The laws of Brownian local time integrals

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Summary. We obtain some identities in law and some limit theorems for integrals of the type $\int_0^t \varphi(s)d\mathrm{L}_s$. Here φ is a positive locally bounded Borel function and L_t denotes the local time at 0 of processes such as Brownian motion, Brownian bridge, Ornstein-Uhlenbeck process, Bessel process or Bessel bridge of dimension $\mathtt{d}$, $0 < \mathtt{d} < 2$.

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Contents:

Introduction

Our purpose is to compute the laws of certain integrals of the local time at level 0 for one-dimensional stochastic processes such as Brownian motion, Brownian bridge, Ornstein-Uhlenbeck process, Bessel process or Bessel bridge of dimension $\mathtt{d}$, $0 < \mathtt{d} < 2$. These functionals are of the type $\int_0^t \varphi(s)d\mathrm{L}_s$, where φ is a positive locally bounded Borel function.

Functionals of the type $\int_0^t \varphi(s)d\mathrm{L}_s$, with the Brownian local time at level

0, appear naturally, for example:

i) if $\varphi \geq 0$ (and regular), $\int_0^t \varphi(s) d\mathrm{L}_s$ is the local time at level 0 of $(\varphi(t)B_t)$ or of $(\varphi(g_t)B_t)$, where $g_t = \sup\{s \leq t : B_s = 0\}$;

ii) in an asymptotical manner,

$$n \int_0^t \varphi(s) f(nB_s)\, ds \to \int_{\mathbb{R}} f(x)dx \int_0^t \varphi(s) d\mathrm{L}_s \ , \text{ as } \ n \uparrow \infty$$

(see for instance [Re-Y], Chap. XIII §2). Here, we assume that the (regular) function φ satisfies $\int_0^t |\varphi(s)|\, ds/\sqrt{s} < \infty$. Clearly, the convergence take place almost surely, for $\varphi(s) = 1/s^\alpha$, with $\alpha < 1/2$.
Consider now $\varphi(s) = 1/s^\alpha$ with $1/2 \leq \alpha$. We may ask what is the asymptotic behaviour of

$$I_n^{(\alpha)}(f) := n \int_0^t \frac{1}{s^\alpha} f(nB_s)\, ds \, ,$$

as $n \uparrow \infty$. By scaling, $I_n^{(\alpha)}(f)$ has the same law as $n^{2\alpha-1} \int_0^{n^2 t} f(B_s) ds/s^\alpha$. Therefore

$$\frac{I_n^{(\alpha)}(f)}{n^{2\alpha-1}} \overset{(\text{law})}{\longrightarrow} \int_0^\infty \frac{ds}{s^\alpha} f(B_s) = \int_{-\infty}^\infty da\, f(a) \left(\int_0^\infty \frac{d\mathrm{L}_s^a(B)}{s^\alpha} \right) \ , \text{ as } \ n \uparrow \infty \, ,$$

for some function f such that $\int_{-\infty}^\infty |f(a)|\, da/|a|^{2\alpha-1} < \infty$. Here, we denote by $\mathrm{L}_t^a(B)$ the Brownian local time at level a. A natural question is to describe the law of the process

$$\left\{ a \mapsto \int_0^\infty \frac{d\mathrm{L}_s^a(B)}{s^\alpha} : a \in \mathbb{R} \right\} .$$

We compute the Laplace transform of $\int_0^\infty d\mathrm{L}_s^a(B)/s^\alpha$, for fixed a and $\alpha > 1/2$. In particular, if $\alpha = 1$, $a \int_0^\infty d\mathrm{L}_s^a(B)/s^\alpha$ follows the exponential distribution with parameter 1 (see Remark 1.25 *iv)*);

iii) the study $\int_0^\infty e^{-s} d\mathrm{L}_s(B)$ is connected to certain discounted arcsine laws of Baxter and Williams (see [Ba-Wi], [Y3] and [S-Y]).

We prove three types of results:

1. Laplace transforms expressed through series

We give the Laplace transform of the following functionals

(0.1) $$\int_0^1 (s + (1-s)t)^{\beta - 1/2} d\mathrm{L}_t(B) \ \text{ and } \ \int_0^1 (s + (1-s)t)^{\beta-1/2} d\mathrm{L}_t(b) \, ,$$

where $\beta \geq 0$, $s \geq 0$ and $\mathrm{L}_t(B)$ and $\mathrm{L}_t(b)$ denote respectively the local times at 0 of the Brownian motion and of the Brownian bridge (see Corollary 1.12 and 1.15). The Laplace transform is expressed as a series expansion.
Note that taking $s = 0$ and $\beta = 0$ leads to $\int_0^1 d\mathrm{L}_t(B)/\sqrt{t} = \infty$ a.s.; we recall (see [Ra-Y] and [J]) that

$$\int_0^1 \varphi(t)\, d\mathrm{L}_t(B) < \infty \Leftrightarrow \int_0^1 \frac{\varphi(t)}{\sqrt{t}} dt < \infty\,. \tag{0.2}$$

The Laplace transform is expressed as a series expansion.
Similarly, let R (respectively r) denote a Bessel process of dimension $0 < \mathtt{d} := 2\mathtt{n} + 2 < 2$ (respectively a Bessel bridge of the same dimension). Then we obtain the Laplace transform of

$$\int_0^1 \frac{d\mathrm{L}_t(R)}{(s + (1-s)t)^{|\mathtt{n}|}} \quad \text{and} \quad \int_0^1 \frac{d\mathrm{L}_t(r)}{(s + (1-s)t)^{|\mathtt{n}|}}\,, \tag{0.3}$$

where $\mathrm{L}_t(R)$ and $\mathrm{L}_t(r)$ denote respectively the local times at 0 of the Bessel process and of the Bessel bridge (see Theorems 2.17 and 2.18). In particular, if $\mathtt{d} = 1$ (that is $\mathtt{n} = -1/2$), (0.3) is precisely (0.1) with $\beta = 0$.
We also obtain a similar formula for the one-dimensional Ornstein-Uhlenbeck process (see Theorem 2.7).

2. Explicit laws with randomization

When the parameter s is replaced by some particular random variable independent of the local time process we obtain well known laws.

We shall denote by $Z_{a,b}$ a beta random variable with parameters $a, b > 0$:

$$P(Z_{a,b} \in du) = \frac{u^{a-1}(1-u)^{b-1}}{\mathrm{B}(a,b)}\ du \ \ (0 < u < 1) \tag{0.4}$$

and we shall assume that $Z_{a,b}$ is independent from the process for which the local time is considered. Here $\mathrm{B}(a,b)$ denotes the Euler function of the first kind. Also, if we write $\xi \sim \mathcal{E}(1)$ if the random variable ξ follows the exponential distribution with parameter 1 and $\xi \sim \zeta$ if the random variables ξ and ζ have the same law.

For the Bessel process R we show that, for every $\alpha > 0$, the random variables

$$(0.5)\qquad R_{Z_{1,\alpha}} \text{ and } \int_0^{Z_{1,\alpha}} \frac{d\mathrm{L}_t(R)}{(1 - Z_{1,\alpha} + t)^{|\mathtt{n}|}}$$

are independent. Moreover,

$$(0.6)\qquad \frac{2^{\mathtt{n}}\Gamma(\alpha - \mathtt{n})\Gamma(\mathtt{n}+1)}{\Gamma(\alpha)\Gamma(|\mathtt{n}|+1)} \int_0^{Z_{1,\alpha}} \frac{d\mathrm{L}_t(R)}{(1 - Z_{1,\alpha} + t)^{|\mathtt{n}|}} \sim \mathcal{E}(1)\,,$$

where $\Gamma(a)$ denotes the Euler function of the second kind. The case of Brownian motion is obtained for $\mathtt{n} = -1/2$ (see Theorem 1.8 and 2.15).

An analogous result is obtained for the Ornstein-Uhlenbeck process (see Theorem 2.6).

Moreover, we also show that for any $\lambda > 0$, there exist some random variables μ_λ, respectively ν_λ, independent of B, respectively b, such that:

$$(0.7)\qquad c_\lambda \int_0^1 \frac{d\mathrm{L}_t(B)}{\sqrt{t + \mu_\lambda}} \sim \mathcal{E}(1)\,,$$

and

$$(0.8)\qquad c_\lambda \int_0^1 \frac{d\mathrm{L}_t(b)}{\sqrt{t + \nu_\lambda}} \sim \mathcal{E}(1)\,,$$

where $c_\lambda = \sqrt{2\pi}/\mathrm{B}(1/2, \lambda)$. Note that, if μ_λ satisfies (0.7), then $\nu_\lambda = \mu_\lambda/g$ satisfies (0.8), where the variables μ_λ and $g = \sup\{s < 1 : B_s = 0\}$ are assumed to be independent.
We have the following examples: $\mu_\lambda^0 = Z_{\lambda,1}/1 - Z_{\lambda,1}$, then $\nu_\lambda^0 = \mu_\lambda^0/g$, and also $\nu_\lambda^1 = Z_{1/2,\lambda}/1 - Z_{1/2,\lambda}$, or $\nu_\lambda^2 = Z_{\lambda,1/2}/1 - Z_{\lambda,1/2}$.
In particular, taking $\lambda = 1$, respectively $\lambda = 1/2$ we get

$$(0.9)\qquad \sqrt{\frac{\pi}{2}} \int_0^1 \frac{d\mathrm{L}_t(B)}{\sqrt{t + \frac{U}{1-U}}} \sim \sqrt{\frac{2}{\pi}} \int_0^1 \frac{d\mathrm{L}_t(b)}{\sqrt{\frac{1}{V} - t}} \sim \mathcal{E}(1)\,,$$

where we denote by $U = Z_{1,1}$ a uniform random variable on $[0,1]$ independent of B and by $V = Z_{1/2,1/2}$ an arcsine random variable independent of b.
We can obtain similar formulas for the Bessel process or Bessel bridge.

3. Limit theorems

On the other hand, using the formulas for the Laplace transform of random variables of type (0.1) we prove some limit theorems (see Theorems 1.26, 1.28 and 1.30). Thus we obtain the convergence in law to $\mathcal{N}(0,1)$, the standard normal distribution, of

$$(0.10)\qquad \sqrt{\frac{\pi}{\log 2}}\left(\sqrt{\beta}\int_0^1 t^{\beta-1/2}d\mathrm{L}_t(B)-\frac{1}{\sqrt{2\pi\beta}}\right)\,,\ \text{as}\ \beta\downarrow 0\,,$$

$$(0.11)\qquad \sqrt{\frac{\pi}{2\log 2}}\left(\frac{1}{\sqrt{\log 1/\varepsilon}}\int_0^1 \frac{d\mathrm{L}_t(B)}{\sqrt{\varepsilon+t}}-\sqrt{\frac{\log 1/\varepsilon}{2\pi}}\right)\,,\ \text{as}\ \varepsilon\downarrow 0\,,$$

or

$$(0.11')\qquad \sqrt{\frac{\pi}{2\log 2}}\left(\frac{1}{\sqrt{\log 1/\varepsilon^2}}\int_0^1 \frac{d\mathrm{L}_t^\varepsilon(B)}{\sqrt{t}}-\sqrt{\frac{\log 1/\varepsilon^2}{2\pi}}\right)\,,\ \text{as}\ \varepsilon\downarrow 0\,,$$

with $\mathrm{L}_t^\varepsilon(B)$ the Brownian local time at level ε, and also

$$(0.12)\qquad \sqrt{\frac{\pi}{2\log 2}}\left(\frac{1}{\sqrt{\log(t/s)}}\int_s^t \frac{d\mathrm{L}_v(B)}{\sqrt{v}}-\sqrt{\frac{\log(t/s)}{2\pi}}\right)\,,\ \text{as}\ s/t\to 0\,.$$

We can prove that (0.10) and (0.11) are also true for the local time of the Brownian bridge (see Remark 1.27).

Our general approach in this paper is based on a probabilistic representation for the solution of a partial differential equation. On one hand we use the backward Kolmogorov representation for a parabolic partial differential equation to get an explicit form of the solution. On the other hand in the Fokker-Planck representation we use the local time of the associated diffusion. From these two expressions we deduce some useful information on certain functionals of local time.

The plan of the paper is as follows. The first part is devoted to the study of the Brownian motion case and general functionals of Brownian local time. In §1.1 we present the general ideas of the method and in §1.2 we make some explicit computations when we take some particular functionals. Thus we obtain results on integrals of local time for the Brownian motion and Brownian bridge with some deterministic or random parameters. §1.3 is devoted to extensions. Formulas for Laplace transforms allow us to obtain some limit theorems in §1.4. The case of more general diffusion processes is studied in the second part. We apply the general results of §2.1 to the Ornstein-Uhlenbeck process (§2.2). Finally, the case of Bessel processes is considered in §2.3. In the Appendix we give different proofs of our results and some comments.

1. Integrals of Brownian local time

1.1. General result for Brownian motion

Let us consider the equation:

$$\frac{\partial \omega}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 \omega}{\partial x^2}(t,x)\,,\; t>0\,,\; x>0\,, \tag{1.1}$$

with

$$\omega(0,x) = \omega_0(x)\,,\; x \geq 0 \tag{1.2}$$

and

$$\omega(t,0) = f(t)\,,\; t \geq 0 \tag{1.3}$$

as initial and boundary conditions, respectively. Our hypotheses on ω_0 and f are those needed to ensure the existence and the uniqueness of the solution for the problem (1.1)-(1.3) (see [K-S], p. 254 for a closely related problem). For instance, we assure that ω_0 is a continuous function with growth less than exponential at infinity, f is a continuous function which is C^1 on $\mathbb{R}^*$ and $\omega_0(0) = f(0)$.

Denote by $\mathrm{L}_t(B)$ the local time at 0 for the linear Brownian motion B_t starting from 0. The main result of this section is the following

Theorem 1.1. *Assume that $f(t) = \omega(t,0) > 0$, for all $t > 0$. Then for every positive Borel function h,*

$$\mathrm{E}_{\omega_0}\left[h(|B_t|)\exp\left(-\int_0^t \frac{\partial\omega/\partial x}{f}(s,0)d\mathrm{L}_s(B)\right)\mathbb{1}_{\{t\geq T_0\}}\right] = \frac{1}{\sqrt{2\pi}}\int_0^\infty y\,h(y)\,dy\int_0^t \frac{f(t-s)}{s^{3/2}}e^{-y^2/2s}ds\,. \tag{1.4}$$

Here, we denote $T_0 := \inf\{t>0 : B_t = 0\}$ and

$$\mathrm{E}_{\omega_0}(A) := \int_0^\infty \omega_0(x)\mathrm{E}_x(A)\,dx\,, \tag{1.5}$$

To prove the equality (1.4) we need some preliminary results concerning the probabilistic interpretation of the problem (1.1)-(1.3).

First, we write the explicit form of the solution $\omega(t,x)$. This is a consequence of the backward Kolmogorov representation for the problem (1.1)-(1.3).

Proposition 1.2. *The solution $\omega(t,x)$ of* (1.1)-(1.3) *is given by*

$$\omega(t,x) = \int_0^\infty \omega_0(y)[p(t,y-x)-p(t,y+x)]\,dy + \frac{1}{\sqrt{2\pi}}\int_0^t \frac{x\,e^{-x^2/2s}}{s^{3/2}} f(t-s)\,ds\,, \tag{1.6}$$

where we denote $p(t,x) := \frac{1}{\sqrt{2\pi\,t}} e^{-x^2/2t}$.

Proof. Consider, for $x \geq 0$ and $t \geq 0$ the space -time process

$$A_s^{t,x} := (x + B_s, t-s)\,, 0 \leq s \leq t\,,$$

associated to the differential operator $\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}$. To obtain the solution of the problem (1.1)-(1.3) it suffices to solve the Dirichlet problem for the (operator $\mathcal{L}$) process A on $[0,\infty[\times[0,t]$.
We obtain

$$\omega(t,x) = \mathrm{E}_x\left[\omega_0(B_t)\mathbb{1}_{\{t<T_0\}}\right] + \mathrm{E}_x\left[f(t-T_0)\mathbb{1}_{\{t\geq T_0\}}\right]\,.$$

We get the second term in (1.6) from the preceding equality, once we recall that

$$\mathrm{P}_x(T_0 \in ds) = \frac{x\,e^{-x^2/2s}}{\sqrt{2\pi\,s^3}}\,\mathbb{1}_{]0,\infty[}(s)\,ds\,.$$

To obtain the first term we use the symmetry principle. □

Before proving Theorem 1.1 let us note the following useful fact. The unique solution $\omega(t,x)$ of (1.1)-(1.3) satisfies the Fokker-Planck representation:

Proposition 1.3. *The solution $\omega(t,x)$ of* (1.1)-(1.3) *satisfies*

$$< h, \omega(t,\cdot) >= \mathrm{E}_{\omega_0}\left[h(|B_t|)\exp\left(-\int_0^t \frac{\partial\omega/\partial x}{f}(s,0)d\mathrm{L}_s(B)\right)\right]\,, \tag{1.7}$$

for every positive Borel function h. Here, we denote

$$(1.8) \qquad < h, \omega(t,\cdot) > := \int_0^\infty h(x)\omega(t,x)\, dx\,.$$

Proof. Let us define:

$$\rho(s) := \frac{\partial\omega/\partial x}{f}(s,0)$$

and

$$M_t := \exp\left(-\int_0^t \rho(s) d\mathrm{L}_s(B)\right)\,.$$

We introduce, for $t \geq 0$, $x \geq 0$, the function $\tilde{\omega}(t,x)$ defined by

$$< h, \tilde{\omega}(t,\cdot) > := \mathrm{E}_{\omega_0}\left[h(|B_t|)\, M_t\right]\,.$$

Consider a smooth function g and apply Ito's formula to get

$$(i) \qquad g(t,|B_t|)\, M_t = g(0,|B_0|) + \int_0^t \frac{\partial g}{\partial s}(s,|B_s|)\, M_s\, ds$$

$$- \int_0^t g(s,|B_s|)\, \rho(s)\, M_s\, d\mathrm{L}_s(B) + \int_0^t \frac{\partial g}{\partial x}(s,|B_s|)\, M_s(d\mathrm{L}_s(B) + d\tilde{B}_s)$$

$$+ \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x^2}(s,|B_s|)\, M_s\, ds\,,$$

where $d\tilde{B}_s = \mathrm{sgn}(B_s)\, dB_s$.
We choose the function g such that, for every $t \geq 0$,

$$g(t,0)\rho(t) = \frac{\partial g}{\partial x}(t,0)\,.$$

Then, by taking E_{ω_0} in the above equality (i) we get, firstly

$$(ii) \qquad < g(t,\cdot), \frac{\partial \tilde{\omega}}{\partial t}(t,\cdot) > = \frac{1}{2} < \frac{\partial^2 g}{\partial x^2}(t,\cdot), \tilde{\omega}(t,\cdot) >\,,$$

and, secondly

$$(iii) \qquad \frac{\partial g}{\partial x}(t,0)\cdot \tilde{\omega}(t,0) = g(t,0)\cdot \frac{\partial \tilde{\omega}}{\partial x}(t,0)\,.$$

Assuming g has compact support disjoint from $\{0\} \times [0,\infty[$, we can deduce from the first equality (ii) that

$$\frac{\partial \tilde{\omega}}{\partial t} = \frac{1}{2}\frac{\partial^2 \tilde{\omega}}{\partial x^2}\,.$$

Moreover, $\tilde{\omega}$ satisfies

$$\tilde{\omega}(0,x) = \omega_0(x)\,,\ \forall x \geq 0\,,$$

and

$$\frac{\partial \tilde{\omega}}{\partial x}(t,0) = \rho(t)\cdot \tilde{\omega}(t,0)\,,\ \forall t \geq 0\,,$$

as we can see combining the second consequence (*iii*) of Ito's formula and the condition on the function g.
But $\omega(t,x)$ verifies the same conditions because it is the solution of (1.1)-(1.3). Hence $\tilde{\omega} = \omega$ (cf [F], p. 147). □

Proof of Theorem 1.1. Combining (1.6) and (1.8), we write the left hand side of (1.7) as

$$\int_0^\infty \int_0^\infty h(x)\,\omega_0(y)[p(t,y-x) - p(t,y+x)]\,dx\,dy$$

$$+\frac{1}{\sqrt{2\pi}}\int_0^\infty h(x)\,dx \int_0^t \frac{x\,e^{-x^2/2s}}{s^{3/2}} f(t-s)\,ds\,.$$

Using (1.5) we write the right hand side of (1.7) as follows:

$$< h, \omega(t,\cdot) >= \int_0^\infty \omega_0(y)\mathrm{E}_y\left[h(|B_t|)\exp\left(-\int_0^t \frac{\partial\omega/\partial x}{f}(s,0)d\mathrm{L}_s(B)\right)\right]\,dy$$

$$= \int_0^\infty \omega_0(y)\mathrm{E}_y[h(|B_t|)\,1\!\!1_{\{t\leq T_0\}}]\,dy$$

$$+\int_0^\infty \omega_0(y)\mathrm{E}_y\left[h(|B_t|)\exp\left(-\int_0^t \frac{\partial\omega/\partial x}{f}(s,0)d\mathrm{L}_s(B)\right)1\!\!1_{\{t\geq T_0\}}\right]\,dy\,.$$

Indeed, before T_0, the Brownian motion does not visit 0. It is well known (see [K-S], p. 265) that

$$\mathrm{E}_y[h(|B_t|)\,1\!\!1_{\{t\leq T_0\}}] = \int_0^\infty h(x)\,[p(t,y-x) - p(t,y+x)]\,dx\,.$$

The result of the theorem follows from the previous equalities. □

Using the scaling property of the local time $\mathrm{L}_t(B)$ and the Markov property we get

Corollary 1.4. *For every positive Borel function h and every fixed $t > 0$,*

$$(1.9)\qquad \int_0^\infty y\, e^{-y^2/2}\, dy \int_0^1 \frac{\omega_0(y\sqrt{t(1-u)})}{\sqrt{1-u}}\, du$$
$$\times E_0\left[h(|B_u|)\exp\left(-\sqrt{t}\int_0^u \frac{\partial\omega/\partial x}{f}(t(1-u+v))d\mathrm{L}_v(B)\right)\right]$$
$$= \int_0^\infty z\, h(z)\, dz \int_0^1 \frac{f(t(1-u))}{u^{3/2}} e^{-z^2/2u} du\,.$$

Proof. Using the Markov property and the expression of the density of T_0 we can write the left hand side of (1.4) as follows

$$\int_0^\infty \omega_0(x)\mathrm{E}_x\left[h(|B_t|)\exp\left(-\int_0^t \frac{\partial\omega/\partial x}{f}(s,0)d\mathrm{L}_s(B)\right)\mathbb{I}_{\{t\geq T_0\}}\right]\, dx$$

$$= \int_0^\infty x\,\omega_0(x)\, dx \int_0^\infty \frac{e^{-x^2/2s}}{\sqrt{2\pi\, s^3}}\mathbb{I}_{\{t\geq s\}}\, ds$$
$$\times \mathrm{E}_0\left[h(|B_{t-s}|)\exp\left(-\int_0^{t-s} \frac{\partial\omega/\partial x}{f}(s+w,0)d\mathrm{L}_w(B)\right)\right]$$
$$= \int_0^\infty y\,\omega_0(\sqrt{s}\, y)\, dy \int_0^t \frac{e^{-y^2/2}}{\sqrt{2\pi\, s}} ds$$
$$\times \mathrm{E}_0\left[h(|B_{t-s}|)\exp\left(-\int_0^{t-s} \frac{\partial\omega/\partial x}{f}(s+w,0)d\mathrm{L}_w(B)\right)\right]$$
$$= \sqrt{t}\int_0^\infty y\, e^{-y^2/2}\, dy \int_0^1 \frac{\omega_0(y\sqrt{t(1-u)})}{\sqrt{1-u}}\, du$$
$$\times E_0\left[h(|B_{tu}|)\exp\left(-\int_0^{tu} \frac{\partial\omega/\partial x}{f}(t(1-u)+w)d\mathrm{L}_w(B)\right)\right]\,.$$

Here we made the two successive changes of variables $x = y\sqrt{s}$ and $s = t(1-u)$.
It is well known that the Brownian motion and its local time have the following scaling property:

$$(1.10)\qquad (B_u, \mathrm{L}_u)_{u\geq 0} \sim (\sqrt{t}\, B_{u/t}, \sqrt{t}\, \mathrm{L}_{u/t})_{u\geq 0}\,.$$

