

Lévy-type processes: convergence and discrete schemes

Mihai Gradinaru and Tristan Haugomat

Université de Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France
{Mihai.Gradinaru,Tristan.Haugomat}@univ-rennes1.fr

Abstract: We characterise the convergence of a certain class of discrete time Markov processes toward locally Feller processes in terms of convergence of associated operators. The theory of locally Feller processes is applied to Lévy-type processes in order to obtain convergence results on discrete and continuous time indexed processes, simulation methods and Euler schemes. We also apply the same theory to a slightly different situation, in order to get results of convergence of diffusions or random walks toward singular diffusions. As a consequence we deduce the convergence of random walks in random medium toward diffusions in random potential.

Key words: Lévy-type processes, random walks and diffusions in random and non-random environment, weak convergence of probability measures, discrete schemes, Skorokhod topology, martingale problem, Feller processes, generators

MSC2010 Subject Classification: Primary 60J25; Secondary 60J75, 60B10, 60G44, 60J35, 60J05, 60K37, 60E07, 47D07

1 Introduction

Lévy-type processes constitute a large class of processes allowing to build models for many phenomena. Heuristically, a Lévy-type process is a Markov process taking its values in the one-point compactification $\mathbb{R}^{d\Delta}$, such that in each point of $a \in \mathbb{R}^d$,

- it is drifted by the value of a vector $\delta(a)$,
- it diffuse with the value $\gamma(a)$, a symmetric positive semi-definite matrix,
- it jumps into a Borel subset B of $\mathbb{R}^{d\Delta}$ with the rate $\nu(a, B)$, a positive measure satisfying

$$\int (1 \wedge |b - a|^2) \nu(a, db) < \infty.$$

For a Lévy-type process we will call (δ, γ, ν) its Lévy triplet.

In general, the study of the convergence of sequences of general Markov processes is an important question. The present paper consider this question, among others, in the setting of the preceding two models. The approximating Markov sequences could

have continuous or discrete time parameter in order to cover scaling transformations or discrete schemes.

A usual way to obtain such results is the use of the theory of Feller processes. In this context there exist two corresponding results of convergence (see, for instance Kallenberg [Kal02], Thms. 19.25, p. 385 and 19.27, p. 387). However, on one hand, when one needs to consider unbounded coefficients, for instance the Lévy triplet (δ, γ, ν) for Lévy-type processes, technical difficulties could appear in the framework of Feller processes. On the other hand the cited results of convergence impose the knowledge of a core of the generator. This could not be the case in some probabilistic constructions.

Our method to tackle these difficulties is to consider the context of the martingale local problems and of locally Feller processes, introduced in [GH17b]. In this general framework we have already analysed the question of convergence of sequences of locally Feller processes. In the present paper we add the study of the convergence for processes indexed by a discrete time parameter toward processes indexed by a continuous time parameter. We obtain the characterisation of the convergence in terms of convergence of associated operators, by using the uniform convergence on compact sets, and hence operators with unbounded coefficients could be considered. Likewise, we do not impose that the operator is a generator, but we assume only the well-posed feature of the associated martingale local problem. Indeed, it could be more easy to verify the well-posed feature (see for instance, Stroock [Str75] for Lévy-type processes, Stroock and Varadhan [SV06] for diffusion processes, Kurtz [Kur11] for Lévy-driven stochastic differential equations and forward equations...).

We apply our abstract results and we obtain sharp results of convergence for discrete and continuous time sequences of processes toward Lévy-type process, in terms of Lévy parameters (δ, γ, ν) . We prefer the use of the Lévy triplet than the symbol associated to the operator, since the results are more precise in the situation of possibly instantaneous explosions. This is due essentially to the fact that the vague convergence of bounded measures cannot be characterised in terms of characteristic function. Our results can also be used to simulate Lévy-type processes and we improve Theorem 7.6 from Böttcher, Schilling and Wang [BSW13], p. 172, which is an approximation result of type Euler scheme. We state the results in terms of convergence of operators, but essentially one can deduce the convergence of the associated processes.

Another well known model is the dynamic of a Brownian particle in a potential. It is often given by the solution of the one-dimensional stochastic differential equation

$$dX_t = dB_t - \frac{1}{2}V'(X_t)dt,$$

where $V : \mathbb{R} \rightarrow \mathbb{R}$. The process X is also a Lévy-type process, and, thanks to the regularising property of the Brownian motion one can consider very general potentials, for example cadlag functions (see Mandel [Man68]). In particular, it can be supposed that the potential is a Brownian path (see Brox [Bro86]), a Lévy paths (see Carmona[Car97]) or other random paths (Gaussian and/or fractional process ...).

