Distribution tails and asymptotics of solutions of SDE driven by an asymmetric stable Lévy process

Richard Eon and Mihai Gradinaru
Institut de Recherche Mathématique de Rennes, Université de Rennes 1,
Campus de Beaulieu, 35042 Rennes Cedex, France
Richard.Eon@univ-rennes1.fr,Mihai.Gradinaru@univ-rennes1.fr

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Abstract: The behaviour of the tails of the invariant distribution for stochastic differential equations driven by an asymmetric stable Lévy process is obtained. The result is then used to deduce a scaling limit for a non-linear Langevin type process whose speed satisfies a stochastic differential equation of this type. These results generalize those contained in [11] and [4] where the stable driven noise was supposed symmetric.

Key words: stochastic differential equation; asymmetric stable Lévy noise; tail behaviour; ergodic processes; stationary distribution; non-linear Langevin type equation; functional central limit theorem for martingales.

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1 Introduction

The first goal of this paper is to extend a result obtained by Samorodnitski and Grigoriu in [11]. They consider the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}L_t - f(X_t)\mathrm{d}t, \tag{1.1}$$

where f is a quickly increasing to infinity function and L is a symmetric Lévy motion and they study the exact rate of decay of the tail probabilities of the random variables X_t, $t > 0$. The proof in [11] is technical and in Remark 3.2, p. 76, the authors conjecture that their main result remains true without the assumption of symmetry of the Lévy process. The first part of the present paper (Section 2) contains a proof of this conjecture and we try to reduce the technical difficulties announced in the cited remark by assuming that the Lévy process is α-stable. More precisely, we assume that X is a solution of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}\ell_t - f(X_t)\mathrm{d}t, \quad X_0 = x, \tag{1.2}$$

where ℓ is the asymmetric α-stable Lévy process having its Lévy measure given by

$$\nu(\mathrm{d}z) = |z|^{-1-\alpha}\big[a_-\mathbb{1}_{\{z<0\}} + a_+\mathbb{1}_{\{z>0\}}\big]\,\mathrm{d}z. \tag{1.3}$$

Here $\alpha \in (0,2)\setminus\{1\}$, $a_+ \neq a_-$ and x is a real number.

Our second objective is to prove a similar result that we obtained recently in [4] about non-linear Langevin dynamics driven by Lévy noises. Consider the dynamics of a particle whose speed satisfies a one-dimensional stochastic differential equation driven by a small α-stable Lévy process in a potential of the form a power function of exponent $\beta+1$. Let us recall that the dynamics of some integrated processes driven by Lévy noises appears in financial mathematics models or in physics. In [4] a scaling limit of the position process having this speed was studied and it was proved that when the driving noise is a symmetric stable process, its limit in distribution is a Brownian motion. Diffusions in heterogeneous materials or prices in finance could be modelled by using stochastic differential equations driven by asymmetric Lévy noises (see for instance [12]). In the second part (Section 3) we explain which are the differences in order to obtain a similar result as in [4] when the driven noise is the asymmetric α-stable Lévy process. More precisely, one considers the stochastic differential equation

$$\mathrm{d}v_t^\varepsilon = \varepsilon \mathrm{d}\ell_t - \frac{1}{2}\mathcal{U}'(v_t^\varepsilon)\mathrm{d}t, \quad v_0^\varepsilon = 0, \tag{1.4}$$

with the potential $\mathcal{U}(v) := \frac{2}{\beta+1}|v|^{\beta+1}$ and assume again that ℓ is the asymmetric α-stable Lévy noise. To get the limit in distribution, as $\varepsilon \to 0$, of the position process $x_t^\varepsilon = \int_0^t v_s^\varepsilon \mathrm{d}s$, one uses the exact rate of decay of the tail probabilities for the speed process obtained in Section 2.

2 Tails for the invariant measure of the solution

2.1 Notations and main result

We will always assume that ℓ is the asymmetric α-stable Lévy process having its Lévy measure given by (1.3), with $\alpha \in (0,2)\setminus\{1\}$, $a_+ \neq a_-$ and $a_+ \neq 0$ and $a_- \neq 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function with $f(0) = 0$ which is regularly varying at infinity with exponent $\beta > 1$: for all $a > 0$, $\lim_{x\to\infty} \frac{f(ax)}{f(x)} = a^\beta$. Recall that the process X satisfies

$$X_t = x + \ell_t - \int_0^t f(X_s)\mathrm{d}s,\ t \geq 0. \tag{2.5}$$

Let us note that the existence and the uniqueness of a global solution for the equation (2.5) is justified in [11] for a general Lévy driven noise and it is a consequence of Theorem 6.2.11, p. 376 in [1] (see also Proposition 1.2.10, p. 27 in [3]). The statement of our main result of this section is the following:

Theorem 2.1. *Denote, for all $u > 0$*

$$h(u) := \int_u^{+\infty} \frac{\nu((y,+\infty))}{f(y)}\mathrm{d}y\,. \tag{2.6}$$

Then

$$\lim_{u\to+\infty} \frac{\mathbb{P}_x(X_t > u)}{h(u)} = 1 \tag{2.7}$$

uniformly with respect to $x \in \mathbb{R}$ and $t \geqslant 1$.

As a consequence we obtain the behaviour of the tail for the invariant measure. According to Proposition 0.1, p. 604 in [8], and under the assumptions on the function f, the exponential ergodicity of the solution X of (1.1) is insured. Moreover its unique invariant measure, denoted $m_{\alpha,\beta}$, satisfies

$$\forall x \in \mathbb{R}, \quad \|\mathbb{P}_x^t - m_{\alpha,\beta}\|_{\mathrm{TV}} = \mathrm{O}(\exp(-Ct)), \quad \text{as } t \to \infty, \tag{2.8}$$

where $\mathbb{P}_x^t$ is the distribution of X_t under $\mathbb{P}_x$ and $\|\cdot\|_{\mathrm{TV}}$ is the norm in total variation. Therefore letting t goes to infinity in Theorem 2.1, we get:

Corollary 2.2. *Under the same assumptions as in* Theorem 2.1*, we have*

$$\lim_{u\to+\infty} \frac{m_{\alpha,\beta}((u,+\infty))}{h(u)} = 1. \tag{2.9}$$

2.2 Proof of the result on tails of solutions of SDE

We split the proof of Theorem 2.1 in several steps.

Step 1. Introduce, for $\sigma > 0$ and for some $c > 0$ to be chosen, the Lévy process $\ell^{(\sigma)}$ with the following small jumps prescribed by the Lévy measure

$$\nu^{(\sigma)}(\mathrm{d}z) = |z|^{-1-\alpha}\big[a_-\mathbb{1}_{\{z<-\sigma\}} + a_+\mathbb{1}_{\{z>c\sigma\}}\big]\,\mathrm{d}z. \tag{2.10}$$

The process $\ell^{(\sigma)}$ has a finite number of jumps on each finite interval of time. Denote by T_j the time when the j-th jump occurs (with the convention $T_0 = 0$) and by $W_j^{(\sigma)}$ its size. The random variables $(W_j^{(\sigma)})$ are i.i.d. We will choose the constant c such that, for all y and σ,

$$\mathbb{E}\big(W_1^{(\sigma)}\mathbb{1}_{\{-y\leqslant W_1^{(\sigma)}\le cy\}}\big) = 0.$$

Since the probability density of $W_1^{(\sigma)}$ is given by

$$z \mapsto \frac{1}{\lambda_\sigma}|z|^{-1-\alpha}\big[a_-\mathbb{1}_{\{z<-\sigma\}} + a_+\mathbb{1}_{\{z>c\sigma\}}\big]\,, \qquad \text{with} \qquad \lambda_\sigma := \frac{\sigma^{-\alpha}}{\alpha}(a_- + a_+c^{-\alpha}), \tag{2.11}$$

we deduce that the only possible value of the constant is

$$c = \Big(\frac{a_-}{a_+}\Big)^{1/(1-\alpha)}. \tag{2.12}$$

Let us point out that, by the definition of $\nu^{(\sigma)}$, for $u > c\sigma > 0$,

$$\nu^{(\sigma)}\big((u,+\infty)\big) = \nu\big((u,+\infty)\big) =: \rho(u). \tag{2.13}$$

Step 2. Let us denote

$$X_t^{(\sigma)} = x + \ell_t^{(\sigma)} - \int_0^t f(X_s^{(\sigma)})\,\mathrm{d}s,\ t \geqslant 0. \tag{2.14}$$

According to Theorem 19.25 in [7], p. 385, $X^{(\sigma)}$ converges in distribution to X, as σ tends to 0. To get (2.7) it is enough to prove that there exists σ_0, such that,

$$\Big|\frac{\mathbb{P}_x(X_t^{(\sigma)} > u)}{h(u)} - 1\Big| \leqslant \mathrm{o}(1), \qquad \text{as } u \to +\infty, \tag{2.15}$$

uniformly in $x \in \mathbb{R}$, $\sigma \leqslant \sigma_0$ and $t \geqslant 1$.