Then the expectation in the above formula can be written as

$$E_0\left[h(|B_{tu}|)\exp\left(-\int_0^{tu} \frac{\partial\omega/\partial x}{f}(t(1-u)+s)d\mathrm{L}_s(B)\right)\right]$$

$$= E_0\left[h(\sqrt{t}\,|B_u|)\exp\left(-\sqrt{t}\int_0^{tu}\frac{\partial\omega/\partial x}{f}(t(1-u)+s)d\mathrm{L}_{s/t}(B)\right)\right]$$

$$= E_0\left[h(\sqrt{t}\,|B_u|)\exp\left(-\sqrt{t}\int_0^{u}\frac{\partial\omega/\partial x}{f}(t(1-u+v))d\mathrm{L}_{v}(B)\right)\right],$$

by the change of variable $v = s/t$ in the integral with respect to the local time. Hence, from the left hand side of (1.4) we obtain the left hand side of (1.9), up to multiplication by $\sqrt{2\pi t}$. To get (1.9), we make the change of variables $s = tu$ and $y = z\sqrt{t}$ on the right hand side of (1.4), and we simplify the factor $\sqrt{2\pi t}$. □

From (1.6) we calculate the derivative $(\partial\omega/\partial x)(t,0)$:

Corollary 1.5. *For every $t > 0$,*

(1.11)
$$\frac{\partial\omega}{\partial x}(t,0) = 2\int_0^\infty \frac{y\,\omega_0(y)}{t}\,p(t,y)\,dy - \frac{2f(t)}{\sqrt{2\pi\,t}} + \frac{1}{\sqrt{2\pi}}\int_0^t \frac{f(t-s)-f(t)}{s^{3/2}}\,ds\,.$$

Proof. A direct calculation shows that the first term in (1.11) is the value at $(t,0)$ of the derivative with respect to x of the first term in (1.6).
We need to compute the derivative of the second term in (1.6). t being fixed, we denote, for $x > 0$,

$$\psi(x) := \frac{1}{\sqrt{2\pi}}\int_0^t \frac{x\,e^{-x^2/2s}}{s^{3/2}}f(t-s)\,ds\,.$$

Then

$$\psi(x) = \mathrm{E}_0[f(t-T_x)\,\mathbb{1}_{\{T_x\le t\}}]\,,$$

where, as usual $T_x := \inf\{t > 0 : B_t = x\}$. Since $\lim_{x\downarrow 0} T_x = 0$, P_0-a.s., we get

$$\lim_{x\downarrow 0}\psi(x) = f(t) = \psi(0)\,.$$

We compute $\lim_{x\downarrow 0}(\psi(x) - f(t))/x$. Let us denote

$$\psi_1(x) := \mathrm{E}_0[(f(t-T_x) - f(t))\,\mathbb{1}_{\{T_x\le t\}}]/x$$

and

$$\psi_2(x) := f(t)\,\mathrm{P}_0(T_x > t)/x\,.$$

Then

$$(\psi(x) - f(t))/x = \psi_1(x) - \psi_2(x)\,.$$

Since

$$\psi_1(x) = \frac{1}{\sqrt{2\pi}} \int_0^t \frac{f(t-s) - f(t)}{s^{3/2}}\, e^{-x^2/2s}\, ds\,,$$

using the dominated convergence theorem and the fact that $f \in C^1(\mathbb{R}^*)$, we get

$$\lim_{x \downarrow 0} \psi_1(x) = \frac{1}{\sqrt{2\pi}} \int_0^t \frac{f(t-s) - f(t)}{s^{3/2}}\, ds\,,$$

which is the third term in (1.11).
We show that the second term in (1.11) is the limit of ψ_2, as $x \downarrow 0$. Indeed, it follows from [K-S], p. 96, that

$$T_x \sim x^2/\eta^2\,, \text{ with } \eta \sim \mathcal{N}(0,1)\,.$$

Therefore

$$\mathrm{P}_0(T_x > t) = \mathrm{P}(\sqrt{t}\,|\eta| < x)\,,$$

which yields

$$\lim_{x \downarrow 0} \psi_2(x) = f(t) \left(\frac{2\, e^{-x^2/2t}}{\sqrt{2\pi\, t}} \right)_{x=0}\,. \quad \square$$

1.2. Explicit calculations in some particular cases

In the particular case where the initial and boundary conditions in the problem (1.1)-(1.3) are powers, we can perform explicit calculations. Moreover if we replace some deterministic parameters by beta random variables we get some classical distributions.

First, we explain the idea. To find the law of an integral $\int_0^t \varphi(s) d\mathrm{L}_s(B)$, we look for some functions f and ω_0 such that $\varphi(t) = \frac{\partial\omega/\partial x}{f}(t,0)$ and then use Theorem 1.1. Moreover, taking into account the third term in (1.11) we see that a suitable type of functions are the power functions.

Let us introduce, for $\alpha > 0$,

$$\text{(1.12)} \qquad c_\alpha := 2\,\alpha\,\mathrm{B}(1/2,\alpha) = \frac{2\pi}{\mathrm{B}(1/2, \alpha + 1/2)} = 2 + \int_0^1 \frac{1 - (1-u)^\alpha}{u^{3/2}} du\,.$$

We consider

$$(1.13)\qquad f(t) := t^\alpha \ \text{ and } \ \omega_0(x) := \frac{c_\alpha + \mu}{\theta_\alpha} x^{2\alpha}\,,$$

where $\mu \geq 0$ and

$$(1.14)\qquad \theta_\alpha := 2^{\alpha+1}\Gamma(\alpha+1) = 2\int_0^\infty x^{2\alpha+1} e^{-x^2/2}\,dx\,.$$

These functions satisfy the hypothesis which ensures the existence of the solution $\omega(t,x)$ of (1.1)-(1.3) (see the commentary after (1.3)). The main results of this section give some relations involving the Laplace transforms of some particular integrals of local time.

To justify the particular form of the functionals in the statement of our results, let us establish the following

Lemma 1.6. *Consider the functions f and ω_0 given by* (1.13). *Then*

$$(1.15)\qquad \frac{\partial\omega/\partial x}{f}(t,0) = \frac{\mu}{\sqrt{2\pi\, t}}\,.$$

Proof. The result is a consequence of Corollary 1.5. We calculate (1.15) directly from (1.11). Indeed, using (1.13) and (1.14), we obtain:

$$\frac{\partial\omega}{\partial x}(t,0) = \frac{2}{\sqrt{2\pi\, t}}\frac{c_\alpha+\mu}{\theta_\alpha}\frac{1}{t}\int_0^\infty x^{2\alpha+1}e^{-x^2/2t}\,dx - \frac{2}{\sqrt{2\pi}}t^{\alpha-1/2}$$

$$+\frac{1}{\sqrt{2\pi}}\int_0^t \frac{(t-s)^\alpha - t^\alpha}{s^{3/2}}\,ds = \frac{1}{\sqrt{2\pi}}(c_\alpha+\mu)t^{\alpha-1/2} - \frac{1}{\sqrt{2\pi}}c_\alpha t^{\alpha-1/2}\,.$$

by (1.12), and (1.15) follows at once. □

Our first result is a direct consequence of Corollary 1.4.

Theorem 1.7. *For every positive Borel function h, and for every $\alpha > -1/2$, $\mu \geq 0$,*

$$(1.16)\qquad \int_0^1 (1-u)^{\alpha-1/2} E_0\left[h(|B_u|)\,\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^u \frac{d\mathrm{L}_v(B)}{\sqrt{1-u+v}}\right)\right] du$$

$$= \frac{2}{c_\alpha+\mu}\int_0^\infty y\,h(y)\,dy\int_0^1 \frac{(1-u)^\alpha}{u^{3/2}}e^{-y^2/2u}\,du\,.$$

Remark. For $\alpha > -1/2$, c_α is well defined by the second equality in (1.12).

As a consequence we can prove one of the main results of this section. To state the result we introduce some notations.

Recall that, here and everywhere else, $Z_{a,b}$ denotes a beta random variable with parameters $a, b > 0$ (see (0.4)), independent from the Brownian motion.

The next result is a translation of Theorem 1.7 in terms of independence of Brownian functionals.

Theorem 1.8. *For* $\alpha > -1/2$, *the random variables*

$$B_{Z_{1,\alpha+1/2}} \text{ and } \int_0^{Z_{1,\alpha+1/2}} \frac{d\mathrm{L}_v(B)}{\sqrt{1-Z_{1,\alpha+1/2}+v}}\,, \tag{1.17}$$

are independent. Moreover,

$$\frac{c_\alpha}{\sqrt{2\pi}} \int_0^{Z_{1,\alpha+1/2}} \frac{d\mathrm{L}_v(B)}{\sqrt{1-Z_{1,\alpha+1/2}+v}} \sim \mathcal{E}(1)\,, \tag{1.18}$$

the exponential distribution with parameter 1.

Proof of Theorem 1.7. It suffices to prove (1.16) for $\alpha > 0$. Then, by analyticity, it can be extended for $\alpha > -1/2$.
We write the equality (1.9) with $t = 1$ and the functions f and ω_0 given by (1.13):

$$\begin{aligned} &\frac{c_\alpha+\mu}{\theta_\alpha} \int_0^\infty y\, e^{-y^2/2}\, dy \int_0^1 \frac{y^{2\alpha}(1-u)^\alpha}{(1-u)^{1/2}}\, du \\ &\times \mathrm{E}_0\left[h(|B_u|) \exp\left(-\int_0^u \frac{\mu}{\sqrt{2\pi\,(1-u+v)}} d\mathrm{L}_v(B)\right)\right] \\ &= \int_0^\infty z\, h(z)\, dz \int_0^1 \frac{(1-u)^\alpha}{u^{3/2}} e^{-z^2/2u} du\,. \end{aligned}$$

(1.16) is then obtained by (1.14). □

Proof of Theorem 1.8. The first part is clear from the result of Theorem 1.7, noting that the positive Borel function h is arbitrary and that

$$\frac{1}{c_\alpha+\mu} = \int_0^\infty e^{-(c_\alpha+\mu)s} ds\,.$$

To prove the second part we take $h \equiv 1$ in (1.16) and we get

$$\int_0^1 \frac{(1-u)^{\alpha-1/2}}{\mathrm{B}(1,\alpha+1/2)} \mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}} \int_0^u \frac{d\mathrm{L}_v(B)}{\sqrt{1-u+v}}\right)\right] du$$

$$= \frac{2\,\mathrm{B}(1/2,\alpha+1)}{\mathrm{B}(1,\alpha+1/2)} \int_0^\infty e^{-\mu\, s} e^{-c_\alpha\, s} ds = \int_0^\infty e^{-\mu\, s} c_\alpha e^{-c_\alpha\, s} ds\,.$$

We deduce (1.18) from the last equality. □

Remark. In the Appendix we give another proof for the first part of Theorem 1.8. We prove that the result is connected with a simple property of independence for beta-gamma random variables.

In the proof of Theorem 1.8 we used the Laplace transform with respect to the parameter μ. In the second main result we shall use the "Laplace transform" with respect to the parameter α. We shall use Theorem 1.7 and Corollary 1.5. To state the result we introduce some notations.

Let $V = Z_{1/2,1/2}$ be a random variable which is arcsine distributed. Consider $V_1, \dots, V_n$ independent copies of V. We shall denote

$$P_n = \prod_{j=1}^n V_j\,. \tag{1.19}$$

Theorem 1.9. *For every $\mu \geq 0$, $0 < u \leq 1$, $y \geq 0$,*

$$\begin{aligned} &\mathrm{E}_0 \left[\exp\left(-\frac{\mu}{\sqrt{2\pi}} \sqrt{1-u} \int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{u+(1-u)v}} \right) \mid |B_1| = y \right] \\ &= \sqrt{1-u}\, e^{y^2/2} \sum_{n\geq 0} \left(\frac{-\mu}{2} \right)^n \mathrm{E}\left[\frac{\mathbb{1}_{\{P_n \geq u\}}}{\sqrt{P_n - u}} \exp\left(-\frac{y^2}{2} \frac{(1-u)P_n}{P_n - u} \right) \right], \end{aligned} \tag{1.20}$$

or, equivalently, denoting $w = u/(1-u)$,

$$\begin{aligned} &\mathrm{E}_0 \left[\exp\left(-\frac{\mu}{\sqrt{2\pi}} \int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{w+v}} \right) \mid |B_1| = y \right] \\ &= e^{y^2/2} \sum_{n\geq 0} \left(\frac{-\mu}{2} \right)^n \mathrm{E}\left[\frac{\mathbb{1}_{\{(w+1)P_n \geq w\}}}{\sqrt{(w+1)P_n - w}} \exp\left(-\frac{y^2}{2} \frac{P_n}{(w+1)P_n - w} \right) \right]. \end{aligned} \tag{1.20'}$$

For the proof of Theorem 1.9 we shall use the following two lemmas. The first one is a consequence of Theorem 1.7.

Lemma 1.10 *For every $\alpha > -1/2$, $\mu \geq 0$, $y \geq 0$,*

$$\int_0^1 \frac{u^{\alpha-1/2}}{\sqrt{1-u}} e^{-y^2/2(1-u)} du \tag{1.21}$$

$$\times\,\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1\frac{d\mathrm{L}_v}{\sqrt{u+(1-u)v}}\right)\mid |B_1|=\frac{y}{\sqrt{1-u}}\right]$$

$$=\frac{\sqrt{2\pi}}{c_\alpha+\mu}\int_0^1\frac{y\,u^\alpha e^{-y^2/2(1-u)}}{(1-u)^{3/2}}du\,.$$

Proof. First we transform (1.16). On the left hand side we apply again the scaling property (1.10). Therefore, for every positive Borel function h,

$$\int_0^1(1-u)^{\alpha-1/2}\mathrm{E}_0\left[h(\sqrt{u}\,|B_1|)\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{u}\int_0^1\frac{d\mathrm{L}_w}{\sqrt{1-u+uw}}\right)\right]\,du$$

$$=\frac{2}{c_\alpha+\mu}\int_0^\infty z\,h(z)\,dz\int_0^1\frac{(1-u)^\alpha}{u^{3/2}}e^{-z^2/2u}du\,,$$

also by the change of variable $v=uw$ in the integral with respect to the local time, or

$$\int_0^1 u^{\alpha-1/2}du$$

$$\times\,\mathrm{E}_0\left[h(\sqrt{1-u}\,|B_1|)\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1\frac{d\mathrm{L}_w}{\sqrt{u+(1-u)w}}\right)\right]$$

$$=\frac{2}{c_\alpha+\mu}\int_0^\infty z\,h(z)\,dz\int_0^1\frac{u^\alpha}{(1-u)^{3/2}}e^{-z^2/2(1-u)}du\,.$$

By conditioning with respect to $|B_1|=z$ in the left hand side of the above equality we obtain

$$\int_0^\infty e^{-z^2/2}dz\int_0^1 u^{\alpha-1/2}du$$

$$\times\,\mathrm{E}_0\left[h(\sqrt{1-u}\,z)\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1\frac{d\mathrm{L}_w}{\sqrt{u+(1-u)w}}\right)\mid |B_1|=z\right]$$

$$=\frac{\sqrt{2\pi}}{c_\alpha+\mu}\int_0^\infty h(z)\,dz\int_0^1\frac{z\,u^\alpha e^{-z^2/2(1-u)}}{(1-u)^{3/2}}du\,,$$

or, putting $y=\sqrt{1-u}\,z$ on the left hand side,

$$\int_0^\infty h(y)\,dy\int_0^1\frac{u^{\alpha-1/2}}{\sqrt{1-u}}e^{-y^2/2(1-u)}du$$

$$\times\,\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1\frac{d\mathrm{L}_w}{\sqrt{u+(1-u)w}}\right)\mid |B_1|=\frac{y}{\sqrt{1-u}}\right]$$

$$= \frac{\sqrt{2\pi}}{c_\alpha + \mu} \int_0^\infty h(y)\, dy \int_0^1 \frac{y\, u^\alpha e^{-y^2/2(1-u)}}{(1-u)^{3/2}} du\,.$$

□

It is not difficult to deduce from this result the Laplace transforms with respect to α by a simple change of variable.

Lemma 1.11. *For every $\mu \geq 0$, $0 < u \leq 1$, $y \geq 0$,*

$$\text{(1.22)} \qquad \mathrm{E}_0 \left[\exp\left(-\frac{\mu}{\sqrt{2\pi}} \sqrt{1-u} \int_0^1 \frac{d\mathrm{L}_v}{\sqrt{u + (1-u)v}} \right) \mid |B_1| = y \right]$$
$$= \frac{\sqrt{2\pi\, u}}{2} (1-u)\, e^{y^2/2} \sum_{n \geq 0} \left(\frac{-\mu}{2} \right)^n$$
$$\times\, \mathrm{E} \left[\frac{y \sqrt{P_{n+1}}}{(P_{n+1} - u)^{3/2}} \exp\left(-\frac{y^2 (1-u) P_{n+1}}{2(P_{n+1} - u)} \right) \mathbb{1}_{\{P_{n+1} \geq u\}} \right] .$$

Proof. Put

$$F(\mu, y, u) := \mathrm{E}_0 \left[\exp\left(-\frac{\mu}{\sqrt{2\pi}} \sqrt{1-u} \right.\right.$$
$$\left.\left. \times \int_0^1 \frac{d\mathrm{L}_v}{\sqrt{u + (1-u)v}} \right) \mid |B_1| = \frac{y}{\sqrt{1-u}} \right] .$$

Performing in (1.21) the change of variable $u = e^{-w}$, we get

$$\int_0^\infty \frac{e^{-w\alpha - w/2}}{\sqrt{1 - e^{-w}}} e^{-y^2/2(1-e^{-w})} F(\mu, y, e^{-w})\, dw$$
$$= \frac{\sqrt{2\pi}}{c_\alpha + \mu} \int_0^\infty \frac{y\, e^{-w\alpha} e^{-y^2/2(1-e^{-w})}}{(1 - e^{-w})^{3/2}}\, e^{-w}\, dw\,.$$

Let us remark that $2/c_\alpha$ is a Laplace transform with respect to α. Indeed, using (1.12) and the same change of variable $u = e^{-w}$, we get

$$\frac{2}{c_\alpha} = \frac{1}{\pi} \int_0^1 \frac{u^{\alpha - 1/2}}{\sqrt{1-u}}\, du = \frac{1}{\pi} \int_0^\infty e^{-w\alpha} \frac{e^{-w/2}}{\sqrt{1 - e^{-w}}}\, dw\,.$$

This is the Laplace transform of the random variable $\log 1/V$ (recall that $V = Z_{1/2,1/2}$ is an arcsine random variable), with density

$$\chi(w) = \frac{e^{-w/2}}{\pi \sqrt{1 - e^{-w}}} \mathbb{1}_{[0,\infty[}(w)\,.$$

Then we can write,

$$\frac{\sqrt{2\pi}}{c_\alpha+\mu}\int_0^\infty \frac{y\,e^{-w\alpha}e^{-y^2/2(1-e^{-w})}}{(1-e^{-w})^{3/2}}\,e^{-w}\,dw$$

$$=\frac{\sqrt{2\pi}}{c_\alpha}\left(\sum_{n\geq 0}\left(\frac{-\mu}{c_\alpha}\right)^n\right)\int_0^\infty e^{-w\alpha}g(w)\,dw$$

$$=\frac{\sqrt{2\pi}}{2}\left(\sum_{n\geq 0}\left(\frac{-\mu}{2}\right)^n\left(\frac{2}{c_\alpha}\right)^{n+1}\right)\int_0^\infty e^{-w\alpha}g(w)\,dw\,,$$

where $g(w):=y\,e^{-w}e^{-y^2/2(1-e^{-w})}/(1-e^{-w})^{3/2}\,\mathbb{I}_{]0,\infty[}(w)$.
Therefore we obtain the following equality where we denote the convolution product by $*$:

$$\frac{e^{-w/2}}{\sqrt{1-e^{-w}}}e^{-y^2/2(1-e^{-w})}F(\mu,y,e^{-w})$$

$$=g*\left(\frac{\sqrt{2\pi}}{2}\sum_{n\geq 0}\left(\frac{-\mu}{2}\right)^n\chi^{*(n+1)}\right)(w)$$

$$=\frac{\sqrt{2\pi}}{2}\sum_{n\geq 0}\left(\frac{-\mu}{2}\right)^n \mathrm{E}[g(w+\log P_{n+1})]\,,$$

because $\left(g*\chi^{*(n+1)}\right)(w)=\mathrm{E}[g(w+\log(V_1\ldots V_{n+1}))]$. We denote $e^{-w}=u$ and we get

$$e^{-y^2/2(1-u)}F(\mu,y,u)=\frac{\sqrt{2\pi}}{2}\sqrt{u(1-u)}\sum_{n\geq 0}\left(\frac{-\mu}{2}\right)^n$$

$$\times\,\mathrm{E}\left[\frac{y\sqrt{P_{n+1}}}{(P_{n+1}-u)^{3/2}}\exp\left(-\frac{y^2P_{n+1}}{2(P_{n+1}-u)}\right)\mathbb{I}_{\{P_{n+1}\geq u\}}\right],$$

using also the expression of the function g. The result of the lemma is obtained replacing y by $y\sqrt{1-u}$. □

Proof of Theorem 1.9. We only need to transform

$$\mathrm{E}\left[\frac{y\sqrt{P_{n+1}}}{(P_{n+1}-u)^{3/2}}\exp\left(-\frac{y^2(1-u)P_{n+1}}{2(P_{n+1}-u)}\right)\mathbb{I}_{\{P_{n+1}\geq u\}}\right]$$

$$= \mathrm{E}\left[\frac{y}{P_n}\frac{V_{n+1}}{(V_{n+1}-u/P_n)^{3/2}}\exp\left(-\frac{y^2(1-u)}{2}\frac{V_{n+1}}{V_{n+1}-u/P_n}\right)\mathbb{1}_{\{P_n\,V_{n+1}\geq u\}}\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{1-u}}\mathrm{E}\left[\frac{1}{P_n}\frac{\exp\left(-\frac{y^2(1-u)}{2(1-u/P_n)}\right)}{\sqrt{\frac{u}{P_n}(1-u/P_n)}}\mathbb{1}_{\{P_n\geq u\}}\right],$$

as follows from Lemma A.2. Hence we get

$$\mathrm{E}\left[\frac{y\sqrt{P_{n+1}}}{(P_{n+1}-u)^{3/2}}\exp\left(-\frac{y^2(1-u)P_{n+1}}{2(P_{n+1}-u)}\right)\mathbb{1}_{\{P_{n+1}\geq u\}}\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{u(1-u)}}\mathrm{E}\left[\frac{1}{\sqrt{P_n-u}}\exp\left(-\frac{y^2}{2}\frac{(1-u)P_n}{P_n-u}\right)\mathbb{1}_{\{P_n\geq u\}}\right].$$

Replacing this equality in (1.22) we obtain the result of the theorem. The proof is complete except for the proof of Lemma A.2 in the Appendix. □

As simple consequences of Theorem 1.9 we get results without conditioning and for the Brownian bridge.

First, we state the result without conditioning:

Corollary 1.12. *For every* $\mu \geq 0$, $0 < u \leq 1$,

(1.23)
$$\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1\frac{d\mathrm{L}_v(B)}{\sqrt{u+(1-u)v}}\right)\right] = \sum_{n\geq 0}\left(\frac{-\mu}{2}\right)^n\mathrm{E}\left[\frac{\mathbb{1}_{\{P_n\geq u\}}}{\sqrt{P_n}}\right],$$

or, denoting $w = u/(1-u)$,

(1.23')
$$\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^1\frac{d\mathrm{L}_v(B)}{\sqrt{w+v}}\right)\right] = \sum_{n\geq 0}\left(\frac{-\mu}{2}\right)^n\mathrm{E}\left[\frac{\mathbb{1}_{\{(w+1)P_n\geq w\}}}{\sqrt{P_n}}\right],$$

Remark 1.13. We cannot take $u = 0$ in (1.23) because

(1.24)
$$\int_0^1\varphi(t)\,d\mathrm{L}_t(B) < \infty \Leftrightarrow \int_0^1\frac{\varphi(t)}{\sqrt{t}}dt < \infty$$

(see [Ra-Y], p. 655 and [J], p. 44).

