ALEA, Lat. Am. J. Probab. Math. Stat. 21, 815–835 (2024)
DOI: 10.30757/ALEA.v21-33

Kinetic time-inhomogeneous Lévy-driven model

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Abstract. We study a one-dimensional kinetic stochastic model driven by a Lévy process with a non-linear time-inhomogeneous drift. More precisely, the process (V, X) is considered, where X is the position of the particle and its velocity V is the solution to a stochastic differential equation with a drift of the form $t^{-\beta}F(v)$. The driving process is a stable Lévy process of index α, the function F satisfies a homogeneity condition and β is a real number. The behaviour in large time of the process (V, X) is analysed. A result concerning the moments estimates of the velocity process is one of main tools.

1. Introduction

In this paper, we consider a one-dimensional stochastic kinetic model driven by a Lévy process.

$$\mathrm{d}V_t = \mathrm{d}L_t - F(V_t)t^{-\beta}\,\mathrm{d}t \quad \text{and} \quad X_t = X_0 + \int_0^t V_s\,\mathrm{d}s. \tag{1.1}$$

The process $(V_t, X_t)_{t>0}$ may be thought of as the velocity and position processes of a particle subject to a friction force $F(v)t^{-\beta}$ and interacting with its environment. Our purpose is to study the long-time behaviour of solutions to (1.1) where L is an α-stable (non-symmetric) Lévy process. More precisely, we look for the convergence in distribution of the process $(V_{t/\varepsilon}, X_{t/\varepsilon})_{t>0}$, as $\varepsilon \to 0$, with an appropriate rate.

It is a simple observation, when $F = 0$, to see that the rescaled process $(\varepsilon^{\frac{1}{\alpha}}V_{t/\varepsilon}, \varepsilon^{1+\frac{1}{\alpha}}X_{t/\varepsilon})_{t>0}$ converges in distribution towards the Kolmogorov process $(L_t, \int_0^t L_s\,\mathrm{d}s)_{t>0}$. The goal of the present paper is to extend the results obtained for the Brownian motin driven noise in Gradinaru and Luirard (2023), to an α-stable driven motion, with $\alpha \in (0, 2)$, when F is homogeneous of some degree.

The study of stochastic differential equations (SDEs) driven by a Lévy process is a topic of great interest (see Bass (2003) for a survey). The α-stable perturbation is a generalization of

Received by the editors January 30th, 2023; accepted March 20th, 2024.
2010 *Mathematics Subject Classification.* 60H10, 60F17, 60G52, 60G18, 60J65.
Key words and phrases. kinetic stochastic equation; time-inhomogeneous stochastic differential equation; Lévy process; explosion time; scaling transformation; asymptotic distribution; ergodicity; tightness.

the Gaussian case, and it is also motivated by some Langevin-type models in stochastic climate dynamics (see Ditlevsen (1999)). So far, most of the papers present results about existence and uniqueness of solution, see for instance Applebaum and Siakalli (2009), Dong (2018), Kurenok (2007), Pilipenko (2012), Chen et al. (2018) and Chen et al. (2021). The coefficients of the studied SDE are often supposed to be time-homogeneous (see for instance Applebaum and Siakalli (2009) and Dong (2018)). Accordingly, the case of time-dependent coefficients is scarcely studied (see Chen et al. (2021), Kurenok (2007) and Zhang (2013)). In this situation, the usual tools associated with time-homogeneous equation may no longer be employed.

Furthermore, few papers (see Applebaum and Siakalli (2009), Priola et al. (2012), Reker (2023)) present results about the asymptotic behaviour of the solution of such SDEs. For instance, in Applebaum and Siakalli (2009) the authors give conditions for asymptotic stability of the solutions to a SDE driven by a Brownian motion and a compensated Poisson process, with coefficients that are supposed to satisfy usual global Lipschitz and growth assumptions. In Priola et al. (2012), the authors establish the exponential ergodicity of the solutions to a SDE driven by an α-stable process, where the drift coefficient is supposed to be the sum of two components, one linear and the other bounded. In a number of articles, the small noise influence of the solutions is . To our knowledge, the only work before ours considering the long-time behaviour is Fournier and Tardif (2021). In all cited papers, coefficients are time-homogeneous.

Let us explain heuristically what the intuition of our analysis is. In long-time regime, we observe three schemes, depending on the balance between the space and time coefficients of the drift function with respect to α, the parameter of stability of the driving process. When the drag force is sufficiently "small at infinity", the convergence towards the Kolmogorov process $(\mathcal{S}, \int_0^{\cdot} \mathcal{S})$ still holds. When the two terms in the stochastic equation of the velocity process offset, we still get a kinetic process of the form $(\mathcal{V}, \int_0^{\cdot} \mathcal{V})$, as limiting process. Though the process $\mathcal{V}$ no longer has the same distribution as the driving Lévy process. Alternatively, when the drift swings with the random noise, the limiting process is no longer kinetic.

The main ingredients of the proofs are moments estimates and the self-similarity of the driving process. By their scaling property, Lévy stable processes are natural extensions of the Brownian motion, but the jump component of the Lévy noise brings some difficulties. Indeed, by contrast with a Brownian motion, an α-stable Lévy process can only have moments of order $\kappa \in [0, \alpha)$. Thus, moments estimation of the velocity process stands as a significant part of our study (see Section 4). Moment estimates of Lévy and Lévy-type processes were studied in Luschgy and Pagès (2008), Kühn (2017) and Deng and Schilling (2015), nevertheless the methods used by those authors can not be easily adapted to the solutions to a SDE. The key idea will be to cut the jumps of the driving process in a non-homogeneous manner.

The proof for the critical (see Theorem 2.4) significantly relies on a change in both space and time, taking advantage of the scaling property of the driving process, to be close to a stationary time-homogeneous SDE, as performed in Appleby and Wu (2009) and Gradinaru and Offret (2013). The same strategy applies for the sub-critical case and gives only a partial result concerning the behavior of the velocity process. As in the Gaussian situation the study of the position process seems more difficult and stays an open problem.

Here is the structure of the paper: in Section 2, we introduce some notations and we state our main results. We study the existence of the solution to the system (1.1) in Section 3. In Section 4, we give estimates of the moments, which also ensure the non-explosion of the velocity process. The proofs of our main results are presented in Section 5. Some useful auxiliary results are presented in the Appendix.

2. Notations and statements of main results

Throughout the paper, we deal with $L = (L_t)_{t\geq 0}$ as an α-stable Lévy process with $\alpha \in (0,2)$. We call ν its Lévy measure, given by

$$\nu(\mathrm{d}z) = \frac{a_+\mathbf{1}_{\{z>0\}} + a_-\mathbf{1}_{\{z<0\}}\,\mathrm{d}z}{|z|^{1+\alpha}}, \quad \text{with } a_+, a_- \geq 0 \text{ and } a_+ + a_- > 0. \tag{2.1}$$

As a Lévy measure, ν satisfies $\int_{\mathbb{R}^*}(1\wedge z^2)\nu(\mathrm{d}z) < +\infty$. By Lévy-Itô's decomposition, L is a pure-jump Lévy process and there exists a Poisson point measure N and the associated compensated Poisson measure $\widetilde{N}$ such that, for all $t \geq 0$,

$$L_t = \begin{cases} \displaystyle\int_0^t\int_{\mathbb{R}^*} zN(\mathrm{d}s,\mathrm{d}z) & \text{if } \alpha \in (0,1),\\ \displaystyle\int_0^t\int_{\{0<|z|<1\}} z\widetilde{N}(\mathrm{d}s,\mathrm{d}z) + \int_0^t\int_{\{|z|\geq 1\}} zN(\mathrm{d}s,\mathrm{d}z) & \text{if } \alpha = 1,\\ \displaystyle\int_0^t\int_{\mathbb{R}^*} z\widetilde{N}(\mathrm{d}s,\mathrm{d}z) & \text{if } \alpha \in (1,2). \end{cases} \tag{2.2}$$

The space of continuous functions $\mathcal{C}((0,+\infty),\mathbb{R})$ is endowed with the uniform topology given by the metric d_u defined for $(f,g) \in \mathcal{C}((0,+\infty),\mathbb{R})^2$ by

$$\mathrm{d}_u(f,g) := \sum_{n=1}^{+\infty}\frac{1}{2^n}\min\Big(1, \sup_{[\frac{1}{n},n]}|f-g|\Big). \tag{2.3}$$

Set $\Lambda := \{\lambda : \mathbb{R}^+ \to \mathbb{R}^+, \text{continuous and increasing function s.t. } \lambda(0) = 0,\ \lim_{t\to+\infty}\lambda(t) = +\infty\}$ and set

$$k_n(t) := \begin{cases} 1 & \text{if } \frac{1}{n} \leq t \leq n,\\ n+1-t & \text{if } n < t < n+1,\\ 0 & \text{if } n+1 \leq t. \end{cases}$$

The space of right-continuous with left limits (càdlàg) functions $\mathcal{D}((0,+\infty),\mathbb{R})$ is endowed with the Skorokhod topology given by the metric d_s defined for $(f,g) \in \mathcal{D}((0,+\infty),\mathbb{R})^2$ by

$$\mathrm{d}_s(f,g) := \sum_{n=1}^{+\infty}\frac{1}{2^n}\min\left(1, \inf_{\lambda\in\Lambda}\left\{\sup_{s\neq t}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right| \bigvee \sup_{t\geq\frac{1}{n}}\left|k_n(t)\left(f\circ\lambda(t)-g(t)\right)\right|\right\}\right). \tag{2.4}$$

For simplicity, we will write $\mathcal{C}$ and $\mathcal{D}$ for $\mathcal{C}((0,+\infty),\mathbb{R})$ and $\mathcal{D}((0,+\infty),\mathbb{R})$, respectively.

For a family $((Z_t^{(\varepsilon)})_{t>0})_{\varepsilon>0}$ of càdlàg processes, we write

$$(Z_t^{(\varepsilon)})_{t>0} \underset{\varepsilon\to 0}{\Longrightarrow} (Z_t)_{t>0},$$

provided $(Z_t^{(\varepsilon)})_{t>0}$ converges in distribution to $(Z_t)_{t>0}$ in $\mathcal{D}$, as $\varepsilon \to 0$, and we write

$$(Z_t^{(\varepsilon)})_{t>0} \overset{\text{f.d.d.}}{\underset{\varepsilon\to 0}{\Longrightarrow}} (Z_t)_{t>0},$$

provided, for any finite subset $S \subset (0,+\infty)$, the vector $(Z_t^{(\varepsilon)})_{t\in S}$ converges in distribution to $(Z_t)_{t\in S}$ in $\mathbb{R}^S$, as $\varepsilon \to 0$.

Let β a real number and F a continuous function satisfying

$$\text{for some } \gamma \in \mathbb{R},\ \forall v \in \mathbb{R},\ \lambda > 0,\ F(\lambda v) = \lambda^\gamma F(v). \tag{H_γ}$$

We introduce another assumption on F, which will sometimes be in force in the sequel.

$$\begin{gathered}\text{When (i) } \alpha \in (0,1] \text{ or (ii) } \alpha \in (1,2) \text{ and } \gamma \geq 1,\\ \text{we suppose furthermore that for all } v \in \mathbb{R},\ vF(v) \geq 0.\end{gathered} \tag{H_{sgn}}$$

Our main interest is taking into the following one-dimensional stochastic kinetic model given, for $t \geq t_0 > 0$, by

$$\mathrm{d}V_t = \mathrm{d}L_t - t^{-\beta} F(V_t)\,\mathrm{d}t, \text{ with } V_{t_0} = v_0 > 0, \quad \text{and} \quad \mathrm{d}X_t = V_t\,\mathrm{d}t, \text{ with } X_{t_0} = x_0 \in \mathbb{R}. \tag{SKE}$$

In the following, sgn is the sign function with the convention that $\mathrm{sgn}(0) = 0$. We denote by C some positive constants, which may change from line to line. We use the subscripts to indicate the parameters on which the constant depends. For instance, $C_{t_0,\alpha}$ denotes a constant depending on the parameters t_0 and α.