Step 3. The ordinary differential equation

$$x(t) = x - \int^t f(x(s))\mathrm{d}s, \quad t \geqslant 0 \tag{2.16}$$

has a unique solution. As in [11], p. 93, we introduce, for all $u > 0$

$$g(u) := \int_u^{+\infty} \frac{1}{f(y)}\mathrm{d}y\,. \tag{2.17}$$

This function is clearly finite, non-negative, continuous and strictly decreasing for large u. Let us fix $1 \leqslant s \leqslant t$. It is no difficult to see that the solution of (2.16) satisfies $g(x(t)) = g(x(s)) + t - s$ and in particular, for any $u > 0$, if $x(t) > u$ then $g(u) > g(x(t)) \geqslant t - s$. We deduce that the solution of (2.16) on $[t - g(u), t]$ will end up, at time t, not higher than u.

At this level let us recall an important result from [11] (see Lemma 5.1, p. 94). Let $A > 0$ and denote by y, the solution of the deterministic equation (2.16) on each interval of the form (S_{i-1}, S_i) with $0 = S_0 < \cdots < S_n < A$ but with jumps at time S_i of a size j_i. More precisely

$$y'(t) = -f(y(t)), \quad \text{on } (S_{i-1}, S_i) \text{ and } \quad y(S_i) = y(S_i^-) + j_i, \quad y(0) = x. \tag{2.18}$$

As previously, it is not difficult to see that y satisfies $g(y(A)) = g(y(S_n)) + A - S_n$ and in particular, for any $u > 0$, if $y(A) > u$, then $A - S_n \leqslant g(u)$. Moreover, one can compare the solution x of (2.16) with y:

$$-\max_{k=1,\dots,n} \Big(\sum_{i=k}^{n} j_i \Big)_- \leqslant y(A) - x(A) \leqslant \max_{k=1,\dots,n} \Big(\sum_{i=k}^{n} j_i \Big)_+ .$$

In particular, if we set, for $a > 0$, $N(a) = \sup\{i \leqslant n : \, j_i + \cdots + j_n > a\}$ (=0 if the set is empty), then

$$\text{for } \ t \in [S_{N(a)}, A] \ \text{ such that } \ y(t) \leqslant b, \ \text{ we have } \ y(A) \leqslant a + b. \tag{2.19}$$

Step 4. For $t \geqslant 1$, denote by $N_t^{(\sigma)}$ the number of jumps of $\ell^{(\sigma)}$ during the interval $[0, t]$ and define, for all $a < 0$ and $b > 0$,

$$M_1^{(\sigma)}(a, b) := \sup\{j \leqslant N_t^{(\sigma)} : W_j^{(\sigma)} \notin [a, b]\}, \quad \text{ and } = 0 \text{ if the set is empty.} \tag{2.20}$$

To simplify notations we will denote by $\tau_1 := T_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}$ the time of the jump with index $M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)$. We can write

$$\begin{aligned} \mathbb{P}_x(X_t^{(\sigma)} > u) = \mathbb{P}_x \Big(X_t^{(\sigma)} > u, \, \tau_1 < t - g(\delta u) \Big) + \mathbb{P}_x \Big(X_t^{(\sigma)} > u, \, \tau_1 \in [t - g(\delta u), t] \Big) \\ := p_1(u) + p_2(u). \end{aligned} \tag{2.21}$$

Let us fix $s \leqslant t$ and for $\varepsilon, \gamma, \delta > 0$ and $u > 0$, introduce the event

$$A_{\varepsilon,\gamma,\delta,u,s} := \left\{ \sup_{\substack{1 \leqslant i \leqslant N_t^{(\sigma)} \\ s - g(\delta u) \leqslant T_i \leqslant s}} \sum_{i \leqslant j \leqslant N_t^{(\sigma)}} W_j^{(\sigma)} \mathbb{1}_{\{-\varepsilon u \leqslant W_j^{(\sigma)} \leqslant c\varepsilon u\}} \geqslant \gamma u \right\} . \tag{2.22}$$

We can state the following lemma:

Lemma 2.3. *If $(1 \vee c)\varepsilon \leqslant \gamma/4$ then there exist $u_0(\varepsilon, \gamma, \delta)$, σ_0 and a positive constant $C(\beta, \gamma)$ such that, for all $u \geqslant u_0(\varepsilon, \gamma, \delta)$ and $\sigma \leqslant \sigma_0$,*

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s}) \leqslant C(\varepsilon, \gamma) g(\delta u) \rho(u)^{\gamma/(4\varepsilon(1 \vee c))}. \tag{2.23}$$

Remark 2.4. *Let us point out that all the constants in (2.23) do not depend on t.*

We postpone the proof of Lemma 2.3 and we proceed with the proof of our main result.

Step 5. To begin with, we study the term p_1 in (2.21). We can write

$$p_1(u) \leqslant \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) + \mathbb{P}_x(A^c_{\varepsilon,\gamma,\delta,u,t} \cap \{X^{(\sigma)}_t > u,\, \tau_1 < t - g(\delta u)\}). \tag{2.24}$$

Since the solution of (2.16) on $[t - g(\delta u), t]$ will end up, at time t, not higher than δu, by using (2.19) we get,

$$X^{(\sigma)}_t \leqslant \delta u + \gamma u \quad \text{ on the event } \quad A^c_{\varepsilon,\gamma,\delta,u,t} \cap \{\tau_1 < t - g(\delta u)\}.$$

By choosing $\delta+\gamma \leqslant 1$, the second term on the right hand side of (2.24) is equal to 0. Furthermore, assuming that $(1 \vee c)\varepsilon \leqslant \gamma/4$, using Lemma 2.3, we see that there exist $u_0(\varepsilon,\gamma,\delta)$ and σ_0 such that, for all $u \geqslant u_0(\varepsilon,\gamma,\delta)$ and $\sigma \leqslant \sigma_0$,

$$p_1(u) \leqslant \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) \leqslant C(\varepsilon,\gamma) g(\delta u)\rho(u)^{\gamma/(4\varepsilon(1\vee c))}. \tag{2.25}$$

We analyse now the term p_2 in (2.21). Let us introduce, for all $a < 0$ and $b > 0$,

$$M^{(\sigma)}_2(a,b) := \sup\{j < M^{(\sigma)}_1(a,b) : W^{(\sigma)}_j \notin [a,b]\}, \tag{2.26}$$

and again, to simplify, we set $\tau_2 := T_{M^{(\sigma)}_2(-\varepsilon u, c\varepsilon u)}$ the time of the jump with index $M^{(\sigma)}_2(-\varepsilon u, c\varepsilon u)$. We can write

$$\begin{aligned} p_2(u) = \mathbb{P}_x\Big(X^{(\sigma)}_t > u,\, \tau_1 \in [t - g(\delta u), t]\Big) &\leqslant \mathbb{P}\Big(t - \tau_1 \leqslant g(\delta u),\, \tau_1 - \tau_2 \leqslant g(\delta u)\Big) \\ &+ \mathbb{P}_x\Big(X^{(\sigma)}_t > u,\, t - \tau_1 \leqslant g(\delta u),\, \tau_1 - \tau_2 > g(\delta u)\Big) =: p_{21}(u) + p_{22}(u). \end{aligned} \tag{2.27}$$

Step 6. First, we estimate p_{21}. Since $N^{(\sigma)}_{g(\delta u)}$ has the same distribution as the number of jumps of $\ell^{(\sigma)}$ in the interval $[t - g(\delta u), t]$, we get

$$\mathbb{P}(\tau_1 \leqslant t - g(\delta u)) = \mathbb{P}\Big(\forall j \in \{1, \ldots, N^{(\sigma)}_{g(\delta u)}\},\, -\varepsilon u \leqslant W^{(\sigma)}_j \leqslant c\varepsilon u\Big).$$

By using the fact that $N^{(\sigma)}_{g(\delta u)}$ is a Poisson distributed random variable of parameter $\lambda_\sigma g(\delta u)$ and is independent of the $W^{(\sigma)}_i$, we deduce

$$\begin{aligned} \mathbb{P}(\tau_1 \leqslant t - g(\delta u)) &= e^{-\lambda_\sigma g(\delta u)} \sum_{n=0}^{+\infty} \frac{(\lambda_\sigma g(\delta u))^n}{n!} \mathbb{P}(-\varepsilon u \leqslant W^{(\sigma)}_1 \leqslant c\varepsilon u)^n \\ &= e^{-\lambda_\sigma g(\delta u)(1 - \mathbb{P}(-\varepsilon u \leqslant W^{(\sigma)}_1 \leqslant c\varepsilon u))} = e^{-\lambda_\sigma g(\delta u)\mathbb{P}(W^{(\sigma)}_1 \notin [-\varepsilon u, c\varepsilon u])}. \end{aligned}$$