Before proving Corollary 1.12, let us state an important consequence of this result.

Proposition 1.14. *For* $\lambda > 0$,

$$\frac{\sqrt{2\pi}}{\mathrm{B}(\lambda, 1/2)} \int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{v + \frac{Z_{\lambda,1}}{1-Z_{\lambda,1}}}} \sim \mathcal{E}(1)\,. \tag{1.25}$$

In particular, for $\lambda = 1$, $Z_{1,1} = U$ *is a uniform random variable on* $[0,1]$ *independent from the Brownian motion and*

$$\sqrt{\frac{\pi}{2}} \int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{v + \frac{U}{1-U}}} \sim \mathcal{E}(1)\,. \tag{1.26}$$

Remark. The second statement of Theorem 1.8 can be obtained using (1.25). Indeed, by the scaling property, for every $c > 0$,

$$\int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{v + \frac{Z_{\lambda,1}}{1-Z_{\lambda,1}}}} \sim \int_0^1 \sqrt{1 - Z_{\lambda,1}} \frac{d\mathrm{L}_{c\,v}(B)}{\sqrt{c}\sqrt{Z_{\lambda,1} + v(1 - Z_{\lambda,1})}}$$

$$\sim \int_0^c \sqrt{1 - Z_{\lambda,1}} \frac{d\mathrm{L}_w(B)}{\sqrt{c}\sqrt{Z_{\lambda,1} + \frac{w}{c}(1 - Z_{\lambda,1})}} \sim \int_0^{1-Z_{\lambda,1}} \frac{d\mathrm{L}_w(B)}{\sqrt{Z_{\lambda,1} + w}}\,,$$

with $c = 1 - Z_{\lambda,1}$. It suffices to take $\lambda = \alpha + 1/2$ and to observe that

$$1 - Z_{\lambda,1} \sim Z_{1,\lambda}\,.$$

Proof of Corollary 1.12. By conditioning,

$$\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{u + (1-u)v}}\right)\right] =$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-y^2/2} dy$$

$$\times \mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{u + (1-u)v}}\right) \;\middle|\; |B_1| = y\right]\,.$$

Using (1.20) and the obvious equality $\int_0^\infty e^{-y^2 a/2}\, dy = \sqrt{\pi/2a}$, we get the result of the corollary. $\square$

Proof of Proposition 1.14. First, let us note that

$$\mathrm{E}\left[\frac{\mathbb{1}_{\{P_n \geq Z_{\lambda,1}\}}}{\sqrt{P_n}}\right] = \mathrm{E}\left(P_n^{\lambda-1/2}\right) = \left[\mathrm{E}\left(V^{\lambda-1/2}\right)\right]^n = \left[\frac{\mathrm{B}(\lambda, 1/2)}{\pi}\right]^n .$$

Therefore the right hand side of (1.23) can be written as

$$\sum_{n\geq 0}\left(\frac{-\mu}{2}\right)^n \frac{\mathrm{B}(\lambda,1/2)^n}{\pi^n} = \frac{1}{1+\mu\,\mathrm{B}(\lambda,1/2)/2\pi} .$$

$\square$

Remark. The random variable $Z_{\lambda,1}/(1-Z_{\lambda,1})$ has the density $\lambda\, u^{\lambda-1}/(1+u)^{1+\lambda}\mathbb{1}_{]0,\infty[}(u)$.

Taking $y = 0$ in Theorem 1.9 we get the result for the Brownian bridge:

Corollary 1.15. *For every $\mu \geq 0$, $0 < u \leq 1$,*

$$\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{u+(1-u)v}}\right)\right] = \sqrt{1-u}\sum_{n\geq 0}\left(\frac{-\mu}{2}\right)^n \mathrm{E}\left[\frac{\mathbb{1}_{\{P_n\geq u\}}}{\sqrt{P_n - u}}\right] . \tag{1.27}$$

or, denoting $w = u/(1-u)$,

$$\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{w+v}}\right)\right] = \sum_{n\geq 0}\left(\frac{-\mu}{2}\right)^n \mathrm{E}\left[\frac{\mathbb{1}_{\{(w+1)P_n\geq w\}}}{\sqrt{(w+1)P_n - w}}\right] . \tag{1.27'}$$

Remark 1.16. We can obtain (1.23) from (1.27) and vice-versa. Indeed, it is classical (and it goes back to Lévy, [L]) that the local time of the Brownian bridge can be represented as:

$$(\mathrm{L}_t(b))_{t\in[0,1]} \sim (\mathrm{L}_{gt}(B)/\sqrt{g})_{t\in[0,1]} . \tag{1.28}$$

Here, we denote $g = \sup\{s < 1 : B_s = 0\}$ which is independent from the process $(\mathrm{L}_{gt}(B)/\sqrt{g})_{t\in[0,1]}$. From (1.28) we obtain

$$\sqrt{g}\int_0^1 \varphi(g\,v)\,d\mathrm{L}_v(b) \sim \int_0^1 \varphi(v)\,d\mathrm{L}_v(B)\,, \tag{1.29}$$

for any positive Borel function φ. Therefore

$$\int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{\frac{u}{g}+(1-u)v}} \sim \int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{u+(1-u)v}}\,. \tag{1.30}$$

We prove that (1.27') implies (1.23'). By (1.27'), with w replaced by w/g,

$$\mathrm{E}\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{\frac{w}{g}+v}}\right)\right]$$

$$= \sum_{n\ge 0}\left(\frac{-\mu}{2}\right)^n \mathrm{E}\left[\frac{\sqrt{g}}{\sqrt{(w+g)P_n-w}}\mathbb{1}_{\{P_n\ge\frac{w}{w+g}\}}\right].$$

Conditionally on P_n, taking the expectation with respect to g in the n-th term on the right hand side of the previous equality, we can write

$$\frac{1}{\pi}\mathrm{E}\left[\left(\int_{\frac{w(1-P_n)}{P_n}}^1 \frac{dx}{\sqrt{1-x}\sqrt{(w+x)P_n-w}}\right)\mathbb{1}_{\{(w+1)P_n\ge w\}}\right]$$

$$= \frac{1}{\pi}\mathrm{E}\left[\frac{\mathbb{1}_{\{(w+1)P_n\ge w\}}}{\sqrt{P_n}}\int_{\frac{w(1-P_n)}{P_n}}^1 \frac{dx}{\sqrt{1-x}\sqrt{x-\frac{w(1-P_n)}{P_n}}}\right]$$

$$= \frac{1}{\pi}\mathrm{E}\left[\frac{\mathbb{1}_{\{(w+1)P_n\ge w\}}}{\sqrt{P_n}}\right]\int_0^1 \frac{dy}{\sqrt{y(1-y)}} = \mathrm{E}\left[\frac{\mathbb{1}_{\{(w+1)P_n\ge w\}}}{\sqrt{P_n}}\right],$$

by the change of variable $y=(x-w(1-P_n)/P_n)/(1-w(1-P_n)/P_n)$. (1.23') follows immediately. □

Similarly to Proposition 1.14, we can prove

Proposition 1.17. *For $\lambda>0$,*

$$\frac{\sqrt{2\pi}}{\mathrm{B}(\lambda,1/2)}\int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{v+\frac{Z_{\lambda,1/2}}{1-Z_{\lambda,1/2}}}} \sim \mathcal{E}(1)\,. \tag{1.31}$$

In particular, for $\lambda=1/2$, $Z_{1/2,1/2}=V$ is an arcsine random variable *independent from the Brownian motion and*

$$\sqrt{\frac{2}{\pi}} \int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{v + \frac{V}{1-V}}} \sim \sqrt{\frac{2}{\pi}} \int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{\frac{1}{V} - v}} \sim \mathcal{E}(1)\,. \tag{1.32}$$

Proof. The reasoning is similar to that made in the proof of Proposition 1.14. First, we calculate

$$\mathrm{E}\left[\sqrt{1 - Z_{\lambda,1/2}}\frac{\mathbb{1}_{\{P_n \geq Z_{\lambda,1/2}\}}}{\sqrt{P_n - Z_{\lambda,1/2}}}\right] = \mathrm{E}\left(P_n^{\lambda-1/2}\right)$$

$$= \left[\mathrm{E}\left(V^{\lambda-1/2}\right)\right]^n = \left[\frac{\mathrm{B}(\lambda, 1/2)}{\pi}\right]^n .$$

Then we use the right hand side of (1.27) to obtain (1.31). The particular case $\lambda = 1/2$ gives the first identity in (1.32). For the second one we simply change v in $1 - v$ and we use the identity in law

$$(b_{1-u})_{0 \leq u \leq 1} \sim (b_u)_{0 \leq u \leq 1}\,.$$

□

Remark 1.18. By the identity in law for the Brownian bridge used in the proof of (1.32) and taking $\varphi(v) = 1/\sqrt{1-v}$ in (1.29), we see that

$$\int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{\frac{1-g}{g} + v}} \sim \int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{1-v}}\,. \tag{1.33}$$

Recall that g is arcsine distributed and independent from b. Therefore, by (1.32) we obtain

$$\sqrt{\frac{2}{\pi}} \int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{1-v}} \sim \mathcal{E}(1)\,. \tag{1.34}$$

This known result is a particular case of Azéma's result (see [A] and [J]) asserting that, if $(A_t^\Lambda, t \geq 0)$ is the dual predictable projection of Λ, the end of a predictable set such that $\mathrm{P}(\Lambda = T) = 0$ for every stopping time T, then $A_\infty^\Lambda \sim \mathcal{E}(1)$; here $\Lambda = g = \sup\{t < 1 : B_t = 0\}$.

Finally, our method allows to obtain other distributions than the exponential:

Proposition 1.19. *Consider Z a random variable of density*

$$\frac{1}{k\pi}\frac{\log 1/u}{\sqrt{u(1-u)}}\mathbb{1}_{[0,1]}(u)\,, \ \text{where}\ k = \frac{1}{\pi}\int_0^1 \frac{\log 1/u}{\sqrt{u(1-u)}}du\,,$$

and Z is independent from the Brownian motion. Then

$$\sqrt{\frac{2}{\pi}}\int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{v+\frac{Z}{1-Z}}} \sim \gamma(2)\,, \tag{1.35}$$

the gamma distribution of parameter 2, having the density $x\,e^{-x}\,\mathbb{1}_{[0,\infty[}(x)$.

Proof. As previously, we calculate the right hand side of (1.27). First,

$$\mathrm{E}\left[\sqrt{1-Z}\,\frac{\mathbb{1}_{\{P_n\geq Z\}}}{\sqrt{P_n-Z}}\right] = \frac{1}{k\pi}\,\mathrm{E}\left[\int_0^{P_n}\sqrt{\frac{1-u}{P_n-u}}\,\frac{\log 1/u}{\sqrt{u(1-u)}}\,du\right]$$

$$= \frac{1}{k\pi}\,\mathrm{E}\left[\int_0^1 \frac{\log 1/v + \log 1/P_n}{\sqrt{v(1-v)}}\,dv\right] = \mathrm{E}\left(1-\frac{\log P_n}{k}\right) = 1+n\,,$$

as we can see by the change of variable $u = vP_n$ and direct integration. Then the right hand side of (1.27) is $1/(1+\mu/2)^2$, which is the Laplace transform of the gamma distribution of parameter 2. □

Remark. Using Proposition 1.17 we can prove a more general result. Indeed, by (1.31) we can write, for an arbitrary positive Borel function h,

$$\int_0^1 du\frac{u^{\lambda-1}}{\sqrt{1-u}}\,\mathrm{E}\,h\left(\sqrt{\frac{2}{\pi}}\int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{v+\frac{u}{1-u}}}\right)$$

$$= \mathrm{B}(\lambda,1/2)\int_0^\infty h\left(\frac{\mathrm{B}(\lambda,1/2)}{\pi}\,x\right)e^{-x}dx = \pi\int_0^\infty h(y)\exp\left(-\frac{\pi\,y}{\mathrm{B}(\lambda,1/2)}\right)dy\,.$$

By derivation with respect to λ, we get

$$\int_0^1 du\frac{u^{\lambda-1}\log u}{\sqrt{1-u}}\,\mathrm{E}\,h\left(\sqrt{\frac{2}{\pi}}\int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{v+\frac{u}{1-u}}}\right)$$

$$= -\pi^2\frac{\mathrm{B}'(\lambda,1/2)}{\mathrm{B}(\lambda,1/2)}\int_0^\infty y\,h(y)\,\exp\left(-\frac{\pi\,y}{\mathrm{B}(\lambda,1/2)}\right)dy\,.$$

This a generalisation of (1.35). Indeed, (1.35) can be obtained taking $\lambda = 1/2$ (recall that $\mathrm{B}(1/2,1/2) = \pi$).

Moreover, by successive derivations with respect to λ, we can obtain that, for every integer $p \geq 1$, the density of

$$\sqrt{\frac{2}{\pi}} \int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{v + \frac{Z_p}{1-Z_p}}}$$

is a linear combination of densities of gamma distributions of parameters $2, 3, \ldots, p+1$. Here, the random variable Z_p is independent of B and its density is

$$\frac{1}{k_p \pi} \frac{(\log u)^p}{\sqrt{u(1-u)}} \mathbb{I}_{[0,1]}(u), \quad k_p \text{ the normalisation constant}.$$

1.3. Extensions of the particular cases

We show that a simple superposition of power functions as initial conditions allows the same type of calculations.

Let us consider, for $\alpha > 0$ $\beta \geq 0$,

$$f(t) := t^\alpha \text{ and } \omega_0(x) := \frac{c_\alpha}{\theta_\alpha} x^{2\alpha} + \frac{\mu}{\theta_{\alpha+\beta}} x^{2\alpha+2\beta} \tag{1.36}$$

where $\mu \geq 0$ and the constants c_α and θ_α are given by (1.12) and (1.14). Then the main result of this section is contained in the following

Theorem 1.20. *For every* $\mu \geq 0$, $\beta \geq 0$, $0 \leq u \leq 1$,

$$\begin{aligned} &\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1 (u+(1-u)v)^{\beta-1/2} d\mathrm{L}_v(B)\right)\right] \\ &\quad = \sum_{n\geq 0} \left(\frac{-\mu}{2}\right)^n u^{n\beta} \mathrm{E}\left[\frac{(P_1 \ldots P_{n-1})^\beta}{P_n^{n\beta}} \frac{\mathbb{I}_{\{P_n \geq u\}}}{\sqrt{P_n}}\right] \end{aligned} \tag{1.37}$$

and

$$\begin{aligned} &\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1 (u+(1-u)v)^{\beta-1/2} d\mathrm{L}_v(b)\right)\right] \\ &\quad = \sqrt{1-u} \sum_{n\geq 0} \left(\frac{-\mu}{2}\right)^n u^{n\beta} \mathrm{E}\left[\frac{(P_1 \ldots P_{n-1})^\beta}{P_n^{n\beta}} \frac{\mathbb{I}_{\{P_n > u\}}}{\sqrt{P_n - u}}\right]. \end{aligned} \tag{1.38}$$

Proof. i) By Corollary 1.5, we note that, for $t > 0$,

$$\text{(1.39)} \qquad \frac{\partial\omega/\partial x}{f}(t,0) = \frac{\mu}{\sqrt{2\pi}} t^{\beta-1/2}\,.$$

ii) By the general formula (1.9) we can write, for $t = 1$,

$$\int_0^1 \left[(1-u)^{\alpha-1/2} + \frac{\mu}{c_\alpha}(1-u)^{\alpha+\beta-1/2}\right] du$$
$$\times\, \mathrm{E}_0\left[h(|B_u|)\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^u (1-u+v)^{\beta-1/2} d\mathrm{L}_v(B)\right)\right]$$
$$= \frac{2}{c_\alpha}\int_0^\infty z\,h(z)\,dz \int_0^1 \frac{(1-u)^\alpha}{u^{3/2}} e^{-z^2/2u} du\,.$$

Using the scaling property (1.10), and making the change of variable $v = uw$ in the integral with respect to the local time, we get, for every positive Borel function h,

$$\int_0^1 \left[(1-u)^{\alpha-1/2} + \frac{\mu}{c_\alpha}(1-u)^{\alpha+\beta-1/2}\right] du$$
$$\times\, \mathrm{E}_0\left[h(\sqrt{u}\,|B_1|)\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{u}\int_0^1 (1-u+uw)^{\beta-1/2} d\mathrm{L}_w(B)\right)\right]$$
$$= \frac{2}{c_\alpha}\int_0^\infty z\,h(z)\,dz \int_0^1 \frac{(1-u)^\alpha}{u^{3/2}} e^{-z^2/2u} du\,,$$

or, replacing u by $1-u$,

$$\text{(1.40)} \qquad \int_0^1 \left[u^{\alpha-1/2} + \frac{\mu}{c_\alpha}u^{\alpha+\beta-1/2}\right] du$$
$$\times\, \mathrm{E}_0\left[h(\sqrt{1-u}\,|B_1|)\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1 (u+(1-u)w)^{\beta-1/2} d\mathrm{L}_w(B)\right)\right]$$
$$= \frac{2}{c_\alpha}\int_0^\infty h(y)\,dy \int_0^1 \frac{y\,u^\alpha\, e^{-y^2/2(1-u)}}{(1-u)^{3/2}} du\,.$$

iii) Proof of (1.38). By conditioning in (1.40) with respect to $|B_1|$ we get

$$\int_0^\infty h(y)\,dy \int_0^1 \frac{u^{\alpha-1/2} + \frac{\mu}{c_\alpha}u^{\alpha+\beta-1/2}}{\sqrt{1-u}} e^{-y^2/2(1-u)} du$$
$$\times\, \mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1 (u+(1-u)w)^{\beta-1/2} d\mathrm{L}_w(B)\right) \,\Big|\, |B_1| = \frac{y}{\sqrt{1-u}}\right]$$
$$= \frac{\sqrt{2\pi}}{c_\alpha}\int_0^\infty h(y)\,dy \int_0^1 \frac{y\,u^\alpha\, e^{-y^2/2(1-u)}}{(1-u)^{3/2}} du\,.$$

Hence,

$$\int_0^1 \frac{u^{\alpha-1/2}+\frac{\mu}{c_\alpha}u^{\alpha+\beta-1/2}}{\sqrt{1-u}}e^{-y^2/2(1-u)}du$$

$$\times\,\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1 (u+(1-u)w)^{\beta-1/2}d\mathrm{L}_w(B)\right)\,|\,|B_1|=\frac{y}{\sqrt{1-u}}\right]$$

$$=\frac{\sqrt{2\pi}}{c_\alpha}\int_0^1 \frac{y\,u^\alpha\,e^{-y^2/2(1-u)}}{(1-u)^{3/2}}du\,.$$

iv) We proceed now as in the proof of Lemma 1.11. Let us denote

$$F(\mu,y,u):=$$

$$\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1 (u+(1-u)w)^{\beta-1/2}d\mathrm{L}_w(B)\right)\,|\,|B_1|=\frac{y}{\sqrt{1-u}}\right]$$

and we put $u=e^{-w}$ in the preceding equality. Then

$$\int_0^\infty e^{-w\alpha}e^{-w/2}\frac{e^{-y^2/2(1-e^{-w})}}{\sqrt{1-e^{-w}}}F(\mu,y,e^{-w})\,dw$$

$$+\frac{\mu}{2}\,\frac{2}{c_\alpha}\int_0^\infty e^{-w\alpha}e^{-w\beta-w/2}\frac{e^{-y^2/2(1-e^{-w})}}{\sqrt{1-e^{-w}}}F(\mu,y,e^{-w})\,dw$$

$$=\sqrt{\frac{\pi}{2}}\,\frac{2}{c_\alpha}\int_0^\infty e^{-w\alpha}e^{-w}\frac{y\,e^{-y^2/2(1-e^{-w})}}{(1-e^{-w})^{3/2}}dw\,.$$

We recall that $\alpha\mapsto 2/c_\alpha$ is the Laplace transform of the random variable $\log 1/V$ with density

$$\chi(w)=\frac{e^{-w/2}}{\pi\sqrt{1-e^{-w}}}\mathbb{1}_{[0,\infty[}(w)$$

(V is as usual an arcsine random variable). Therefore, for every $w\geq 0$,

$$e^{-w/2}\frac{e^{-y^2/2(1-e^{-w})}}{\sqrt{1-e^{-w}}}F(\mu,y,e^{-w})$$

$$+\frac{\mu}{2}\left(e^{-\bullet\beta-\bullet/2}\frac{e^{-y^2/2(1-e^{-\bullet})}}{\sqrt{1-e^{-\bullet}}}F(\mu,y,e^{-\bullet})*\chi\right)(w)$$

$$=\sqrt{\frac{\pi}{2}}\left(e^{-\bullet}\frac{y\,e^{-y^2/2(1-e^{-\bullet})}}{(1-e^{-\bullet})^{3/2}}*\chi\right)(w)\,,$$

or, by calculating the convolutions, for every $u = e^{-w} \in [0,1]$,

$$\frac{e^{-y^2/2(1-u)}}{\sqrt{1-u}} F(\mu, y, u) + \frac{\mu}{2} \, \mathrm{E}\left[\frac{u^\beta}{V^\beta} \, \frac{e^{-y^2 V/2(V-u)}}{\sqrt{V-u}} F(\mu, y, u/V) \, 1\!\!\mathrm{I}_{\{V>u\}}\right]$$

$$= \sqrt{\frac{\pi}{2}} \, \sqrt{u} \, \mathrm{E}\left[\frac{y\sqrt{V} e^{-y^2 V/2(V-u)}}{(V-u)^{3/2}} 1\!\!\mathrm{I}_{\{V>u\}}\right],$$

where we simplified by $u^{1/2}$. But the right hand side of the last equality is $e^{-y^2/2(1-u)}/\sqrt{1-u}$ by the result of Lemma A.2. Hence, denoting $y = z\sqrt{1-u}$ we get

$$e^{-z^2/2} F(\mu, z\sqrt{1-u}, u) + \tag{1.41}$$

$$\frac{\mu}{2}\sqrt{1-u} \, \mathrm{E}\left[\frac{u^\beta}{V^\beta} \, \frac{e^{-z^2 V(1-u)/2(V-u)}}{\sqrt{V-u}} F(\mu, z\sqrt{1-u}, u/V) \, 1\!\!\mathrm{I}_{\{V>u\}}\right] = e^{-z^2/2}.$$

with

$$F(\mu, z\sqrt{1-u}, u) :=$$

$$\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1 (u + (1-u)w)^{\beta-1/2} d\mathrm{L}_w(B)\right) \mid |B_1| = z\right].$$

To obtain the result for the Brownian bridge we take $z = 0$ in (1.41):

$$F(\mu, 0, u) + \frac{\mu}{2}\sqrt{1-u} \, \mathrm{E}\left[\frac{u^\beta}{V^\beta} \, \frac{F(\mu, 0, u/V)}{\sqrt{V-u}} 1\!\!\mathrm{I}_{\{V>u\}}\right] = 1.$$

Thus, we need to solve the functional equation

$$(I + \frac{\mu}{2}\mathcal{B})F = 1,$$

where

$$\mathcal{B}\psi(u) := \sqrt{1-u} \, u^\beta \, \mathrm{E}\left[\frac{\psi(u/V)}{V^\beta \sqrt{V-u}} 1\!\!\mathrm{I}_{\{V>u\}}\right].$$

Hence

$$F(\mu, 0, u) = \sum_{n \geq 0} \left(\frac{-\mu}{2}\right)^n (\mathcal{B}^n 1)(u),$$

with

$$(\mathcal{B}^n 1)(u) = \sqrt{1-u} \, u^{n\beta} \, \mathrm{E}\left[\frac{(P_1 \ldots P_{n-1})^\beta}{P_n^{n\beta}} \, \frac{1\!\!\mathrm{I}_{\{P_n > u\}}}{\sqrt{P_n - u}}\right],$$

as we can see by induction. (1.38) is proven.

v) Proof of (1.37). Unfortunately, we cannot use (1.41) to get, by integration with respect to z, the result for the Brownian motion without conditioning as was done in Corollary 1.12.
To obtain (1.37) we use again (1.40), taking $h \equiv 1$:

$$\int_0^1 \left[u^{\alpha-1/2} + \frac{\mu}{c_\alpha} u^{\alpha+\beta-1/2}\right] du$$

$$\times \mathrm{E}_0 \left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1 (u+(1-u)w)^{\beta-1/2} d\mathrm{L}_w(B)\right)\right]$$

$$= \frac{2}{c_\alpha}\int_0^\infty y\, e^{-y^2/2(1-u)} dy \int_0^1 \frac{u^\alpha}{(1-u)^{3/2}} du\,,$$

or, by integrating with respect to y on the right hand side,

$$\int_0^1 \left[u^{\alpha-1/2} + \frac{\mu}{c_\alpha} u^{\alpha+\beta-1/2}\right] G(\mu,u)\, du = \frac{2}{c_\alpha}\int_0^1 \frac{u^\alpha}{\sqrt{1-u}} du\,,$$

where

$$G(\mu,u) := \mathrm{E}_0 \left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^1 (u+(1-u)v)^{\beta-1/2} d\mathrm{L}_v(B)\right)\right].$$

We make the change of variable $u = e^{-w}$:

$$\int_0^\infty e^{-w\alpha} e^{-w/2} G(\mu, e^{-w})\, dw + \frac{\mu}{2}\frac{2}{c_\alpha}\int_0^\infty e^{-w\alpha} e^{-w\beta - w/2} G(\mu, e^{-w})\, dw =$$

$$\frac{2}{c_\alpha}\int_0^\infty e^{-w\alpha}\frac{e^{-w}}{\sqrt{1-e^{-w}}} dw\,,$$

or, for every $w \geq 0$,

$$e^{-w/2} G(\mu, e^{-w}) + \frac{\mu}{2}\left(e^{-\bullet\beta - \bullet/2} G(\mu, e^{-\bullet}) * \chi\right)(w) = \left(\frac{e^{-\bullet}}{\sqrt{1-e^{-\bullet}}} * \chi\right)(w)\,,$$

χ being, as usual, the density of $\log 1/V$. By calculating the convolutions, for every $u = e^{-w} \in [0,1]$,

$$G(\mu,u) + \frac{\mu}{2} u^\beta\, \mathrm{E}\left[\frac{G(\mu, u/V)}{V^{\beta+1/2}} \mathbb{I}_{\{V \geq u\}}\right] = \sqrt{u}\, \mathrm{E}\left[\frac{\mathbb{I}_{\{V>u\}}}{\sqrt{V(V-u)}}\right].$$

By (A.8) in Lemma A.3 the right hand side of the last equality is 1. Hence, we need to solve the functional equation

$$(I + \frac{\mu}{2}\mathcal{A})G = 1\,,$$

where

$$\mathcal{A}\psi(u) := u^\beta\,\mathrm{E}\left[\frac{\psi(u/V)}{V^{\beta+1/2}}\mathbb{1}_{\{V\geq u\}}\right].$$

Hence

$$G(\mu,u) = \sum_{n\geq 0}\left(\frac{-\mu}{2}\right)^n(\mathcal{A}^n 1)(u)\,,$$

with

$$(\mathcal{A}^n 1)(u) = u^{n\beta}\mathrm{E}\left[\frac{(P_1\ldots P_{n-1})^\beta}{P_n^{n\beta}}\,\frac{\mathbb{1}_{\{P_n\geq u\}}}{\sqrt{P_n}}\right],$$

as we can see again by induction. (1.37) is proven.

