On the derivative with respect to a function with applications to Riemann-Stieltjes integral

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In [2]-[4], William Feller introduced the derivative of a function with respect to another function, strictly increasing, in connection with a second order differential operator.

In this paper, we shall use as the definition of the derivative the unilateral limits of the functions in a point, instead of the values, and we shall study some properties to obtain a Leibniz-Newton formula for Riemann-Stieltjes integral.

In the sequel, I is an interval of the real line, $x_0 \in \mathring{\mathrm{I}}$, and f, g are real functions defined on I. We shall suppose that the function g is strictly increasing on I and the unilateral limits $f(x_0 - 0)$, $f(x_0 + 0)$ exist.

Definition 1 *We define the left derivative of f with respect to g in x_0, by*

$$f_g^{\prime s}(x_0) := \lim_{x \uparrow x_0} \frac{f(x) - f(x_0 + 0)}{g(x) - g(x_0 + 0)},$$

if x_0 is a point of continuity of g (provided the limit exists), and

$$f_g^{\prime s}(x_0) := \frac{f(x_0 - 0) - f(x_0 + 0)}{g(x_0 - 0) - g(x_0 + 0)},$$

if x_0 is a point of discontinuity of g. The right derivative $f_g^{\prime d}$ is defined symmetrically,

$$f_g^{\prime d}(x_0) := \lim_{x \downarrow x_0} \frac{f(x) - f(x_0 - 0)}{g(x) - g(x_0 - 0)},$$

if x_0 is a point of continuity of g (provided the limit exists), and

$$f_g^{\prime d}(x_0) := \frac{f(x_0 + 0) - f(x_0 - 0)}{g(x_0 + 0) - g(x_0 - 0)},$$

if x_0 is a point of discontinuity of g. If $f_g^{\prime s}$ and $f_g^{\prime d}$ are finite, we say that f is left respectively, right differentiable with respect to g in x_0.

Remark 2 It is clear that $f_g^{\prime s}(x_0) = f_g^{\prime d}(x_0)$ in each point of discontinuity of g. Also, if x_0 is a point of continuity of g and f is left or right differentiable function with respect to g in x_0, then the $\lim_{x \to x_0} f(x)$ exists.

Remark 3 We notice that $g_g^{\prime s}$ and $g_g^{\prime d}$ exist and $g_g^{\prime}(x_0) = 1$, $x_0 \in \mathring{\mathrm{I}}$.

Definition 4 *We define the derivative of f with respect to g in x_0, as follows:*

$$f'_g(x_0) := \frac{f(x_0+0) - f(x_0-0)}{g(x_0+0) - g(x_0-0)},$$

if x_0 is a point of discontinuity of g, and

$$f'_g(x_0) := \lim_{x \to x_0} \frac{f(x) - l_0}{g(x) - g(x_0)},$$

(whenever this limit exists), if x_0 is a point of continuity of g and there exists $\lim_{x \to x_0} f(x) = l_0 \in \mathbb{R}$. When the previous limit is finite, we say that f is a differentiable function with respect to g in x_0. As usual f is a differentiable function with respect to g on I *if the function f is differentiable with respect to g in each point of* I.

Remark 5 It is easy to see that a necessary and sufficient condition for f to be a differentiable function with respect to g in x_0 is that f be a left and right differentiable function with respect to g in x_0 and $f'^{s}_{g}(x_0) = f'^{d}_{g}(x_0)$.

Remark 6 If the limit $\lim_{x \to x_0} f(x)$ exists and x_0 is a point of discontinuity of g, then $f'_g(x_0) = 0$.

In the sequel, we give some properties of the differentiable functions in the meaning of our definition, analogous with the properties of the differentiable function with respect to an independent variable.

Theorem 7 *(Fermat)*
Let $f,\, g : \mathrm{I} \to \mathbb{R}$ be two functions which are continuous in $x_0 \in \mathring{\mathrm{I}}$. Assume that x_0 is a point of extremum of f. If f is a differentiable function with respect to g in x_0, then $f'_g(x_0) = 0$.

Proof. Let x_0 be a point of minimum, *i.e.* there exists a neighbourhood V of x_0, so that $f(x) \geq f(x_0)$, $x \in V \cap \mathrm{I}$. By Remark 2, it follows the existence of the limit $\lim_{x \to x_0} f(x)$, which is $f(x_0)$ by the continuity of f in x_0. Therefore we get $f(x) - f(x_0) \geq 0$, $\forall x \in V \cap \mathrm{I}$. Then, because g is a strictly increasing function,

$$\frac{f(x) - f(x_0+0)}{g(x) - g(x_0+0)} \leq 0, \text{ for } x < x_0,\, x \in V \cap \mathrm{I}$$

and it follows $f'^{s}_{g}(x_0) \leq 0$. Similarly $f'^{d}_{g}(x_0) \geq 0$ and we have $f'_g(x_0) = 0$. □

Theorem 8 *(Rolle)*
Let $f,\, g : [a,b] \to \mathbb{R}$ and let g be strictly increasing function. If f is a continuous function on $[a,b]$, differentiable with respect to g on (a,b), and $f(a) = f(b)$, then there is at least one zero of its derivative.