Since

$$\mathbb{P}\big(W^{(\sigma)}_1 \notin [-\varepsilon u, c\varepsilon u]\big) = \frac{c^{1-\alpha} + c^{-\alpha}}{\lambda_\sigma} \rho(\varepsilon u),$$

we get

$$\mathbb{P}(\tau_1 \leqslant t - g(\delta u)) = e^{-(c^{1-\alpha}+c^{-\alpha})g(\delta u)\rho(\varepsilon u)}.$$

Since $t - \tau_1$ and $\tau_1 - \tau_2$ are independent and have the same distribution, we obtain

$$\begin{aligned} p_{21}(u) = \mathbb{P}\Big(t - \tau_1 \leqslant g(\delta u),\, \tau_1 - \tau_2 \leqslant g(\delta u)\Big) &= \Big(1 - e^{-(c^{1-\alpha}+c^{-\alpha})g(\delta u)\rho(\varepsilon u)}\Big)^2 \\ &\leqslant (c^{1-\alpha} + c^{-\alpha})^2 \rho(\varepsilon u)^2 g(\delta u)^2. \end{aligned} \tag{2.28}$$

To estimate p_{22}, we fix η that will be chosen later. We can write

$$p_{22}(u) \leqslant \mathbb{P}_x\Big(X_t^{(\sigma)} > u,\, t-\tau_1 \leqslant g(\delta u),\, X_{\tau_1-}^{(\sigma)} \leqslant \eta u\Big) \\ + \mathbb{P}_x\Big(t-\tau_1 \leqslant g(\delta u),\, X_{\tau_1-}^{(\sigma)} > \eta u,\, \tau_1-\tau_2 > g(\delta u)\Big) =: p_{221}(u) + p_{222}(u). \quad (2.29)$$

Step 7. We begin with the study of p_{221}. We have

$$p_{221}(u) \leqslant \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) + \mathbb{P}_x\Big(A^c_{\varepsilon,\gamma,\delta,u,t} \cap \Big\{X_t^{(\sigma)} > u,\, t-\tau_1 \leqslant g(\delta u),\, X_{\tau_1-}^{(\sigma)} \leqslant \eta u\Big\}\Big) \\ := \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) + p_{\mathrm{main}}(u). \quad (2.30)$$

By using Lemma 2.3, for all $u \geqslant u_0(\varepsilon,\gamma,\delta)$ and $\sigma \leqslant \sigma_0$,

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) \leqslant C(\varepsilon,\gamma) g(\delta u)(\rho(u))^{\gamma/(4\varepsilon(1\vee c))}. \quad (2.31)$$

Furthermore, by the definition of g and (2.19), for all $u \geqslant u_0$, on the event

$$A^c_{\varepsilon,\gamma,\delta,u,t} \cap \Big\{X_t^{(\sigma)} > u,\, t-\tau_1 \leqslant g(\delta u),\, X_{\tau_1-}^{(\sigma)} \leqslant \eta u\Big\},$$

the magnitude $W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}$ of the jump at time τ_1 should satisfy

$$t-\tau_1 + g\Big(\eta u + W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}\Big) \leqslant g((1-\gamma)u).$$

Hence, since g is positive and decreasing, we get

$$t-\tau_1 \leqslant g((1-\gamma)u) \quad \text{and} \quad W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)} \geqslant g^{-1}(g((1-\gamma)u) - (t-\tau_1) - \eta u).$$

At this level, we need to assume that $(1\vee c)\varepsilon + \gamma + \eta < 1$. For all $s \in (0, g((1-\gamma)u))$,

$$\begin{aligned}\mathbb{P}\Big(&W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)} \geqslant g^{-1}(g((1-\gamma)u) - (t-\tau_1) - \eta u)\Big) \\ &= \mathbb{P}\big(W_1^{(\sigma)} \geqslant g^{-1}(g((1-\gamma)u) - (t-\tau_1) - \eta u) \mid W_1^{(\sigma)} \notin [-\varepsilon u, c\varepsilon u]\big) \\ &= \frac{\mathbb{P}\big(W_1^{(\sigma)} \geqslant g^{-1}(g((1-\gamma)u) - (t-\tau_1) - \eta u)\big)}{\mathbb{P}\big(W_1^{(\sigma)} \notin [-\varepsilon u, c\varepsilon u]\big)} = \frac{\rho(g^{-1}(g((1-\gamma)u) - s) - \eta u))}{(c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u)}.\end{aligned}$$

Since $t-\tau_1$ and $W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}$ are independent and recalling that the distribution of $t-\tau_1$ is exponential with parameter $(c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u)$, we obtain

$$\begin{aligned}p_{\mathrm{main}}(u) &= \mathbb{P}_x\Big(A^c_{\varepsilon,\gamma,\delta,u,t} \cap \Big\{X_t^{(\sigma)} > u,\, t-\tau_1 \leqslant g(\delta u),\, X_{\tau_1-}^{(\sigma)} \leqslant \eta u\Big\}\Big) \\ &\leqslant \int_0^{g((1-\gamma)u)} e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s} \rho(g^{-1}(g((1-\gamma)u) - s) - \eta u))\mathrm{d}s \\ &\leqslant \int_0^{g((1-\gamma)u)} \rho(g^{-1}(g((1-\gamma)u) - s) - \eta u))\mathrm{d}s.\end{aligned}$$

We perform the change of variable $y = g^{-1}(g((1-\gamma)u) - s)$ and we get

$$\begin{aligned}p_{\mathrm{main}}(u) &\leqslant \int_{(1-\gamma)u}^{+\infty} \frac{\rho(y-\eta u)}{f(y)}\mathrm{d}y \leqslant \int_{(1-\gamma)u}^{+\infty} \frac{\rho(y(1-\eta/(1-\gamma)))}{f(y)}\mathrm{d}y \\ &= \Big(1 - \frac{\eta}{1-\gamma}\Big)^{-\alpha} \int_{(1-\gamma)u}^{+\infty} \frac{\rho(y)}{f(y)}\mathrm{d}y = \Big(1 - \frac{\eta}{1-\gamma}\Big)^{-\alpha} h((1-\gamma)u). \quad (2.32)\end{aligned}$$

Putting together (2.30), (2.31) and (2.32), we deduce, for all $u \geqslant u_0(\varepsilon, \gamma, \delta)$ and $\sigma \leqslant \sigma_0$,

$$p_{221}(u) \leqslant \left(1 - \frac{\eta}{1-\gamma}\right)^{-\alpha} h((1-\gamma)u) + C(\varepsilon, \gamma) g(\delta u)(\rho(u))^{\gamma/(4\varepsilon(1\vee c))}. \tag{2.33}$$

It remains to estimate p_{222}. Since $\tau_1 - \tau_2$ and $t - \tau_1$ are independent, we can split

$$p_{222}(u) = \mathbb{P}(t - \tau_1 \leqslant g(\delta u)) \cdot \mathbb{P}_x\Big(X^{(\sigma)}_{\tau_1 -} > \eta u,\, \tau_1 - \tau_2 > g(\delta u)\Big).$$

We can write

$$\mathbb{P}_x\Big(X^{(\sigma)}_{\tau_1 -} > \eta u,\, \tau_1 - \tau_2 > g(\delta u)\Big) \leqslant \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,\tau_1}) + \mathbb{P}_x\Big(A^c_{\varepsilon,\gamma,\delta,u,\tau_1} \cap \Big\{X^{(\sigma)}_{\tau_1 -} > \eta u,\, \tau_1 - \tau_2 > g(\delta u)\Big\}\Big).$$

By choosing γ, δ and ε small enough, we can assume that $\delta + \gamma < \eta$. By employing the same argument used to estimate p_1, we deduce

$$\mathbb{P}_x\Big(A^c_{\varepsilon,\gamma,\delta,u,\tau_1} \cap \Big\{X^{(\sigma)}_{\tau_1 -} > \eta u,\, \tau_1 - \tau_2 > g(\delta u)\Big\}\Big) = 0.$$

We use again Lemma 2.3 and the exponential distribution of $t - \tau_1$ with parameter $(c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u)$ to obtain that, for all $u \geqslant u_0(\varepsilon, \delta, \gamma)$ and $\sigma \leqslant \sigma_0$,

$$p_{222}(u) \leqslant C(\varepsilon, \delta, \gamma, \eta) \rho(u)^{(1+\gamma/(4(1\vee c)\varepsilon))} g(u)^2. \tag{2.34}$$

Step 8. Finally, summarizing the inequalities (2.25), (2.28), (2.33) and (2.34), for ε, γ, δ and η such that $\delta + \gamma < \eta < 1$, $(1 \vee c)\varepsilon < \gamma/4$ and $(1 \vee c)\varepsilon + \gamma + \eta < 1$, there exist $u_0(\varepsilon, \gamma, \delta, \eta)$ and σ_0 such that, for all $u \geqslant u_0(\varepsilon, \gamma, \delta, \eta)$ and $\sigma \leqslant \sigma_0$,

$$\begin{aligned}\mathbb{P}_x(X^{(\sigma)}_t > u) \leqslant \left(1 - \frac{\eta}{1-\gamma}\right)^{-\alpha} h((1-\gamma)u) &+ (c^{1-\alpha} + c^{-\alpha})^2 \rho(\varepsilon u)^2 g(\delta u)^2 \\ &+ C(\varepsilon, \gamma, \delta, \eta) g(u) \rho(u)^{\gamma/(4(1\vee c)\varepsilon)}.\end{aligned}$$

Since h is regularly varying at infinity with exponent $1 - \alpha - \beta$, g is regularly varying at infinity with exponent $1 - \beta$ and $\rho(u)$ is regularly varying at infinity with exponent $-\alpha$, choosing ε, γ, δ and η small enough, we get that for all $\xi > 0$, there exists $u_0(\xi)$ such that, for all $u \geqslant u_0(\xi)$, all $x \in \mathbb{R}$ and all $t \geqslant 1$,

$$\frac{\mathbb{P}_x(X^{(\sigma)}_t > u)}{h(u)} \leqslant 1 + \xi,$$

hence we have established the upper bound of the main result.