The proof of Theorem 1.21 is complete except for the proofs of Lemmas A.2 and A.3 which are presented in the Appendix. □

Remark. Clearly, formulas (1.23) and (1.27) are simple consequences of Theorem 1.21 for $\beta = 0$.

As in §1.2 in Theorem 1.20 we can replace u by some particular random variables in (1.37) and (1.38).

Corollary 1.21. *For $\lambda > 0$ and, for every $\mu \geq 0$, $\beta \geq 0$,*

$$\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-Z_{\lambda,1}}\int_0^1 (Z_{\lambda,1}+(1-Z_{\lambda,1})v)^{\beta-1/2}\,d\mathrm{L}_v(B)\right)\right] = \sum_{n\geq 0}\left(\frac{-\mu}{2\pi}\right)^n\frac{\lambda}{n\beta+\lambda}\prod_{j=0}^{n-1}\mathrm{B}(j\beta+\lambda,1/2)\,, \tag{1.42}$$

and

$$\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-Z_{\lambda,1/2}}\times \int_0^1\Big(Z_{\lambda,1/2}+(1-Z_{\lambda,1/2})v\Big)^{\beta-1/2}\,d\mathrm{L}_v(b)\right)\right] = \sum_{n\geq 0}\left(\frac{-\mu}{2\pi}\right)^n\prod_{j=1}^{n}\mathrm{B}(j\beta+\lambda,1/2)\,. \tag{1.43}$$

Proof. To obtain (1.42) we need to calculate, from (1.37)

$$\mathrm{E}\left[(\mathcal{A}^n 1)(Z_{\lambda,1})\right] = \mathrm{E}\left[Z_{\lambda,1}^{n\beta}\frac{(P_1 \dots P_{n-1})^\beta}{P_n^{n\beta}} \frac{\mathbb{1}_{\{P_n \geq Z_{\lambda,1}\}}}{\sqrt{P_n}}\right]$$

$$= \mathrm{E}\left[\frac{(P_1 \dots P_{n-1})^\beta}{P_n^{n\beta}} \frac{1}{\sqrt{P_n}} \int_0^{P_n} \lambda\, u^{n\beta} u^{\lambda-1} du\right]$$

$$= \frac{\lambda}{n\beta+\lambda}\, E\left[(P_1 \dots P_{n-1})^\beta P_n^{\lambda-1/2}\right].$$

It suffices to note that $\mathrm{E}(V^p) = \mathrm{B}(p+1/2, 1/2)/\pi$ and (1.42) follows. Similarly, using (1.38)

$$\mathrm{E}\left[(\mathcal{B}^n 1)(Z_{\lambda,1/2})\right] = \mathrm{E}\left[\sqrt{1-Z_{\lambda,1/2}}\, Z_{\lambda,1/2}^{n\beta} \frac{(P_1 \dots P_{n-1})^\beta}{P_n^{n\beta}} \frac{\mathbb{1}_{\{P_n \geq Z_{\lambda,1/2}\}}}{\sqrt{P_n - Z_{\lambda,1/2}}}\right]$$

$$= \mathrm{E}\left[\frac{(P_1 \dots P_{n-1})^\beta}{P_n^{n\beta}} \int_0^{P_n} \frac{u^{n\beta+\lambda-1}}{\mathrm{B}(\lambda,1/2)\sqrt{P_n - u}}\, du\right]$$

$$= \frac{\mathrm{B}(n\beta+\lambda, 1/2)}{\mathrm{B}(\lambda,1/2)}\, \mathrm{E}\left[\frac{(P_1 \dots P_{n-1})^\beta}{P_n^{n\beta}}\, P_n^{n\beta+\lambda-1/2}\right].$$

We get (1.43) as previously. □

Remark 1.22. Again, (1.25) and (1.31) can be obtained taking $\beta = 0$ in (1.42) and (1.43), respectively.

Remark 1.23. Taking $\varphi(v) = (1-v)^{\beta-1/2}$ in (1.29) and using the time reversal property of the Brownian bridge, we get

$$\int_0^1 (1-v)^{\beta-1/2} d\mathrm{L}_v(B) \sim g^\beta \int_0^1 \left(\frac{1-g}{g} + v\right)^{\beta-1/2} d\mathrm{L}_v(b)\,, \tag{1.44}$$

where g is as usual the last zero before time 1; g has arcsine distribution. Let us note that the Laplace transform of the left hand side of (1.44) can be obtained by the same method as in Corollary 1.21, taking the random variable $1-g$ instead of u in (1.38). Therefore we get, for every $\mu \geq 0$, $\beta \geq 0$,

$$(1.45)\qquad \mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^1 (1-v)^{\beta-1/2} d\mathrm{L}_v(B)\right)\right]$$

$$= \sum_{n\geq 0}\left(\frac{-\mu}{2}\right)^n \mathrm{E}\left[\sqrt{g}\,(1-g)^{n\beta}\,\frac{(P_1\dots P_{n-1})^\beta}{P_n^{n\beta}}\,\frac{\mathbb{1}_{\{P_n>1-g\}}}{\sqrt{P_n-1+g}}\right]$$

$$= \sum_{n\geq 0}\left(\frac{-\mu}{2\pi}\right)^n \prod_{j=1}^n \mathrm{B}(j\beta+1/2,1/2)\,,$$

with similar calculations as in Remark 1.16.

Finally, let us note that for $\beta > 0$ it is possible to let $u \downarrow 0$ in the formulas of Theorem 1.20. This is a different situation than in the case $\beta = 0$ (see also Remark 1.13).

Corollary 1.24. *For every* $\mu \geq 0$, $\beta > 0$,

$$(1.46)\qquad \mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^1 v^{\beta-1/2} d\mathrm{L}_v(B)\right)\right] = \sum_{n\geq 0}\left(\frac{-\mu}{2\pi}\right)^n \frac{1}{n\beta}\prod_{j=1}^{n-1}\mathrm{B}(j\beta,1/2)$$

and

$$(1.47)\qquad \mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^1 v^{\beta-1/2} d\mathrm{L}_v(b)\right)\right] = \sum_{n\geq 0}\left(\frac{-\mu}{2\pi}\right)^n \prod_{j=1}^{n}\mathrm{B}(j\beta,1/2)\,.$$

Proof. To obtain (1.46) and (1.47) we need to calculate the limits

$$\lim_{u\downarrow 0}(\mathcal{A}^n 1)(u) \quad\text{and}\quad \lim_{u\downarrow 0}(\mathcal{B}^n 1)(u)\,,$$

where

$$(\mathcal{A}^n 1)(u) = u^{n\beta}\mathrm{E}\left[\frac{(P_1\dots P_{n-1})^\beta}{P_n^{n\beta}}\,\frac{\mathbb{1}_{\{P_n\geq u\}}}{\sqrt{P_n}}\right]\,,$$

and

$$(\mathcal{B}^n 1)(u) = \sqrt{1-u}\,u^{n\beta}\,\mathrm{E}\left[\frac{(P_1\dots P_{n-1})^\beta}{P_n^{n\beta}}\,\frac{\mathbb{1}_{\{P_n> u\}}}{\sqrt{P_n-u}}\right]\,.$$

It suffices to prove that

$$\lim_{u\downarrow 0}(\mathcal{A}^n 1)(u) = \frac{1}{\pi^n n\beta}\prod_{j=1}^{n-1}\mathrm{B}(j\beta,1/2)$$

and

$$\lim_{u\downarrow 0}(\mathcal{B}^n 1)(u) = \frac{1}{\pi^n}\prod_{j=1}^{n}\mathrm{B}(j\beta,1/2)\,.$$

Let us prove the last equality, the previous one being similar. For $n=1$ we can write

$$\begin{aligned}(\mathcal{B}1)(u) &= u^\beta\sqrt{1-u}\,\mathrm{E}\left[\frac{\mathbb{1}_{\{V_1>u\}}}{V_1^\beta\sqrt{V_1-u}}\right]\\ &= \frac{u^\beta\sqrt{1-u}}{\pi}\int_u^1\frac{dv}{v^{\beta+1/2}\sqrt{v-u}\sqrt{1-v}}\\ &= \frac{\sqrt{1-u}}{\pi}\int_1^{1/u}\frac{dw}{w^{\beta+1/2}\sqrt{w-1}\sqrt{1-wu}}\,.\end{aligned}$$

By the Lebesgue dominated theorem the last integral tends to $\mathrm{B}(\beta,1/2)$, when $u\downarrow 0$. If $n=2$, we obtain

$$\begin{aligned}(\mathcal{B}^2 1)(u) &= \frac{u^{2\beta}\sqrt{1-u}}{\pi^2}\int\int_{\{vw>u\}}\frac{dv\,dw}{v^{2\beta+1/2}w^{\beta+1/2}\sqrt{1-v}\sqrt{1-w}\sqrt{vw-u}}\\ &= \frac{\sqrt{1-u}}{\pi^2}\int_1^{1/u}\frac{dx}{x^{\beta+1/2}\sqrt{1-ux}}\int_1^x\frac{dy}{y^{\beta+1/2}\sqrt{x-y}\sqrt{y-1}}\,,\end{aligned}$$

as we can see by making the changes of variables $w=yu/v$ and $v=ux$. Again by the Lebesgue dominated theorem, the last double integral tends to

$$\frac{1}{\pi^2}\int_1^\infty\frac{dx}{x^{\beta+1/2}}\int_1^x\frac{dy}{y^{\beta+1/2}\sqrt{x-y}\sqrt{y-1}} = \frac{\mathrm{B}(2\beta,1/2)\,\mathrm{B}(\beta,1/2)}{\pi^2}\,.$$

The same reasoning applies for arbitrary n. □

Remark 1.25. *i)* Taking $\beta=1/2$ in (1.46) we recover the well known Laplace transform of $\mathrm{L}_1(B)\sim|B_1|$.

ii) Let us denote $M_n^{(B)}(\beta)$, respectively $M_n^{(b)}(\beta)$ the moments of order n

of the random variables $\int_0^1 v^{\beta-1/2} d\mathrm{L}_v(B)$, respectively $\int_0^1 v^{\beta-1/2} d\mathrm{L}_v(b)$. Using (1.46) and (1.47) we can obtain, by straightforward calculation,

$$M_n^{(B)}(\beta) = \frac{2^{\beta\, n(n-1)}}{(2\pi)^{n/2}\beta^n} \prod_{j=1}^{n-1} \frac{1}{C_{2j\beta}^{j\beta}}\;, \text{ if } \beta \in \mathbb{N}^*\,,$$

$$M_n^{(B)}(\beta) = \frac{(n-1)!}{2^{n/2}\beta}\,\frac{\Gamma(\beta)}{\Gamma((n-1)\beta+\frac{1}{2})} \prod_{j=1}^{n-1}\prod_{i=1}^{\beta-\frac{1}{2}} (j\beta - i)\,, \text{ if } \beta - \frac{1}{2} \in \mathbb{N}\,,$$

$$M_n^{(b)}(\beta) = \frac{2^{\beta\, n(n+1)}}{(2\pi)^{n/2}\beta^n} \prod_{j=1}^{n} \frac{1}{C_{2j\beta}^{j\beta}}\;, \text{ if } \beta \in \mathbb{N}^*$$

and

$$M_n^{(b)}(\beta) = \frac{n!}{2^{n/2}}\,\frac{\Gamma(\beta)}{\Gamma(n\beta+\frac{1}{2})} \prod_{j=1}^{n}\prod_{i=1}^{\beta-\frac{1}{2}} (j\beta - i)\,, \text{ if } \beta - \frac{1}{2} \in \mathbb{N}\,.$$

Note also, looking at the expressions of the moments featured in (1.46) and (1.47), that this agrees with the identity in law

$$\int_0^1 v^{\beta-1/2} d\mathrm{L}_v(B) \sim g^\beta \int_0^1 v^{\beta-1/2} d\mathrm{L}_v(b)\,,$$

where on the right hand side, g and b are independent.

iii) Here is another method to obtain (1.47), or the moments of order n, $M_n^{(b)}(\beta)$ of the random variable $X_{\beta-1/2} := \int_0^1 v^{\beta-1/2} d\mathrm{L}_v(b)$.
Let us apply the balayage formula to the martingale $\psi(g_t)\,|B_t|$, with $g_t = \sup\{s \le t : B_s = 0\}$ and ψ a given function (for the balayage formula, see, for instance, [A-Y]). Then, for some function φ,

$$\varphi\left(\int_0^t \psi(v) d\mathrm{L}_v(B)\right) \psi(g_t)\,|B_t| - \Phi\left(\int_0^t \psi(v) d\mathrm{L}_v(B)\right)$$

is a martingale. Here, we denote $\Phi(x) = \int_0^x \varphi(y)\,dy$. The expectation of the previous expression is identically zero. By projection of $|B_t|$ on $\mathcal{F}_{g_t}$ we get

$$\mathrm{E}\left(\psi(g_t)\,|B_t|\,|\,\mathcal{F}_{g_t}\right) = \psi(g_t)\sqrt{\pi/2}\sqrt{t-g_t}\,.$$

Hence, we can write

$$\mathrm{E}\left[\varphi\left(\int_0^1 \psi(v) d\mathrm{L}_v(B)\right) \psi(g)\sqrt{\pi/2}\sqrt{1-g}\right] = \mathrm{E}\left[\Phi\left(\int_0^1 \psi(v) d\mathrm{L}_v(B)\right)\right]$$

where $g = g_1$. Taking $\psi(v) = v^\alpha$, with $\alpha = \beta - 1/2$, we obtain

$$\mathrm{E}\left[\varphi\left(\int_0^1 v^\alpha d\mathrm{L}_v(B)\right)\sqrt{\pi/2}\sqrt{1-g}\,g^\alpha\right] = \mathrm{E}\left[\Phi\left(\int_0^1 v^\alpha d\mathrm{L}_v(B)\right)\right].$$

To get the moments of order n, we consider $\varphi(x) = x^n$. Then, by scaling, we get

$$\begin{aligned}&\mathrm{E}\left[\sqrt{\pi/2}\sqrt{1-g}\,g^\alpha\,(\sqrt{g}g^\alpha)^n\right]\mathrm{E}\left[(X_\alpha)^n\right]\\ &= \frac{1}{n+1}\,\mathrm{E}\left[\left(g^{\alpha+1/2}\right)^{n+1}\right]\mathrm{E}\left[(X_\alpha)^{n+1}\right]\end{aligned}$$

Therefore,

$$\begin{aligned}&\sqrt{\frac{\pi}{2}}\,\mathrm{E}\left[\sqrt{1-g}\,g^{\alpha+n(\alpha+1/2)}\right]M_n^{(b)}(\alpha+1/2)\\ &= \frac{1}{n+1}\,\mathrm{E}\left[g^{(n+1)(\alpha+1/2)}\right]M_{n+1}^{(b)}(\alpha+1/2)\,,\end{aligned}$$

or, using the fact that g is arcsine distributed,

$$M_{n+1}^{(b)}(\alpha+1/2) = \frac{n+1}{\sqrt{2}}\,\frac{\Gamma((n+1)\beta)}{\Gamma((n+1)\beta+1/2)}\,M_n^{(b)}(\alpha+1/2)\,.$$

This equality agrees with that obtained by (1.47).

iv) Using Corollary 1.24 we can study the law of $\int_0^\infty d\mathrm{L}_s^a(B)/s^\alpha$, with fixed a. Here $1/2 < \alpha$ and $\mathrm{L}_t^a(B)$ denotes the local time at level a. Indeed, we can write (see also (A.16) in the Appendix),

$$(*) \qquad \int_0^\infty \frac{d\mathrm{L}_s^a(B)}{s^\alpha} \sim \frac{1}{T_a^{\alpha-1/2}}\int_0^\infty \frac{d\mathrm{L}_v^0(\tilde{B})}{(1+v)^\alpha} \sim \frac{1}{T_a^{\alpha-1/2}}\int_0^1 v^{\alpha-1}d\mathrm{L}_v^0(\tilde{b})\,,$$

where $\tilde{B}$ and $\tilde{b}$ are independent Brownian motion and Brownian bridge. Let us note $\beta = \alpha - 1/2 > 0$. Taking $\mu = \lambda\sqrt{2\pi}/T_a^\beta$ in (1.47) we obtain

$$\begin{aligned}&\mathrm{E}_0\left[\exp\left(-\frac{\lambda}{T_a^{\alpha-1/2}}\int_0^1 v^{\alpha-1}d\mathrm{L}_v^0(\tilde{b})\right)\right]\\ &= \mathrm{E}_0\left[\exp\left(-\frac{\lambda}{T_a^\beta}\int_0^1 v^{\beta-1/2}d\mathrm{L}_v^0(\tilde{b})\right)\right]\\ &= \sum_{n\geq 0}\left(\frac{-\lambda}{\sqrt{2\pi}}\right)^n\mathrm{E}\left(\frac{1}{T_a^{n\beta}}\right)\prod_{j=1}^n \mathrm{B}(j\beta,1/2)\,.\end{aligned}$$

A simple calculation gives

$$\mathrm{E}\left(\frac{1}{T_a^\gamma}\right) = \frac{2^\gamma}{\sqrt{\pi}\, a^{2\gamma}}\Gamma(\gamma + 1/2)\,.$$

Therefore

$$\mathrm{E}_0\left[\exp\left(-\frac{\lambda}{T_a^{\alpha-1/2}}\int_0^1 v^{\alpha-1} d\mathrm{L}_v^0(\tilde{b})\right)\right]$$

$$= \frac{1}{\sqrt{\pi}}\sum_{n\geq 0}\left(\frac{-2^{\alpha-1}\lambda}{a^{2\alpha-1}}\right)^n \frac{\Gamma(\alpha-1/2)\ldots\Gamma(n(\alpha-1/2))}{\Gamma((\alpha-1/2)+1/2)\ldots\Gamma((n-1)(\alpha-1/2)+1/2)}\,.$$

If $\alpha = 1$ the right hand side of the preceding equality is equal to $a/a+\lambda$, which gives by (*)

$$a\int_0^\infty \frac{d\mathrm{L}_s^a(B)}{s} \sim a\,\frac{L_1^0(\tilde{b})}{\sqrt{T_a}} \sim \mathcal{E}(1)\,.$$

The second identity in law can be also obtained as follows: if $\eta \sim \mathcal{N}(0,1)$ is independent from $\tilde{b}$, then $|\eta|/a \sim 1/\sqrt{T_a}$; moreover, taking $\mu = \lambda\sqrt{2\pi}\,|\eta|$ in (1.47), we prove that

$$|\eta|\, L_1^0(\tilde{b}) \sim \mathcal{E}(1)$$

by calculation of the Laplace transform. Let us note that from the last identity in law we deduce that

$$L_1^0(\tilde{b}) \sim \sqrt{2\,\mathcal{E}(1)}\,,$$

using also the classical and simple fact that $\eta^2 \sim 2\,g\,\mathcal{E}(1)$ (see also the proof of (A.5') in the Appendix). Here, g and η are independent random variables.

1.4. Limit theorems

In this section we show that the Laplace transform formulas we already obtained can be used to get some limit theorems.

For the moment, we illustrate the method with a simple case (obtaining a simple formula). The other cases will be treated in the same manner.

Before stating the first result let us note some simple facts. Let us replace μ in (1.47) by $\mu\beta$:

$$\mathrm{E}_0\left[\exp\left(-\frac{\mu\beta}{\sqrt{2\pi}}\int_0^1 v^{\beta-1/2} d\mathrm{L}_v(b)\right)\right] = \sum_{n\geq 0}\left(\frac{-\mu\beta}{2\pi}\right)^n \prod_{j=1}^n \mathrm{B}(j\beta, 1/2)\,.$$

It is known that $\mathrm{B}(\beta, 1/2) \cong 1/\beta$, as $\beta \downarrow 0$. Therefore, the right hand side of the preceeding equality tends to $\exp(-\mu/2\pi)$, when $\beta \downarrow 0$. Thus we obtained that, for $\beta \downarrow 0$,

$$\sqrt{2\pi}\ \beta \int_0^1 v^{\beta-1/2} d\mathrm{L}_v(b) \overset{\text{(law)}}{\longrightarrow} 1\,. \tag{1.48}$$

In fact we prove the following stronger result:

Theorem 1.26. *For* $\beta \downarrow 0$,

$$\sqrt{\frac{\pi}{\log 2}} \left(\sqrt{\beta} \int_0^1 v^{\beta-1/2} d\mathrm{L}_v(b) - \frac{1}{\sqrt{2\pi\beta}} \right) \overset{\text{(law)}}{\longrightarrow} \mathcal{N}(0,1)\,. \tag{1.49}$$

(Recall that $\mathcal{N}(0,1)$ denotes the standard normal distribution).

Remark 1.27. A similar result can be obtained for the Brownian local time using (1.46). In fact, the contribution of the time t, came from an interval of type $[0,a]$ $(a < 1)$, and it suffices to use the equivalence in law of the Brownian motion and the Brownian bridge on $\sigma(\{B_t; t \le a\})$.