Remark 2.1. If a function π satisfies (H_γ), then for all $x \in \mathbb{R}$, $\pi(x) = \pi(\mathrm{sgn}(x))\,|x|^\gamma$. As an example of a function satisfying (H_γ) one can keep in mind $F : v \mapsto \mathrm{sgn}(v)\,|v|^\gamma$ (see also Gradinaru and Offret (2013)).

Let us state our main results which correspond to the three regimes: super-critical, critical and sub-critical, depending respectively on the position of the exponent β with respect to $1 + \frac{\gamma-1}{\alpha}$.

When $\beta > 1 + \frac{\gamma-1}{\alpha}$ we have a super-critical regime, the friction force is asymptotically negligible and the couple velocity-position process behaves like Kolmogorov diffusion:

Theorem 2.2. *Consider $\gamma \in (1 - \frac{\alpha}{2}, \alpha)$. Assume that ($H_\gamma$) and ($H_{\mathrm{sgn}}$) are satisfied, and suppose that $\beta > 1 + \frac{\gamma-1}{\alpha}$. Let $(V_t, X_t)_{t\geq t_0}$ be the unique global solution to (SKE). Then, in the space $\mathcal{D}$,*

$$\left(\varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon}, \varepsilon^{1+\frac{1}{\alpha}} X_{t/\varepsilon}\right)_{t\geq \varepsilon t_0} \underset{\varepsilon\to 0}{\Longrightarrow} \left(L_t, \int_0^t L_s\,\mathrm{d}s\right)_{t>0}.$$

Remark 2.3. Theorem 2.2 is also true when the following hypothesis holds instead of (H_γ).

$$\begin{gathered} F \text{ is such that (SKE) has a unique solution up to explosion and} \\ |F| \leq G \text{ where } G \text{ is a positive function satisfying } (H_\gamma). \end{gathered} \tag{H'_γ}$$

For instance, the function $F : v \mapsto \frac{v}{(1+v^2)}$ (see also Fournier and Tardif (2021)) satisfies (H'_γ), with $\gamma = 0$.

In the critical regime, i.e. when $\beta = 1 + \frac{\gamma-1}{\alpha}$, the friction force compensates somehow the random force. The limit law is the kinetic law of a "mixture" between the limit laws of two regimes, and it depends only on the parameters of the friction force.

Theorem 2.4. *Consider $\alpha > 1$ and $\gamma \in [1, \alpha)$. Assume that (H_γ) and (H_{sgn}) are satisfied, and suppose that $\beta = 1 + \frac{\gamma-1}{\alpha}$. Let $(V_t, X_t)_{t\geq t_0}$ be the unique global solution to (SKE). Then the time-homogeneous SDE, driven by an α-stable process L,*

$$\mathrm{d}\mathfrak{H}_s = \mathrm{d}L_s - \frac{\mathfrak{H}_s}{\alpha}\,\mathrm{d}s - F\big(\mathfrak{H}_s\big)\,\mathrm{d}s \tag{2.5}$$

admits a unique strong solution which is exponentially ergodic. Denote by H the eternal ergodic process, that is the solution to (2.5) having the invariant measure as the distribution of $H_{-\infty}$ and introduce the process $\mathcal{V} := \left(t^{\frac{1}{\alpha}} H_{\log(t)}\right)_{t\geq 0}$.

Then, under (H_γ), the following convergence holds in the space $\mathcal{D}$

$$(\varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon}, \varepsilon^{1+\frac{1}{\alpha}} X_{t/\varepsilon})_{t\geq \varepsilon t_0} \underset{\varepsilon\to 0}{\Longrightarrow} \left(\mathcal{V}_t, \int_0^t \mathcal{V}_s\,\mathrm{d}s\right)_{t>0}.$$

Remark 2.5. If Λ_F denotes the invariant measure of the eternal process, it can be noticed that the d-dimensional distribution of the process $\mathcal{V}$ is the pushforward measure of the measure $\Lambda_{F,\log(t_1),..,\log(t_d)}$ by the linear map $T(u_1, \cdots, u_d) := (t_1^{1/\alpha} u_1, \cdots, t_d^{1/\alpha} u_d)$.

For the sub-critical regime, when $\beta < 1 + \frac{\gamma-1}{\alpha}$, we obtain only the convergence of the velocity process, the behaviour of the position process being an open problem. The velocity process diverges towards the infinity under the random force, but it is very quickly recalled towards 0 by the drift. Note that the finite dimensional marginals of the velocity process depend on the parameters of the friction force. The study of the homogeneous case seems to indicate that there is no convergence in distribution of the normalised velocity but only for its marginals of finite rank. Even for the simple linear case with Brownian driving noise, it is not possible to prove tightness. We therefore lose the kinetic character of the limit process, and at the same time the regularity of its position component.

Theorem 2.6. *Consider $\alpha > 1$ and $\gamma \geq 1$. Assume that (H_γ) and (H_{sgn}) are satisfied, and suppose that $\beta < 1 + \frac{\gamma-1}{\alpha}$. Let $(V_t, X_t)_{t\geq t_0}$ be the unique global solution to (SKE) and set $q := \frac{\beta}{\alpha+\gamma-1} < \frac{1}{\alpha}$. Then the time-homogeneous SDE, driven by an α-stable process L,*

$$\mathrm{d}\mathfrak{H}_s = \mathrm{d}L_s - F(\mathfrak{H}_s)\,\mathrm{d}s. \tag{2.6}$$

admits a unique strong solution which is ergodic. Denote again by H the eternal ergodic process solution of (2.6) starting at its invariant measure and introduce the process $\mathscr{V} = (t^q H_t)_{t\geq 0}$ Then,

$$\left(\varepsilon^q V_{t/\varepsilon}\right)_{t\geq \varepsilon t_0} \underset{\varepsilon\to 0}{\overset{f.d.d.}{\Longrightarrow}} (\mathscr{V}_t)_{t\geq 0}\,.$$

Remark 2.7. If Π_F denotes the invariant measure of the eternal process then the d-dimensional distribution of $\mathscr{V}$ is the pushforward of the measure $\Pi_F^{\otimes d}$ by the linear map $T(u_1,\cdots,u_d) := ({t_1}^q u_1, \cdots, {t_d}^q u_d)$.

Remark 2.8. Assuming that the driving process is the Brownian motion and considering only the linear situation ($\gamma = 1$), in Gradinaru and Luirard (2023) it was proved that, provided that $\beta \in (-1/2, 1)$, the process $\left(\varepsilon^{\beta+1/2} X_{t/\varepsilon}\right)_{t\geq\varepsilon t_0}$ converges in f.d.d. as ε tends to 0 towards a centered Gaussian process with an explicit covariance function. We conjecture that for the α-stable non-Gaussian Lévy driving process, and for the linear case ($\gamma = 1$) again, provided that $\beta \in (-\frac{1}{\alpha}, 1)$, the family of processes $\left(\varepsilon^{\beta+1/\alpha} X_{t/\varepsilon}\right)_{t\geq\varepsilon t_0}$ should converge in f.d.d. to the process $\mathscr{X} := \int_0^\cdot s^\beta\,\mathrm{d}L_s$. Nevertheless, the method employed in Gradinaru and Luirard (2023) does not succeed and the question is open.

Remark 2.9. As we will see in Section 3, the assumption $\gamma > 1 - \frac{\alpha}{2}$ is needed in order to obtain the existence and uniqueness up to explosion of the solution under the hypothesis (H_γ) and without the hypothesis (H_{sgn}).

Remark 2.10. Let us point out that, during the proof of Theorems 2.2 and 2.4, we employ some moments estimates for the the velocity process V. Assuming that (H_{sgn}) is satisfied, we suppose also that the hypothesis on the sign of F holds for $(\alpha,\gamma,\kappa) \in (1,2)\times[0,1]\times(1,\alpha)$. Then, for any $\alpha \in (0,2)$, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\kappa \in [0,\alpha)$, we will show, in Section 4, that there exists a constant $C_{\gamma,\kappa,\beta,t_0}$ such that

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0} t^{\frac{\kappa}{\alpha}}.$$

Note that the above bounds are the best possible, taking $F = 0$.

3. Existence up to explosion

In this section, we study the existence of the solution to (SKE) up to explosion time.

Remark 3.1. Assume that (H_γ) holds. If $0 < \gamma < 1$, then the function F is γ-Hölder and if $\gamma \geq 1$, it is locally Lipschitz.

Proposition 3.2. *Assume that (H_γ) is satisfied. There exists a pathwise unique strong solution to (SKE), defined up to the explosion time, provided that*

(i) $1-\frac{\alpha}{2}<\gamma<1$ *and* $\beta\geq 0$ *when* $\alpha\in(0,2)$.
(ii) $\gamma\geq 1$ *when* $\alpha>1$.

Proof: If $\gamma\in(0,1)$, the drift coefficient is γ-Hölder (see Remark 3.1) and locally bounded, thereby the conclusion of the first point follows from Remark 1.3 in Chen et al. (2021).

Assume now that $\alpha>1$ and $\gamma\geq 1$. The drift coefficient is locally Lipschitz (see Remark 3.1) and locally bounded, so we can apply Lemma 115 p. 78 in Situ (2005) to get the pathwise uniqueness. Thanks to Theorem 137 p. 104 in Situ (2005), it suffices to prove that there exists a weak solution.

The drift coefficient is continuous with respect to its two variables, so it is a locally bounded and measurable function. By a standard localization argument, since the drift coefficient is locally Lipschitz, by using Theorem 9.1 p. 231 in Ikeda and Watanabe (1981), we deduce that there exists a unique solution defined up to explosion. □

Remark 3.3. If (H_{sgn}) is satisfied, then the function F is increasing and then uniqueness follows. Since the drift coefficient is a continuous function, using a standard localization argument, we can apply Theorem 3.1 p. 866 in Kurenok (2007) to conclude to existence. Hence, we can remove the condition $\gamma>1-\frac{\alpha}{2}$ under this hypothesis.

4. Moment estimates and non-explosion of the velocity process

In this section, we present estimates on moments of the velocity process V solution to (SKE). This will be doubly useful to conclude of the non-explosion of solution to (SKE) with Lemma 4.1, and to control some terms appearing along the proofs of Theorems 2.2 and 2.4 in Section 5.

Let V be the unique solution up to explosion time to (SKE). For all $r\geq 0$, define the stopping time

$$\tau_r:=\inf\{t\geq t_0,\ |V_t|\geq r\}\tag{4.1}$$

and set $\tau_\infty:=\lim_{r\to+\infty}\tau_r$ the explosion time of V. We give first a sufficient condition for the non-explosion of a general process.

Lemma 4.1. *Let $(Y_t)_{t\geq t_0}$ be a càdlàg process and τ_∞ its explosion time. Assume that there exist two measurable and non-negative functions ϕ and b such that*

(i) ϕ is non-decreasing and $\lim\limits_{r\to\infty}\phi(r)=+\infty$,
(ii) b is finite-valued,
(iii) and for all $t\geq t_0$,

$$\sup_{r\geq 0}\mathbb{E}\left[\phi(Y_{t\wedge\tau_r})\right]\leq b(t).\tag{4.2}$$

Then $\tau_\infty=+\infty$ a.s.

Proof: Pick $t\geq t_0$. Using the definition of τ_r, the monotony of ϕ and *(iii)*, we get, for all $r\geq 0$,

$$\phi(r)\mathbb{P}(\tau_r\leq t)\leq\mathbb{E}\left[\phi\left(Y_{\tau_r}\right)\mathbf{1}_{\{\tau_r\leq t\}}\right]\leq\mathbb{E}\left[\phi\left(Y_{\tau_r}\right)\right]\leq b(t).$$

Thus, by Fatou's lemma,

$$0\leq\mathbb{P}\left(\tau_\infty\leq t\right)\leq\liminf_{r\to\infty}\mathbb{P}(\tau_r\leq t)\leq b(t)\lim_{r\to\infty}\frac{1}{\phi(r)}=0.$$

We conclude that

$$0\leq\mathbb{P}\left(\tau_\infty<+\infty\right)\leq\sum_{t\in\mathbb{Q}}\mathbb{P}\left(\tau_\infty\leq t\right)=0.$$

□

With this result in hands we can now state and prove moments estimates for the velocity process V. Recall that $(V_t)_{t\geq t_0}$ is the solution to (SKE). We will split our analysis in several cases following the position of the stability parameter α.