Remark 2.5. *At this level we note that, if instead of the regular variation at infinity of the function f, we made only the assumption $f(x) \geqslant \hat{f}(x)$ for all $x \geqslant A$ for some function $\hat{f}$ which is regularly varying at infinity with exponent greater than one, we would still have the upper bound, for all $\xi > 0$, there exists $u_0(\xi)$ such that, for all $u \geqslant u_0(\xi)$, all $x \in \mathbb{R}$ and all $t \geqslant 1$,*

$$\frac{\mathbb{P}_x(X^{(\sigma)}_t > u)}{\hat{h}(u)} \leqslant 1 + \xi \quad \textit{with} \quad \hat{h}(u) = \int_u^{+\infty} \frac{\nu((y, +\infty))}{\hat{f}(y)} \mathrm{d}y.$$

Step 9. We proceed with the proof of the lower bound. For all $\varepsilon < 1$, $\delta < 1$ and $\eta < 1$, we get, by the strong Markov property and (2.19)

$$\begin{aligned}\mathbb{P}_x(X_t^{(\sigma)} > u) &\geqslant \mathbb{P}_x\Big(X_t^{(\sigma)} > u,\, \tau_1 \geqslant t - g(u(1+\delta)),\, X_{\tau_1-}^{(\sigma)} \geqslant -\eta u\Big)\\ &\geqslant \int_0^{g(u(1+\delta))} (c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u) e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s}\, \mathbb{P}_x(X_{(t-s)-}^{(\sigma)} \geqslant -\eta u)\\ &\qquad\qquad \times \int_{c\varepsilon u}^{+\infty} \mathbb{P}_{y-\eta u}(X_s^{(\varepsilon u)} > u) \frac{\nu(\mathrm{d}y)}{(c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u)} \mathrm{d}s.\end{aligned}$$

Let us observe that $X^{(\sigma)}$ has, under $\mathbb{P}_x$, the same distribution as $-X^{(\sigma)}$ under the distribution $\mathbb{P}_{-x}$, but with a drift $\hat{f}(x) = -f(-x)$ and an asymmetric driving noise where the coefficients a_+, a_- in the expressions of its Lévy measure are inverted. By using the hypothesis on f and Remark 2.5, we obtain that for all $u \geqslant u_0$, for all $\sigma \leqslant \sigma_0$, all $x \in \mathbb{R}$ and all $s < g(u(1+\delta))$,

$$\mathbb{P}_x(X_{(t-s)-}^{(\sigma)} \geqslant -\eta u) \geqslant 1 - r(u),$$

where r is a function converging to zero. In the sequel, the function r can change from line to line. Observe that, according to (2.19), in a similar manner as we studied p_1, if

$$y \geqslant \eta u + g^{-1}(g(u(1+\delta)) - s) \tag{2.35}$$

then, under the distribution $\mathbb{P}_{y-\eta u}$, the event $\left\{X_s^{(\varepsilon u)} > u\right\}$ contains, up to an event of probability zero, the event $A^c_{\varepsilon,\delta,1+\delta,u,t}$. Hence, for all s and y satisfying (2.35), we get

$$\mathbb{P}_{y-\eta u}(X_s^{(\varepsilon u)} > u) \geqslant 1 - \mathbb{P}_x(A_{\varepsilon,\delta,1+\delta,u,t}).$$

Therefore, by using Lemma 2.3, for all $\sigma \leqslant \sigma_0$ and $u \geqslant u_0(\varepsilon, \delta)$,

$$\mathbb{P}_{y-\eta u}(X_s^{(\varepsilon u)} > u) \geqslant 1 - r(u),$$

for all s and y satisfying (2.35), as long as ε is small relatively to δ. So, for all $\varepsilon < 1$, $\delta < 1$ and $\eta < 1$ such that ε is small relatively to δ, for all $\sigma \leqslant \sigma_0$ and all $u \geqslant u_0(\varepsilon, \delta)$,

$$\begin{aligned}\mathbb{P}_x(X_t^{(\sigma)} > u) &\geqslant \int_0^{g(u(1+\delta))} e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s} \mathbb{P}_x(X_{(t-s)-}^{(\sigma)} \geqslant -\eta u)\\ &\qquad \times \int_{\eta u + g^{-1}(g(u(1+\delta))-s)}^{+\infty} \mathbb{P}_{y-\eta u}(X_s^{(\varepsilon u)} > u)\nu(\mathrm{d}y)\mathrm{d}s\\ &\geqslant (1-r(u))^2 \int_0^{g(u(1+\delta))} e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s} \rho(\eta u + g^{-1}(g(u(1+\delta)) - s))\mathrm{d}s\\ &\geqslant (1-r(u))^2 e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u) g(u(1+\delta))} \int_{u(1+\delta)}^{+\infty} \frac{\rho(\eta u + y)}{f(y)} \mathrm{d}y\\ &\geqslant (1-r(u))^3 \int_{u(1+\delta)}^{+\infty} \frac{\rho(y(1 + \eta/(1+\delta)))}{f(y)}\mathrm{d}y = (1-r(u))^3 \left(1 + \frac{\eta}{1+\delta}\right)^{-\alpha} h(u(1+\delta)).\end{aligned}$$

We conclude that, for all $\xi > 0$, choosing η, ε and δ small enough, there exist $u_0(\xi)$ and $\sigma_0(\xi)$ such that

$$\frac{\mathbb{P}_x(X_t^{(\sigma)} > u)}{\hat{h}(u)} \geqslant 1 - \xi,$$

for all $u \geqslant u_0(\xi)$, all $\sigma \leqslant \sigma_0(\xi)$, all $x \in \mathbb{R}$ and $t \geqslant 1$. $\square$

Proof of Lemma 2.3. Recall that we denoted $\rho(u) = \nu\big((u, +\infty)\big)$ and

$$\lambda_\sigma = \frac{\sigma^{-\alpha}}{\alpha}(a_- + a_+ c^{-\alpha}).$$

Set $q := \frac{a_-}{a_- + a_+ c^{-\alpha}}$. For all ε, u and σ, 0 is a quantile of order q for the random variable $W_1^{(\sigma)} \mathbb{1}_{\{W_1^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}}$ since, by using (2.11),

$$\begin{aligned}\mathbb{P}(W_1^{(\sigma)} \mathbb{1}_{\{W_1^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} < 0) = \mathbb{P}(W_1^{(\sigma)} \in [-\varepsilon u, -\sigma]) &= \frac{1}{\lambda_\sigma \alpha}(a_- \sigma^{-\alpha} - a_-(\varepsilon u)^{-\alpha}) \\ &= \frac{q}{\sigma^{-\alpha}}(\sigma^{-\alpha} - (\varepsilon u)^{-\alpha}) \leqslant q,\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(W_1^{(\sigma)} \mathbb{1}_{\{W_1^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} \leqslant 0) &= \mathbb{P}(W_1^{(\sigma)} \leqslant -\sigma) + \mathbb{P}(W_1^{(\sigma)} > c\varepsilon u) \\ &= \frac{1}{\lambda_\sigma \alpha}(a_- \sigma^{-\alpha} + a_+ c^{-\alpha}(\varepsilon u)^{-\alpha}) \geqslant \frac{a_- \sigma^{-\alpha}}{\lambda_\sigma \alpha} = q.\end{aligned}$$

Recall that $N_{g(\delta u)}^{(\sigma)}$ has the same distribution as the number of jumps of $\ell^{(\sigma)}$ in $[s - g(\delta u), s]$. By using Theorem 2.1 p. 50 in [9], we get

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s}) \leqslant \frac{1}{q}\mathbb{P}\Big(\sum_{i=1}^{N_{g(\delta u)}^{(\sigma)}} W_i^{(\sigma)} \mathbb{1}_{\{W_i^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} \geqslant \gamma u\Big).$$

Again we use the fact that $N_{g(\delta u)}^{(\sigma)}$ is a Poisson distributed random variable of parameter $\lambda_\sigma g(\delta u)$ and is independent of the $W_i^{(\sigma)}$. By conditioning, we obtain