Proof of Theorem 1.26. Using (1.47) we can write

$$\mathrm{E}_0 \left[\exp - \frac{\mu}{\sqrt{2\pi}} \left(\sqrt{\beta} \int_0^1 v^{\beta-1/2} d\mathrm{L}_v(b) - \frac{1}{\sqrt{2\pi\beta}} \right) \right]$$

$$= \left[\sum_{n\ge 0} \left(\frac{-\mu}{2\pi} \right)^n \beta^{n/2} \prod_{j=1}^n \mathrm{B}(j\beta, 1/2) \right] \cdot \left[\sum_{n\ge 0} \frac{1}{n!} \left(\frac{\mu}{2\pi} \right)^n \frac{1}{\beta^{n/2}} \right].$$

The general term of the product series, $c_n(\beta)$ can be written as

$$c_n(\beta) = \sum_{k=0}^n \left(\frac{-\mu}{2\pi} \right)^k \beta^{k/2} \prod_{j=1}^k \mathrm{B}(j\beta, 1/2) \times \frac{1}{(n-k)!} \left(\frac{\mu}{2\pi} \right)^{n-k} \frac{1}{\beta^{(n-k)/2}}$$

$$= \left(\frac{\mu}{2\pi} \right)^n \frac{1}{\beta^{n/2}} \sum_{k=0}^n \frac{(-1)^k}{(n-k)!\,k!} \prod_{j=1}^k j\beta\, \mathrm{B}(j\beta, 1/2)\,.$$

We now compute $\lim_{\beta\downarrow 0} c_n(\beta)$. Let us note two important facts. First,

$$\psi(x) := x\,\mathrm{B}(x, 1/2) = 1 + \delta\, x + o(x)\,, \quad \text{where } \delta := \frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(1/2)}{\Gamma(1/2)} = \log 4$$

(see [Leb], p. 5). Second, for every integer n,

$$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,k!}\,k^n = (-1)^n\,, \text{ whereas } \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,k!}\mathcal{P}(k) = 0\,, \tag{1.50}$$

when $\mathcal{P}$ is a polynomial of degree $\leq n-1$. We shall denote $\mathcal{P} \equiv \mathcal{Q}\ (n)$, if $\mathcal{P} - \mathcal{Q}$ is of degree less or equal to $n-1$.
With this remarks in hand, it is not difficult to see that the coefficient of β^r in $\prod_{j=1}^k \psi(j\beta)$ is $\equiv 0\ (n)$, if $r \leq n/2 - 1$. Indeed, the coefficient of greatest degree is that of $\prod_{j=1}^k (1 + \delta\, j\beta)$. Moreover, the coefficient of β^m with $n = 2m$ is $\equiv \delta^m\, k^{2m}/(2^m\, m!)\ (n)$. Clearly, when we take the limit as $\beta \downarrow 0$, the terms in β^r with $r > m = n/2$ have no contribution.
Therefore

$$\lim_{\beta\downarrow 0} c_{2m}(\beta) = \left(\frac{\mu}{2\pi}\right)^{2m} \frac{\delta^m}{2^m\, m!}\,,$$

and thus

$$\lim_{\beta\downarrow 0} \mathrm{E}_0\left[\exp - \frac{\mu}{\sqrt{2\pi}}\left(\sqrt{\beta}\int_0^1 v^{\beta-1/2} d\mathrm{L}_v(b) - \frac{1}{\sqrt{2\pi\beta}}\right)\right]$$

$$= \sum_{m\geq 0} \left(\frac{\mu}{2\pi}\right)^{2m} \frac{\delta^m}{2^m\, m!} = \exp\left(\frac{\mu^2}{(2\pi)^2}\,\frac{\delta}{2}\right)\,.$$

□

Remark. In the Appendix we give another approach of (1.49) based on Ito's formula.

With this method in hand we can prove other limit theorems.

Theorem 1.28. *For* $u \downarrow 0$,

$$\sqrt{\frac{\pi}{2\log 2}}\left(\sqrt{\frac{1-u}{\log 1/u}}\int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{u + (1-u)v}} - \sqrt{\frac{\log 1/u}{2\pi(1-u)}}\right) \xrightarrow{\text{(law)}} \mathcal{N}(0,1)\,, \tag{1.51}$$

or, equivalently, for $\varepsilon \downarrow 0$,

$$\sqrt{\frac{\pi}{2\log 2}}\left(\frac{1}{\sqrt{\log 1/\varepsilon}}\int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{\varepsilon + v}} - \sqrt{\frac{\log 1/\varepsilon}{2\pi}}\right) \xrightarrow{\text{(law)}} \mathcal{N}(0,1)\,. \tag{1.52}$$

Proof. i) To get (1.51) we use the result of Corollary 1.12 with μ replaced by $\mu\sqrt{2\pi}/(\sqrt{1-u}\log 1/u)$:

$$\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\log 1/u}\int_0^1 \frac{d\mathrm{L}_v(B)}{\sqrt{u+(1-u)v}}\right)\right]$$

$$=\sum_{n\geq 0}\left(\frac{-\mu}{\sqrt{1-u}\,\log 1/u}\,\sqrt{\frac{\pi}{2}}\right)^n \alpha_n(u)\,,$$

where $\alpha_n(u) = \mathrm{E}[1/\sqrt{P_n}\,\mathbb{1}_{\{P_n\geq u\}}]$. We prove that, for all integers $n>0$,

$$\alpha_n(u) \cong \frac{(\log 1/u)^n}{\pi^n n!}\,,\ \text{ as } u\downarrow 0\,.$$

Indeed,

$$\alpha_n(u) = \frac{1}{\pi^n}\int\cdots\int_{u_1\ldots u_n\geq u}\frac{1}{u_1\ldots u_n}\,\frac{du_1\ldots du_n}{\sqrt{1-u_1}\ldots\sqrt{1-u_n}}$$

$$=\frac{(\log 1/u)^n}{\pi^n}\int\cdots\int_{x_1+\ldots+x_n\leq 1}\frac{dx_1\ldots dx_n}{\sqrt{1-e^{-x_1(\log 1/u)}}\ldots\sqrt{1-e^{-x_n(\log 1/u)}}}\,,$$

by the change of variables $x_j = (\log 1/u_j)/(\log 1/u)$, $j=1,\ldots,n$. By Lebesgue dominated theorem we get the equivalent for $\alpha_n(u)$, as $u\downarrow 0$. Therefore, we get

$$\lim_{u\downarrow 0}\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\log 1/u}\int_0^1\frac{d\mathrm{L}_v(B)}{\sqrt{u+(1-u)v}}\right)\right] = \exp\left(-\frac{\mu}{\sqrt{2\pi}}\right)\,,$$

or

$$\frac{\sqrt{2\pi}}{\log 1/u}\int_0^1\frac{d\mathrm{L}_v(B)}{\sqrt{u+(1-u)v}}\overset{\text{(law)}}{\longrightarrow}1\,,$$

as $u\downarrow 0$. We proceed now as in the proof of Theorem 1.26:

$$\mathrm{E}_0\left[\exp -\frac{\mu}{\sqrt{2\pi}}\left(\frac{\sqrt{1-u}}{\sqrt{\log 1/u}}\int_0^1\frac{d\mathrm{L}_v(B)}{\sqrt{u+(1-u)v}}-\frac{\sqrt{\log 1/u}}{\sqrt{2\pi}}\right)\right]$$

$$=\left[\sum_{n\geq 0}\left(\frac{-\mu}{2\sqrt{\log 1/u}}\right)^n\alpha_n(u)\right]\cdot\left[\sum_{n\geq 0}\frac{1}{n!}\left(\frac{\mu}{2\pi}\right)^n(\log 1/u)^{n/2}\right]\,.$$

Let us denote $\rho := \log 1/u$. The general term of the product series is

$$c_n(\rho) = \frac{1}{n!}\left(\frac{\mu}{2\pi}\right)^n\rho^{n/2}\sum_{k=0}^n(-1)^k\left(\frac{\pi}{\rho}\right)^k C_n^k\,k!\,\alpha_k(\rho)\,,$$

where

$$\alpha_n(\rho) = \left(\frac{\rho}{\pi}\right)^n \int \dots \int_{x_1+\ldots+x_n \le 1} \frac{dx_1 \dots dx_n}{\sqrt{1-e^{-x_1\rho}} \dots \sqrt{1-e^{-x_n\rho}}}\,.$$

Hence we need to compute the limit as $\rho \uparrow \infty$ of

$$c_n(\rho) = \frac{1}{n!}\left(\frac{\mu}{2\pi}\right)^n \rho^{n/2} \sum_{k=0}^{n} (-1)^k C_n^k\, k!\, I_k(\rho)\,,$$

where

$$I_k(\rho) := \int \dots \int_{x_1+\ldots+x_k \le 1} \frac{dx_1 \dots dx_k}{\sqrt{(1-e^{-x_1\rho}) \dots (1-e^{-x_k\rho})}}\,.$$

ii) Let us note that

$$2\log 2 = \int_0^\infty \left(\frac{1}{\sqrt{1-e^{-x}}} - 1\right) dx = \sum_{n\ge 1} \frac{(2n)!}{n 2^{2n} (n!)^2} = \mathrm{E} \int_0^{\eta^2/2} \frac{e^y - 1}{y}\, dy\,,$$

where $\eta \sim \mathcal{N}(0,1)$. The first equality is obtained by direct integration and the others by series expansion. Moreover,

$$\int_0^\rho \left(\frac{1}{\sqrt{1-e^{-x}}} - 1\right)\, dx = 2\log 2 + O(e^{-\rho})\,,$$

because $(1-e^{-x})^{-1/2} - 1 \le 2e^{-x}$, for large x. We shall use the following important similar fact: there exists two positive constants a, b such that

$$\int \dots \int_{x_1+\ldots+x_k \le \rho} \left(\frac{1}{\sqrt{1-e^{-x_1}}} - 1\right) \dots \left(\frac{1}{\sqrt{1-e^{-x_k}}} - 1\right) \times x_1^{i_1} \dots x_k^{i_k}\, dx_1 \dots dx_k < a\, e^{-b\rho}\,. \tag{1.53}$$

This can be done using the upper bound $\prod_{j=1}^k \int_0^\rho \left((1-e^{-x_j})^{-1/2} - 1\right) x_j^{i_j}\, dx_j$ and the lower bound $\prod_{j=1}^k \int_0^{\rho/k} \left((1-e^{-x_j})^{-1/2} - 1\right) x_j^{i_j}\, dx_j$.
The classical equality

$$\prod_{i=1}^r a_i - 1 = \tag{1.54}$$

$$\prod_{i=1}^r (a_i - 1) + \sum_j \prod_{i\ne j} (a_i - 1) + \ldots + \sum_{j_1,\ldots,j_p} \prod_{i \ne j_1,\ldots,j_p} (a_i - 1) + \ldots + \sum_{i=1}^r (a_i - 1)\,,$$

and the relations (1.50) will be also used.
iii) We illustrate the idea computing $\lim_{\rho\uparrow\infty} c_2(\rho)$. Using (1.50) and (1.54) we can write

$$c_2(\rho) =$$

$$\frac{1}{2!}\left(\frac{\mu}{2\pi}\right)^2 \rho \sum_{k=0}^{2}(-1)^k C_2^k\, k! \int \dots \int_{x_1+\ldots+x_k\leq 1} \frac{dx_1\dots dx_k}{\sqrt{(1-e^{-x_1\rho})\dots(1-e^{-x_k\rho})}}$$

$$= \frac{1}{2!}\left(\frac{\mu}{2\pi}\right)^2 \rho \sum_{k=0}^{2}(-1)^k C_2^k\, k! \int \dots \int_{x_1+\ldots+x_k\leq 1} dx_1\dots dx_k$$

$$\times \left(\frac{1}{\sqrt{(1-e^{-x_1\rho})\dots(1-e^{-x_k\rho})}} - 1\right)$$

$$= \frac{1}{2!}\left(\frac{\mu}{2\pi}\right)^2 \rho \left[-2\int_0^1 \left(\frac{1}{\sqrt{1-e^{-x\rho}}} - 1\right) dx\right.$$

$$+2\int_{x_1+x_2\leq 1}\left(\frac{1}{\sqrt{1-e^{-x_1\rho}}} - 1\right)\left(\frac{1}{\sqrt{1-e^{-x_2\rho}}} - 1\right) dx_1\, dx_2$$

$$\left.+4\int\int_{x_1+x_2\leq 1}\left(\frac{1}{\sqrt{1-e^{-x_2\rho}}} - 1\right) dx_1\, dx_2\right]$$

$$= \frac{1}{2!}\left(\frac{\mu}{2\pi}\right)^2 \rho \left[-\frac{2}{\rho}\int_0^\rho \left(\frac{1}{\sqrt{1-e^{-y}}} - 1\right) dy\right.$$

$$+\frac{2}{\rho^2}\int\int_{y_1+y_2\leq \rho}\left(\frac{1}{\sqrt{1-e^{-y_1}}} - 1\right)\left(\frac{1}{\sqrt{1-e^{-y_2}}} - 1\right) dy_1\, dy_2$$

$$\left.+\frac{4}{\rho}\int_0^1 \left(\frac{1}{\sqrt{1-e^{-y}}} - 1\right)\left(1-\frac{y}{\rho}\right) dy\right].$$

The last equality is obtained making the change of variables $x_i\rho = y_i$. Using (1.53) we get

$$\lim_{\rho\uparrow\infty} c_2(\rho) = \frac{1}{2!}\left(\frac{\mu}{2\pi}\right)^2 4\log 2\,.$$

iv) In the general case we perform a similar calculation:

$$c_{2n}(\rho) =$$

$$\frac{1}{(2n)!}\left(\frac{\mu}{2\pi}\right)^{2n} \rho^n \sum_{k=0}^{2n}(-1)^k C_{2n}^k k! \int \dots \int_{x_1+\ldots+x_k\leq 1} dx_1\dots dx_k$$

$$\times \left(\frac{1}{\sqrt{(1-e^{-x_1\rho})\dots(1-e^{-x_k\rho})}} - 1\right)$$

$$= \frac{1}{(2n)!}\left(\frac{\mu}{2\pi}\right)^{2n}\rho^n\sum_{k=0}^{2n}(-1)^k C_{2n}^k k!\sum_{l=1}^{k} C_k^l$$

$$\times\int\ldots\int_{x_1+\ldots+x_k\le 1}\left(\frac{1}{\sqrt{1-e^{-x_1\rho}}}-1\right)\ldots\left(\frac{1}{\sqrt{1-e^{-x_l\rho}}}-1\right)dx_1\ldots dx_k$$

$$= \frac{1}{(2n)!}\left(\frac{\mu}{2\pi}\right)^{2n}\rho^n\sum_{k=0}^{2n}(-1)^k C_{2n}^k k!\sum_{l=1}^{k} C_k^l$$

$$\times\int\ldots\int_{x_1+\ldots+x_l\le 1}\left(\frac{1}{\sqrt{1-e^{-x_1\rho}}}-1\right)\ldots\left(\frac{1}{\sqrt{1-e^{-x_l\rho}}}-1\right)$$

$$\times\frac{(1-x_1-\ldots-x_l)^{k-l}}{(k-l)!}dx_1\ldots dx_l$$

$$= \frac{1}{(2n)!}\left(\frac{\mu}{2\pi}\right)^{2n}\rho^n\sum_{k=0}^{2n}(-1)^k C_{2n}^k k!\sum_{l=1}^{k} C_k^l\frac{1}{\rho^l}$$

$$\times\int\ldots\int_{y_1+\ldots+y_l\le \rho}\left(\frac{1}{\sqrt{1-e^{-y_1}}}-1\right)\ldots\left(\frac{1}{\sqrt{1-e^{-y_l}}}-1\right)$$

$$\times\frac{1}{(k-l)!}\left(1-\frac{y_1}{\rho}-\ldots-\frac{y_l}{\rho}\right)^{k-l}dx_1\ldots dx_l\,.$$

We shall make $\rho\uparrow\infty$. Replacing in the previous expression the integrals on $\{x_1+\ldots x_l\le\rho\}$ by integrals on $\mathbb{R}_+^l$ we make an exponentially small error. We look in the preceeding sum for the coefficient of the term in $1/\rho^p$, with $p\le n$. The contribution of the constant terms of the development of $(1-y_1/\rho-\ldots-y_l/\rho)^{k-l}$ is

$$\frac{1}{(2n)!}\left(\frac{\mu}{2\pi}\right)^{2n}\rho^n\sum_{k=p}^{2n}(-1)^k C_{2n}^k k!\,C_k^p\frac{1}{\rho^p}(2\log 2)^p\frac{1}{(k-p)!}$$

$$= \frac{1}{(2n)!}\left(\frac{\mu}{2\pi}\right)^{2n}\rho^{n-p}\sum_{k=p}^{2n}(-1)^k C_{2n}^k(2\log 2)^p\frac{(k(k-1)\ldots(k-p+1))^2}{p!}\,.$$

Using (1.50) we see that this contribution is zero if $2p<2n$, because $(k(k-1)\ldots(k-p+1))^2$ is a polynomial of degree less than $2n$. If $p=n$ the coefficient is

$$\frac{1}{(2n)!}\left(\frac{\mu}{2\pi}\right)^{2n}(2\log 2)^n\frac{(2n)!}{n!}\,.$$

The coefficient of $1/\rho^{p+1}$ is given by the contribution of the terms with degree 1 in y_i of the development of $(1-y_1/\rho-\ldots-y_l/\rho)^{k-l}$:

$$\frac{1}{(2n)!}\left(\frac{\mu}{2\pi}\right)^{2n}\rho^n\sum_{k=p}^{2n}(-1)^k C_{2n}^k k!\, C_k^p\frac{1}{\rho^{p+1}}(2\log 2)^p c\frac{p(k-p)}{(k-p)!}\,,$$

where $c=\int_0^\infty x\left((1-e^{-x})^{-1/2}-1\right)dx$. But

$$k!\, C_k^p\frac{p(k-p)}{(k-p)!}=\left(k(k-1)\ldots(k-p+1)\right)^2 p(k-p)$$

is a polynomial of degree $2p+1$ in k. We consider only $p+1\le n$, that is $2p+1\le 2n-1$, and by (1.50) the sum of these coefficients is zero.
In a similar way, the coefficients of $1/\rho^{p+i}$ are given by the terms of degree i of the development of $(1-y_1/\rho-\ldots-y_l/\rho)^{k-l}$ and are polynomials of degree less than $2n-i$, so they are zero.
Finally,

$$\lim_{\rho\uparrow\infty}c_{2n}(\rho)=\left(\frac{\mu}{2\pi}\right)^{2n}\frac{(2\log 2)^n}{n!}$$

and, clearly,

$$\lim_{\rho\uparrow\infty}c_{2n+1}(\rho)=0\,.$$

Therefore

$$\mathrm{E}_0\left[\exp-\frac{\mu}{\sqrt{2\pi}}\left(\frac{\sqrt{1-u}}{\sqrt{\log 1/u}}\int_0^1\frac{d\mathrm{L}_v(B)}{\sqrt{u+(1-u)v}}-\frac{\sqrt{\log 1/u}}{\sqrt{2\pi(1-u)}}\right)\right]$$

$$\to\sum_{n\ge 0}\left(\frac{\mu}{2\pi}\right)^{2n}\frac{(2\log 2)^n}{n!}=\exp\left[2\log 2\left(\frac{\mu}{2\pi}\right)^2\right]\,,\ \text{as}\ u\downarrow 0\,.$$

□

Corollary 1.29 *We denote, for* $\delta>0$, $\mathrm{L}_t^\delta(B)$ *the local time at level* δ. *Then, for* $\delta\downarrow 0$,

$$\sqrt{\frac{\pi}{2\log 2}}\left(\frac{1}{\sqrt{\log 1/\delta^2}}\int_0^1\frac{d\mathrm{L}_v^\delta(B)}{\sqrt{v}}-\sqrt{\frac{\log 1/\delta^2}{2\pi}}\right)\xrightarrow{\text{(law)}}\mathcal{N}(0,1)\,. \tag{1.55}$$

Proof. We can write

$$\int_0^1\frac{d\mathrm{L}_v^\delta(B)}{\sqrt{v}}=\int_{T_\delta}^1\frac{d\mathrm{L}_v^\delta(B)}{\sqrt{v}}=\int_0^{1-T_\delta}\frac{d\mathrm{L}_v^0(\tilde{B})}{\sqrt{v+T_\delta}}\,,$$

where $T_\delta = \inf\{t > 0 : B_t = \delta\}$ and $\tilde{B}_v := B_{T_\delta + v} - B_{T_\delta}$. Hence

$$\int_0^1 \frac{d\mathrm{L}_v^\delta(B)}{\sqrt{v}} \sim \int_0^{1-\delta^2 T_1} \frac{d\mathrm{L}_v^0(\tilde{B})}{\sqrt{v + \delta^2 T_1}} \,.$$

By the independence we can take $\varepsilon = \delta^2 T_1$ and we apply (1.52) to obtain (1.55). □

We finish this section with the following:

Theorem 1.30. *For $s/t \to 0$,*

$$\frac{\sqrt{2\pi}}{\log(t/s)} \int_s^t \frac{d\mathrm{L}_v(B)}{\sqrt{v}} \overset{(\text{prob.})}{\longrightarrow} 1\,. \tag{1.56}$$

Moreover, for $s/t \to 0$,

$$\sqrt{\frac{\pi}{2\log 2}} \left(\frac{1}{\sqrt{\log(t/s)}} \int_s^t \frac{d\mathrm{L}_v(B)}{\sqrt{v}} - \sqrt{\frac{\log(t/s)}{2\pi}} \right) \overset{(\text{law})}{\longrightarrow} \mathcal{N}(0,1)\,. \tag{1.57}$$

Proof. The proof of this result is similar to the proofs of Theorems 1.26 and 1.28. We only point out the main ideas.
We can write

$$\begin{aligned}
\mathrm{E}_0\left[\exp -\frac{\mu}{\sqrt{2\pi}} \int_s^t \frac{d\mathrm{L}_v(B)}{\sqrt{v}}\right] &= \mathrm{E}_0\left[\mathrm{E}_{B_s}\left[\exp -\frac{\mu}{\sqrt{2\pi}} \int_0^{t-s} \frac{d\mathrm{L}_v(B)}{\sqrt{s+v}}\right]\right] \\
&= \mathrm{E}_0\left[g(B_s)\right]\,.
\end{aligned}$$

Here

$$g(x) := \mathrm{E}_x\left[\exp -\frac{\mu}{\sqrt{2\pi}} \int_0^{t-s} \frac{d\mathrm{L}_v(B)}{\sqrt{s+v}}\right] = g_1(x) + g_2(x)\,,$$

with

$$g_1(x) := \mathrm{P}_x(T_0 > t - s)$$

and

$$g_2(x) := \int_0^{t-s} \frac{|x|}{\sqrt{2\pi u^3}} e^{-x^2/2u} \mathrm{E}_0\left[\exp -\frac{\mu}{\sqrt{2\pi}} \int_0^{t-s-u} \frac{d\mathrm{L}_v(B)}{\sqrt{s+v+u}}\right] du\,.$$

We can easily see that

$$\mathrm{E}_0\left[g_1(B_s)\right] \to g_1(0)\,, \text{ as } s/t \to 0\,.$$

On the other hand, by straightforward calculation we get

$$\mathrm{E}_0\left[g_2(B_s)\right] = \frac{2}{\pi}\sum_{n\geq 0}\left(\frac{-\mu}{2}\right)^n \mathrm{E}\left[\frac{\mathbb{1}_{\{P_n\geq s/t\}}}{\sqrt{P_n}}\arctan\sqrt{\frac{P_n - s/t}{s/t}}\right].$$

Therefore, the n-th moment of $1/\sqrt{2\pi}\int_s^t d\mathrm{L}_v(B)/\sqrt{v}$ behaves, for $s/t \to 0$, as $\mathrm{E}\left[\mathbb{1}_{\{P_n\geq s/t\}}/\sqrt{P_n}\right]$. But this is exactly $\alpha_n(s/t)$ in the proof of the Theorem 1.28 (this explain the same constant $2\log 2$ in (1.57)). The rest of the proof is similar to the proof of (1.51) and we leave it to the reader. □

2. Integrals of local time for other diffusion processes

2.1 General results for diffusion processes

The purpose of this section is to develop the same method for other diffusion processes. We follow the strategy developed in the previous sections.