Proposition 4.2. *Pick $\alpha \in (0,1)$ and assume that (H_{sgn}) holds. For any γ, β, the explosion time τ_∞ is a.s. infinite and for all $\kappa \in [0,\alpha)$, there exists a constant C_{κ,t_0} such that, we have*

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\kappa,t_0} t^{\frac{\kappa}{\alpha}}. \tag{4.3}$$

Proof: Fix $t \geq t_0$. Since $\alpha < 1$, the stable process can be written as

$$L_t = \int_0^t \int_{\mathbb{R}^*} z N(\mathrm{d}s, \mathrm{d}z) = \sum_{s \leq t} \Delta L_s.$$

Fix $\kappa \in [0,\alpha)$. Pick the sequence of $\mathcal{C}^2$-functions $f_n : x \mapsto \sqrt{x^2 + \frac{1}{n}}$, which converges uniformly to the function $x \mapsto |x|$ on $\mathbb{R}$. Then, for all $n \geq 1$, we apply Itô's formula (see Theorem 32 p. 78 in Protter (2005)) to get

$$\begin{aligned} f_n(V_{t\wedge\tau_r}) &= f_n(v_0) - \int_{t_0}^{t\wedge\tau_r} f_n'(V_s) F(V_s) s^{-\beta}\,\mathrm{d}s + \int_{t_0}^{t\wedge\tau_r} \int_{\mathbb{R}^*} \left(f_n(V_{s-} + z) - f_n(V_{s-})\right) N(\mathrm{d}s, \mathrm{d}z) \\ &\leq f_n(v_0) + \sum_{s \leq t\wedge\tau_r} \left(f_n(V_{s-} + \Delta L_s) - f_n(V_{s-})\right). \end{aligned}$$

The term $\int_{t_0}^{t\wedge\tau_r} f_n'(V_s) F(V_s) s^{-\beta}\,\mathrm{d}s$ is non-negative, since (H_{sgn}) holds.
Hence, the previous inequality can be written as

$$f_n(V_{t\wedge\tau_r}) \leq f_n(v_0) + \sum_{s \leq t\wedge\tau_r} \left(f_n(V_s) - f_n(V_{s-})\right).$$

Since $\|f_n'\|_\infty \leq 1$, we deduce that $(f_n(V_s) - f_n(V_{s-})) \leq |\Delta V_s| = |\Delta L_s|$, hence,

$$|V_{t\wedge\tau_r}| \leq f_n(V_{t\wedge\tau_r}) \leq f_n(v_0) + \sum_{s \leq t\wedge\tau_r} |\Delta L_s|.$$

Furthermore, since $\kappa < \alpha < 1$, we have

$$|V_{t\wedge\tau_r}|^\kappa \leq f_n(v_0)^\kappa + \Big(\sum_{s \leq t\wedge\tau_r} |\Delta L_s|\Big)^\kappa.$$

Taking the expectation, we get

$$\mathbb{E}\left[|V_{t\wedge\tau_r}|^\kappa\right] \leq \mathbb{E}\left[f_n(v_0)^\kappa\right] + \mathbb{E}\Big[\Big(\sum_{s \leq t} |\Delta L_s|\Big)^\kappa\Big]. \tag{4.4}$$

Notice that the process $L_t^+ := \sum_{s\leq t} |\Delta L_s|$ is also a pure-jump Lévy process and an α-stable process. By owing to (2.1), its Lévy measure is given, for any Borel subset $A \subset (0,\infty)$, by $\nu^+(A) = \nu(A) + \nu(-A) = (a_+ + a_-) \int_A \frac{dz}{|z|^{1+\alpha}}$. In other words, the Lévy measure of L_t^+ is the Lévy measure of an α-stable subordinator and is given by

$$\nu^+(dz) = (a_+ + a_-)\mathbf{1}_{\{z>0\}} \frac{dz}{|z|^{1+\alpha}}.$$

Invoking Kingman's formula, the same conclusion can be readily obtained by computing the characteristic function of L_t^+. With this observation in hand, since $\kappa < \alpha$, letting $n \to +\infty$ in (4.4) we obtain

$$\mathbb{E}\left[|V_{t\wedge\tau_r}|^\kappa\right] \leq |v_0|^\kappa + \mathbb{E}\left[\left|L_t^+\right|^\kappa\right] \leq C_{t_0,\kappa} t^{\frac{\kappa}{\alpha}}.$$

Thanks to Lemma 4.1, we can conclude that the explosion time of V is a.s. infinite, and (4.3) follows, letting $r \to \infty$. □

Proposition 4.3. *Pick $\alpha \in (1,2)$. For any $\gamma \in [0,1)$ and any $\beta \in \mathbb{R}$, the explosion time τ_∞ is a.s. infinite and for all $\kappa \in [0,1]$, there exists $C_{\gamma,\kappa,\beta,t_0}$ such that we have*

$$\forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\gamma,\kappa,\beta,t_0} \begin{cases} t^{\frac{\kappa}{\alpha}} & \text{if } \frac{\gamma-1}{\alpha}+1 \leq \beta, \\ t^{\kappa\frac{1-\beta}{1-\gamma}} & \text{else.} \end{cases} \tag{4.5}$$

Proof of Proposition 4.3: Assume that $\gamma \in [0,1)$ and fix $\kappa \in [0,1]$. Then Jensen's inequality yields, for all $t \geq t_0$, $\mathbb{E}\left[|V_t|^\kappa\right] \leq \mathbb{E}\left[|V_t|\right]^\kappa$, hence it suffices to verify (4.5) only for $\kappa = 1$.
Recall that under (H_γ), there exists a positive constant K, such that for all $v \in \mathbb{R}$, $|F(v)| \leq K|v|^\gamma$. Hence, we can write, for any $t \geq t_0$ and $r \geq 0$,

$$\begin{aligned} \left|V_{(t\wedge\tau_r)-}\right| &\leq |v_0 - L_{t_0}| + \left|L_{(t\wedge\tau_r)-}\right| + \int_{t_0}^{t\wedge\tau_r} s^{-\beta}\left|F(V_{s\wedge\tau_r})\right| \mathrm{d}s \\ &\leq |v_0 - L_{t_0}| + \left|L_{(t\wedge\tau_r)-}\right| + K\int_{t_0}^{t\wedge\tau_r} s^{-\beta}\left|V_{s\wedge\tau_r}\right|^\gamma \mathrm{d}s. \end{aligned}$$

Since L is an α-stable process, it has a finite first moment, which can be computed. Taking the expectation in the above inequality, we get, by choosing C_{t_0} big enough,

$$\begin{aligned} \mathbb{E}\left[\left|V_{(t\wedge\tau_r)-}\right|\right] &\leq \mathbb{E}\left[|v_0 - L_{t_0}|\right] + \mathbb{E}\left[\left|L_{(t\wedge\tau_r)-}\right|\right] + K\int_{t_0}^{t} s^{-\beta}\mathbb{E}\left[|V_{s\wedge\tau_r}|^\gamma\right] \mathrm{d}s \\ &\leq C_{t_0}t^{\frac{1}{\alpha}} + K\int_{t_0}^{t} s^{-\beta}\mathbb{E}\left[|V_{s\wedge\tau_r}|\right]^\gamma \mathrm{d}s. \end{aligned}$$

Recalling that τ_r is given by (4.1), the function $g_r : t \mapsto \mathbb{E}\left[\left|V_{(t\wedge\tau_r)-}\right|\right]$ is bounded by r. Applying a Grönwall-type lemma (see Lemma A.1), we end up, for $\beta \neq 1$, with

$$\forall t \geq t_0,\ \mathbb{E}\left[\left|V_{(t\wedge\tau_r)-}\right|\right] \leq C_\gamma\left[C_{t_0}t^{\frac{1}{\alpha}} + \left(\frac{1-\gamma}{1-\beta}K(t^{1-\beta} - t_0^{1-\beta})\right)^{\frac{1}{1-\gamma}}\right].$$

The case $\beta = 1$ can be treated similarly. Thanks to Lemma 4.1, we conclude that the explosion time of V is a.s. infinite, and (4.5) follows from Fatou's lemma. □

Proposition 4.4. *Pick $\alpha \in [1,2)$ and assume that for all $v \in \mathbb{R}$, $vF(v) \geq 0$. For any $\gamma \in \mathbb{R}$ and any $\beta \in \mathbb{R}$, the explosion time τ_∞ is a.s. infinite and there exists C_{κ,t_0} such that*

$$\text{for } \kappa \in (0,\alpha),\ \forall t \geq t_0,\ \mathbb{E}\left[|V_t|^\kappa\right] \leq C_{\kappa,t_0}t^{\frac{\kappa}{\alpha}}. \tag{4.6}$$

Proof: The key idea is to slice the small and big jumps in a non-homogeneous way with respect to the function $\xi \mapsto \xi^{\frac{1}{\alpha}}$. We write the proof in the general setting of $\alpha \in (1,2)$. When $\alpha = 1$, the proof is similar since ν is symmetric.

Pick $\xi \geq t_0$. The α-stable Lévy driving process can be written, by using this cutting threshold (see for instance Chaudru de Raynal and Menozzi (2022) and references therein for similar ideas), as

$$L_t - L_{t_0} = \int_{t_0}^t\int_{|z|\leq\xi^{\frac{1}{\alpha}}} z\widetilde{N}(\mathrm{d}s,\mathrm{d}z) + \int_{t_0}^t\int_{|z|>\xi^{\frac{1}{\alpha}}} zN(\mathrm{d}s,\mathrm{d}z) - \int_{t_0}^t\int_{|z|>\xi^{\frac{1}{\alpha}}} z\nu(\mathrm{d}z)\,\mathrm{d}s.$$

The two first integrals satisfy the same scaling property as the α-stable Lévy driving process. It is a direct and simple computation to see that

$$\int_{|z|>\xi^{\frac{1}{\alpha}}} z\nu(\mathrm{d}z) = \frac{a_+ - a_-}{\alpha-1}\xi^{\frac{1}{\alpha}-1}.$$

Step 1. We first apply Itô's formula and estimate the expectation of each term, in order to get (4.6).

Fix $\eta > 0$ and define the $\mathcal{C}^2$-function $f : v \mapsto (\eta + v^2)^{\kappa/2}$. For all $t \geq t_0$, by Itô's formula, using that for all $v \in \mathbb{R}$, $vF(v) \geq 0$, we have

$$f(V_{t\wedge\tau_r}) \leq f(V_0) - \frac{a_+ - a_-}{\alpha - 1}\xi^{\frac{1}{\alpha}-1}\int_{t_0}^t \mathbf{1}_{\{s\leq\tau_r\}} f'(V_s)\,\mathrm{d}s + M_t + R_t + S_t, \tag{4.7}$$

where the terms M, R and S are respectively given by

$$M_t := \int_{t_0}^t \int_{0<|z|<\xi^{\frac{1}{\alpha}}} \mathbf{1}_{\{s\leq\tau_r\}}\left[f(V_{s-} + z) - f(V_{s-})\right]\widetilde{N}(\mathrm{d}s, \mathrm{d}z), \tag{4.8}$$

$$R_t := \int_{t_0}^t \int_{|z|\geq\xi^{\frac{1}{\alpha}}} \mathbf{1}_{\{s\leq\tau_r\}}\left[f(V_{s-} + z) - f(V_{s-})\right])N(\mathrm{d}s, \mathrm{d}z), \tag{4.9}$$

$$S_t := \int_{t_0}^t \int_{0<|z|<\xi^{\frac{1}{\alpha}}} \mathbf{1}_{\{s\leq\tau_r\}}\left[f(V_s + z) - f(V_s) - zf'(V_s)\right]\nu(\mathrm{d}z)\,\mathrm{d}s. \tag{4.10}$$