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s}) \leqslant \frac{1}{q}\exp(-\lambda_\sigma g(\delta u)) \sum_{n \geqslant 1} \frac{\big(\lambda_\sigma g(\delta u)\big)^n}{n!} \mathbb{P}\Big(\sum_{i=1}^{n} W_i^{(\sigma)} \mathbb{1}_{\{W_i^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} \geqslant \gamma u\Big). \quad (2.36)$$

Recall that $W_i^{(\sigma)} \mathbb{1}_{\{W_i^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}}$ are i.i.d. random variables with expectation 0, bounded by $(1 \vee c)\varepsilon u$, we can use Theorem 1 in [10], p. 201. We get

$$\mathbb{P}\Big(\sum_{i=1}^{n} W_i^{(\sigma)} \mathbb{1}_{\{W_i^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} \geqslant \gamma u\Big) \leqslant \exp\left[-\frac{\gamma}{2\varepsilon(1 \vee c)} \operatorname{arcsinh}\Big(\frac{\gamma u^2 \varepsilon (1 \vee c)}{n \mathbb{V}\mathrm{ar}\big(\mathrm{W}_1^{(\sigma)} \mathbb{1}_{\{\mathrm{W}_1^{(\sigma)} \in [-\varepsilon \mathrm{u}, c\varepsilon \mathrm{u}]\}}\big)}\Big)\right].$$

Furthermore, we can estimate

$$\begin{aligned}\mathbb{V}\mathrm{ar}\big(\mathrm{W}_1^{(\sigma)} \mathbb{1}_{\{\mathrm{W}_1^{(\sigma)} \in [-\varepsilon \mathrm{u}, c\varepsilon \mathrm{u}]\}}\big) &= \mathbb{E}\big((\mathrm{W}_1^{(\sigma)})^2 \mathbb{1}_{\{\mathrm{W}_1^{(\sigma)} \in [-\varepsilon \mathrm{u}, c\varepsilon \mathrm{u}]\}}\big) \\ &= \frac{1}{\lambda_\sigma}\left(\int_{-\varepsilon u}^{-\sigma} a_- |z|^{1-\alpha} \mathrm{d}z + \int_{c\sigma}^{c\varepsilon u} a_+ z^{1-\alpha} \mathrm{d}z\right) \leqslant \frac{\alpha(c^{1-\alpha} + c^{2-\alpha})}{\lambda_\sigma (2-\alpha)} \varepsilon^{2-\alpha} u^2 \rho(u).\end{aligned}$$

Setting $\hat{C} := \frac{(1 \vee c)(2-\alpha)}{\alpha(c^{1-\alpha} + c^{2-\alpha})}$, we can write

$$\mathbb{P}\Big(\sum_{i=1}^{n} W_i^{(\sigma)} \mathbb{1}_{\{W_i^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} \geqslant \gamma u\Big) \leqslant \exp\left[-\frac{\gamma}{2\varepsilon(1 \vee c)} \operatorname{arcsinh}\Big(\frac{\hat{C} \varepsilon^{\alpha-1} \gamma \lambda_\sigma}{n \rho(u)}\Big)\right].$$

Since $\operatorname{arcsinh}(x) \sim \log(x)$ when $x \to +\infty$, there exists $a > 0$ such that for all $x \geqslant a$, $\operatorname{arcsinh}(x) \geqslant \frac{1}{2}\log(x)$. Therefore, if $n \leqslant \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\,\rho(u)}$, we get

$$\mathbb{P}_x\Big(\sum_{i=1}^{n} W_i^{(\sigma)} \mathbb{1}_{\{W_i^{(\sigma)}\in[-\varepsilon u, c\varepsilon u]\}} \geqslant \gamma u\Big)$$
$$\leqslant \exp\Big[-\frac{\gamma}{4\varepsilon(1\vee c)}\log\Big(\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{n\rho(u)}\Big)\Big] = \Big(\frac{n\rho(u)}{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}\Big)^{\gamma/(4\varepsilon(1\vee c))}.$$

By injecting this result in (2.36), we obtain

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s})$$
$$\leqslant \frac{1}{q}\Big(\frac{\rho(u)}{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}\Big)^{\gamma/(4\varepsilon(1\vee c))}\mathbb{E}\Big((N_{g(\delta u)}^{(\sigma)})^{\gamma/(4\varepsilon(1\vee c))}\Big) + \frac{1}{q}\mathbb{P}\Big(N_{g(\delta u)}^{(\sigma)} > \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\Big). \quad (2.37)$$

It is no difficult to see that, if ξ is a Poisson distributed random variable, for all $p \geqslant 1$, there exists C_p such that

$$\mathbb{E}(\xi^p) \leqslant C_p\big(\mathbb{E}(\xi) + \mathbb{E}(\xi)^p\big).$$

Since $(1\vee c)\varepsilon \leqslant \gamma/4$, we can apply this result to $N_{g(\delta u)}^{(\sigma)}$ and we deduce

$$\mathbb{E}\Big((N_{g(\delta u)}^{(\sigma)})^{\gamma/(4\varepsilon(1\vee c))}\Big) \leqslant C'_{\varepsilon,\gamma}\Big(\lambda_\sigma g(\delta u) + (\lambda_\sigma g(\delta u))^{\gamma/(4\varepsilon(1\vee c))}\Big).$$

We obtain an estimate for the first term on the right hand side of (2.37): there exists $C(\varepsilon,\gamma)$ such that

$$\frac{1}{q}\Big(\frac{\rho(u)}{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}\Big)^{\gamma/(4\varepsilon(1\vee c))}\mathbb{E}\Big((N_{g(\delta u)}^{(\sigma)})^{\gamma/(4\varepsilon(1\vee c))}\Big) \leqslant C(\varepsilon,\gamma) g(\delta u)(\rho(u))^{\gamma/(4\varepsilon(1\vee c))}. \quad (2.38)$$

To study the second term on the right hand side of (2.37), we set $\vartheta := \log\Big(\frac{\varepsilon^{\alpha-1}\gamma}{g(\delta u)\rho(u)}\Big)$. There exists $u_0(\varepsilon,\gamma,\delta)$ such that for all $u \geqslant u_0(\varepsilon,\gamma,\delta)$, ϑ is strictly positive. We get, for all $u \geqslant u_0(\varepsilon,\gamma,\delta)$,

$$\mathbb{P}\Big(N_{g(\delta u)}^{(\sigma)} > \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\Big) = \mathbb{P}\Big(e^{\vartheta N_{g(\delta u)}^{(\sigma)}} > \exp\Big(\vartheta\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\Big)\Big) \leqslant \exp\Big((e^{\vartheta}-1)\lambda_\sigma g(\delta u) - \vartheta\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\Big),$$

by using Markov's inequality. By using the expression of ϑ and choosing $C(\varepsilon,\gamma)$ and $u_0(\varepsilon,\gamma,\delta)$ large enough, we obtain

$$\mathbb{P}\Big(N_{g(\delta u)}^{(\sigma)} > \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\Big) \leqslant C(\varepsilon,\gamma)\,(g(\delta u)\rho(u))^{C(\varepsilon,\gamma)\lambda_\sigma/\rho(u)}. \quad (2.39)$$

Replacing (2.38) and (2.39) in (2.37), we get (2.23). □

3 Asymptotic stability for SDEs driven by the asymmetric α-stable noise

Recall that ℓ is an α-stable Lévy process with $\alpha \in]0;2[\backslash\{1\}$. Consider the equation for the speed process starting from 0 and the corresponding position process

$$v_t^\varepsilon = \varepsilon\ell_t - \int_0^t \operatorname{sgn}(v_s^\varepsilon)|v_s^\varepsilon|^\beta \mathrm{d}s \quad \text{and} \quad x_t^\varepsilon = \int_0^t v_s^\varepsilon \mathrm{d}s. \quad (3.1)$$

We denote

$$\mathcal{V}_t^\varepsilon := v_{\varepsilon^{-\alpha}t}^\varepsilon \quad \text{and} \quad \mathcal{X}_t^\varepsilon := x_{\varepsilon^{-\alpha}t}^\varepsilon\,, \tag{3.2}$$

satisfying

$$\mathcal{V}_t^\varepsilon = \mathcal{L}_t^\varepsilon - \frac{1}{\varepsilon^\alpha}\int_0^t \mathrm{sgn}(\mathcal{V}_s^\varepsilon)|\mathcal{V}_s^\varepsilon|^\beta \mathrm{d}s \quad \text{and} \quad \mathcal{X}_t^\varepsilon = \frac{1}{\varepsilon^\alpha}\int_0^t \mathcal{V}_s^\varepsilon \mathrm{d}s.$$

To simplify, denote

$$\theta = \theta_{\alpha,\beta} := \frac{\alpha}{\alpha+\beta-1} > 0, \quad \text{if } \alpha+\beta-1>0.$$

