepsilon)\right)}{t^{(1-\varepsilon)^2}(1-\varepsilon)\sqrt{2\ln(t)}} \notin L^1(\infty).\end{aligned}$$

By applying Motoo's theorem to $V(u)$ and using (25) we get

$$1 \leq \limsup_{t\to\infty} \frac{V_t(u)}{\sqrt{2\ln(t)}} \leq \limsup_{t\to\infty} \frac{V_t^{(e)}}{\sqrt{2\ln(t)}} \text{ a.s. on } \Omega_u, \text{ and } \mathbb{P}\left(\cup_{u\geq 0}\Omega_u\right) = 1.$$

This concludes the proof of (21).

Step 3. We prove now (22). Fix $\eta > 0$. thanks to (21), there exists $T_0 > 1$ such that

$$\forall T \geq T_0,\ \sup_{t\geq T} \frac{|V_t|}{\sqrt{2\ln\ln t}} \leq 1 + \eta. \tag{27}$$

Fix $T \geq T_0$, then, denoting $H : t \in (1,\infty) \mapsto \int_1^t \sqrt{2\ln\ln s}\,\mathrm{d}s$,

$$\begin{aligned}\sup_{t\geq T} \frac{|X_t|}{H(t)} &\leq (T_0 - t_0) \sup_{s\in[t_0,T_0]} |V_s| \sup_{t\geq T} \frac{1}{H(t)} + \sup_{t\geq T} \sup_{s\in[T_0,t]} \frac{|V_s|}{\sqrt{2\ln\ln(s)}} \frac{1}{H(t)} \int_{T_0}^t \sqrt{2\ln\ln(s)}\,\mathrm{d}s \\ &\leq (T_0 - t_0) \sup_{s\in[t_0,T_0]} |V_s| \sup_{t\geq T} \frac{1}{H(t)} + (1+\eta) \sup_{t\geq T} \frac{1}{H(t)} \int_{T_0}^t \sqrt{2\ln\ln(s)}\,\mathrm{d}s \underset{T\to\infty}{\longrightarrow} 1 + \eta.\end{aligned}$$

□

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