We estimate expectations of M, R and S.
Observe that, for $\kappa < 1$ and for all $v \in \mathbb{R}$,

$$\left|f'(v)\right| \leq \kappa\eta^{\frac{\kappa-1}{2}}. \tag{4.11}$$

Again, as previously, by direct computation, for all $k > \alpha$,

$$\int_{0<|z|<\xi^{\frac{1}{\alpha}}} |z|^k\,\nu(\mathrm{d}z) = \frac{a_+ + a_-}{k - \alpha}\xi^{\frac{k}{\alpha}-1}, \tag{4.12}$$

and for all $k < \alpha$,

$$\int_{|z|\geq\xi^{\frac{1}{\alpha}}} |z|^k\,\nu(\mathrm{d}z) = \frac{a_+ + a_-}{\alpha - k}\xi^{\frac{k}{\alpha}-1}. \tag{4.13}$$

Firstly, we show that the local martingale $(M_t)_{t\geq t_0}$ is a martingale. Fix $q \geq 2$ and $r \geq 0$. Set

$$I_t(q) := \int_{t_0}^t \int_{0<|z|<\xi^{\frac{1}{\alpha}}} \mathbf{1}_{\{s\leq\tau_r\}}\left|f(V_{s-} + z) - f(V_{s-})\right|^q\nu(\mathrm{d}z)\,\mathrm{d}s.$$

Notice that, since for all $|v| \leq r$ and $|z| \leq \xi^{\frac{1}{\alpha}}$, $|f(v + z) - f(v)| \leq \left\|f'\mathbf{1}_{[-(r+\xi^{\frac{1}{\alpha}}),r+\xi^{\frac{1}{\alpha}}]}\right\|_\infty |z|$, so we have

$$I_t(q) \leq \left\|f'\mathbf{1}_{[-(r+\xi^{\frac{1}{\alpha}}),r+\xi^{\frac{1}{\alpha}}]}\right\|_\infty^q \int_{t_0}^t \int_{0<|z|<\xi^{\frac{1}{\alpha}}} \mathbf{1}_{\{s\leq\tau_r\}}|z|^q\,\nu(\mathrm{d}z)\,\mathrm{d}s.$$

The right-hand side of this last inequality is a finite quantity, since (4.12) holds and $q \geq 2$. Therefore, for $q \geq 2$, by Kunita's inequality (see Theorem 4.4.23 p. 265 in Applebaum (2009)), there exists $D_q > 0$ such that

$$\mathbb{E}\left[\sup_{t_0\leq s\leq t}|M_s|^q\right] \leq D_q\left(\mathbb{E}\left[I_t(2)^{\frac{q}{2}}\right] + \mathbb{E}\left[I_t(q)\right]\right) < +\infty.$$

Hence, by Theorem 51 p. 38 in Protter (2005), M is a martingale.
We estimate now the finite variation part S defined in (4.10). We use a similar idea as in the proof of Theorem 3.1 p. 3863 in Deng and Schilling (2015). Note that for all $v \in \mathbb{R}$,

$$\begin{aligned}\left|f''(v)\right| &= \kappa(2-\kappa)v^2(v^2+\eta)^{\frac{\kappa}{2}-2} + \kappa(v^2+\eta)^{\frac{\kappa}{2}-1}\\ &= \kappa(2-\kappa)v^2(v^2+\eta)^{-1}(v^2+\eta)^{\frac{\kappa}{2}-1} + \kappa(v^2+\eta)^{\frac{\kappa}{2}-1}\\ &\leq \kappa(3-\kappa)(v^2+\eta)^{\frac{\kappa}{2}-1} \leq \kappa(3-\kappa)\eta^{\frac{\kappa}{2}-1},\end{aligned}$$

where we used the fact that $\frac{\kappa}{2} - 1 < 0$. Assume that $|z| < \xi^{\frac{1}{\alpha}}$. Using Taylor's formula, we get a.s.

$$\left|f(V_s + z) - f(V_s) - zf'(V_s)\right| \leq \frac{1}{2}\kappa(3-\kappa)\eta^{\frac{\kappa}{2}-1}z^2.$$

By integrating, we get the almost sure following bound

$$\left|\int_{0<|z|<\xi^{\frac{1}{\alpha}}} \big(f(V_s+z)-f(V_s)-zf'(V_s)\big)\,\nu(\mathrm{d}z)\right| \le \frac{1}{2}\kappa(3-\kappa)\eta^{\frac{\kappa}{2}-1}\int_{0<|z|<\xi^{\frac{1}{\alpha}}} z^2\nu(\mathrm{d}z).$$

Gathering (4.12) into this last inequality, we end up with

$$\left|\int_{0<|z|<\xi^{\frac{1}{\alpha}}} \big(f(V_s+z)-f(V_s)-zf'(V_s)\big)\,\nu(\mathrm{d}z)\right| \le \frac{1}{2}\kappa(3-\kappa)\eta^{\frac{\kappa}{2}-1}\frac{a_+ + a_-}{2-\alpha}\xi^{\frac{2}{\alpha}-1}. \tag{4.14}$$

It remains to study the Poisson integral R defined in (4.9), using Theorem 2.3.7 p. 106 in Applebaum (2009). Pick $\kappa \le 1$. By Hölder property of power functions, we can write,

$$\begin{aligned}|f(v+z)-f(v)| &= \left|\big(\eta+(v+z)^2\big)^{\frac{\kappa}{2}} - \big((v+z)^2\big)^{\frac{\kappa}{2}} + (v+z)^\kappa - v^\kappa + \big(v^2\big)^{\frac{\kappa}{2}} - \big(\eta+v^2\big)^{\frac{\kappa}{2}}\right| \\ &\le 2\eta^{\frac{\kappa}{2}} + |z|^\kappa .\end{aligned}$$

By integration we deduce that

$$\int_{|z|\ge\xi^{\frac{1}{\alpha}}} |f(V_s+z)-f(V_s)|\,\nu(\mathrm{d}z) \le \eta^{\frac{\kappa}{2}}\nu(|z|\ge\xi^{\frac{1}{\alpha}}) + \int_{|z|\ge\xi^{\frac{1}{\alpha}}} |z|^\kappa\,\nu(\mathrm{d}z).$$

Replacing (4.13) into the latter relation, we obtain

$$\int_{|z|\ge\xi^{\frac{1}{\alpha}}} |f(V_s+z)-f(V_s)|\,\nu(\mathrm{d}z) \le \eta^{\frac{\kappa}{2}}\frac{a_+ + a_-}{\alpha}\xi^{-1} + \frac{a_+ + a_-}{\alpha-\kappa}\xi^{\frac{\kappa}{\alpha}-1}. \tag{4.15}$$

Gathering (4.11), (4.15) and (4.14), we get from (4.7),

$$\begin{aligned}\mathbb{E}\left[|V_{t\wedge\tau_r}|^\kappa\right] &\le \mathbb{E}\left[f(V_{t\wedge\tau_r})\right] \le \mathbb{E}\left[f(V_{t_0})\right] + t\xi^{-1} \\ &\times\left(\kappa\eta^{\frac{\kappa-1}{2}}\frac{a_+ - a_-}{\alpha-1}\xi^{\frac{1}{\alpha}} + \eta^{\kappa/2}\frac{a_+ + a_-}{\alpha} + \frac{a_+ + a_-}{\alpha-\kappa}\xi^{\frac{\kappa}{\alpha}} + \frac{1}{2}\kappa(3-\kappa)\eta^{\frac{\kappa}{2}-1}\frac{a_+ + a_-}{2-\alpha}\xi^{\frac{2}{\alpha}}\right).\end{aligned} \tag{4.16}$$

It suffices to choose $\eta = t^{\frac{2}{\alpha}}$ and $\xi = t$ on the right hand side of the previous inequality to obtain

$$\begin{aligned}\mathbb{E}\left[|V_{t\wedge\tau_r}|^\kappa\right] &\le \mathbb{E}\left[f(V_{t_0})\right] + t^{\frac{\kappa}{\alpha}}\times\left(\kappa\frac{a_+ - a_-}{\alpha-1} + \frac{a_+ + a_-}{\alpha} + \frac{a_+ + a_-}{\alpha-\kappa} + \frac{1}{2}\kappa(3-\kappa)\frac{a_+ + a_-}{2-\alpha}\right) \\ &\le C_{\kappa,t_0}t^{\frac{\kappa}{\alpha}}.\end{aligned} \tag{4.17}$$

Thanks to Lemma 4.1, we can conclude that the explosion time of V is a.s. infinite and letting $r\to+\infty$, for all $\kappa\in[0,1]$,

$$\mathbb{E}\left[|V_t|^\kappa\right] \le C_{\kappa,t_0}t^{\frac{\kappa}{\alpha}}. \tag{4.18}$$

Step 2. Pick $\kappa\in(1,\alpha)$. We estimate R in another manner, using again Theorem 2.3.7 p. 106 in Applebaum (2009).

By Hölder property of power function and (4.13), we get

$$\begin{aligned}\int_{|z|\ge\xi^{\frac{1}{\alpha}}} |f(V_s+z)-f(V_s)|\,\nu(\mathrm{d}z) &\le \int_{|z|\ge\xi^{\frac{1}{\alpha}}} \left|2zV_s+z^2\right|^{\frac{\kappa}{2}}\nu(\mathrm{d}z) \\ &\le C_\kappa\left(\frac{a_+ + a_-}{\alpha-\kappa}\xi^{\frac{\kappa}{\alpha}-1} + |V_s|^{\frac{\kappa}{2}}\frac{a_+ + a_-}{\alpha-\frac{\kappa}{2}}\xi^{\frac{\kappa}{2\alpha}-1}\right).\end{aligned} \tag{4.19}$$

Gathering (4.14), (4.19) and then using that for all $v\in\mathbb{R}$, $|f'(v)|\le\kappa\,|v|^{\kappa-1}$, we deduce that

$$\mathbb{E}\left[|V_{t\wedge\tau_r}|^\kappa\right] \le \mathbb{E}\left[f(V_{t_0})\right] + \left(C_\kappa \frac{a_+ + a_-}{\alpha-\kappa}\xi^{\frac{\kappa}{\alpha}-1} + \frac{1}{2}\kappa(3-\kappa)\eta^{\frac{\kappa}{2}-1}\frac{a_+ + a_-}{2-\alpha}\xi^{\frac{2}{\alpha}-1}\right)t$$
$$+ \kappa\frac{a_+ - a_-}{\alpha-1}\xi^{\frac{1}{\alpha}-1}\int_{t_0}^t \mathbb{E}\left[|V_s|^{\kappa-1}\right]\mathrm{d}s + C_\kappa\frac{a_+ + a_-}{\alpha-\frac{\kappa}{2}}\xi^{\frac{\kappa}{2\alpha}-1}\int_{t_0}^t \mathbb{E}\left[|V_s|^{\frac{\kappa}{2}}\right]\mathrm{d}s. \quad (4.20)$$

Injecting (4.18) and by choosing $\eta = t^{\frac{2}{\alpha}}$ and $\xi = t$, we conclude that

$$\mathbb{E}\left[|V_{t\wedge\tau_r}|^\kappa\right] \le C_{\kappa,t_0,\alpha}t^{\frac{\kappa}{\alpha}}.$$

Letting $r \to +\infty$, the announced inequality (4.6) follows.

□

Example 4.5. Remark that the velocity process V is explicit in the linear case ($\gamma = 1$), and that the moments estimate is as best as possible. Choose $F(1) = \rho > 0$, $F(-1) = -\rho$. Pick $\beta \neq 1$, so

$$V_t = v_0 + \exp\left(-\rho\frac{t^{1-\beta}}{1-\beta}\right)\int_{t_0}^t \exp\left(\rho\frac{s^{1-\beta}}{1-\beta}\right)\mathrm{d}L_s$$

is solution to (SKE). Hence, by an integration by parts,

$$V_t = v_0 + L_t - e^{\rho\frac{1}{1-\beta}(t_0^{1-\beta}-t^{1-\beta})}L_{t_0} - e^{-\rho\frac{t^{1-\beta}}{1-\beta}}\int_{t_0}^t \rho s^{-\beta}e^{\rho\frac{s^{1-\beta}}{1-\beta}}L_s\,\mathrm{d}s,$$

and, we end up with

$$\mathbb{E}\left[|V_t|\right] \le C_{t_0}\left(t^{\frac{1}{\alpha}} + t^{1-\beta+\frac{1}{\alpha}}\right) \le C_{t_0}t^{\frac{1}{\alpha}}.$$

The case $\beta = 1$ can be treated similarly.