Proof. If there are points $x_0 \in (a,b)$ where g is a discontinuous function then the conclusion is true by Remark 6. Now, we prove that the assertion is true in the case when g is a continuous function on (a,b). If f is a constant function, it follows at once, by Definition 4, that $f'_g(x_0) = 0$, $x_0 \in (a,b)$. Assume that f is not a constant function. Since f is continuous on $[a,b]$ it involves that f has an extremum $c \in (a,b)$ and by Theorem 7, we have $f'_g(c) = 0$. □

Theorem 9 *(Cauchy)*
Let f, g, $h : [a,b] \to \mathbb{R}$ and suppose that g is strictly increasing. If f, g, h are continuous functions on $[a,b]$, f, h are differentiable with respect to g on (a,b), and $h'_g(x) \neq 0$, for each $x \in (a,b)$, then $h(a) \neq h(b)$ and there is at least one point $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{h(b) - h(a)} = \frac{f'_g(c)}{h'_g(c)}.$$

Proof. There would exist $c \in (a,b)$ such that $h'_g(c) = 0$, if we had $h(a) = h(b)$ (by Theorem 8). There is a contradiction and so $h(a) \neq h(b)$. Consider $\varphi : [a,b] \to \mathbb{R}$, with $\varphi(x) = f(x) + \lambda\, h(x)$, $x \in [a,b]$. It follows that φ is a continuous function on $[a,b]$, a differentiable function with respect to g on (a,b). From $\varphi(a) = \varphi(b)$ we find $\lambda_0 = (f(b) - f(a))/(h(b) - h(a))$ and $\varphi(x) = f(x) + \lambda_0 h(x)$ satisfies the conditions of Theorem 8, *i.e.* there is at least one $c \in (a,b)$ such that $f'_g(c) + \lambda_0 h'_g(c) = 0$. The equality follows at once. □

Theorem 10 *(Lagrange)*
Let f, $g : [a,b] \to \mathbb{R}$ and suppose that g is strictly increasing. If f, g are continuous functions on $[a,b]$, f is differentiable with respect to g on (a,b), then there is at least one $c \in (a,b)$ such that

$$f(b) - f(a) = f'_g(c)[g(b) - g(a)].$$

Proof. It follows immediately, applying Theorem 9, taking $h = g$ and observing that $g'_g(x) = 1$, for each $x \in (a,b)$. □

Proposition 11 *Let $g : [a,b] \to \mathbb{R}$ be a strictly increasing function and assume that g has a finite number of points of discontinuity. Let $s : [a,b] \to \mathbb{R}$ be the jump component of g, i.e. $s(a) = 0$ and for $a < x \leq b$,*

$$s(x) = [g(a+0) - g(a)] + \sum_{x_k < x} [g(x_k + 0) - g(x_k - 0)] + [g(x) - g(x-0)].$$

Then, the function s is a differentiable function with respect to g on (a,b).

Proof. For simplicity, we assume that g has only one point of discontinuity $x_0 \in (a,b)$ because, in the general case the reasoning is analogous. We have

$$s(x) = \begin{cases} 0, & x \in [a, x_0) \\ g(x_0) - g(x_0 - 0), & x = x_0 \\ g(x_0 + 0) - g(x_0 - 0), & x \in (x_0, b]. \end{cases}$$

Since s is a continuous function on $(a,b) \setminus \{x_0\}$ it is clear that there exists s'_g and $s'_g(x) = 0$ for each $x \in (a,b) \setminus \{x_0\}$. Then

$$s'_g(x_0) = \frac{s(x_0+0) - s(x_0-0)}{g(x_0+0) - g(x_0-0)} = \frac{g(x_0+0) - g(x_0-0) - 0}{g(x_0+0) - g(x_0-0)} = 1.$$

Consequently,

$$s'_g(x) = \begin{cases} 0, & \text{if } x \text{ is a point of continuity of } g \\ 1, & \text{if } x \text{ is a point of discontinuity of } g. \end{cases}$$

□

Let us denote $\mathcal{D}_{[a.b]} = \{g : [a,b] \to \mathbb{R} : g$ is a strictly increasing function and the jump component s of g is a differentiable function with respect to g on (a,b) $\}$.