Let us consider the problem

$$\frac{\partial\omega}{\partial t}(t,x) = \frac{1}{2}\left(\frac{\partial^2\omega}{\partial x^2}(t,x) - \mathtt{b}(x)\,\frac{\partial\omega}{\partial x}(t,x)\right),\ t>0,\ x>0, \tag{2.1}$$

with

$$\omega(0,x) = \omega_0(x),\ x\geq 0 \tag{2.2}$$

and

$$\omega(t,0) = f(t),\ t\geq 0. \tag{2.3}$$

Here $\mathtt{b}(-x) = -\mathtt{b}(x)$. Under reasonable hypothesis on $\mathtt{b}$, ω_0 and f we have the existence and the uniqueness of the solution for the problem (2.1)-(2.3).

Let X_t be the diffusion starting from 0 and generated by the differential operator $\mathcal{L} := \frac{1}{2}(\frac{d^2}{dx^2} - \mathtt{b}(x)\frac{d}{dx})$. We shall denote $\mathrm{L}_t(X)$ its local time at level 0,

$$\phi(x,t) := \frac{d}{dt}\mathrm{P}_x(t\geq T_0), \tag{2.4}$$

the density of $T_0 := \inf\{t>0 : X_t = 0\}$ and by ν the following density of probability

$$\nu(x) := \frac{1}{k}\exp\left(-\int_0^{|x|} \mathtt{b}(y)\,dy\right)\ ,\ \text{for}\ \ x \neq 0\ \ \text{and}\ \ \nu(0) \neq 0 \tag{2.5}$$

(we assume that $\mathtt{b}$ is locally integrable and k is the normalisation constant).

We can state the following general result

Theorem 2.1. *For every positive Borel function h,*

$$\begin{aligned} \mathrm{E}_{\omega_0\nu}\left[h(|X_t|)\exp\left(-\int_0^t \frac{\partial\omega/\partial x}{f}(s,0)d\mathrm{L}_s(X)\right)\mathbb{1}_{\{t\geq T_0\}}\right] \\ = \int_0^\infty h(y)\nu(y)\,dy \int_0^t f(t-s)\phi(y,s)\,ds\,, \end{aligned} \tag{2.6}$$

where

$$\mathrm{E}_{\omega_0\nu}(A) := \int_0^\infty \omega_0(x)\,\nu(x)\,\mathrm{E}_x(A)\,dx\,. \tag{2.7}$$

The proof is similar to that of Theorem 1.1. Precisely, we use the backward Kolmogorov representation for the problem (2.1)-(2.3) to obtain the explicit form of the solution $\omega(t,x)$. Then, from the Fokker-Planck representation we get an equality containing this solution and the local time $\mathrm{L}_t(X)$.

Proposition 2.2. *The solution $\omega(t,x)$ of* (2.1)-(2.3) *is given by*

$$\omega(t,x) = \mathrm{E}_x(\omega_0(X_t)\mathbb{1}_{\{t<T_0\}}) + \int_0^t \phi(x,s)f(t-s)\,ds\,. \tag{2.8}$$

Proposition 2.3. *For every positive Borel function h, the solution $\omega(t,x)$ satisfies*

$$< h,\omega(t,\cdot) >_\nu = \mathrm{E}_{\omega_0\nu}\left[h(|X_t|)\exp\left(-\int_0^t \frac{\partial\omega/\partial x}{f}(s,0)d\mathrm{L}_s(X)\right)\right]\ . \tag{2.9}$$

Here, we denote

$$< h,\omega(t,\cdot) >_\nu := \int_0^\infty h(x)\omega(t,x)\nu(x)\,dx\,. \tag{2.10}$$

The proofs of Theorem 2.1 and of Propositions 2.2 and 2.3 are entirely similar to the proofs of Theorem 1.1, of Propositions 1.2 and 1.3. Let us note that the symmetry of the operator $\mathcal{L}$ with respect to ν plays a central role. The proofs are omitted.

2.2. Integrals of local time for Ornstein-Uhlenbeck process

Clearly, the results in §2.1 are more explicit for particular processes. We illustrate this for the case of the Ornstein-Uhlenbeck process O_t.

Let us note that $\mathtt{b}(x) = x$ and that the density ν is given by $\nu(x) = 2/\sqrt{2\pi}\, e^{-x^2/2}1\!\!1_{[0,\infty[}(x)$. Moreover, using both, the symmetry principle for O and the explicit form of the density of the Ornstein-Uhlenbeck semi-group, we obtain

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}} \frac{xe^{-t/2}}{(1-e^{-t})^{3/2}} \exp\left(-\frac{x^2 e^{-t}}{2(1-e^{-t})}\right), \; x \geq 0\,. \tag{2.11}$$

(for an introduction to the Ornstein-Uhlenbeck process see [Bo-Sa]).

Therefore, using the result of Proposition 2.2, we obtain,

$$\begin{aligned} \frac{\partial\omega}{\partial x}(t,0) &= Q_t\tilde{\omega}_0'(0) - \frac{2f(t)}{\sqrt{2\pi}\sqrt{e^t-1}} \\ &+ \frac{1}{\sqrt{2\pi}}\int_0^t (f(t-s)-f(t))\frac{e^{-s/2}}{(1-e^{-s})^{3/2}}ds\,, \end{aligned} \tag{2.12}$$

where Q_t denotes the Ornstein-Uhlenbeck semi-group and

$$\tilde{\omega}_0(x) := \begin{cases} \omega_0(x)\,, & \text{if } x \geq 0 \\ -\omega_0(-x)\,, & \text{if } x \leq 0\,, \end{cases} \tag{2.13}$$

As for the Brownian motion we take some particular initial and boundary conditions. Let us consider, for $\alpha > 0$,

$$f(t) := (1-e^{-t})^\alpha \;\text{ and }\; \omega_0(x) := \frac{c_\alpha + \mu}{\theta_\alpha}x^{2\alpha}\,, \tag{2.14}$$

where $\mu \geq 0$ and

$$c_\alpha := 2\,\alpha\,\mathrm{B}(1/2,\alpha)\,,\; \theta_\alpha := 2^{\alpha+1}\Gamma(\alpha+1) \tag{2.15}$$

(see (1.12) and (1.14)).

To state the results we need the following:

Lemma 2.4. *Consider the functions* f *and* ω_0 *given by* (2.14). *Then*

$$\frac{\partial\omega/\partial x}{f}(t,0) = \frac{\mu}{\sqrt{2\pi}}\frac{1}{\sqrt{e^t-1}} \tag{2.16}$$

Proof. We use (2.12). By the Mehler-Heine formula we can write

$$Q_t\tilde{\omega}_0(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{\omega}_0(x\,e^{-t/2} + \sqrt{1-e^{-t}}y)\,e^{-y^2/2}dy\,.$$

Hence

$$\begin{aligned} Q_t\tilde{\omega}_0'(0) &= \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-t/2}\omega_0'(\sqrt{1-e^{-t}}y)e^{-y^2/2}dy \\ &= \frac{4\alpha}{\sqrt{2\pi}}\,\frac{c_\alpha+\mu}{\theta_\alpha}\int_0^\infty e^{-t/2}(\sqrt{1-e^{-t}}y)^{2\alpha-1}e^{-y^2/2}dy\,. \end{aligned}$$

From this we get

$$\begin{aligned} \frac{Q_t\tilde{\omega}_0'(0)}{f(t)} &= \frac{4\alpha}{\sqrt{2\pi}}\,\frac{c_\alpha+\mu}{\theta_\alpha}\,\frac{e^{-t/2}}{\sqrt{1-e^{-t}}}\int_0^\infty y^{2\alpha-1}e^{-y^2/2}dy \\ &= \frac{c_\alpha+\mu}{\sqrt{2\pi}}\,\frac{1}{\sqrt{e^t-1}}\,. \end{aligned}$$

The third term in (2.12) gives

$$\frac{1}{f(t)}\,\frac{1}{\sqrt{2\pi}}\int_0^t (f(t-s)-f(t))\frac{e^{-s/2}}{(1-e^{-s})^{3/2}}ds = \frac{-c_\alpha+2}{\sqrt{2\pi}\sqrt{e^t-1}}\,.$$

Thus (2.16) follows at once. □

A direct consequence of Theorem 2.1 is the following

Theorem 2.5. *For every positive Borel function* h *and every* $\mu \geq 0$, $\alpha > 0$,

$$\begin{aligned} &\int_0^t e^{-s/2}(1-e^{-s})^{\alpha-1/2}ds \\ &\quad\times \mathrm{E}_0\left[h(|O_{t-s}|)\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^{t-s}\frac{d\mathrm{L}_v(O)}{\sqrt{e^{s+v}-1}}\right)\right] \\ &= \frac{2}{c_\alpha+\mu}\int_0^\infty y\,h(y)e^{-y^2/2}dy\int_0^t \frac{(1-e^{-(t-s)})^\alpha e^{-s/2}}{(1-e^{-s})^{3/2}}\exp\left(-\frac{y^2e^{-s}}{2(1-e^{-s})}\right)ds\,. \end{aligned} \tag{2.17}$$

Proof. Using (2.6), (2.14) and (2.16) we get

$$\int_0^\infty e^{-x^2/2}\frac{c_\alpha+\mu}{\theta_\alpha}x^{2\alpha}dx$$

$$\times \mathrm{E}_x\left[h(|O_t|)\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^t\frac{d\mathrm{L}_s(O)}{\sqrt{e^s-1}}\right)\mathbb{1}_{\{t\geq T_0\}}\right]$$

$$=\int_0^\infty x\,h(x)e^{-x^2/2}dx$$

$$\times\int_0^t(1-e^{-(t-s)})^\alpha\frac{e^{-s/2}}{\sqrt{2\pi}(1-e^{-s})^{3/2}}\exp\left(-\frac{x^2e^{-s}}{2(1-e^{-s})}\right)ds\,,$$

or, by (2.11),

$$\frac{c_\alpha+\mu}{\theta_\alpha}\int_0^\infty x^{2\alpha+1}e^{-x^2/2}dx\int_0^t\frac{e^{-s/2}}{(1-e^{-s})^{3/2}}\exp\left(-\frac{x^2e^{-s}}{2(1-e^{-s})}\right)ds$$

$$\times \mathrm{E}_0\left[h(|O_{t-s}|)\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^{t-s}\frac{d\mathrm{L}_v(O)}{\sqrt{e^{s+v}-1}}\right)\right]$$

$$=\int_0^\infty x\,h(x)e^{-x^2/2}dx\int_0^t(1-e^{-(t-s)})^\alpha\frac{e^{-s/2}}{(1-e^{-s})^{3/2}}\exp\left(-\frac{x^2e^{-s}}{2(1-e^{-s})}\right)ds\,.$$

We deduce (2.17) from (2.15), after making the change of variable $y=x\,e^{-s/2}$ on the left hand side. □

To state the main result of this section we introduce some notation. For $t,\alpha>0$ we denote by $Y_{t,\alpha}$ a random variable with density

$$k_{t,\alpha}e^{u/2}(1-e^{u-t})^{\alpha-1/2}\mathbb{1}_{[0,t]}(u)$$

($k_{\alpha,t}$ is a normalisation constant) independent from the Ornstein-Uhlenbeck process.

Theorem 2.6. *For $t,\alpha>0$, the random variables*

$$O_{Y_{t,\alpha}} \text{ and } \int_0^{Y_{t,\alpha}}\frac{d\mathrm{L}_v(O)}{\sqrt{e^{t-Y_{t,\alpha}+v}-1}} \tag{2.18}$$

are independent. Moreover,

$$\text{(2.19)} \qquad \frac{c_\alpha}{\sqrt{2\pi}} \int_0^{Y_{t,\alpha}} \frac{d\mathrm{L}_v(O)}{\sqrt{e^{t-Y_{t,\alpha}+v}-1}} \sim \mathcal{E}(1)\,.$$

Proof. We perform the change of variable $t-s=u$ in (2.17):

$$\int_0^t e^{u/2}(1-e^{-(t-u)})^{\alpha-1/2}du$$
$$\times\, \mathrm{E}_0\left[h(|O_u|)\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^u \frac{d\mathrm{L}_v(O)}{\sqrt{e^{t-u+v}-1}}\right)\right]$$
$$= \int_0^\infty e^{-\mu\, s}e^{-c_\alpha\, s}ds$$
$$\times\, 2\int_0^\infty y\, h(y)e^{-y^2/2}dy \int_0^t \frac{(1-e^{-u})^\alpha e^{u/2}}{(1-e^{-(t-u)})^{3/2}} \exp\left(-\frac{y^2e^{-(t-u)}}{2(1-e^{-(t-u)})}\right) du\,.$$

We obtain the first part of the theorem noting that the positive Borel function h is arbitrary. To prove the second part it suffices to take $h\equiv 1$ in the previous equality. □

As for the Brownian motion we shall use the "Laplace transform" with respect to α. Recall that the random variable P_n was defined as the product of n independent copies of an arcsine random variable (see (1.19)).

Theorem 2.7. *For every $\mu \ge 0$, $w>0$,*

$$\text{(2.20)} \qquad \mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^\infty \frac{d\mathrm{L}_v(O)}{\sqrt{\frac{e^v}{1-e^{-w}}-1}}\right)\right]$$
$$= \sum_{n\ge 0}\left(\frac{-\mu}{2}\right)^n \mathrm{E}\left[\sqrt{\frac{1-e^{-w}}{P_n-e^{-w}}}\,\mathbb{1}_{\{P_n>e^{-w}\}}\right].$$

Proof. Let us take $h\equiv 1$ in (2.17):

$$\int_0^t e^{-s/2}(1-e^{-s})^{\alpha-1/2}\,\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^{t-s}\frac{d\mathrm{L}_v(O)}{\sqrt{e^{s+v}-1}}\right)\right] ds$$
$$= \frac{2}{c_\alpha+\mu}\int_0^t \frac{(1-e^{-t+s})^\alpha e^{-s/2}}{\sqrt{1-e^{-s}}}ds\,.$$

Letting $t\uparrow\infty$ we get

$$\int_0^\infty e^{-s/2}(1-e^{-s})^{\alpha-1/2}\,\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\int_0^\infty\frac{d\mathrm{L}_v(O)}{\sqrt{e^{s+v}-1}}\right)\right] ds$$

$$= \frac{2}{c_\alpha + \mu} \int_0^\infty \frac{e^{-s/2}}{\sqrt{1-e^{-s}}}\, ds\,,$$

or, performing the integration on the right hand side and denoting $1-e^{-s} = e^{-w}$ on the left hand side,

$$\int_0^\infty \frac{e^{-w}}{\sqrt{1-e^{-w}}} e^{-w\,(\alpha-1/2)}\, \mathrm{E}_0 \left[\exp\left(-\frac{\mu}{\sqrt{2\pi}} \int_0^\infty \frac{d\mathrm{L}_v(O)}{\sqrt{\frac{e^v}{1-e^{-w}} - 1}} \right) \right] dw$$

$$= \frac{2\pi}{c_\alpha + \mu}\,.$$

Recall that $2/c_\alpha$ is the Laplace transform with respect to α of the random variable $\log 1/V$; V is an arcsine random variable (see also the proof of Lemma 1.11). Moreover, the right hand side of the previous equality can be written as

$$\frac{2\pi}{c_\alpha} \sum_{n\geq 0} \left(\frac{-\mu}{2}\right)^n \left(\frac{2}{c_\alpha}\right)^n .$$

Therefore,

$$\frac{e^{-w/2}}{\sqrt{1-e^{-w}}} \mathrm{E}_0 \left[\exp\left(-\frac{\mu}{\sqrt{2\pi}} \int_0^\infty \frac{d\mathrm{L}_v(O)}{\sqrt{\frac{e^v}{1-e^{-w}} - 1}} \right) \right]$$

$$= \pi \left(\chi * \sum_{n\geq 0} \left(\frac{-\mu}{2}\right)^n \chi^{*\,n} \right)(w)\,,$$

where again $\chi(w) = e^{-w/2}/\pi\sqrt{1-e^{-w}}\, \mathbb{I}_{[0,\infty[}(w)$. But, the right hand side of the last equality is

$$\pi \sum_{n\geq 0} \left(\frac{-\mu}{2}\right)^n \mathrm{E}\left[\chi(w + \log P_n)\right]$$

$$= \pi \sum_{n\geq 0} \left(\frac{-\mu}{2}\right)^n \mathrm{E}\left[\frac{e^{-w/2}}{\sqrt{P_n - e^{-w}}} \mathbb{I}_{\{P_n > e^{-w}\}} \right] ,$$

and (2.20) follows at once. □

The first consequence of Theorem 2.7 is contained in the following result. The idea is simple: we replace the deterministic parameter e^{-w} by a random variable. Recall that $Z_{a,b}$ denote a beta random variable independent from the Ornstein-Uhlenbeck process (see (0.4)).

Proposition 2.8. *For every* $\lambda > 0$,

$$\frac{\sqrt{2\pi}}{\mathrm{B}(\lambda, 1/2)} \int_0^\infty \frac{d\mathrm{L}_v(O)}{\sqrt{\frac{e^v}{1-Z_{\lambda,1/2}} - 1}} \sim \mathcal{E}(1)\,. \tag{2.21}$$

Proof. We need to calculate, by (2.20),

$$\mathrm{E}\left[\sqrt{\frac{1-Z_{\lambda,1/2}}{P_n - Z_{\lambda,1/2}}}\,\mathbb{1}_{\{P_n > Z_{\lambda,1/2}\}}\right]$$

$$= \frac{1}{\mathrm{B}(\lambda,1/2)}\mathrm{E}\left[\int_0^{P_n} \frac{u^{\lambda-1}}{\sqrt{P_n - u}}\,du\right] = \mathrm{E}\left[P_n^{\lambda-1/2}\right] = \left(\frac{\mathrm{B}(\lambda,1/2)}{\pi}\right)^n\,.$$

□

Another consequence is the relation with the results on Brownian bridge.

Proposition 2.9. *For every* $0 \le u \le 1$,

$$\int_0^\infty \frac{d\mathrm{L}_v(O)}{\sqrt{e^v - (1-u)}} \sim \int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{u + (1-u)v}}\,. \tag{2.22}$$

Moreover, if $V = Z_{1/2,1/2}$ *is an arcsine random variable independent from the Brownian motion, then*

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sqrt{V}\,d\mathrm{L}_v(O)}{\sqrt{e^v - V}} \sim \mathcal{E}(1)\,. \tag{2.23}$$

Proof. To get (2.22) it suffices to replace e^{-w} by u in (2.20) and to compare the relation obtained with (1.27). Then we use (1.32) to deduce (2.23).

Remark 2.10. Another proof of the equality (2.22) is given in the Appendix, where we use some relations between the local times for Ornstein-Uhlenbeck process, Brownian motion and Brownian bridge. We can deduce then, as consequences, the following equalities. For every $0 < u \le 1$,

$$\begin{aligned} &\mathrm{E}_0\left[\exp\left(-\frac{\mu}{\sqrt{2\pi}}\sqrt{1-u}\int_0^\infty \frac{d\mathrm{L}_v(B)}{\sqrt{(1+v)\,(v+u)}}\right)\right] \\ &= \sum_{n\ge 0}\left(\frac{-\mu}{2}\right)^n \mathrm{E}\left[\sqrt{\frac{1-u}{P_n - u}}\,\mathbb{1}_{\{P_n \ge u\}}\right], \end{aligned} \tag{2.24}$$

and, for every $\lambda > 0$,

$$\frac{\sqrt{2\pi}}{\mathrm{B}(\lambda,1/2)}\sqrt{1-Z_{\lambda,1/2}}\int_0^\infty \frac{d\mathrm{L}_v(B)}{\sqrt{(1+v)\,(v+Z_{\lambda,1/2})}} \sim \mathcal{E}(1)\,. \tag{2.25}$$

This is a different type of formula than those of §1.2, since here the function $v \mapsto 1/\sqrt{(1+v)(v+u)}$ appears instead of $v \mapsto 1/\sqrt{v+u}$.

2.3. Integrals of local time for Bessel processes

The purpose of this section is to apply our previous results to Bessel processes. Let us consider R_t a Bessel process of dimension $\mathtt{d}$, $0 < \mathtt{d} < 2$ (for an introduction to Bessel processes, see [Re-Y], Chap. XI). Let $\mathtt{n} := \mathtt{d}/2 - 1$ be the index of R, so that $-1 < \mathtt{n} < 0$. The assumption $0 < \mathtt{d} < 2$ implies that R may be chosen to be recurrent at 0, hence the existence of a local time for R at 0, denoted $\mathrm{L}_t(R)$. The invariant measure associated to R is

$$\nu(dx) := x^{\mathtt{d}-1}\mathbb{1}_{[0,\infty[}(dx) \tag{2.26}$$

Recall that the local times for the Bessel process may be chosen to satisfy, for any positive Borel function h,

$$\int_0^t h(R_s)\,ds = 2\int_0^\infty h(x)x^{\mathtt{d}-1}\mathrm{L}_t^x(R)\,dx \tag{2.27}$$

Note that we use a different normalisation than the formula in [Re-Y], p. 308, ex. (4.16), 4°. This choice of local times agrees with the fact that $R_t^{2-\mathtt{d}} - (2-\mathtt{d})\mathrm{L}_t^0(R)$ is a martingale.

As previously, we denote $T_0 := \inf\{t > 0 : R_t = 0\}$.

Lemma 2.11. *Let $\phi(x,t)$ be the density of T_0 (see (2.4)). Then*

$$\phi(x,t) = \frac{2^{\mathtt{n}}t^{\mathtt{n}-1}}{\Gamma(|\mathtt{n}|)}x^{-2\mathtt{n}}e^{-x^2/2t}\,. \tag{2.28}$$

Proof. Consider X_t a Bessel process of index $|\mathtt{n}|$ and note $S_x = \sup\{t : X_t = x\}$. Then the law of the process $\{X_{S_x - t}\,,\, t < S_x\}$ under $\mathrm{P}_0^{|\mathtt{n}|}$ is the same as the law of $\{R_t\,,\, t < T_0\}$ under $\mathrm{P}_x^{\mathtt{n}}$ (see [Re-Y], p. 309, ex. (4.18), with Q_t the Bessel semi-group of index $\mathtt{n}$). We deduce that

$$\mathrm{P}_x^{\mathtt{n}}(T_0 \in dt) = \mathrm{P}_0^{|\mathtt{n}|}(S_x \in dt)\,.$$

Using a result in [Re-Y], p. 308, ex. (4.16), 5°, we can write:

$$\mathrm{P}_0^{|\mathtt{n}|}(S_x \in dt) = \frac{|\mathtt{n}|}{x^{-2|\mathtt{n}|}} \frac{q^{|\mathtt{n}|}(t,0,x)}{x^{2|\mathtt{n}|+1}}\, dt\,,$$

where by $q^{|\mathtt{n}|}(t,y,x)$ we denote the density with respect to Lebesgue measure of the Bessel semi-group of index $|\mathtt{n}|$. It is known that for a Bessel semi-group of index $\mathtt{n}$

$$q^{\mathtt{n}}(t,0,x) = \frac{2^{-\mathtt{n}}}{\Gamma(\mathtt{n}+1)} \frac{x^{2\mathtt{n}+1}}{t^{\mathtt{n}+1}} e^{-x^2/2t}\,,$$

(see [Re-Y], p. 426). Replacing this in the previous equality we obtain (2.28). □

Remark. The Laplace transform of T_0 is

$$\mathrm{E}_x\left[e^{-\lambda T_0}\right] = \int_0^\infty e^{-\lambda s}\phi(x,s)\, ds = \frac{(2\lambda)^{|\mathtt{n}|/2}}{2^{|\mathtt{n}|-1}\Gamma(|\mathtt{n}|)} x^{|\mathtt{n}|}\mathrm{K}_{|\mathtt{n}|}(x\sqrt{2\lambda})\,,$$

(see [Leb], p. 119 (5.10.25) and [Ken], p. 762; see also [P-Y1] for a general discussion of first and last passage times distributions for Bessel processes). Here $\mathrm{K}_{|\mathtt{n}|}$ is the modified Bessel function of parameter $|\mathtt{n}|$; $K_{\mathtt{n}} = K_{-\mathtt{n}}$, $\forall \mathtt{n}$.