5. Proof of the asymptotic behavior of the solution

This section is devoted to the proofs of our main results, Theorems 2.2, 2.4 and 2.6.

Notice that, in the super-critical and critical regimes, it is enough to prove the convergence of the rescaled velocity process $(\varepsilon^{\frac{1}{\alpha}}V_{t/\varepsilon})_{t\ge\varepsilon t_0}$ in the space $\mathcal{D}$ endowed with the Skorokhod topology. We recall that in the sub-critical situation we obtained the convergence only of the f.d.d. It was pointed out in Gradinaru and Luirard (2023), Remark 2.6 that even for the simple linear case with Brownian driving noise, it is not possible to prove tightness.

Let us briefly explain how the convergence of the rescaled position process can be obtained by assuming the convergence of the rescaled velocity process. For $\varepsilon \in (0,1]$ and $t \ge \varepsilon t_0$, we can write

$$\varepsilon^{1+\frac{1}{\alpha}}X_{t/\varepsilon} = \varepsilon^{1+\frac{1}{\alpha}}x_0 + \int_{\varepsilon t_0}^t V_s^{(\varepsilon)}\,\mathrm{d}s.$$

Let us introduce the mapping $g_\varepsilon : V \mapsto \left(V_t, \int_{\varepsilon t_0}^t V_s\,\mathrm{d}s\right)_{t>0}$ defined and valued on $\mathcal{D}$. It converges, as $\varepsilon \to 0$, to the continuous mapping $g : V \mapsto \left(V_t, \int_0^t V_s\,\mathrm{d}s\right)_{t>0}$.

In order to obtain the desired result, it suffices to show that $g_\varepsilon(V_\bullet^{(\varepsilon)})$ converges weakly in $\mathcal{D}$, endowed with the Skorokhod topology. To see $V^{(\varepsilon)}$ as a process of $\mathcal{D}([0,+\infty))$, we state, for all $s \in [0,\varepsilon t_0]$, $V_s^{(\varepsilon)} := V_{\varepsilon t_0}^{(\varepsilon)} = \varepsilon^{\frac{1}{\alpha}}v_0$. Call P_ε, P the distribution of $V^{(\varepsilon)}$, $\mathcal{V}$, respectively. Invoking the Portmanteau theorem (see Theorem 2.1 p. 16 in Billingsley (1999)), it suffices to prove that for all bounded and uniformly continuous function $h : \mathcal{D}([0,+\infty)) \times \mathcal{D}([0,+\infty)) \to \mathbb{R}$,

$$\int_{\mathcal{D}([0,+\infty))^2} h(g_\varepsilon(\omega))\,\mathrm{d}P_\varepsilon(\mathrm{d}\omega) \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad \int_{\mathcal{D}([0,+\infty))^2} h(g(\omega))\,\mathrm{d}P(\mathrm{d}\omega).$$

Pick such a function h. By assumption, the convergence $P_\varepsilon \underset{\varepsilon\to 0}{\Longrightarrow} P$ holds, hence, using Lemma A.4, it suffices to prove that the uniformly bounded sequence $(h\circ g_\varepsilon)_\varepsilon$ of continuous functions on $\mathcal{D}([0,+\infty))$

converges to the continuous function $h \circ g$ uniformly on compact subsets of $\mathcal{D}([0,+\infty))$. This can be verified exploiting the properties of h, g_ε and ω in compact subset of $\mathcal{D}([0,+\infty))$. Therefore, it suffices to prove the convergence of the rescaled velocity process $(\varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon})_{t\geq \varepsilon t_0}$ in order to prove Theorems 2.2 and 2.4.

In Sections 5.1 and 5.2, the aim is to prove the convergence of the velocity process.

5.1. *Asymptotic behavior in the super-critical regime.* In the remainder of this section, we assume that $\gamma \geq 0$ and $\beta > 1 + \frac{\gamma-1}{\alpha}$.

Proof of Theorem 2.2: Thanks to a change of variables, we have, for all $\varepsilon \in (0,1]$ and $t \geq \varepsilon t_0$,

$$\begin{aligned}\varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon} =& \varepsilon^{\frac{1}{\alpha}}(v_0 - L_{t_0}) + \varepsilon^{\frac{1}{\alpha}} L_{t/\varepsilon} - \varepsilon^{\frac{1}{\alpha}} \int_{t_0}^{t/\varepsilon} F(V_s) s^{-\beta}\,\mathrm{d}s \\ =& \varepsilon^{\frac{1}{\alpha}}(v_0 - L_{t_0}) + \varepsilon^{\frac{1}{\alpha}} L_{t/\varepsilon} - \varepsilon^{\beta-1+\frac{1}{\alpha}} \int_{\varepsilon t_0}^{t} F(V_{u/\varepsilon}) u^{-\beta}\,\mathrm{d}u.\end{aligned}$$

By self-similarity, $L^{(\varepsilon)} := (\varepsilon^{\frac{1}{\alpha}} L_{t/\varepsilon})_{t\geq 0}$ has the same distribution as an α-stable process.

As a consequence, thanks to Theorem 3.1 p. 27 in Billingsley (1999) and Lemma A.3, it suffices to prove

$$\forall T > 0 \sup_{\varepsilon t_0 \leq t \leq T} \left| V_t^{(\varepsilon)} - L_t^{(\varepsilon)} \right| \xrightarrow{\mathbb{P}} 0, \quad \text{as } \varepsilon \to 0. \tag{5.1}$$

Recall that under (H_γ), there exists a positive constant K, such that

$$\varepsilon^{\frac{\gamma}{\alpha}} \left| F\left(\frac{V_\bullet^{(\varepsilon)}}{\varepsilon^{\frac{1}{\alpha}}}\right)\right| \leq K \left| V_\bullet^{(\varepsilon)} \right|^{\gamma}. \tag{5.2}$$

Modifying the factor in front of the integral, we get

$$V_t^{(\varepsilon)} = \varepsilon^{\frac{1}{\alpha}}(v_0 - L_{t_0}) + L_t^{(\varepsilon)} - \varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} \int_{\varepsilon t_0}^{t} \varepsilon^{\frac{\gamma}{\alpha}} F\left(\frac{V_u^{(\varepsilon)}}{\varepsilon^{\frac{1}{\alpha}}}\right) u^{-\beta}\,\mathrm{d}u. \tag{5.3}$$

Gathering (5.3) and (5.2), for all $T > 0$, we have, test

$$\begin{aligned}\sup_{\varepsilon t_0 \leq t \leq T} \left| V_t^{(\varepsilon)} - L_t^{(\varepsilon)} \right| \leq& \varepsilon^{\frac{1}{\alpha}}(v_0 - L_{t_0}) + \varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} \sup_{\varepsilon t_0 \leq t \leq T} \left| \int_{\varepsilon t_0}^{t} \varepsilon^{\frac{\gamma}{\alpha}} F\left(\frac{V_u^{(\varepsilon)}}{\varepsilon^{\frac{1}{\alpha}}}\right) u^{-\beta}\,\mathrm{d}u \right| \\ \leq& \varepsilon^{\frac{1}{\alpha}}(v_0 - L_{t_0}) + \varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} \int_{\varepsilon t_0}^{T} K \left| V_u^{(\varepsilon)} \right|^{\gamma} u^{-\beta}\,\mathrm{d}u.\end{aligned} \tag{5.4}$$

Taking the expectation and using the moments estimates on V (see Remark 2.10), we obtain, when $\beta \neq \frac{\gamma}{\alpha} + 1$,

$$\begin{aligned}&\varepsilon^{\beta-1+\frac{(1-\gamma)}{\alpha}} \mathbb{E}\left[\int_{\varepsilon t_0}^{T} K \left| V_u^{(\varepsilon)} \right|^{\gamma} u^{-\beta}\,\mathrm{d}u\right] = \varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} \int_{\varepsilon t_0}^{T} K \mathbb{E}\left[\left| V_u^{(\varepsilon)} \right|^{\gamma}\right] u^{-\beta}\,\mathrm{d}u \\ &= \varepsilon^{\beta-1+\frac{1}{\alpha}} \int_{\varepsilon t_0}^{T} K \mathbb{E}\left[\left| V_{u/\varepsilon} \right|^{\gamma}\right] u^{-\beta}\,\mathrm{d}u \leq \varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} \int_{\varepsilon t_0}^{T} C_{\alpha,\beta,t_0} u^{\frac{\gamma}{\alpha}-\beta}\,\mathrm{d}u \\ &= C_{\alpha,\beta,t_0}\left(\varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} T^{\frac{\gamma}{\alpha}-\beta+1} - t_0^{\frac{\gamma}{\alpha}-\beta+1} \varepsilon^{\frac{1}{\alpha}}\right).\end{aligned} \tag{5.5}$$

Gathering (5.4) and (5.5) and setting $r := \min\left(\beta - 1 + \frac{1-\gamma}{\alpha}, \frac{1}{\alpha}\right) > 0$, we get

$$\mathbb{E}\Big[\sup_{\varepsilon t_0 \leq t \leq T} \left| V_t^{(\varepsilon)} - L_t^{(\varepsilon)} \right|\Big] = \underset{\varepsilon\to 0}{O}(\varepsilon^r).$$

The case $\beta = 1 + \frac{\gamma}{\alpha}$ can be treated similarly and we end up with

$$\mathbb{E}\Big[\sup_{\varepsilon t_0 \le t \le T}\Big|V_t^{(\varepsilon)} - L_t^{(\varepsilon)}\Big|\Big] = \underset{\varepsilon\to 0}{O}(\varepsilon^{\frac{1}{\alpha}}\ln(\varepsilon)).$$

This concludes the proof. □

5.2. *Asymptotic behaviour in the critical regime.* We adapt the strategy in Gradinaru and Offret (2013) to the α-stable Lévy case. Pick a $\mathcal{C}^2$-diffeomorphism $\varphi : [0, t_1) \to [t_0, +\infty)$. Let V be the solution to the equation (SKE). Thanks to Proposition 3.4.1 p. 124 in Samorodnitsky and Taqqu (1994), the following process is also an α-stable process

$$(R_t)_{t\ge 0} := \left(\int_0^t \frac{\mathrm{d}L_{\varphi(s)}}{\varphi'(s)^{\frac{1}{\alpha}}}\right)_{t\ge 0}. \tag{5.6}$$

Then, by the change of variables $t = \varphi(s)$, we get

$$V_{\varphi(t)} - V_{\varphi(0)} = \int_0^t \varphi'(s)^{\frac{1}{\alpha}}\,\mathrm{d}R_s - \int_0^t \frac{F(V_{\varphi(s)})}{\varphi(s)^\beta}\varphi'(s)\,\mathrm{d}s.$$

By integration by parts, we get

$$\mathrm{d}\left(\frac{V_{\varphi(s)}}{\varphi'(s)^{\frac{1}{\alpha}}}\right) = \mathrm{d}R_s - \frac{\varphi'(s)^{1-\frac{1}{\alpha}}}{\varphi(s)^\beta}F(V_{\varphi(s)})\,\mathrm{d}s - \frac{\varphi''(s)}{\alpha\varphi'(s)}\frac{V_{\varphi(s)}}{\varphi'(s)^{\frac{1}{\alpha}}}\,\mathrm{d}s.$$

Set $\Omega = \overline{\mathcal{D}}([t_0, \infty))$ the set of càdlàg functions that are equal ∞ after their (possibly infinite) explosion time. Introduce the scaling transformation Φ_φ defined, for $\omega \in \Omega$, by

$$\Phi_\varphi(\omega)(s) := \frac{\omega(\varphi(s))}{\varphi'(s)^{\frac{1}{\alpha}}}, \text{ with } s \in [0, t_1).$$

As a consequence, we obtain the following result (see Proposition 2.1, p. 187 in Gradinaru and Offret (2013)).