We introduce

$$L_t^\varepsilon := \frac{\mathcal{L}_{t\,\varepsilon^{\alpha\theta}}^\varepsilon}{\varepsilon^\theta} = \frac{\ell_{t\,\varepsilon^{-(\beta-1)\theta}}}{\varepsilon^{(\beta-1)\theta/\alpha}} \quad \text{and} \quad V_t^\varepsilon := \frac{\mathcal{V}_{t\,\varepsilon^{\alpha\theta}}^\varepsilon}{\varepsilon^\theta}. \tag{3.3}$$

Since ℓ is self-similar it can be easily verified that L^ε is a α-stable Lévy process with the same Lévy measure as ℓ, so there is no dependence on the parameter ε. Moreover, we have

$$V_t^\varepsilon = L_t^\varepsilon - \int_0^t \mathrm{sgn}(V_s^\varepsilon)|V_s^\varepsilon|^\beta \mathrm{d}s \quad \text{and} \quad \mathcal{X}_t^\varepsilon = \varepsilon^{(2-\beta)\theta}\int_0^{t\varepsilon^{-\alpha\theta}} V_s^\varepsilon \mathrm{d}s. \tag{3.4}$$

Again, there is no dependence on the parameter ε for the speed process. We will skip indices ε for the processes L and V. Our second main result is the following:

Theorem 3.1. *Assume that $\alpha \in (0,2)$ and $\beta + \frac{\alpha}{2} > 2$. Then there exists a positive constant $\kappa_{\alpha,\beta}$ such that the process*

$$\left\{\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)}\left(x_{\varepsilon^{-\alpha}t}^\varepsilon - \varepsilon^{\theta-\alpha}t\int_{\mathbb{R}} x\, m_{\alpha,\beta}(\mathrm{d}x)\right) : t \geqslant 0\right\} \tag{3.5}$$

converges in distribution toward a Brownian motion with diffusion coefficient $\kappa_{\alpha,\beta}$, when $\varepsilon \to 0$. Here $m_{\alpha,\beta}$ is the invariant measure of the speed process V which is exponentially ergodic. The constant $\kappa_{\alpha,\beta}$ has the following integral representation

$$\kappa_{\alpha,\beta} = -2\int_{\mathbb{R}} \big(x - \int_{\mathbb{R}} y\, m_{\alpha,\beta}(\mathrm{d}y)\big) g_{\alpha,\beta}(x)\, m_{\alpha,\beta}(\mathrm{d}x) > 0. \tag{3.6}$$

To prove this result we need some important results concerning the speed process.

3.1 The speed process V

If $\beta > 1$ the drift coefficient in the equation (3.4_1) is a locally Lipschitz function hence there exists a locally path-wise unique strong solution V, defined up to an explosion random time (see, for instance [1], Thm. 6.2.11, p. 376). As in [4] Lemma 4.1, p. 7, it can be proved that the explosion time is infinite almost surely hence the solution V is global.

Provided that $\beta > 1$, the drift coefficient $-f(x) := -\mathrm{sgn}(x)|x|^\beta$ and the jump measure $\nu(\mathrm{d}z) = |z|^{-1-\alpha}\big[a_-\mathbb{1}_{\{z<0\}} + a_+\mathbb{1}_{\{z>0\}}\big]\,\mathrm{d}z$ clearly satisfy the conditions of Proposition 0.1, p. 604 in [8]. Hence V is an exponential ergodic and Harris recurrent process having an unique invariant distribution, denoted by $m_{\alpha,\beta}$.

At this level we use the result obtained in Section 2: thanks to Corollary 2.2 $m_{\alpha,\beta}$ satisfies

$$m_{\alpha,\beta}((x,+\infty)) \underset{x\to+\infty}{\sim} \frac{\theta a_+}{\alpha^2 x^{\alpha+\beta-1}} \quad \text{and} \quad m_{\alpha,\beta}((-\infty,-x)) \underset{x\to+\infty}{\sim} \frac{\theta a_-}{\alpha^2 x^{\alpha+\beta-1}}\,. \tag{3.7}$$

These equivalents will play an essential role during the proof of Theorem 3.1.

By the classical ergodic theorem, for all $f \in \mathrm{L}^1(m_{\alpha,\beta})$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(V_s)\mathrm{d}s = \int_{\mathbb{R}} f(x) m_{\alpha,\beta}(\mathrm{d}x), \text{ a.s.} \tag{3.8}$$

Under the hypothesis of Theorem 3.1, the identity function id belongs to $\mathrm{L}^1(m_{\alpha,\beta})$, so

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha\theta} \int_0^{t\varepsilon^{-\alpha\theta}} \big(V_s - \int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)\big)\mathrm{d}s = 0, \text{ a.s.}$$

Since we are interested on the behaviour as $\varepsilon \to 0$ of

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)}\Big(x^{\varepsilon}_{\varepsilon^{-\alpha}t} - \varepsilon^{\theta-\alpha} t\int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)\Big) = \varepsilon^{-\frac{\alpha\theta}{2}}\Big[\varepsilon^{\alpha\theta}\int_0^{t\varepsilon^{-\alpha\theta}} \big(V_s - \int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)\big)\mathrm{d}s\Big] \tag{3.9}$$

we are looking for the behaviour of a functional of V in large time, hence it is quite natural to perform the study in steady state. In other words, we should assume that V is starting with $m_{\alpha,\beta}$ as an initial distribution. This fact is contained in the following

Lemma 3.2. *Suppose that $\beta + \frac{\alpha}{2} - 2 > 0$ and assume that the process*

$$\Big\{Z_{\varepsilon,0}(t) : t \geqslant 0\Big\} := \Big\{\varepsilon^{\frac{\alpha\theta}{2}}\int_0^{t\varepsilon^{-\alpha\theta}} \big(V_s - \int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)\big)\mathrm{d}s : t \geqslant 0\Big\}$$

converges in distribution to a Brownian motion, as $\varepsilon \to 0$, provided that the initial speed V_0 has the distribution $m_{\alpha,\beta}$. Then the process $Z_{\varepsilon,0}$ converges in distribution to a Brownian motion, as $\varepsilon \to 0$, when $V_0 = 0$.

The proof works in exactly the same manner as for Lemma 4.2, p.9 in [4], by replacing V_s by

$$V_s^{\mathrm{cen}} := V_s - \int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x),\ s \geqslant 0. \tag{3.10}$$

Note that this process is centred under the invariant measure $m_{\alpha,\beta}$.

Proof of Lemma 3.2. For $t, \Delta \geqslant 0$ we set

$$Z_{\varepsilon,\Delta}(t) := \varepsilon^{-\frac{\alpha\theta}{2}}\Big[\varepsilon^{\alpha\theta}\int_{\Delta}^{t\varepsilon^{-\alpha\theta}+\Delta} \big(V_s - \int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)\big)\mathrm{d}s\Big] = \varepsilon^{\frac{\alpha\theta}{2}}\int_{\Delta}^{t\varepsilon^{-\alpha\theta}+\Delta} V_s^{\mathrm{cen}}\mathrm{d}s.$$

As in [4] we prove that $Z_{\varepsilon,\Delta}(\cdot)$ converges in distribution to a Brownian motion, as $\Delta \to \infty$ and $\varepsilon \to 0$, when $V_0 = 0$, and, by choosing $\Delta = \Delta(\varepsilon) = \varepsilon^{-\alpha\theta/4}$ we prove that

$$\Big\{\varepsilon^{\frac{\alpha\theta}{2}}\int_0^{\bullet\varepsilon^{-\alpha\theta}+\Delta(\varepsilon)} V_s^{\mathrm{cen}}\mathrm{d}s : t \geqslant 0\Big\}$$

converges in distribution to a Brownian motion, as $\varepsilon \to 0$, when $V_0 = 0$. We finish by applying a consequence of the continuous mapping theorem for the composition function stated in Lemma p. 151 in [2], to deduce that

$$\varepsilon^{\frac{\alpha\theta}{2}}\int_0^{\bullet\varepsilon^{-\alpha\theta}} V_s^{\mathrm{cen}}\mathrm{d}s + \Delta(\varepsilon)\varepsilon^{\frac{\alpha\theta}{2}}\int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x)$$

converges in distribution to a Brownian motion, as $\varepsilon \to 0$, when $V_0 = 0$. We can conclude since $\lim_{\varepsilon\to 0} \Delta(\varepsilon)\varepsilon^{\alpha\theta/2}\int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x) = 0$. □

Although in the asymmetric case the expression of the infinitesimal generator $\mathcal{A}_{\alpha,\beta}$ of V is slightly different, several of its properties remain true. Indeed, since the process L is asymmetric, it is classical (see for instance [1], Thm. 6.7.4, p. 402) that there exists $b \in \mathbb{R}$ such that

$$(\mathcal{A}_{\alpha,\beta}\, g)(x) = (-\mathrm{sgn}(x)|x|^{\beta} + b)g'(x) + \int_{\mathbb{R}} \Big[g(x+y) - g(x) - yg'(x)\mathbb{1}_{|y|\leqslant 1}\Big]\nu(\mathrm{d}y). \tag{3.11}$$