Let u and v be two real smooth functions defined on $[0,\infty[$ and let us consider $g(t,x) := u(t)x^{2-\mathtt{d}} + v(t)$. Then Ito's formula yields

$$g(t,R_t) = g(0,R_0) + \int_0^t \frac{\partial g}{\partial x}(s,R_s)\, dB_s$$

$$+(2-\mathtt{d})\int_0^t u(s)\, d\mathrm{L}_s(R) + \int_0^t \frac{\partial g}{\partial s}(s,R_s)\, ds\,.$$

Here and elsewhere we denote by $\mathrm{L}_t(R) = \mathrm{L}_t^0(R)$ the local time at level 0.

We cannot apply directly Theorem 2.1 to Bessel processes: the drift and the invariant measure admit a singularity at 0. We need to modify the proof of Theorem 2.1, and in particular, we need to change the boundary conditions. The new proof is based on the above Ito's formula. Let us note that the local time process is normalised (by (2.27)).

Let us denote by $\omega(t,x)$ the solution of (2.1)-(2.3) with $\mathtt{b}(x) = (1-\mathtt{d})/x$ (the drift term is equal to $-\mathtt{b}(x)/2 = (\mathtt{d}-1)/2x$).

Theorem 2.12. *For every bounded positive Borel function h,*

$$\text{(2.29)} \qquad \mathrm{E}_{\omega_0\nu}\left[h(R_t)\exp\left(-\int_0^t \varphi(s)\, d\mathrm{L}_s(R)\right)\mathbb{1}_{\{t\geq T_0\}}\right]$$
$$= \int_0^\infty h(x)\, x^{\mathtt{d}-1}dx \int_0^t \phi(x,s) f(t-s)\, ds\,,$$

where

$$\text{(2.30)} \qquad \varphi(t) := \frac{1}{f(t)} \lim_{x\downarrow 0} x^{\mathtt{d}-1}\, \partial\omega/\partial x\,(t,x)\,.$$

Here, $\mathrm{E}_{\omega_0\nu}$ *is given by* (2.7).

We leave the proof to the reader. □

Again we shall take some particular functions f and ω_0. Let us introduce, for $\alpha > 0$,

$$\text{(2.31)} \qquad c_{\alpha,\mathtt{n}} := \frac{\alpha}{|\mathtt{n}|}\,\mathrm{B}(\mathtt{n}+1,\alpha)\,.$$

We consider

$$\text{(2.32)} \qquad f(t) := t^\alpha \ \text{ and } \ \omega_0(x) := \frac{c_{\alpha,\mathtt{n}}+\mu}{\theta_{\alpha,\mathtt{n}}} x^{2\alpha}\,,$$

where $\mu \geq 0$ and

$$\text{(2.33)} \qquad \theta_{\alpha,\mathtt{n}} := \frac{2^\alpha\Gamma(\alpha+1)}{|\mathtt{n}|} = c_{\alpha,\mathtt{n}}\,\mathrm{E}_0\left[R_1^{2\alpha}\right]\,.$$

Before stating the first important result for the particular functions we establish the following

Lemma 2.13. *Consider the functions given by* (2.32). *Then*

$$\text{(2.34)} \qquad \varphi(t) = \frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)}\, t^{\mathtt{n}}\,.$$

Proof. i) As in Proposition 2.8, the solution of (2.1)-(2.3) can be written as

$$\omega(t,x) = \mathrm{E}_x\left[\omega_0(R_t)\mathbb{1}_{\{t<T_0\}}\right] + \int_0^t \phi(x,s)(t-s)^\alpha ds := \omega_1(t,x)+\omega_2(t,x)\,.$$

Let us denote

$$\varphi_1(t) := \frac{1}{t^\alpha}\lim_{x\downarrow 0} x^{\mathtt{d}-2}\omega_1(t,x)\,,$$

and

$$\varphi_2(t) := \frac{1}{t^\alpha} \lim_{x\downarrow 0} x^{\mathtt{d}-1} \frac{\omega_2(t,x) - \omega_2(t,0)}{x} \,,$$

(because $\omega_1(t,0) = 0$). Then

$$\varphi(t) = \frac{c_{\alpha,\mathtt{n}} + \mu}{\theta_{\alpha,\mathtt{n}}} \varphi_1(t) + \varphi_2(t) \,,$$

so, to get (2.34) we need to compute φ_1 and φ_2.
ii) Let us calculate φ_2. Since $\int_0^\infty \phi(x,t)\,dt = 1$ and $\omega_2(t,0) = t^\alpha$, we can write

$$\rho_2(t,x) := \frac{x^{\mathtt{d}-1}}{t^\alpha} \frac{\omega_2(t,x) - \omega_2(t,0)}{x} = \frac{x^{\mathtt{d}-2}}{t^\alpha}(\omega_2(t,x) - t^\alpha)$$
$$= x^{\mathtt{d}-2} t \int_0^1 \phi(x,tu)((1-u)^\alpha - 1)\,du - x^{\mathtt{d}-2} \int_t^\infty \phi(x,s)\,ds$$

Using the explicit form of $\phi(x,t)$ (2.28) we obtain, as $x \downarrow 0$,

$$\varphi_2(t) = \lim_{x\downarrow 0} \rho_2(t,x) = -\frac{2^{\mathtt{n}} c_{\alpha,\mathtt{n}}\, t^{\mathtt{n}}}{\Gamma(|\mathtt{n}|)} \,,$$

with $c_{\alpha,\mathtt{n}}$ given by (2.31).
iii) Now we calculate φ_1. We can write

$$\rho_1(t,x) := \frac{x^{\mathtt{d}-2}}{t^\alpha} \omega_1(t,x) = \frac{x^{\mathtt{d}-2}}{t^\alpha} \mathrm{E}_x \left[R_t^{2\alpha} 1\!\!1_{\{t<T_0\}} \right]$$
$$= \frac{x^{\mathtt{d}-2}}{t^\alpha} \mathrm{E}_x \left[R_t^{2\alpha} \right] - \frac{x^{\mathtt{d}-2}}{t^\alpha} \mathrm{E}_x \left[R_t^{2\alpha} 1\!\!1_{\{t\geq T_0\}} \right]$$
$$= \frac{x^{\mathtt{d}-2}}{t^\alpha} \left(\mathrm{E}_x \left[R_t^{2\alpha} \right] - \mathrm{E}_0 \left[R_t^{2\alpha} \right] \right)$$
$$+ \frac{x^{\mathtt{d}-2}}{t^\alpha} \left(\mathrm{E}_0 \left[R_t^{2\alpha} \right] - \frac{x^{\mathtt{d}-2}}{t^\alpha} \mathrm{E}_0 \left[R_1^{2\alpha} \right] \int_0^t \phi(x,s)\,(t-s)^\alpha ds \right)$$
$$= \frac{x^{\mathtt{d}-2}}{t^\alpha} \left(\mathrm{E}_x \left[R_t^{2\alpha} \right] - \mathrm{E}_0 \left[R_t^{2\alpha} \right] \right)$$
$$+ \frac{x^{\mathtt{d}-2}}{t^\alpha} \mathrm{E}_0 \left[R_1^{2\alpha} \right] \left(\int_0^t (t^\alpha - (t-s)^\alpha) \phi(x,s)\,ds + t^\alpha \int_t^\infty \phi(x,s)\,ds \right) \,,$$

as we can see by the scaling property of the Bessel process. We can verify that the first term in the last expression tends to 0, as $x \downarrow 0$. Then, as for φ_2 we get

$$\varphi_1(t) = \lim_{x\downarrow 0} \rho_1(t,x) = \frac{(2t)^{\mathtt{n}}}{\Gamma(|\mathtt{n}|)} c_{\alpha,\mathtt{n}}\, \mathrm{E}_0 \left[R_1^{2\alpha} \right] = \frac{(2t)^{\mathtt{n}}}{\Gamma(|\mathtt{n}|)} \theta_{\alpha,\mathtt{n}} \,,$$

by (2.33).
We replace φ_1 and φ_2 in the expression of φ and the lemma is proven. □

Now we can state the first consequence of Theorem 2.12.

Theorem 2.14. *For every bounded positive Borel function h and for every $\mu \geq 0$,*

$$\begin{aligned}(2.35)\quad &\int_0^1 (1-u)^{\alpha+\mathtt{n}} \mathrm{E}_0 \left[h(R_u) \exp\left(-\frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)} \int_0^u (1-u+v)^{\mathtt{n}} d\mathrm{L}_v(R) \right) \right] du \\ &= \frac{\Gamma(|\mathtt{n}|)}{2^{\mathtt{n}}|\mathtt{n}|\,(c_{\alpha,\mathtt{n}}+\mu)} \int_0^\infty x^{\mathtt{d}-1} h(x) dx \int_0^1 (1-s)^\alpha \phi(x,s)\, ds\,.\end{aligned}$$

As a consequence we can prove one of the main results of this section. Recall that $Z_{a,b}$ denotes a beta random variable independent from the Bessel process (see (0.4)).

Theorem 2.15. *For every $\alpha > 0$, the random variables*

$$(2.36)\qquad R_{Z_{1,\alpha}} \text{ and } \int_0^{Z_{1,\alpha}} (1-Z_{1,\alpha}+v)^{\mathtt{n}} d\mathrm{L}_v(R)\,,$$

are independent. Moreover,

$$(2.37)\qquad \frac{2^{\mathtt{n}}\Gamma(\alpha+|\mathtt{n}|)\Gamma(\mathtt{n}+1)}{\Gamma(\alpha)\Gamma(|\mathtt{n}|+1)} \int_0^{Z_{1,\alpha}} (1-Z_{1,\alpha}+v)^{\mathtt{n}} d\mathrm{L}_v(R) \sim \mathcal{E}(1)\,.$$

Proof of Theorem 2.14. The left hand side of (2.29) can be written as

$$\begin{aligned} &\mathrm{E}_{\omega_0\nu}\left[h(R_t) \exp\left(-\frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)} \int_0^t v^{\mathtt{n}}\, d\mathrm{L}_v(R) \right) \right] \\ &= \frac{c_{\alpha,\mathtt{n}}+\mu}{\theta_{\alpha,\mathtt{n}}} \int_0^\infty x^{2\alpha+\mathtt{d}-1} \phi(x,s)\, dx \\ &\times \int_0^t \mathrm{E}_0 \left[h(R_{t-s}) \exp\left(-\frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)} \int_0^{t-s} (s+v)^{\mathtt{n}}\, d\mathrm{L}_v(R) \right) \right] ds\,. \end{aligned}$$

To calculate the integral in x we use the expression of the density (2.28) and we get

$$(2.38)\qquad \int_0^\infty x^{2\alpha+\mathtt{d}-1}\phi(x,s)\,dx = \frac{2^{\alpha+\mathtt{n}}\Gamma(\alpha+1)}{\Gamma(|\mathtt{n}|)}\, s^{\alpha+\mathtt{n}} = \theta_{\alpha,\mathtt{n}} \frac{2^{\mathtt{n}}|\mathtt{n}|}{\Gamma(|\mathtt{n}|)}\, s^{\alpha+\mathtt{n}}\,,$$

by (2.33). We replace this in the transformed expression of the left hand side of (2.29) and we get the following equality

$$\begin{aligned}&\frac{2^{\mathtt{n}}|\mathtt{n}|\,(c_{\alpha,\mathtt{n}}+\mu)}{\Gamma(|\mathtt{n}|)}\int_0^t s^{\alpha+\mathtt{n}}ds\\ &\times \mathrm{E}_0\left[h(R_{t-s})\exp\left(-\frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)}\int_0^{t-s}(s+v)^{\mathtt{n}}\,d\mathrm{L}_v(R)\right)\right]\\ &=\int_0^\infty x^{\mathtt{d}-1}h(x)dx\int_0^t(t-s)^\alpha\phi(x,s)\,ds\,.\end{aligned}$$

We make the change of variable $s=t-u$ on the left hand side, we take $t=1$ and we get (2.35). □

Proof of Theorem 2.15. We use the same idea as for Theorem 1.8, namely we write

$$\frac{1}{c_{\alpha,\mathtt{n}}+\mu}=\int_0^\infty e^{-s\,c_{\alpha,\mathtt{n}}-s\,\mu}ds\,.$$

The end of the proof is similar to that of Theorem 1.8. □

Remark. In the Appendix we prove that the first part of the Theorem 2.15 is connected with the property of independence for beta-gamma random variables.

Remark. We obtain the results of §1.2 taking $\mathtt{n}=-1/2$, that is $\mathtt{d}=1$.

Remark 2.16. It follows from the scaling property of R and from the density of occupation formula (2.27) that,

$$(\mathrm{L}_{c\,t}(R))_{t\geq 0}\sim(c^{|\mathtt{n}|}\mathrm{L}_t(R))_{t\geq 0} \tag{2.39}$$

(see [B-P-Y], p. 299). Then, it is not difficult to verify that

$$\int_0^u\frac{d\mathrm{L}_v(R)}{(1-u+v)^{|\mathtt{n}|}}\sim u^{|\mathtt{n}|}\int_0^1\frac{d\mathrm{L}_v(R)}{(1-u+uv)^{|\mathtt{n}|}}\,. \tag{2.40}$$

We carry on with the use of the "Laplace transform" with respect to α. We introduce

$$P_m := \prod_{j=1}^{m} V_j\,, \tag{2.41}$$

where $V_1, \dots, V_m$ are independent copies of a beta random variable $Z_{1+\mathtt{n},|\mathtt{n}|}$.

Theorem 2.17. *For every $\mu \geq 0$, $0 < u \leq 1$,*

$$\begin{aligned}\mathrm{E}_0\left[\exp\left(-\frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)}(1-u)^{|\mathtt{n}|}\int_0^1 \frac{d\mathrm{L}_v(R)}{(u+(1-u)v)^{|\mathtt{n}|}}\right)\right] \\ = \sum_{m\geq 0}(\mu\,\mathtt{n})^m \mathrm{E}\left[\frac{\mathbb{I}_{\{P_m\geq u\}}}{P_m^{1+\mathtt{n}}}\right].\end{aligned} \tag{2.42}$$

Proof. The ideas are essentially the same as for the proof of Corollary 1.12 (see also Lemma 1.11).
i) Let us note that the random variable $\log 1/Z_{1+\mathtt{n},|\mathtt{n}|}$ has the density

$$\chi(w) = \frac{e^{-w(\mathtt{n}+1)}(1-e^{-w})^{|\mathtt{n}|-1}}{\mathrm{B}(|\mathtt{n}|,\mathtt{n}+1)}\,\mathbb{I}_{[0,\infty[}(w)\,,$$

and its Laplace transform with respect to α is equal to

$$\int_0^\infty e^{-\alpha\, w}\chi(w)\,dw = \frac{1}{|\mathtt{n}|\,c_{\alpha,\mathtt{n}}}\,.$$

ii) Taking $\alpha = 0$ in (2.38) we get

$$\int_0^\infty x^{\mathtt{d}-1}\phi(x,u)\,dx = \frac{2^{\mathtt{n}}}{\Gamma(|\mathtt{n}|)}\,u^{\mathtt{n}}\,.$$

iii) We put this in (2.35) after we made $h \equiv 1$ and we replaced $1-u$ by u:

$$\begin{aligned}\int_0^1 u^{\alpha+\mathtt{n}}\mathrm{E}_0\left[\exp\left(-\frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)}\int_0^{1-u}(u+v)^{\mathtt{n}}d\mathrm{L}_v(R)\right)\right]du \\ = \left(\sum_{m\geq 0}\frac{(\mu\,\mathtt{n})^m}{(|\mathtt{n}|\,c_{\alpha,\mathtt{n}})^{m+1}}\right)\int_0^1 u^\alpha(1-u)^{\mathtt{n}}du\,,\end{aligned}$$

or, by the scaling property (2.39),

$$\begin{aligned}\int_0^1 u^{\alpha+\mathtt{n}}\mathrm{E}_0\left[\exp\left(-\frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)}(1-u)^{|\mathtt{n}|}\int_0^1\frac{d\mathrm{L}_v(R)}{(u+(1-u)v)^{|\mathtt{n}|}}\right)\right]du \\ = \left(\sum_{m\geq 0}\frac{(\mu\,\mathtt{n})^m}{(|\mathtt{n}|\,c_{\alpha,\mathtt{n}})^{m+1}}\right)\int_0^1 u^\alpha(1-u)^{\mathtt{n}}du\,.\end{aligned}$$

Let us denote

$$F(\mu,u) := \mathrm{E}_0\left[\exp\left(-\frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)}(1-u)^{|\mathtt{n}|}\int_0^1 \frac{d\mathrm{L}_v(R)}{(u+(1-u)v)^{|\mathtt{n}|}}\right)\right]$$

and we put $u = e^{-w}$. Therefore,

$$e^{-w(\mathtt{n}+1)}F(\mu,e^{-w}) = \left(e^{-\bullet}(1-e^{-\bullet})^{\mathtt{n}} * \sum_{m\geq 0}(\mu\,\mathtt{n})^m\chi^{*(m+1)}\right)(w)\,,$$

or,

$$e^{-w(\mathtt{n}+1)}F(\mu,e^{-w})$$
$$= \sum_{m\geq 0}(\mu\,\mathtt{n})^m \mathrm{E}\left[e^{-(w+\log P_{m+1})}(1-e^{-(w+\log P_{m+1})})^{\mathtt{n}}\mathbb{1}_{\{P_{m+1}>e^{-w}\}}\right]\,.$$

We denote $e^{-w} = u$ and we get

$$F(\mu,u) = u^{|\mathtt{n}|}\sum_{m\geq 0}(\mu\,\mathtt{n})^m \mathrm{E}\left[\frac{(P_{m+1}-u)^{\mathtt{n}}}{P_{m+1}^{1+\mathtt{n}}}\mathbb{1}_{\{P_{m+1}>u\}}\right]\,.$$

iv) To obtain (2.42) we need to calculate the expectation on the right hand side of the last equality.

$$\mathrm{E}\left[\frac{(P_{m+1}-u)^{\mathtt{n}}}{P_{m+1}^{1+\mathtt{n}}}\mathbb{1}_{\{P_{m+1}>u\}}\right]$$
$$= \mathrm{E}\left[\frac{1}{P_m}\frac{(V_{m+1}-u/P_m)^{\mathtt{n}}}{V_{m+1}^{1+\mathtt{n}}}\mathbb{1}_{\{V_{m+1}>u/P_m\}}\,\mathbb{1}_{\{P_m>u\}}\right]$$
$$= \mathrm{E}\left[\frac{1}{P_m}\left(\frac{P_m}{u}\right)^{|\mathtt{n}|}\mathbb{1}_{\{P_m>u\}}\right]\,,$$

as follows by Lemma A.3. The proof of the theorem is complete except for the proof of Lemma A.3 found in the Appendix. □

We finish with a similar result for the local time at 0 of the Bessel bridge r of dimension $0 < \mathtt{d} < 2$, which goes from 0 to 0, and is indexed by $[0,1]$.

Theorem 2.18. *For every* $\mu \geq 0$, $0 < u \leq 1$,

$$\text{(2.43)}\qquad \mathrm{E}_0\left[\exp\left(-\frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)}(1-u)^{|\mathtt{n}|}\int_0^1 \frac{d\mathrm{L}_v(r)}{(u+(1-u)v)^{|\mathtt{n}|}}\right)\right]$$
$$= (1-u)^{1+\mathtt{n}}\sum_{m\geq 0}(\mu\,\mathtt{n})^m \mathrm{E}\left[(P_m-u)^{|\mathtt{n}|-1}\mathbb{I}_{\{P_m\geq u\}}\right].$$

Proof. The proof is similar to the proof of Theorem 2.17. We only point out the essential formulas.
By (2.35), by conditioning and scaling we get, for every $y \geq 0$,

$$\int_0^1 \left(u^{\alpha+\mathtt{n}}/\sqrt{1-u}\right) q(1,0,y/\sqrt{1-u})\,du$$
$$\times\,\mathrm{E}_0\left[\exp\left(-\frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)}(1-u)^{|\mathtt{n}|}\int_0^1 \frac{d\mathrm{L}_v(R)}{(u+(1-u)v)^{|\mathtt{n}|}}\right)\ \Big|\ R_1 = y/\sqrt{1-u}\right]$$
$$= \frac{\Gamma(|\mathtt{n}|)}{2^{\mathtt{n}}|\mathtt{n}|\,(c_{\alpha,\mathtt{n}}+\mu)}y^{\mathtt{d}-1}\int_0^1 (1-s)^{\alpha}\phi(y,s)\,ds\,,$$

where by $q(t,x,y)$ we denoted the density with respect to Lebesgue measure of the Bessel semi-group. It is known that

$$q(1,0,y) = \frac{2^{-\mathtt{n}}}{\Gamma(\mathtt{n}+1)}y^{2\mathtt{n}+1}e^{-y^2/2}$$

(see the calculation preceding (2.28)). Replacing this in the previous equality and taking $y=0$, we obtain

$$\int_0^1 u^{\alpha+\mathtt{n}}(1-u)^{-\mathtt{n}-1}du$$
$$\times\,\mathrm{E}_0\left[\exp\left(-\frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)}(1-u)^{|\mathtt{n}|}\int_0^1 \frac{d\mathrm{L}_v(R)}{(u+(1-u)v)^{|\mathtt{n}|}}\right)\ \Big|\ R_1 = 0\right]$$
$$= \frac{\Gamma(|\mathtt{n}|)\Gamma(1+\mathtt{n})}{|\mathtt{n}|\,c_{\alpha,\mathtt{n}}}\sum_{m\geq 0}\frac{(\mu\,\mathtt{n})^m}{(|\mathtt{n}|\,c_{\alpha,\mathtt{n}})^m}\,.$$

We denote

$$F(\mu,u) := \mathrm{E}_0\left[\exp\left(-\frac{2^{\mathtt{n}}\mu}{\Gamma(|\mathtt{n}|)}(1-u)^{|\mathtt{n}|}\int_0^1 \frac{d\mathrm{L}_v(r)}{(u+(1-u)v)^{|\mathtt{n}|}}\right)\right],$$

we make the change of variable $u = e^{-w}$ and finally we get, after inverting the Laplace transform with respect to α,

$$(1-u)^{-\mathtt{n}-1}F(\mu,u) = \sum_{m\geq 0}(\mu\,\mathtt{n})^m \mathrm{E}\left[\frac{1}{P_m^{1+\mathtt{n}}}\left(1-\frac{u}{P_m}\right)^{|\mathtt{n}|-1}\mathbb{I}_{\{P_m>u\}}\right].$$

Here we denoted again $e^{-w} = u$. $\square$

Remark. It is clear that from Theorems 2.17 and 2.18 we can obtain similar results as Propositions 1.14 and 1.17.

Appendix

A.1. *Another proof for* (1.17) *and* (2.36).

We give here a different proof of the first statements of Theorems 1.8 and 2.15. That is, for $\alpha > 0$, if $Z_{1,\alpha+1/2}$ denotes a beta random variable which is independent from the Brownian motion, respectively $Z_{1,\alpha}$ is independent from the Bessel process, then the random variables

$$B_{Z_{1,\alpha+1/2}} \quad \text{and} \quad \int_0^{Z_{1,\alpha+1/2}} \frac{d\mathrm{L}_v(B)}{\sqrt{1 - Z_{1,\alpha+1/2} + v}}\,,$$

respectively

$$R_{Z_{1,\alpha}} \quad \text{and} \quad \int_0^{Z_{1,\alpha}} (1 - Z_{1,\alpha} + v)^{\mathtt{n}} d\mathrm{L}_v(R)\,,$$

are independent.

We now give the details for Brownian motion. For the Bessel process the proof is similar once we note that $\sup\{s \leq 1 : R_s = 0\} \sim Z_{\mathtt{n},1-\mathtt{n}}$ (see [Dy] and [B-P-Y]).