Proposition 5.1. *If V is a solution to the equation (SKE), then $V^{(\varphi)} := \Phi_\varphi(V)$ is a solution to*

$$\mathrm{d}V_s^{(\varphi)} = \mathrm{d}R_s - \frac{\varphi'(s)^{1-\frac{1}{\alpha}}}{\varphi(s)^\beta}F(\varphi'(s)^{\frac{1}{\alpha}}V_s^{(\varphi)})\,\mathrm{d}s - \frac{\varphi''(s)}{\varphi'(s)}\frac{V_s^{(\varphi)}}{\alpha}\,\mathrm{d}s, \qquad \text{with } V_0^{(\varphi)} = \frac{V_{\varphi(0)}}{\varphi'(0)^{\frac{1}{\alpha}}}, \tag{5.7}$$

where R is an α-stable process given by (5.6).
Conversely, if $V^{(\varphi)}$ is a solution to (5.7), then $\Phi_\varphi^{-1}(V^{(\varphi)})$ is a solution to the equation (SKE), where

$$L_t - L_{t_0} := \int_{t_0}^t (\varphi' \circ \varphi^{-1})^{\frac{1}{\alpha}}(s)\,\mathrm{d}R_{\varphi^{-1}(s)}$$

is an α-stable process.
Furthermore, uniqueness in law, pathwise uniqueness, strong existence hold for the equation (SKE) if and only if they hold for the equation (5.7).

Let us focus firstly on the exponential change of time $\varphi_e : t \mapsto t_0 e^t$. This scaling is convenient since it allows to produce a time-homogeneous term in (5.7). Thanks to Proposition 5.1, the process $V^{(e)} := \Phi_e(V)$ satisfies the SDE driven by the α-stable process $(R_t)_{t\ge 0}$.

$$\mathrm{d}V_s^{(e)} = \mathrm{d}R_s - \frac{V_s^{(e)}}{\alpha}\,\mathrm{d}s - t_0^{1-\frac{1}{\alpha}-\beta}e^{(1-\frac{1}{\alpha}-\beta)s}F\big(t_0^{\frac{1}{\alpha}}e^{\frac{s}{\alpha}}V_s^{(e)}\big)\,\mathrm{d}s. \tag{5.8}$$

Proof of Theorem 2.4: Assume in the sequel that $\beta = 1 + \frac{\gamma-1}{\alpha}$.

STEP 1. Firstly we prove the finite-dimensional convergence of the rescaled velocity process. To that end, we reduce the problem to the convergence of a time-homogenous process.
Since (H_γ) holds, (5.8) becomes exactly the equation (2.5), driven by the α-stable process R:

$$\mathrm{d}V_s^{(e)} = \mathrm{d}R_s - \frac{V_s^{(e)}}{\alpha}\,\mathrm{d}s - F\big(V_s^{(e)}\big)\,\mathrm{d}s. \tag{5.9}$$

Using the bijection Φ_e induced by the exponential change of time (see Proposition 5.1), and the unique strong existence of the velocity process V (see Proposition 3.2 and Remark 2.10), there exists a pathwise unique strong solution $\mathfrak{H}$ to the time-homogeneous equation (5.9). Hence, we have the equality

$$\left(\frac{V_{t_0e^t}}{(t_0e^t)^{1/\alpha}}\right)_{t\geq 0} = (\mathfrak{H}_t)_{t\geq 0},$$

as two solutions to the same SDE, starting from the same point. We rewrite the above equality as

$$\left(\frac{V_t}{t^{\frac{1}{\alpha}}}\right)_{t\geq t_0} = (\mathfrak{H}_{\log(t/t_0)})_{t\geq t_0}.$$

So, we have, for all $\varepsilon>0$, $d\in\mathbb{N}^*$, and $(t_1,\cdots,t_d)\in[\varepsilon t_0,+\infty)^d$,

$$\left(\frac{V_{\varepsilon^{-1}t_1}}{(\varepsilon^{-1}t_1)^{1/\alpha}},\cdots,\frac{V_{\varepsilon^{-1}t_d}}{(\varepsilon^{-1}t_d)^{1/\alpha}}\right) = \left(\mathfrak{H}_{\log(t_1)+\log((\varepsilon t_0)^{-1})},\cdots,\mathfrak{H}_{\log(t_d)+\log((\varepsilon t_0)^{-1})}\right). \tag{5.10}$$

Since $\limsup_{|x|\to+\infty}\frac{-F(x)-x/\alpha}{x}<0$, it follows from Proposition 0.1 in Kulik (2009) that the process $(\mathfrak{H}_t)_{t\geq 0}$ is exponentially ergodic. We denote its invariant measure by Λ_F. Call H the solution to the time homogeneous equation (5.9), such that the initial condition $H_{-\infty}$ has the distribution Λ_F. For $(t_1,\cdots,t_d)\in\mathbb{R}^d$, let $\Lambda_{F,t_1,\cdots,t_d}:=\mathcal{L}(H_{t_1},\cdots,H_{t_d})$ be the distribution of $(H_{t_1},\cdots,H_{t_d})$. Then, for all $s\geq 0$, $\Lambda_{F,t_1,\cdots,t_d}=\Lambda_{F,t_1+s,\cdots,t_d+s}$. Indeed, thanks to the invariance property of Λ_F, $(H_\bullet)$ and $(H_{\bullet+s})$ satisfy the same SDE, starting from the same point. As a consequence, we get the stationary limit

$$\lim_{\varepsilon\to 0}\mathcal{L}\left(H_{\log(t_1)+\log((\varepsilon t_0)^{-1})},\cdots,H_{\log(t_d)+\log((\varepsilon t_0)^{-1})}\right) = \Lambda_{F,\log(t_1),\cdots,\log(t_d)}. \tag{5.11}$$

Moreover, by exponential ergodicity, we have for every continuous and bounded function $\psi:\mathbb{R}^d\to\mathbb{R}$,

$$\mathbb{E}\left[\psi\left(\mathfrak{H}_{\log(t_1/(t_0\varepsilon))},\cdots,\mathfrak{H}_{\log(t_d/(t_0\varepsilon))}\right)\right] - \mathbb{E}\left[\psi\left(H_{\log(t_1/(t_0\varepsilon))},\cdots,H_{\log(t_d/(t_0\varepsilon))}\right)\right] \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{5.12}$$

We postpone the proof of this convergence in Step 2.
To conclude this step, we gather (5.10), (5.11) and (5.12) to get

$$\left(\frac{V_{\varepsilon^{-1}t_1}}{(\varepsilon^{-1}t_1)^{1/\alpha}},\cdots,\frac{V_{\varepsilon^{-1}t_d}}{(\varepsilon^{-1}t_d)^{1/\alpha}}\right) \underset{\varepsilon\to 0}{\Longrightarrow} \Lambda_{F,\log(t_1),\cdots,\log(t_d)}.$$

This can also be written as

$$\left(\varepsilon^{\frac{1}{\alpha}}V_{t_1/\varepsilon},\cdots,\varepsilon^{\frac{1}{\alpha}}V_{t_d/\varepsilon}\right) \underset{\varepsilon\to 0}{\Longrightarrow} T*\Lambda_{F,\log(t_1),\cdots,\log(t_d)},$$

where $T*\Lambda_{F,\log(t_1),\cdots,\log(t_d)}$ denotes the pushforward of the measure $\Lambda_{F,\log(t_1),\cdots,\log(t_d)}$ by the linear map $T(u_1,\cdots,u_d):=(t_1^{1/\alpha}u_1,\cdots,t_d^{1/\alpha}u_d)$.

STEP 2. Let us now prove (5.12).
For the sake of clarity, let us give a proof for $d=2$, the general case $d\geq 2$ being similar.
Let $\psi:\mathbb{R}^2\to\mathbb{R}$ be a continuous and bounded function. Pick $\varepsilon t_0\leq s\leq t$ and set $h_0=v_0t_0^{-\frac{1}{\alpha}}$, (5.12) is now equivalent to

$$\mathbb{E}\left[\psi\left(\mathfrak{H}_{\log(s/(t_0\varepsilon))},\mathfrak{H}_{\log(t/(t_0\varepsilon))}\right)\Big|\mathfrak{H}_0=h_0\right] - \mathbb{E}\left[\psi\left(H_{\log(s/(t_0\varepsilon))},H_{\log(t/(t_0\varepsilon))}\right)\Big|H_0\sim\Lambda_F\right] \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

We introduce $\mu_\varepsilon := \mathcal{L}\left(\mathfrak{H}_{\log(s/(t_0\varepsilon))}\middle|\mathfrak{H}_0 = h_0\right)$. We now use the generalized Markov property of solutions to SDE driven by Lévy process (for the sake of completeness, we state and prove it in our context in Appendix, see Lemma A.6.). This leads to

$$\mathbb{E}\left[\psi\left(\mathfrak{H}_{\log(s/(t_0\varepsilon))}, \mathfrak{H}_{\log(t/(t_0\varepsilon))}\right)\middle|\mathfrak{H}_0 = h_0\right] = \mathbb{E}\left[\psi\left(\mathfrak{H}_0, \mathfrak{H}_{\log(t/s)}\right)\middle|\mathfrak{H}_0 \sim \mu_\varepsilon\right]$$

and, since Λ_F is invariant,

$$\mathbb{E}\left[\psi\left(H_{\log(s/(t_0\varepsilon))}, H_{\log(t/(t_0\varepsilon))}\right)\middle|H_0 \sim \Lambda_F\right] = \mathbb{E}\left[\psi\left(H_0, H_{\log(t/s)}\right)\middle|H_0 \sim \Lambda_F\right].$$

Then, we are reduced to prove

$$\mathbb{E}\left[\psi\left(\mathfrak{H}_0, \mathfrak{H}_{\log(t/s)}\right)\middle|\mathfrak{H}_0 \sim \mu_\varepsilon\right] - \mathbb{E}\left[\psi\left(H_0, H_{\log(t/s)}\right)\middle|H_0 \sim \Lambda_F\right] \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

The left-hand side can be written as,

$$\int_{\mathbb{R}} \mathbb{E}\left[\psi\left(\mathfrak{H}_0, \mathfrak{H}_{\log(t/s)}\right)\middle|\mathfrak{H}_0 = y\right](\mu_\varepsilon(\mathrm{d}y) - \Lambda_F(\mathrm{d}y)).$$

Hence, setting $p(t, x, \mathrm{d}y) := \mathbb{P}_x(\mathfrak{H}_t \in \mathrm{d}y)$ and $\|.\|_{TV}$ for the total variation norm, we get

$$\begin{aligned}
&\left|\mathbb{E}\left[\psi\left(\mathfrak{H}_0, \mathfrak{H}_{\log(t/s)}\right)\middle|\mathfrak{H}_0 \sim \mu_\varepsilon\right] - \mathbb{E}\left[\psi\left(H_0, H_{\log(t/s)}\right)\middle|H_0 \sim \Lambda_F\right]\right| \\
&\le \|\psi\|_\infty \int_{\mathbb{R}} |p(\log(s/(t_0\varepsilon)), h_0, \mathrm{d}y) - \Lambda_F(\mathrm{d}y)| \\
&\le \|\psi\|_\infty \|p(\log(s/(t_0\varepsilon)), h_0, \cdot) - \Lambda_F\|_{TV}.
\end{aligned}$$

This converges to 0, as $\varepsilon \to 0$, by the exponential ergodicity of $\mathfrak{H}$.