By a similar reasoning as in [4], Lemma 4.3, p. 10, one can prove that its domain $D_{\mathcal{A}_{\alpha,\beta}}$ contains the space of bounded twice differentiable functions $\mathrm{C}_b^2(\mathbb{R})$. Suppose that $\beta + \frac{\alpha}{2} > 2$ and choose p and γ such that

$$p > 1,\ \ p\gamma > 2,\ \ 2 - \beta < \gamma < \alpha.$$

Then the function

$$h_{p,\gamma}(x) := (1 + |x|^{p\gamma})^{1/p} \tag{3.12}$$

is a Lyapunov function for $\mathcal{A}_{\alpha,\beta}$: there exist a continuous function $f_{p,\alpha,\beta,\gamma}$, a compact set K and a constant d, depending only on p, α, β, γ, such that

$$\forall x \in \mathbb{R},\ \ f_{p,\alpha,\beta,\gamma}(x) \geqslant 1 + |x|, \quad f_{p,\alpha,\beta,\gamma}(x) \underset{|x|\to\infty}{\sim} \gamma|x|^{\gamma+\beta-1}, \tag{3.13}$$

and

$$(\mathcal{A}_{\alpha,\beta}\, h_{p,\gamma})(x) \leqslant -f_{p,\alpha,\beta,\gamma}(x) + d\,\mathbb{1}_K(x). \tag{3.14}$$

To prove this statement it suffices to follow the proof of the second part Lemma 4.5 in [4], p. 18, and use the expression of the operator $\mathcal{A}_{\alpha,\beta}$. Moreover, by using the first part of the same lemma (see p. 11 in [4]) we can obtain the behaviour of the speed process:

Proposition 3.3. *Suppose that $\beta + \frac{\alpha}{2} > 2$ and let p and γ such that*

$$p > 1,\ \ p\gamma > 2,\ \ 2 - \beta < \gamma < \frac{\alpha}{2}. \tag{3.15}$$

Then, as $\varepsilon \to 0$, $\{\varepsilon^{\alpha\theta/2} h_{p,\gamma}(\varepsilon^{-\theta}\mathcal{V}_t^{\varepsilon}) : t \geqslant 0\}$ converges to 0 in probability uniformly on each compact time interval. More precisely, there exists $q > 2$ such that, for any fixed $T > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\Big(\varepsilon^{-\theta}\mathcal{V}_t^{\varepsilon}\Big)\Big)^q\Big] = \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \varepsilon^{\frac{\alpha\theta}{2}} h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big)\Big)^q\Big] = 0. \tag{3.16}$$

The proof of this proposition is exactly the same as in for the symmetric driving noise (see [4], pp. 11-12).

3.2 The position process $\mathcal{X}^{\varepsilon}$

We are ready to sketch the proof of our second main result concerning the behaviour of the position process. The proof of the first part is similar to the proof of Theorem 2.1, pp. 12-15, in [4] while proof of the second part is similar to the proof of Proposition 4.6, pp. 15-16, of the same article. Here we will point out the differences for this asymmetric case. Recall that, thanks to Lemma 3.2, we will assume that V is starting with $m_{\alpha,\beta}$ as an initial distribution.

Proof of Theorem 3.1.
Step 1) As for the symmetric case, thanks to (3.13) we can apply Theorem 3.2, p. 924 in [5]. The Poisson equation is slightly different

$$\mathcal{A}_{\alpha,\beta} g = \mathrm{id} - \int_{\mathbb{R}} x m_{\alpha,\beta}(\mathrm{d}x) \tag{3.17}$$

with $\mathcal{A}_{\alpha,\beta}$ given by (3.11). This equation admits a solution $\mathcal{g}_{\alpha,\beta}$ satisfying $|\mathcal{g}_{\alpha,\beta}| \leqslant c(h_{p,\gamma}+1)$, with c a positive constant. Applying Itô-Levy's formula with $\mathcal{g}_{\alpha,\beta}$, we get

$$\mathcal{g}_{\alpha,\beta}(V_t) - \mathcal{g}_{\alpha,\beta}(V_0) = \int_0^t \big(V_s - \int_{\mathbb{R}} x \mathcal{m}_{\alpha,\beta}(\mathrm{d}x)\big)\mathrm{d}s + M_t = \int_0^t V_s^{\mathrm{cen}} \mathrm{d}s + M_t, \tag{3.18}$$

where

$$M_t := \int_0^t \int_{\mathbb{R}} [\mathcal{g}_{\alpha,\beta}(z+V_s) - \mathcal{g}_{\alpha,\beta}(V_s)] \tilde{N}(\mathrm{d}s, \mathrm{d}z). \tag{3.19}$$

Here $\tilde{N}$ is the compensated Poisson measure which appears in the Lévy-Itô decomposition of L. As in [4], it can be proved that M given by the latter formula is a square integrable true martingale. Note that $h_{p,\gamma}^2$ is continuous, it behaves as $|x|^{2\gamma}$ in the neighbourhood of the infinity and γ was chosen such that $\frac{4}{p} \vee (4-2\beta) < 2\gamma < \alpha$. By using (3.7), we see that

$$\int_{\mathbb{R}} h_{p,\gamma}(x)^2 \mathcal{m}_{\alpha,\beta}(\mathrm{d}x) < \infty. \tag{3.20}$$

We point out that the assumption $\beta + \frac{\alpha}{2} > 2$ is essential in proving the quadratic integrability of the martingale M. The quadratic variation of M is given by

$$\langle M \rangle_t = \int_0^t \int_{\mathbb{R}} [\mathcal{g}_{\alpha,\beta}(y+V_s) - \mathcal{g}_{\alpha,\beta}(V_s)]^2 \nu(\mathrm{d}y)\mathrm{d}s, \tag{3.21}$$

and furthermore, we get

$$\mathbb{E}[\langle M \rangle_t] = \Big(\iint_{\mathbb{R}^2} [\mathcal{g}_{\alpha,\beta}(x+y) - \mathcal{g}_{\alpha,\beta}(x)]^2 \nu(\mathrm{d}y) \mathcal{m}_{\alpha,\beta}(\mathrm{d}x) \Big)\, t =: \kappa_{\alpha,\beta}\, t < \infty. \tag{3.22}$$

Since $\mathcal{g}_{\alpha,\beta}$ satisfies (3.17) it could not be a constant function. Moreover, $\mathcal{m}_{\alpha,\beta}$ has a non-empty support and ν is absolutely continuous with respect to the Lebesgue measure. Hence the constant $\kappa_{\alpha,\beta}$ is positive.

Step 2) Performing a simple time change in (3.18), we see that the process in (3.5) can be written

$$\varepsilon^{\theta(\beta+\frac{\alpha}{2}-2)} \Big(X_t^{\varepsilon} - \varepsilon^{\theta-\alpha} t \int_{\mathbb{R}} x \mathcal{m}_{\alpha,\beta}(\mathrm{d}x) \Big) = \varepsilon^{\frac{\alpha\theta}{2}} \Big[\mathcal{g}_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - \mathcal{g}_{\alpha,\beta}(V_0) \Big] - \varepsilon^{\frac{\alpha\theta}{2}} M_{t\,\varepsilon^{-\alpha\theta}}. \tag{3.23}$$

On one hand, we can show that the martingale term on the right hand side of the latter equality converges to a Brownian motion by using Whitt's theorem (see Theorem 2.1 (ii) in [13], pp. 270-271). As in [4] the hypotheses of this result can be verified. We illustrate only the convergence of the quadratic variation of $\varepsilon^{\frac{\alpha\theta}{2}} M_{\bullet\, \varepsilon^{-\alpha\theta}}$. The essential estimates (3.7) allow to verify that the function

$$x \mapsto \int_{\mathbb{R}} [\mathcal{g}_{\alpha,\beta}(x+y) - \mathcal{g}_{\alpha,\beta}(x)]^2 \nu(\mathrm{d}y)$$

belongs to $\mathrm{L}^1(\mathcal{m}_{\alpha,\beta})$. By using (3.21) and the ergodic theorem (3.8), we deduce that

$$\lim_{\varepsilon\to 0} \langle \varepsilon^{\frac{\alpha\theta}{2}} M_{\bullet\, \varepsilon^{-\alpha\theta}} \rangle_t = \lim_{\varepsilon\to 0} \varepsilon^{\frac{\alpha\theta}{2}} \int_0^{t\varepsilon^{-\alpha\theta}} \int_{\mathbb{R}} [\mathcal{g}_{\alpha,\beta}(y+V_s) - \mathcal{g}_{\alpha,\beta}(V_s)]^2 \nu(\mathrm{d}y)\mathrm{d}s = \kappa_{\alpha,\beta}\, t.$$

Finally, the process $\big\{\varepsilon^{(\alpha\theta)/2} M_{t\,\varepsilon^{-\alpha\theta}} : t \geqslant 0\big\}$ converges in distribution toward $\kappa_{\alpha,\beta}^{1/2} B$, where B is a standard Brownian motion. On the other hand, by using that $|\mathcal{g}_{\alpha,\beta}| \leqslant c(h_{p,\gamma}+1)$, we get

$$\Big|\mathcal{g}_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big) - \mathcal{g}_{\alpha,\beta}(V_0)\Big|^2 \leqslant 4c^2 \Big(\Big|h_{p,\gamma}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big)\Big|^2 + \Big|h_{p,\gamma}(V_0)\Big|^2 + 2 \Big).$$

By using Proposition 3.3,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\varepsilon^{\alpha\theta}\sup_{t\in[0,T]}\Big|\mathcal{g}_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big)-\mathcal{g}_{\alpha,\beta}(V_0)\Big|^2\Big]=0,$$

hence $\big\{\varepsilon^{(\alpha\theta)/2}\big[\mathcal{g}_{\alpha,\beta}\big(V_{t\,\varepsilon^{-\alpha\theta}}\big)-\mathcal{g}_{\alpha,\beta}(V_0)\big]:t\geqslant 0\big\}$ converges in probability toward 0, uniformly on compact sets. The proof of the first part of Theorem 3.1 could be finished exactly in the same manner as in [4], by using the joint convergence theorem (Theorem 11.4.5, p. 379 in [14]) and the continuous-mapping theorem (Theorem 3.4.3, p. 86 in [14]).