For $t \in]0,1]$, let us consider

$$m_t(u) = \frac{1}{\sqrt{t - g_t}} \left|B_{g_t + (t-g_t)\,u}\right| \text{ and } b_t(u) = \frac{B_{g_t u}}{\sqrt{g_t}}\,,\ u \in [0,1]\,,$$

where $g_t = \sup\{s \leq t : B_s = 0\}$. m_t (respectively b_t) is a Brownian meander (respectively a Brownian bridge). m_t and b_t are two independent processes, also independent of g_t. Moreover, their laws do not depend on t. We can write

$$|B_t| = m_t(1)\sqrt{t - g_t} \text{ and } \int_0^t \frac{d\mathrm{L}_v(B)}{\sqrt{1 - t + v}} = \int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{\frac{1-t}{g_t} + v}}\,.$$

Using Brownian scaling, it is easy to check that, for any $c > 0$,

$$(g_{ct})_{t>0} \sim (c\, g_t)_{t>0}\,.$$

We replace t by $Z_{1,\alpha+1/2}$. To obtain (1.17) it suffices to prove that the random variables

$$Z_{1,\alpha+1/2} - g_{Z_{1,\alpha+1/2}} \text{ and } \frac{1 - Z_{1,\alpha+1/2}}{g_{Z_{1,\alpha+1/2}}} \text{ are independent.} \tag{A.1}$$

To prove (A.1) it is sufficient to prove that the random variables

$$Z_{1,\alpha+1/2}(1-g_1) \text{ and } \frac{1}{g_1}\left(\frac{1}{Z_{1,\alpha+1/2}} - 1\right) \text{ are independent.} \tag{A.2}$$

Let us note that $g_1 = g = Z_{1/2,1/2}$ is an arcsine distributed random variable. In fact, we can prove the following general result

Lemma A.1. *Let $p \in]0,1[$. If $Z_{p,1-p}$ and $Z_{1,1-p+\alpha}$ are independent beta random variables then the random variables*

$$Z_{1,1-p+\alpha}(1-Z_{p,1-p}) \textit{ and } \frac{1}{Z_{p,1-p}}\left(\frac{1}{Z_{1,1-p+\alpha}} - 1\right) \tag{A.3}$$

are independent.

Remark. To obtain (A.2) it suffices to take $p = 1/2$.

Proof. *i)* Let us denote by Z_a a gamma random variable with parameter a, having the density $x^{a-1}e^{-x}/\Gamma(a)\ \mathbb{1}_{[0,\infty[}(x)$.
Recall a classical result (see also [Du] for more recent related developments): if the random variables Z_a and Z_b are independent, then the random variables

$$\frac{Z_a}{Z_a + Z_b} \sim Z_{a,b} \text{ and } Z_a + Z_b \sim Z_{a+b} \tag{A.4}$$

are independent.
ii) Let Z_p, Z_{1-p} and $Z_{1-p+\alpha}$ be three independent gamma random variables. Then

$$\xi := \frac{Z_p}{Z_p + Z_{1-p}} \sim Z_{p,1-p}\,,$$

$$\chi := Z_p + Z_{1-p} \sim Z_1\,,$$

$$\zeta := \frac{\chi}{\chi + Z_{1-p+\alpha}} \sim Z_{1,1-p+\alpha}\,.$$

Moreover, by (A.4), the random variables ξ and ζ are independent.

We can write

$$\frac{1}{\xi}\left(\frac{1}{\zeta}-1\right) = \frac{Z_p + Z_{1-p}}{Z_p}\,\frac{Z_{1-p+\alpha}}{\chi} = \frac{Z_{1-p+\alpha}}{Z_p}\,,$$

and

$$\zeta(1-\xi) = \frac{\chi}{Z_1 + Z_{1-p+\alpha}}\,\frac{Z_{1-p}}{\chi} = \frac{Z_{1-p}}{Z_{1-p}+Z_p+Z_{1-p+\alpha}}\,.$$

But, again by (A.4), the random variable

$$\frac{Z_{1-p+\alpha}}{Z_p} = \frac{Z_{1-p+\alpha}+Z_p}{Z_p} - 1$$

is independent from $Z_p + Z_{1-p+\alpha}$, hence, from

$$\frac{Z_{1-p}}{Z_{1-p}+Z_p+Z_{1-p+\alpha}}\,.$$

□

A.2 *Some computations of integrals.*

Here we carry out some integral computations which we used in the proofs of Theorems 1.9, 1.20 and 2.17 (see also [P] and [Y2]).

Lemma A.2. *If V is an arcsine random variable then, for every $0 < u < 1$, $y \geq 0$,*

$$\text{(A.5)} \qquad \mathrm{E}\left[\frac{y\sqrt{V}}{(V-u)^{3/2}}\exp\left(-\frac{y^2\,V}{2(V-u)}\right)\mathbb{1}_{\{V>u\}}\right] = \sqrt{\frac{2}{\pi}}\,\frac{e^{-y^2/2(1-u)}}{\sqrt{u(1-u)}}\,.$$

Proof. The result is a consequence of the following equality:

$$\text{(A.6)} \qquad \frac{y}{\pi}\int_u^1 \frac{\sqrt{v}}{(v-u)^{3/2}}\exp\left(-\frac{y^2\,v(1-u)}{2(v-u)}\right)\frac{dv}{\sqrt{v(1-v)}} = \sqrt{\frac{2}{\pi}}\,\frac{e^{-y^2/2}}{(1-u)\sqrt{u}}\,,$$

replacing y by $y/\sqrt{1-u}$.

It suffices to calculate the above integral. Put $v = u(w+1)/(w+u)$

$$I = \int_u^1 \frac{1}{(v-u)^{3/2}}\exp\left(-\frac{y^2\,v(1-u)}{2(v-u)}\right)\frac{dv}{\sqrt{1-v}} =$$

$$\frac{1}{(1-u)\sqrt{u}}\int_0^\infty \exp\left(-\frac{y^2\,(w+1)}{2}\right)\frac{dw}{\sqrt{w}} = \frac{\sqrt{2}}{y}\,\frac{e^{-y^2/2}}{(1-u)\sqrt{u}}\,\Gamma\left(\frac{1}{2}\right)\,.$$

From this we get (A.6). $\square$

Remark. Here is another proof of this result. Let us consider the primitives with respect to y on both sides of (A.5), which tend to zero at infinity:

$$\text{(A.5')} \qquad \mathrm{E}\left[\exp\left(-\frac{z^2 V(1-u)}{2(V-u)}\right)\frac{\mathbb{1}_{\{V>u\}}}{\sqrt{V(V-u)}}\right] = \sqrt{\frac{2}{\pi}}\int_z^\infty e^{-x^2/2}\,\frac{dx}{\sqrt{u}}\,,$$

where we denote $z = y/\sqrt{1-u}$ (see also [Sh-W], p. 400).
It is known that, for $\eta \sim \mathcal{N}(0,1)$,

$$\eta^2 \sim 2\,V\mathcal{E}(1)\,,$$

and we deduce

$$\sqrt{\frac{2}{\pi}}\int_z^\infty e^{-x^2/2}\,dx = \mathrm{E}\left[e^{-z^2/2V}\right]\,.$$

Hence, to prove (A.5'), we need to show that, for every $z \geq 0$,

$$\mathrm{E}\left[\exp\left(-\frac{z^2 V(1-u)}{2(V-u)}\right)\frac{\sqrt{u}\,\mathbb{1}_{\{V>u\}}}{\sqrt{V(V-u)}}\right] = \mathrm{E}\left[e^{-z^2/2V}\right]\,.$$

By the injectivity of the Laplace transform, it suffices to prove that, for every function $\psi : [0,1] \to [0,\infty[$,

$$(*) \qquad \mathrm{E}\left[\psi\left(\frac{V-u}{V(1-u)}\right)\frac{\sqrt{u}\,\mathbb{1}_{\{V>u\}}}{\sqrt{V(V-u)}}\right] = \mathrm{E}\left[\psi(V)\right]\,.$$

We can verify (*) using the fact that V is arcsine distributed and making a simple change of variable.

Lemma A.3. *If $Z = Z_{1+\mathtt{n},|\mathtt{n}|}$ is a beta random variable, with $\mathtt{n} \in]-1,0[$, then, for every $0 < u \leq 1$,*

$$\text{(A.7)} \qquad \mathrm{E}\left[\frac{(Z-u)^{\mathtt{n}}}{Z^{1+\mathtt{n}}}\mathbb{1}_{\{Z>u\}}\right] = \frac{1}{u^{|\mathtt{n}|}}\,.$$

In particular, for $\mathtt{n} = -1/2$, (A.7) yields: if $V = Z_{1/2,1/2}$ is an arcsine random variable then, for every $0 < u \leq 1$

$$\text{(A.8)} \qquad \mathrm{E}\left[\frac{\mathbb{1}_{\{V>u\}}}{\sqrt{V(V-u)}}\right] = \frac{1}{\sqrt{u}}\,.$$

Proof. (A.7) is a consequence of the following equality:

$$(A.9)\qquad \frac{1}{\Gamma(1+\mathtt{n})\,\Gamma(|\mathtt{n}|)}\int_u^1 \frac{(v-u)^{\mathtt{n}}(1-v)^{|\mathtt{n}|-1}}{v}\,dv = \frac{1}{u^{|\mathtt{n}|}}\,.$$

To obtain (A.9) it suffices to make the change of variables

$$v \mapsto (v-u)/(v-uv)\,.$$

□

Remark. (A.8) can be obtained taking $\psi \equiv 1$ in the equality (*) (see the Remark preceding Lemma A.3).

A.3. *Remarks on the proofs of some limit theorems.*

The celebrated Stroock-Varadhan-Papanicolaou approach to second order limit theorems concerning the occupation measure of Brownian motion make essential use of Ito's formula to replace fluctuating (signed)additive functionals with bounded variation by stochastic integrals (see, for instance, [Re-Y], Chap. XIII). Here, we show that, for integrals of local times, this approach is too crude, and we use in fact Ito's integral representation of L^2 variables as stochastic integrals. This method has already been applied to yield a proof of Varadhan's normalization for planar Brownian motion (see [Y1]).

"Proof" of Theorem 1.26. *i)* We can write, by Ito's formula,

$$(A.10)\qquad \sqrt{\beta}\int_0^1 v^{\beta-1/2} d\mathrm{L}_v(B)$$

$$= \sqrt{\beta}|B_1| - \sqrt{\beta}\int_0^1 v^{\beta-1/2} d\tilde{B}_v - \sqrt{\beta}(\beta-1/2)\int_0^1 v^{\beta-3/2}|B_v|\,dv\,.$$

Let us note that

$$\sqrt{\beta}|B_1| \to 0\,,\ \text{as}\ \beta\downarrow 0\ \text{ in }\ L^1\,,$$

$$\sqrt{\beta}\int_0^1 v^{\beta-1/2} d\tilde{B}_v$$

is a centered Gaussian random variable having the variance 1/2,

$$\beta\sqrt{\beta}\int_0^1 v^{\beta-3/2}|B_v|\,dv \to 0\,,\ \text{as}\ \beta\downarrow 0\ \text{ in }\ L^1\,,$$

and

$$\mathrm{E}_0\left[\frac{\sqrt{\beta}}{2}\int_0^1 v^{\beta-3/2}|B_v|\,dv\right]=\frac{\sqrt{\beta}}{2}\mathrm{E}_0\left[|B_1|\right]\int_0^1 v^{\beta-1}dv=\frac{1}{\sqrt{2\pi\beta}}\,.$$

We subtract in (A.10) $1/\sqrt{2\pi\beta}$ and we would obtain a central limit theorem if we can prove that

$$a(\beta):=\mathrm{E}\left[\beta\int_0^1 v^{\beta-3/2}(|B_v|-\mathrm{E}[|B_v|])\,dv\right]^2\to 0\,,\ \text{ as }\ \beta\downarrow 0\,. \tag{A.11}$$

We shall compute $a(\beta)$.
ii) If X and Y are centered Gaussian random variables such that $\mathrm{E}[X^2]=\mathrm{E}[Y^2]=1$ and $\mathrm{E}[XY]=\rho$, it is not difficult to see that

$$\mathrm{E}[|XY|]=\frac{2}{\pi}\left((1-\rho^2)^{1/2}+\rho\ \arctan\frac{\rho}{\sqrt{1-\rho^2}}\right)\,.$$

Then,

$$\mathrm{E}[|B_uB_v|]=\frac{2}{\pi}\left(\sqrt{u(v-u)}+u\ \arctan\sqrt{\frac{u}{v-u}}\right)\,. \tag{A.12}$$

So, we can write

$$a(\beta)=\beta\int_0^1\int_0^1 du\,dv\,u^{\beta-3/2}v^{\beta-3/2}(\mathrm{E}[|B_uB_v|]-\mathrm{E}[|B_u|]\,\mathrm{E}[|B_v|])$$

$$=2\beta\int_0^1 dv\int_0^v du\,u^{\beta-3/2}v^{\beta-3/2}\frac{2}{\pi}\left(\sqrt{u(v-u)}+u\ \arctan\sqrt{\frac{u}{v-u}}-\sqrt{uv}\right)$$

$$=\frac{4\beta}{\pi}\int_0^1 dv\,v^{\beta-3/2}\,b(v)\,,$$

where

$$b(v):=\int_0^v du\,u^{\beta-1}\left(\sqrt{v-u}+\sqrt{u}\ \arctan\sqrt{\frac{u}{v-u}}-\sqrt{v}\right)=v^{\beta+1/2}\,c(\beta)\,,$$

and

$$c(\beta):=\int_0^1 dw\,w^{\beta-1}\left(\sqrt{1-w}+\sqrt{w}\ \arctan\sqrt{\frac{w}{1-w}}-1\right)\,.$$

We obtain $a(\beta)=c(\beta)/2\pi$. But,

$$c(\beta)\to\int_0^1\frac{dw}{w}\left(\sqrt{1-w}+\sqrt{w}\ \arctan\sqrt{\frac{w}{1-w}}-1\right):=c(0)\ \text{ as }\ \beta\downarrow 0\,,$$

with

$$c(0) = \int_0^\infty \frac{du}{u(1+u)^{3/2}} \left(1 + \sqrt{u}\,\arctan\sqrt{u} - \sqrt{1+u}\right) ,$$

and $c(0) > 0$, since the function $u \mapsto (1 + \sqrt{u}\,\arctan\sqrt{u} - \sqrt{1+u})$ is increasing. We obtain that

$$a(\beta) \to c(0)/2\pi > 0 .$$

This is in contradiction with (A.11) □

In a similar way we can show that proofs of Theorems 1.28 and 1.30 based on direct application of Ito's formula do not lead to the result. In fact we need to be more careful in applying this idea. We illustrate the approach in the following:

Proof of Theorem 1.30. We only point out the essential steps. Take, for simplicity, $s = 1$ and $t \uparrow \infty$. We can write, by Ito's formula,

$$\text{(A.13)} \qquad \frac{1}{\sqrt{\log t}} \left(\int_1^t \frac{d\mathrm{L}_v(B)}{\sqrt{v}} - \frac{\log t}{\sqrt{2\pi}} \right) = \frac{1}{\sqrt{\log t}} \left(\frac{|B_t|}{\sqrt{t}} - |B_1| \right)$$
$$- \frac{1}{\sqrt{\log t}} \int_1^t \frac{\mathrm{sgn}(B_v)\, dB_v}{\sqrt{v}} + \frac{1}{2\sqrt{\log t}} \int_1^t \frac{dv}{v^{3/2}} \left(|B_v| - \mathrm{E}(|B_v|)\right) .$$

The first term on the right hand side of (A.13) converges in law towards zero, as $t \uparrow \infty$. We need to study the convergence in law of the couple

$$\left(\frac{1}{\sqrt{\log t}} \int_1^t \frac{\mathrm{sgn}(B_v)\, dB_v}{\sqrt{v}} \, , \, \frac{1}{\sqrt{\log t}} \int_1^t \frac{dv}{v^{3/2}} \left(|B_v| - \mathrm{E}(|B_v|)\right) \right) .$$

We shall use the following important result:

Lemma A.4. *i) Consider a process* $\{\Phi_v(B); v \ge 0\}$ *adapted to the Brownian filtration, such that* $\mathrm{E}(|\Phi_1(B)|) < \infty$ *and, for each* $a > 0$,

$$\text{(A.14)} \qquad \Phi_{a^2 v}(B) = \Phi_v(B^{(a)}) \, , \; v \ge 0 \, ,$$

where $B_t^{(a)} = B_{a^2 t}/a$. *Then*

$$\lim_{t \uparrow \infty} \frac{1}{\log t} \int_1^t \frac{dv}{v} \Phi_v(B) = \mathrm{E}(\Phi_1(B)) \, , \text{ a.s.}$$

ii) Assume that the processes $\{\Phi_v(B); v \ge 0\}$ *and* $\{\Psi_v(B); v \ge 0\}$ *are adapted to the Brownian filtration, and satisfy* $\mathrm{E}(|\Phi_1(B)|^2)\, , \, \mathrm{E}(|\Psi_1(B)|^2) <$

∞, *as well as* (A.14). *Then*

$$\left(\frac{1}{\sqrt{\log t}}\int_1^t \frac{dB_v}{\sqrt{v}}\Phi_v(B), \frac{1}{\sqrt{\log t}}\int_1^t \frac{dB_v}{\sqrt{v}}\Psi_v(B)\right)$$

converges in law, as $t \uparrow \infty$, *towards a centered Gaussian couple with covariance matrix given by*

$$\begin{pmatrix} \mathrm{E}\left[(\Phi_1(B))^2\right] & \mathrm{E}\left[(\Phi_1(B)\Psi_1(B))\right] \\ \mathrm{E}\left[(\Phi_1(B)\Psi_1(B))\right] & \mathrm{E}\left[(\Psi_1(B))^2\right] \end{pmatrix} .$$

We sketch the proof: to prove *i)* we use Birkhoff theorem (see for instance [Be-We], pp. 150-151; see also [Bi] and [C-F]). The second part of the lemma is a consequence of the first part and of the Dubins-Schwarz representation of continuous martingales as time-changed Brownian motion.

We shall apply Lemma A.4 to our case. Clearly,

$$\Phi_v(B) := \mathrm{sgn}(B_v)$$

satisfies the hypothesis of the lemma. We shall now find out $\Psi_v(B)$.
We can write, from the Markov property and Ito's formula, for $s \le t$,

$$\mathrm{E}\left(|B_t| \mid \mathcal{F}_s\right) = Q_{t-s}\phi(B_s) = \mathrm{E}(|B_t|) + \int_0^s \frac{\partial}{\partial x} Q_{t-u}\phi(B_u)\, dB_u$$

$$= \mathrm{E}(|B_t|) - \int_0^s \frac{dB_u}{t-u}\,\tilde{\mathrm{E}}\left(\tilde{B}_{t-u}\,|B_u - \tilde{B}_{t-u}|\right) ,$$

as we can see by a simple calculation. Here Q_t denotes the heat semi-group, $\tilde{B}$ is an independent Brownian motion and $\phi(v) = |v|$.
Therefore,

$$|B_t| - \mathrm{E}(|B_t|) = -\int_0^t \frac{dB_u}{t-u}\,\tilde{\mathrm{E}}\left(\tilde{B}_{t-u}\,|B_u - \tilde{B}_{t-u}|\right) ,$$

or

$$\frac{1}{\sqrt{\log t}}\int_1^t \frac{dv}{v^{3/2}}\left(|B_v| - \mathrm{E}(|B_v|)\right) = \frac{1}{\sqrt{\log t}}\int_1^t \frac{dv}{v^{3/2}}\left(|B_1| - \mathrm{E}(|B_1|)\right)$$

$$-\frac{1}{\sqrt{\log t}}\int_1^t \frac{dv}{v^{3/2}}\int_1^v \frac{dB_u}{v-u}\,\tilde{\mathrm{E}}\left(\tilde{B}_{v-u}\,|B_u - \tilde{B}_{v-u}|\right) .$$

The first term on the right hand side of the preceeding equality converges in law towards zero, as $t \uparrow \infty$, whereas the second term can be written as

$$\frac{1}{\sqrt{\log t}} \int_1^t dB_u \int_u^t \frac{dv}{v^{3/2}(v-u)} \tilde{\mathrm{E}}\left(\tilde{B}_{v-u}\,|B_u - \tilde{B}_{v-u}|\right) .$$

Since we shall let $t \uparrow \infty$, it suffices to consider only

$$\frac{1}{\sqrt{\log t}} \int_1^t dB_u \int_u^\infty \frac{dv}{v^{3/2}(v-u)} \tilde{\mathrm{E}}\left(\tilde{B}_{v-u}\,|B_u - \tilde{B}_{v-u}|\right)$$

$$= \frac{1}{\sqrt{\log t}} \int_1^t \frac{dB_u}{u^{3/2}} \int_1^\infty \frac{dv}{v^{3/2}(v-1)} \tilde{\mathrm{E}}\left(\tilde{B}_{u(v-1)}\,|B_u - \tilde{B}_{u(v-1)}|\right)$$

$$= \frac{1}{\sqrt{\log t}} \int_1^t \frac{dB_u}{\sqrt{u}} \Psi_u(B) .$$

Therefore,

$$\Psi_u(B) := \frac{1}{u} \int_1^\infty \frac{dv}{v^{3/2}(v-1)} \tilde{\mathrm{E}}\left(\tilde{B}_{u(v-1)}\,|B_u - \tilde{B}_{u(v-1)}|\right) ,$$

and it is not difficult to see that it satisfies the hypothesis of Lemma A.4. Finally, to obtain (1.57) we need to compute the variance of the sum of the components of the Gaussian limit vector. This can be done using *ii)* of Lemma A.4 and (A.12), which is left to the reader. □

A.4 *Another proof for* (2.22).

We give here another proof for the first statement of Proposition 2.9. We leave as an exercice for the reader the proof of the following:

Lemma A.5. *Consider two smooth functions* $u, v : [0,\infty[\to [0,\infty[$. *We suppose that* u *strictly increasing,* $u(0) = 0$ *, and* $v(t) \neq 0$ *for any* $t > 0$. *Denote* $X_t := v(t)\, B_{u(t)}$ *and, as usual, by* $\mathrm{L}_t(B)$ *and* $\mathrm{L}_t(X)$ *the local times at level* 0 *of the linear Brownian motion* B *and of* X*, respectively. Then*

$$\mathrm{L}_t(X) = \int_0^{u(t)} v(u^{-1}(s))\, d\mathrm{L}_s(B) . \tag{A.15}$$

Moreover, if b *denotes the Brownian bridge, and if* $\mathrm{L}_t(b)$ *denotes its local time at level* 0*, then*

(A.16) $$\left(\int_0^t \frac{d\mathrm{L}_s(B)}{1+s}\right)_{t\geq 0} \sim \left(\mathrm{L}_{\frac{t}{1+t}}(b)\right)_{t\geq 0} .$$

Consequently,

$$\int_0^\infty \varphi(s) d\mathrm{L}_s(B) \sim \int_0^1 \tilde{\varphi}(s) d\mathrm{L}_s(b) ,$$

where $\tilde{\varphi}(s) = \varphi(s/1-s)/1-s$.

Proof of (2.22). We can write

$$O_t = e^{t/2} \tilde{B}_{1-e^{-t}} ,$$

where $\tilde{B}$ is a standard Brownian motion starting at 0. So, by (A.15),

$$d\mathrm{L}_t(O) = e^{-t/2}\, d\mathrm{L}_{e^t-1}(\tilde{B}) .$$

We denote

(A.17) $$\delta(w) := \int_0^\infty \frac{d\mathrm{L}_t(O)}{\sqrt{(w+1)e^t - 1}} .$$

Therefore

$$\delta(w) = \int_0^\infty \frac{e^{-t/2}\, d\mathrm{L}_{e^t-1}(\tilde{B})}{\sqrt{(w+1)e^t-1}} = \int_0^\infty \frac{d\mathrm{L}_s(\tilde{B})}{\sqrt{1+s}\sqrt{(w+1)(1+s)-1}} ,$$

making the change of variable $s = e^t - 1$. On the other hand, by (A.16),

$$d\mathrm{L}_t(\tilde{B}) \sim (1+t) d\mathrm{L}_{\frac{t}{1+t}}(b) ,$$

so,

$$\delta(w) \sim \int_0^\infty \frac{\sqrt{1+s}}{\sqrt{(w+1)(1+s)-1}}\, d\mathrm{L}_{\frac{s}{1+s}}(b) .$$

We make the change of variable $v = s/(1+s)$ and we get

$$\delta(w) \sim \int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{1-v}\sqrt{\frac{w+1}{1-v}-1}} = \int_0^1 \frac{d\mathrm{L}_v(b)}{\sqrt{w+v}} ,$$

which is (2.22) with the usual notation $w = u/(1-u)$.

Finally, let us note that (2.24) is a consequence of Lemma A.5.

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