Step 3. Let us prove now the tightness of the family of distributions of the càdlàg process $\left(V^{(\varepsilon)}\right)_{t\ge\varepsilon t_0} = \left(\varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon}\right)_{t\ge\varepsilon t_0}$ on every compact interval $[m, M]$, $0 < m \le M$.
We check the Aldous criterion for tightness stated in Theorem 16.10 p.178 in Billingsley (1999). Let a, η, T be positive real numbers. Let τ be a discrete stopping time with finite range $\mathcal{T}$, bounded by T. Choose $\delta > 0$ and $\varepsilon > 0$ small enough.
We have, by Jensen's inequality, for $r = \frac{\alpha}{2}$,

$$\mathbb{E}\left[\left|V_{\tau+\delta}^{(\varepsilon)} - V_\tau^{(\varepsilon)}\right|^r\right] \le \mathbb{E}\left[\left|L_{\tau+\delta}^{(\varepsilon)} - L_\tau^{(\varepsilon)}\right|^r\right] + \mathbb{E}\left[\int_\tau^{\tau+\delta} K\left|V_u^{(\varepsilon)}\right|^\gamma u^{-\beta}\,\mathrm{d}u\right]^r.$$

Since $L^{(\varepsilon)}$ is an α-stable process, by the strong Markov property,

$$\mathbb{E}\left[\left|L_{\tau+\delta}^{(\varepsilon)} - L_\tau^{(\varepsilon)}\right|^r\right] = \mathbb{E}\left[\mathbb{E}_{L_\tau}\left[|L_\delta - L_0|^r\right]\right] \le C\delta^{\frac{r}{\alpha}}.$$

The stopping time has a finite range $\mathcal{T}$. Hence, we can write

$$\begin{aligned}
\mathbb{E}\left[\int_\tau^{\tau+\delta} K\left|V_u^{(\varepsilon)}\right|^\gamma u^{-\beta}\,\mathrm{d}u\right] &= \mathbb{E}\left[\mathbb{E}\left[\int_\tau^{\tau+\delta} K\left|V_u^{(\varepsilon)}\right|^\gamma u^{-\beta}\,\mathrm{d}u\middle|\tau\right]\right] \\
&= \mathbb{E}\left[\sum_{\tau_i\in\tau}\frac{1}{\mathbb{P}(\tau=\tau_i)}\mathbb{E}\left[\mathbf{1}_{\{\tau=\tau_i\}}\int_{\tau_i}^{\tau_i+\delta} K\left|V_u^{(\varepsilon)}\right|^\gamma u^{-\beta}\,\mathrm{d}u\right]\mathbf{1}_{\{\tau=\tau_i\}}\right] \\
&\le \mathbb{E}\left[\sum_{\tau_i\in\tau}\frac{1}{\mathbb{P}(\tau=\tau_i)}\mathbb{E}\left[\int_{\tau_i}^{\tau_i+\delta} K\left|V_u^{(\varepsilon)}\right|^\gamma u^{-\beta}\,\mathrm{d}u\right]\mathbf{1}_{\{\tau=\tau_i\}}\right].
\end{aligned}$$

For any $\tau_i \in \mathcal{T}$, using the relation $\beta = 1 + \frac{(\gamma-1)}{\alpha}$ and the moments estimates on V (see Remark 2.10), we obtain

$$\begin{aligned}\mathbb{E}\left[\int_{\tau_i}^{\tau_i+\delta} K\left|V_u^{(\varepsilon)}\right|^\gamma u^{-\beta}\,\mathrm{d}u\right] &= \int_{\tau_i}^{\tau_i+\delta} K\mathbb{E}\left[\left|V_u^{(\varepsilon)}\right|^\gamma\right] u^{-\beta}\,\mathrm{d}u \\ &\leq K\int_{\tau_i}^{\tau_i+\delta} u^{\frac{\gamma}{\alpha}-\beta}\,\mathrm{d}u = K\left[(\tau_i+\delta)^{\frac{1}{\alpha}} - \tau_i^{\frac{1}{\alpha}}\right] \\ &\leq K\delta^{1,\frac{1}{\alpha}}.\end{aligned}$$

The term $\delta^{1,\frac{1}{\alpha}}$ has to be read as δ or $\delta^{\frac{1}{\alpha}}$ depending on the fact that $x \mapsto x^{\frac{1}{\alpha}}$ is a Lipschitz continuous function on $[0, T+\delta]$, if $\alpha < 1$, or a $\frac{1}{\alpha}$-Hölder function, if $\alpha > 1$.
By Markov's inequality, for δ small enough, we have

$$\mathbb{P}\left(\left|V_{\tau+\delta}^{(\varepsilon)} - V_\tau^{(\varepsilon)}\right| \geq a\right) \leq \frac{K\delta^{r,\frac{r}{\alpha}}}{a^r} \leq \eta.$$

Furthermore, by moments estimates (see Propositions 4.2, 4.3 and 4.4), for all $t \geq \varepsilon t_0$,

$$\sup_\varepsilon \left[\left|V_t^{(\varepsilon)}\right|^r\right] \leq Ct^{\frac{r}{\alpha}}.$$

Hence, using again Markov's inequality, by Corollary and Theorem 16.8 p. 175 in Billingsley (1999), this concludes the proof of the tightness of the velocity process and therefore the proof of Theorem 2.4.

□

5.3. *The velocity process in the sub-critical regime.* Assume in this section that $\beta < 1 + \frac{\gamma-1}{\alpha}$ and $\alpha > 1$. Recall that we set $q := \frac{\beta}{\alpha+\gamma-1} < \frac{1}{\alpha}$. As a consequence, $\alpha q < 1$. This time we take an interest into the power change of time $\varphi_q : t \mapsto \left(t_0^{1-\alpha q} + (1-\alpha q)t\right)^{\frac{1}{1-\alpha q}}$. Thanks to Proposition 5.1, the process $V^{(q)} := \Phi_q(V)$ satisfies the SDE driven by an α-stable process R,

$$\mathrm{d}V_s^{(q)} = \mathrm{d}R_s - F\big(V_s^{(q)}\big)\,\mathrm{d}s - q\varphi_q^{\alpha q-1}V_s^{(q)}\,\mathrm{d}s. \tag{5.13}$$

For simplicity, we shall write φ instead of φ_q.

Proof of Theorem 2.6: Step 1. We first prove the finite dimensional convergence of the velocity process $(V_t^{(\varepsilon)})_{t\geq\varepsilon t_0} := (\varepsilon^q V_{t/\varepsilon})_{t\geq\varepsilon t_0}$. We give a proof for $d = 2$, the general case $d \geq 2$ being similar. We call $\mathfrak{H}$ the ergodic process solution to

$$\mathrm{d}\mathfrak{H}_s = \mathrm{d}L_s - F\Big(\mathfrak{H}_s\Big)\,\mathrm{d}s, \quad \text{with } \mathfrak{H}_0 = h_0 := v_0 t_0^{-q}, \tag{5.14}$$

where, as previously, L is an α-stable process. We denote by Π_F its invariant measure. Using the bijection induced by the power change of time (Proposition 5.1), as solutions to the same SDE starting at the same point, we have, for all $\varepsilon > 0$, and $(s,t) \in [\varepsilon t_0, +\infty)^2$,

$$\left(\varepsilon^q \frac{V_{\varepsilon^{-1}s}}{s^q}, \varepsilon^q \frac{V_{\varepsilon^{-1}t}}{t^q}\right) = \left(V_{\varphi^{-1}(\varepsilon^{-1}s)}^{(q)}, V_{\varphi^{-1}(\varepsilon^{-1}t)}^{(q)}\right).$$

Using Theorem 3.1 p. 27 in Billingsley (1999), it suffices to prove that for all $(s,t) \in [\varepsilon t_0, +\infty)^2$,

- $\left\|\left(\left(\mathfrak{H}_{\varphi^{-1}(\varepsilon^{-1}s)}, \mathfrak{H}_{\varphi^{-1}(\varepsilon^{-1}t)}\right) - \left(V_{\varphi^{-1}(\varepsilon^{-1}s)}^{(q)}, V_{\varphi^{-1}(\varepsilon^{-1}t)}^{(q)}\right)\right)\right\| \underset{\varepsilon\to 0}{\longrightarrow} 0$, where $\|\cdot\|$ is a norm on $\mathbb{R}^2$.
- $\left(\mathfrak{H}_{\varphi^{-1}(\varepsilon^{-1}s)}, \mathfrak{H}_{\varphi^{-1}(\varepsilon^{-1}t)}\right) \underset{\varepsilon\to 0}{\Longrightarrow} \Pi_F \otimes \Pi_F$.

Step 2. Pick $\kappa \in (1,\alpha)$. We prove that $\mathbb{E}\left[\left|\mathfrak{H}_t - V_t^{(q)}\right|^\kappa\right] \underset{t\to+\infty}{\longrightarrow} 0$.
We can write

$$\mathrm{d}\left(\mathfrak{H} - V^{(q)}\right)_t = -\left(F(\mathfrak{H}_t) - F(V_t^{(q)})\right)\mathrm{d}t + q\varphi^{\alpha q-1}(t)V_t^{(q)}\,\mathrm{d}t.$$

By straightforward differentiation, we can deduce

$$\begin{aligned}\mathrm{d}\left|\mathfrak{H} - V^{(q)}\right|_t^\kappa &= -\kappa\left|F(\mathfrak{H}_t) - F(V_t^{(q)})\right|\left|\mathfrak{H}_t - V_t^{(q)}\right|^{\kappa-1}\mathrm{d}t\\ &\quad + \kappa q\varphi^{\alpha q-1}(t)V_t^{(q)}\mathrm{sgn}\left(\mathfrak{H}_t - V_t^{(q)}\right)\left|\mathfrak{H}_t - V_t^{(q)}\right|^{\kappa-1}\mathrm{d}t.\end{aligned} \tag{5.15}$$

We set

$$g(t) := \mathbb{E}\left[\left|\mathfrak{H}_t - V_t^{(q)}\right|^\kappa\right], \quad t \geq 0.$$

Taking expectation in (5.15), we get

$$\begin{aligned}g'(t) = -\kappa\mathbb{E}&\left[\left|F(\mathfrak{H}_t) - F(V_t^{(q)})\right|\left|\mathfrak{H}_t - V_t^{(q)}\right|^{\kappa-1}\right]\\ &+ \kappa q\varphi^{\alpha q-1}(t)\mathbb{E}\left[V_t^{(q)}\mathrm{sgn}\left(\mathfrak{H}_t - V_t^{(q)}\right)\left|\mathfrak{H}_t - V_t^{(q)}\right|^{\kappa-1}\right].\end{aligned}$$

Since $\gamma \geq 1$, the function F^{-1} is $\frac{1}{\gamma}$-Hölder, there exists $C_\gamma > 0$ such that,

$$g'(t) \leq -C_\gamma\mathbb{E}\left[\left|\mathfrak{H}_t - V_t^{(q)}\right|^{\kappa-1+\gamma}\right] + \kappa\,|q|\,\varphi^{\alpha q-1}(t)\mathbb{E}\left[\left|V_t^{(q)}\right|\left|\mathfrak{H}_t - V_t^{(q)}\right|^{\kappa-1}\right].$$

Then, by Jensen's inequality, since $\gamma \geq 1$,

$$g'(t) \leq -C_\gamma g(t)^{\frac{\kappa-1+\gamma}{\kappa}} + \kappa\,|q|\,\varphi^{\alpha q-1}(t)\mathbb{E}\left[\left|V_t^{(q)}\right|\left|\mathfrak{H}_t - V_t^{(q)}\right|^{\kappa-1}\right].$$

Using Hölder's inequality and moments estimates (Proposition 4.2), we have

$$g'(t) \leq -C_\gamma g(t)^{\frac{\kappa-1+\gamma}{\kappa}} + C\,|q|\,\varphi^{(\alpha q-1)(1-\frac{1}{\alpha})}(t)g(t)^{\frac{\kappa-1}{\kappa}}, \quad g(0) = 0.$$

Recall that $\alpha > 1$ and $\alpha q < 1$, so $\varphi^{(\alpha q-1)(1-\frac{1}{\alpha})}(t) \underset{t\to+\infty}{\longrightarrow} 0$, therefore the conclusion follows from Lemma A.7.
Besides, for all $t \geq \varepsilon t_0$, $\mathbb{E}\left[\left|\mathfrak{H}_{\varphi^{-1}(\varepsilon^{-1}t)} - V^{(q)}_{\varphi^{-1}(\varepsilon^{-1}t)}\right|^\kappa\right] = g\left(\varphi^{-1}(\varepsilon^{-1}t)\right) \underset{\varepsilon\to 0}{\longrightarrow} 0$.

Step 3. Pick $(s,t) \in [\varepsilon t_0, +\infty)^2$. Similarly, as in Gradinaru and Luirard (2023), one can prove that the solution $\mathfrak{H}$ to (5.14) satisfies

$$\left(\mathfrak{H}_{\varphi^{-1}(\varepsilon^{-1}s)}, \mathfrak{H}_{\varphi^{-1}(\varepsilon^{-1}t)}\right) \quad \underset{\varepsilon\to 0}{\Longrightarrow} \quad \Pi_F \otimes \Pi_F. \tag{5.16}$$

$\square$

Appendix A. Some auxiliary results

We collect in this section several auxiliary results. To begin with, let us state a Grönwall-type lemma which has been used to get moments estimates. The proof can be found in Gradinaru and Luirard (2023).