Proposition 12 *If $g \in \mathcal{D}_{[a,b]}$, then the continuous component of g, $\bar{g} : [a,b] \to \mathbb{R}$, $\bar{g}(x) = g(x) - s(x)$, where s is the jump component of g is a differentiable function with respect to g on (a,b).*

Proof. It is known (see [5], p. 269), that $\bar{g}$ is an increasing continuous function on $[a,b]$. We have

$$\bar{g}'_g(x_0) = \lim_{x \to x_0} \frac{\bar{g}(x) - \bar{g}(x_0)}{g(x) - g(x_0)} = \lim_{x \to x_0} \left(1 - \frac{s(x) - s(x_0)}{g(x) - g(x_0)}\right) = 1 - s'_g(x_0),$$

if x_0 is a point of continuity of g and $\bar{g}'_g(x_0) = 0$, if x_0 is a point of discontinuity of g. □

Remark 13 If g satisfies the assumption of Proposition 11, then

$$\bar{g}'_g(x) = \begin{cases} 1, & \text{if } x \text{ is a point of continuity of } g \\ 0, & \text{if } x \text{ is a point of discontinuity of } g. \end{cases}$$

Definition 14 *Let f, $g : \mathrm{I} \to \mathbb{R}$ be two functions such that g is strictly increasing. A primitive f with respect to g is any function $F : \mathrm{I} \to \mathbb{R}$ such that $F'_g(x) = f(x)$ for each $x \in \mathrm{I}$.*

Proposition 15 *If f is a continuous function on $[a,b]$ and if g is strictly increasing finction on $[a,b]$, then the function*

$$F(x) = \int_a^x f(t)dg(t), \ \forall x \in [a,b]$$

(we understand the integral in the sense Riemann-Stieltjes) is a primitive of f with respect to g on $[a,b]$.

Proof. Clearly, f is a Riemann-Stieltjes integrable function with respect to g on each interval $[a, x] \subset [a, b]$. If x_0 is a point of discontinuity of g, then applying the mean value theorem, we have:

$$F'_g(x_0) = \frac{F(x_0+0) - F(x_0-0)}{g(x_0+0) - g(x_0-0)} = \lim_{h\to 0} \frac{F(x_0+h) - F(x_0-h)}{g(x_0+h) - g(x_0-h)}$$

$$= \lim_{h\to 0} \frac{\int_{x_0-h}^{x_0+h} f(t)dg(t)}{g(x_0+h) - g(x_0-h)} = \lim_{h\to 0} \frac{f(\xi)[g(x_0+h) - g(x_0-h)]}{g(x_0+h) - g(x_0-h)}$$

$$= \lim_{h\to 0} f(\xi) = f(x_0).$$

We observe now that F is a continuous function on each point x_0 of continuity of g. Indeed, we have

$$|F(x) - F(x_0)| = \left| \int_x^{x_0} f(t)dg(t) \right| \leq M|g(x) - g(x_0)| \ \ (M = \max_{x\in[a,b]} |f(x)|).$$

Like in [6], we have:

$$F'_g(x_0) = \lim_{x\to x_0} \frac{F(x) - F(x_0)}{g(x) - g(x_0)} = \lim_{x\to x_0} \frac{\int_x^{x_0} f(t)dg(t)}{g(x) - g(x_0)}$$

$$= \lim_{x\to x_0} \frac{f(\xi)[g(x) - g(x_0)]}{g(x) - g(x_0)} = \lim_{x\to x_0} f(\xi) = f(x_0).$$

□

Theorem 16 *(Leibniz-Newton formula)*
Assume that f is a Riemann-Stieltjes integrable function and it has primitives on $[a, b]$ with respect to the strictly increasing continuous function g. Il F is a continuous function on $[a, b]$ and it is a primitive of the function f with respect to g, then we get

$$\int_a^b f(x)dg(x) = F(b) - F(a).$$

Proof. Applying Theorem 10, we have

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} F'_g(\xi_i)[g(x_i) - g(x_{i-1})]$$

$$= \sum_{i=1}^{n} f(\xi_i)[g(x_i) - g(x_{i-1})] = S_g(f, \Delta, \xi),$$

where $S_g(f, \Delta, \xi)$ is the Stieltjes sum for the functions f, g, for the sub-division Δ and the intermediate points (ξ_i). Since f is a Riemann-Stieltjes integrable function, the assertion of the theorem follows at once. □

In [1] is proved the following

Theorem 17 *Let $f, g : [a, b] \to \mathbb{R}$ be two functions. Assume that:*

1. *g is an increasing function and $(a_k)_{k\geq 1}$ are its points of discontinuity;*
2. *f is a Riemann-Stieltjes integrable function with respect to g on $[a, b]$.*

Let us denote s and $\bar{g}$ the jump component and the continuous component of g. Then f is Riemann-Stieltjes integrable with respect to $\bar{g}$ on $[a, b]$ and we have

$$\int_a^b f(x)dg(x) = \int_a^b f(x)d\bar{g}(x) + \sum_{k=1}^{\infty} f(a_k)(s(a_k + 0) - s(a_k - 0)).$$

Remark 18 Assume that the hypotheses of Theorem 17 are satisfied, and g is a strictly increasing function. If f has a primitive F with respect to $\bar{g}$ and F is a continuous function on $[a, b]$, then using Theorem 16 we get

$$\int_a^b f(x)dg(x) = F(b) - F(a) + \sum_{k=1}^{\infty} f(a_k)(s(a_k + 0) - s(a_k - 0)).$$

If f is a continuous function on $[a, b]$, then we can take, by Propostion 15

$$F(x) = \int_a^x f(t)d\bar{g}(t).$$

References

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