Step 3) Recall that, by (3.22), $\kappa_{\alpha,\beta}=\frac{1}{t}\mathbb{E}\big[M_t^2\big]$, for all $t>0$. By taking $t=\varepsilon^{\alpha\theta}$, using (3.18) and performing the similar computations as in [4], p. 15, we get

$$\begin{aligned}\kappa_{\alpha,\beta}=\varepsilon^{-\alpha\theta}\mathbb{E}\Big[\Big(\mathcal{g}_{\alpha,\beta}\big(V_{\varepsilon^{\alpha\theta}}\big)-\mathcal{g}_{\alpha,\beta}(V_0)-\int_0^{\varepsilon^{\alpha\theta}}V_s^{\mathrm{cen}}\mathrm{d}s\Big)^2\Big]=\varepsilon^{-\alpha\theta}\Big\{\mathbb{E}\Big[\Big(\mathcal{g}_{\alpha,\beta}\big(V_{\varepsilon^{\alpha\theta}}\big)-\mathcal{g}_{\alpha,\beta}(V_0)\Big)^2\Big]\\+\mathbb{E}\Big[\Big(\int_0^{\varepsilon^{\alpha\theta}}V_s^{\mathrm{cen}}\mathrm{d}s\Big)^2\Big]-2\mathbb{E}\Big[\Big(\mathcal{g}_{\alpha,\beta}\big(V_{\varepsilon^{\alpha\theta}}\big)-\mathcal{g}_{\alpha,\beta}(V_0)\Big)\int_0^{\varepsilon^{\alpha\theta}}V_s^{\mathrm{cen}}\mathrm{d}s\Big]\Big\}.\end{aligned}\tag{3.24}$$

The first term on the right hand side of (3.24) can be written, by successive transformations:

$$\begin{aligned}2\int_{\mathbb{R}}\mathcal{g}_{\alpha,\beta}(x)^2\mathcal{m}_{\alpha,\beta}(\mathrm{d}x)-2\mathbb{E}\Big[\mathcal{g}_{\alpha,\beta}(V_0)\mathcal{g}_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}})\Big]=2\int\mathcal{g}_{\alpha,\beta}(x)^2\mathcal{m}_{\alpha,\beta}(\mathrm{d}x)\\-2\mathbb{E}\Big[\mathcal{g}_{\alpha,\beta}(V_0)\big(\mathcal{T}_{\varepsilon^{\alpha\theta}}\mathcal{g}_{\alpha,\beta}\big)(V_0)\Big]=-2\mathbb{E}\Big[\mathcal{g}_{\alpha,\beta}(V_0)\int_0^{\varepsilon^{\alpha\theta}}\mathcal{T}_s\big(\mathrm{id}-\int_{\mathbb{R}}x\mathcal{m}_{\alpha,\beta}(\mathrm{d}x)\big)(V_0)\mathrm{d}s\Big]\\=-2\,\varepsilon^{\alpha\theta}\int(x-\int x\mathcal{m}_{\alpha,\beta}(\mathrm{d}x))\mathcal{g}_{\alpha,\beta}(x)\mathcal{m}_{\alpha,\beta}(\mathrm{d}x)-2\int\mathcal{g}_{\alpha,\beta}(x)\,\mathcal{m}_{\alpha,\beta}(\mathrm{d}x)\int_0^{\varepsilon^{\alpha\theta}}\big((\mathcal{T}_s\mathrm{id})-\mathrm{id}\big)(x)\mathrm{d}s.\end{aligned}$$

Here $(\mathcal{T}_t)_{t\geqslant 0}$ is the semi-group associated to the operator $\mathcal{A}_{\alpha,\beta}$ and we used the Poisson equation (3.17) to see that

$$\mathcal{T}_t\mathcal{g}_{\alpha,\beta}(V_0)-\mathcal{g}_{\alpha,\beta}(V_0)=\int_0^t\mathcal{T}_s\big(\mathrm{id}-\int_{\mathbb{R}}x\mathcal{m}_{\alpha,\beta}(\mathrm{d}x)\big)(V_0)\mathrm{d}s.$$

By using the Hölder inequality, we prove that, as $\varepsilon\to 0$,

$$\mathbb{E}\left[\Big(\mathcal{g}_{\alpha,\beta}\big(V_{\varepsilon^{\alpha\theta}}\big)-\mathcal{g}_{\alpha,\beta}(V_0)\Big)^2\right]\sim-2\,\varepsilon^{\alpha\theta}\int_{\mathbb{R}}(x-\int_{\mathbb{R}}y\mathcal{m}_{\alpha,\beta}(\mathrm{d}y))\mathcal{g}_{\alpha,\beta}(x)\mathcal{m}_{\alpha,\beta}(\mathrm{d}x).\tag{3.25}$$

Similarly, by successive transformations using the Markov property, conditioning and Fubini's theorem, the second term on the right hand side of (3.24) can be written

$$\begin{aligned}\int_0^{\varepsilon^{\alpha\theta}}\mathrm{d}s\int_0^s\mathbb{E}\Big(V_{s-u}^{\mathrm{cen}}V_0^{\mathrm{cen}}\Big)\mathrm{d}u=\int_0^{\varepsilon^{\alpha\theta}}\mathrm{d}s\int_0^s\mathbb{E}\Big(V_0^{\mathrm{cen}}\,\mathcal{T}_{s-u}\big(\mathrm{id}-\int_{\mathbb{R}}x\mathcal{m}_{\alpha,\beta}(\mathrm{d}x)\big)(V_0)\Big)\mathrm{d}u\\=\int_0^{\varepsilon^{\alpha\theta}}\mathbb{E}\left[V_0^{\mathrm{cen}}\Big(\big(\mathcal{T}_{\varepsilon^{\alpha\theta}-u}\mathcal{g}_{\alpha,\beta}\big)(V_0)-\mathcal{g}_{\alpha,\beta}(V_0)\Big)\right]\mathrm{d}u=\int_0^{\varepsilon^{\alpha\theta}}\mathrm{d}u\int_{\mathbb{R}}(x-\int_{\mathbb{R}}y\mathcal{m}_{\alpha,\beta}(\mathrm{d}y))\mathcal{m}_{\alpha,\beta}(\mathrm{d}x)\\\times\Big(\big(\mathcal{T}_{\varepsilon^{\alpha\theta}-u}\mathcal{g}_{\alpha,\beta}\big)-\mathcal{g}_{\alpha,\beta}\Big)(x).\end{aligned}$$

Once again by the Hölder inequality, we prove that, as $\varepsilon\to 0$,

$$\mathbb{E}\left[\Big(\int_0^{\varepsilon^{\alpha\theta}}V_s^{\mathrm{cen}}\mathrm{d}s\Big)^2\right]=\mathrm{o}(\varepsilon^{\alpha\theta}).\tag{3.26}$$

Finally, the third term in (3.24) is analysed by using the Cauchy-Schwarz inequality and the behaviour of the other terms. We get that, as $\varepsilon \to 0$,

$$-2\,\mathbb{E}\Big[\Big(g_{\alpha,\beta}(V_{\varepsilon^{\alpha\theta}}) - g_{\alpha,\beta}(V_0)\Big)\int_0^{\varepsilon^{\alpha\theta}} V_s^{\mathrm{cen}}\mathrm{d}s\Big] = \mathrm{o}(\varepsilon^{\alpha\theta}). \tag{3.27}$$

Putting together (3.24)-(3.26), we obtain that, as $\varepsilon \to 0$,

$$\kappa_{\alpha,\beta} = -2\int_{\mathbb{R}}(x - \int_{\mathbb{R}} y\,m_{\alpha,\beta}(\mathrm{d}y))g_{\alpha,\beta}(x)m_{\alpha,\beta}(\mathrm{d}x) + \mathrm{o}(1).$$

and the result is proved. The interested reader may consult [3] for all detailed computations. □

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