Lemma A.1 (Grönwall-type lemma)**.** *Fix $r \in [0,1)$ and $t_0 \in \mathbb{R}$. Assume that g is a non-negative real-valued function, b is a positive function and a is a differentiable real-valued function. Moreover, suppose that the function bg^r is a continuous function.*
Assume that

$$\forall t \geq t_0,\ g(t) \leq a(t) + \int_{t_0}^t b(s)g(s)^r\,\mathrm{d}s. \tag{A.1}$$

Then, setting $C_r := 2^{\frac{1}{1-r}}$,

$$\forall t \geq t_0,\ g(t) \leq C_r \left[a(t) + \left((1-r) \int_{t_0}^{t} b(s)\,\mathrm{d}s \right)^{\frac{1}{1-r}} \right].$$

Remark A.2. Call G the right-hand side of (A.1). Even if g is not continuous, notice that the function G remains continuous and satisfies (A.1) (since b is positive and $g \leq G$). So, one can apply the lemma to G and thereafter use the inequality $g \leq G$.

We state now a technical lemma concerning the convergence in the spaces $\mathcal{C}$ and $\mathcal{D}$. We recall that the spaces of continuous functions $\mathcal{C}$ and of càdlàg functions were endowed with metrics d_u and d_s respectively given by (2.3) and (2.4).

Lemma A.3.

(i) The uniform distance is finer than the Skorokhod one i.e. $\mathrm{d}_s \leq \mathrm{d}_u$.

(ii) Let $(f_\varepsilon)_{\varepsilon \geq 0}, (h_\varepsilon)_{\varepsilon \geq 0}$ *be two sequences of functions of* $\mathcal{D}$. *If for all* $n \geq 1$,

$$\lim_{\varepsilon \to 0} \sup_{t \in [\frac{1}{n}, n]} |f_\varepsilon(t) - h_\varepsilon(t)| = 0$$

in probability, then $\lim_{\varepsilon \to 0} \mathrm{d}(f_\varepsilon, h_\varepsilon) = 0$ *in probability, where* $\mathrm{d} \in \{\mathrm{d}_u, \mathrm{d}_s\}$.

Proof: Let f, g be two càdlàg functions. The first point is true by using the definition of the metrics d_s and d_u and by noting that

$$\inf_{\lambda \in \Lambda} \left\{ \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \bigvee \sup_{t \geq \frac{1}{n}} |k_n(t)\,(f(\lambda(t) - g(t))| \right\} \leq \sup_{t \in [\frac{1}{n+1}, n+1]} |f(t) - g(t)|\,.$$

Let us now prove the second part. Assume that for all $n \geq 1$, $\sup_{[\frac{1}{n}, n]} |f_\varepsilon - h_\varepsilon| \xrightarrow[\varepsilon \to 0]{\mathbb{P}} 0$, as $\varepsilon \to 0$. Fix $\eta > 0$ and choose $N > 0$ such that $\sum\limits_{n=N+1}^{+\infty} \frac{1}{2^n} \leq \frac{\eta}{2}$. Then,

$$\mathrm{d}_s\left(f_\varepsilon, h_\varepsilon\right) \leq \mathrm{d}_u(f_\varepsilon, h_\varepsilon) \leq \frac{\eta}{2} + \sum_{n=1}^{N} \frac{1}{2^n} \sup_{[\frac{1}{n}, n]} |f_\varepsilon - h_\varepsilon|\,.$$

By setting $\eta' := \frac{\eta}{2} \left(\sum\limits_{n=1}^{+\infty} \frac{1}{2^n} \right)^{-1}$, it follows that

$$\mathbb{P}\left(\mathrm{d}\left(f_\varepsilon, h_\varepsilon\right) > \eta\right) \leq \sum_{n=1}^{N} \mathbb{P}\Big(\sup_{[\frac{1}{n}, n]} |f_\varepsilon - h_\varepsilon| > \eta' \Big) \quad \underset{\varepsilon \to 0}{\longrightarrow} \quad 0.$$

□

For the sake of completeness, we state and improve the result of Problem 4.12 p. 64 in Karatzas and Shreve (1991), on a general metric space.

Lemma A.4. *Let S be a Polish metric space endowed with a Borel σ-field $\mathcal{S}$. Suppose that $(P_n)_{n \geq 1}$ is a sequence of probability measures on $(S, \mathcal{S})$ which converges weakly to a probability measure P. Suppose, in addition, that the sequence $(f_n)_{n \geq 1}$ of real-valued continuous functions on S is uniformly bounded and converges to a continuous function f, the convergence being uniform on compact subsets of S. Then, we have*

$$\lim_{n \to +\infty} \int_S f_n(\omega)\,\mathrm{d}P_n(\omega) = \int_S f(\omega)\,\mathrm{d}P(\omega).$$

Proof: Notice that, since $(P_n)_{n\geq 1}$ converges weakly thus, it is tight. So, for any $\varepsilon > 0$, there exists a compact subset K of S such that for any $n \geq 1$, $P_n(K) \geq 1-\varepsilon$.
Let us decompose

$$\int_S f_n \,\mathrm{d}P_n - \int_S f \,\mathrm{d}P = A + B + C + D,$$

where

$$A := \int_{S\setminus K} f_n \,\mathrm{d}P_n,\ B := \int_K (f_n - f)\,\mathrm{d}P_n,\ C := \int_{S\setminus K} f \,\mathrm{d}P_n,\ \text{and } D := \int_S f\,\mathrm{d}P_n - \int_S f\,\mathrm{d}P.$$

Let M be a bound for the sequence (f_n). Thus, by the choice of K,

$$|A| \leq M P_n(S\setminus K) \leq M\varepsilon.$$

The third integral can be treated analogously. Besides, since the sequence (f_n) converges uniformly on K to f, there exists n_ε such that for all $n \geq n_\varepsilon$, $\sup_K |f_n - f| \leq \varepsilon$. Thereby, we get

$$|B| \leq \varepsilon P_n(K).$$

The last integral is smaller than ε for n large enough, since P_n converges weakly to P, and this concludes the proof. □

Remark A.5. Lemma A.4 could be applied with $S = \mathcal{C}([0,+\infty))$ or $\mathcal{D}([0,+\infty))$. However, the result for $S = \mathcal{C}([0,+\infty))$ is already contained in Problem 4.12 p. 64 in Karatzas and Shreve (1991).

Lemma A.6. *Consider b a measurable function such that the following time-homogeneous SDE driven by an α-stable process*

$$\mathrm{d}Y_t = \mathrm{d}L_t + b(Y_t)\,\mathrm{d}t \quad \textit{with} \quad Y_0 = y, \tag{A.2}$$

has a pathwise unique strong solution. Denote by $(Y_t^y)_{t\geq 0}$ this solution. Then $(Y_t)_{t\geq 0}$ is a Markov process.
Namely, for any $d \geq 1$, $0 \leq t_1 \leq \cdots \leq t_d$, $u \geq 0$ and any bounded measurable function $\phi : \mathbb{R}^d \to \mathbb{R}$,

$$\mathbb{E}\left[\phi(Y^y_{t_1+u}, \cdots, Y^y_{t_d+u}) \middle| \mathcal{F}_u\right] = \mathbb{E}\left[\phi(Y^z_{t_1}, \cdots, Y^z_{t_d})\right]_{z=Y^y_u}. \tag{A.3}$$

Proof: For simplicity, we give a proof for $d = 2$, the general case being similar. Call $(Y_t^{s,y})$ the solution to $\mathrm{d}Y_t = \mathrm{d}L_t + b(Y_t)\,\mathrm{d}t$, satisfying $Y_s = y$. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded measurable function. Pick $u \geq 0$ and consider, for $y \in \mathbb{R}$ and $u \leq s \leq t$ the function

$$G(y,s,t,u) := (Y_s^{u,y}, Y_t^{u,y}) = \left(y + L_s - L_u + \int_u^s b(Y_h)\,\mathrm{d}h,\ y + L_t - L_u + \int_u^t b(Y_h)\,\mathrm{d}h\right).$$

Pick $0 \leq s \leq t$. Using pathwise uniqueness, $\left(Y^y_{s+u}, Y^y_{t+u}\right) = G\left(Y^y_u, s+u, t+u, u\right)$. Moreover, by time-homogeneity of the SDE, $(Y^{u,y}_{s+u})_{s\geq 0}$ and $(Y^y_s)_{s\geq 0}$ have the same distribution. As a consequence, $G(y, s+u, t+u, u) = G(y,s,t,0)$. Besides, by Markov property of Lévy processes, the function G is independent of $\mathcal{F}_u$. Hence,

$$\begin{aligned}\mathbb{E}\left[\phi(Y^y_{s+u}, Y^y_{t+u})\middle|\mathcal{F}_u\right] &= \mathbb{E}\left[\phi\circ G\left(Y^y_u, s+u, t+u, u\right)\middle|\mathcal{F}_u\right] = \mathbb{E}\left[\phi\circ G\left(z,s,t,0\right)\middle|\mathcal{F}_u\right]_{z=Y^y_u}\\ &= \mathbb{E}\left[\phi\circ G\left(z,s,t,0\right)\right]_{z=Y^y_u} = \mathbb{E}\left[\phi(Y^z_s, Y^z_t)\right]_{z=Y^y_u}\end{aligned}$$

This concludes the proof. □

Our last technical result is the following.

Lemma A.7. *Let b be a function such that $\lim_{t\to+\infty} b(t) = 0$. Pick $a > 0$, $\gamma \geq 1$ and $\kappa > 1$. Let g be a continuously differentiable positive function satisfying*

$$g'(t) \leq -a g(t)^{\frac{\kappa+\gamma-1}{\kappa}} + b(t) g(t)^{\frac{\kappa-1}{\kappa}}, \quad t \geq 0. \tag{A.4}$$

Then, $g(t) \underset{t\to+\infty}{\longrightarrow} 0$.

Proof: Pick $\varepsilon > 0$. Let t_1 be a positive real such that for all $t \geq t_1$, $|b(t)| \leq \frac{a}{2}\varepsilon^{\frac{\gamma}{\kappa}}$.

STEP 1. We first show that there exists $t^* \geq t_1$, such that $g(t^*) \leq \varepsilon$. Assume, by way of contradiction, that it is not the case. Thus, one can consider the function $y = g^{\frac{1}{\kappa}}$, which satisfies

$$\kappa y'(t) \leq -ay(t)^\gamma + b(t), \quad t \geq t_1. \tag{A.5}$$

For all $t \geq t_1$, we have

$$\kappa y'(t) \leq -a\varepsilon^{\frac{\gamma}{\kappa}} + \frac{a}{2}\varepsilon^{\frac{\gamma}{\kappa}} \leq -\frac{a}{2}\varepsilon^{\frac{\gamma}{\kappa}}.$$

As a consequence, for all $t \geq t_1$,

$$\kappa\varepsilon^{\frac{1}{\kappa}} < \kappa y(t) \leq \kappa y(t_1) - (t - t_1)\frac{a}{2}\varepsilon^{\frac{\gamma}{\kappa}} \underset{t\to+\infty}{\longrightarrow} -\infty.$$

This is a contradiction.

STEP 2. We show that for all $t \geq t^*$, $g(t) \leq \varepsilon$.

Define $T = \inf\{t \geq t^*,\ g(t) > \varepsilon\}$. By continuity of the function g, we have $g(T) = \varepsilon$. Hence,

$$g'(T) \leq -a\varepsilon^{\frac{\kappa+\gamma-1}{\kappa}} + \frac{a}{2}\varepsilon^{\frac{\gamma}{\kappa}}\varepsilon^{\frac{\kappa-1}{\kappa}} < -\frac{a}{2}\varepsilon^{\frac{\kappa+\gamma-1}{\kappa}} < 0.$$

Therefore, there exists $\delta_0 > 0$, such that for all $0 < \delta \leq \delta_0$, $g(T + \delta) < g(T) = \varepsilon$. This is a contradiction with the definition of T.

□

Acknowledgements The authors would like to thank Nicolas Fournier and Thomas Cavallazzi for helpful discussions about moments estimates. We would also like to thank the Referees for their careful reading of the manuscript and useful advice.

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