When studying a Brownian particle in a potential, we prove the continuous dependence of the diffusion with respect to the potential, using again our abstract results, even

this is a different situation. We point out that it can be possible to consider potentials with very few constraints. In particular we consider diffusions in random potentials as limits of random walks in random mediums, as an application of an approximation of the diffusion by random walks on \mathbb{Z} . An important example is the convergence of Sinai's random walk [Sin82] toward the diffusion in a Poisson potential (recovering Thm. 2 from Seignourel [Sei00], p. 296), toward the diffusion in a Brownian potential, also called Brox's diffusion (improving Thm. 1 from Seignourel [Sei00], p. 295) and, more generally, toward the diffusion in a Lévy potential.

Let us describe the organisation of the paper. The next section contains notations and statements from our previous paper [GH17b], which are useful for an easy reading of the present paper. In particular, we give the statements concerning the existence of solutions for martingale local problems and concerning the convergence of continuous time locally Feller processes. Section 3 is devoted to the limits of sequences of discrete time processes, while Section 4 contains two results of convergence toward general Lévy-type processes. The diffusions evolving in a potential are studied in Section 5. The appendix contains the statements of several technical results already proved in [GH17b].

2 Martingale local problem setting and related results

Let S be a locally compact Polish space. Take $\Delta \notin S$, and we will denote by $S^\Delta \supset S$ the one-point compactification of S , if S is not compact, or the topological sum $S \sqcup \{\Delta\}$, if S is compact (so Δ is an isolated point). The fact that a subset A is compactly embedded in an open subset $U \subset S^\Delta$ will be denoted by $A \Subset U$. If $x \in (S^\Delta)^{\mathbb{R}_+}$ we denote the explosion time by

$$\xi(x) := \inf\{t \geq 0 \mid \{x_s\}_{s \leq t} \notin S\}.$$

The set of exploding cadlag paths is defined by

$$\mathbb{D}_{\text{loc}}(S) := \left\{ x \in (S^\Delta)^{\mathbb{R}_+} \left| \begin{array}{l} \forall t \geq \xi(x), x_t = \Delta, \\ \forall t \geq 0, x_t = \lim_{s \downarrow t} x_s, \\ \forall t > 0 \text{ s.t. } \{x_s\}_{s < t} \Subset S, x_{t-} := \lim_{s \uparrow t} x_s \text{ exists} \end{array} \right. \right\},$$

and is endowed with the local Skorokhod topology (see Theorem 2.4 in [GH17a]) which is also Polish. A sequence $(x^k)_{k \in \mathbb{N}}$ in $\mathbb{D}_{\text{loc}}(S)$ converges to x for the local Skorokhod topology if and only if there exists a sequence $(\lambda^k)_k$ of increasing homeomorphisms on \mathbb{R}_+ satisfying

$$\forall t \geq 0 \text{ s.t. } \{x_s\}_{s < t} \Subset S, \quad \limsup_{k \rightarrow \infty} \sup_{s \leq t} d(x_s, x_{\lambda_s^k}^k) = 0 \quad \text{and} \quad \limsup_{k \rightarrow \infty} \sup_{s \leq t} |\lambda_s^k - s| = 0.$$

The local Skorokhod topology does not depend on the arbitrary metric d on S^Δ , but only on the topology on S . We will always denote by X the canonical process on $\mathbb{D}_{\text{loc}}(S)$. We endow $\mathbb{D}_{\text{loc}}(S)$ with the Borel σ -algebra $\mathcal{F} := \sigma(X_s, 0 \leq s < \infty)$ and a filtration $\mathcal{F}_t := \sigma(X_s, 0 \leq s \leq t)$.

Denote by $C(S) := C(S, \mathbb{R})$, respectively by $C(S^\Delta) := C(S^\Delta, \mathbb{R})$, the set of real continuous functions on S , respectively on S^Δ , and by $C_0(S)$ the set of functions $f \in$

$C(S)$ vanishing in Δ . We endow the set $C(S)$ with the topology of uniform convergence on compact sets and $C_0(S)$ with the topology of uniform convergence.

We proceed by recalling the notion of martingale local problem. Let L be a subset of $C_0(S) \times C(S)$. The set $\mathcal{M}(L)$ of solutions of the martingale local problem associated to L is the set of probabilities $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\text{loc}}(S))$ such that for all $(f, g) \in L$ and open subset $U \Subset S$:

$$f(X_{t \wedge \tau^U}) - \int_0^{t \wedge \tau^U} g(X_s) ds \text{ is a } \mathbf{P}\text{-martingale}$$

with respect to the filtration $(\mathcal{F}_t)_t$ or, equivalent, to the filtration $(\mathcal{F}_{t+})_t$. Here τ^U is the stopping time given by

$$\tau^U := \inf \{t \geq 0 \mid X_t \notin U \text{ or } X_{t-} \notin U\}. \quad (2.1)$$

In [GH17b] the following result of existence of solutions for martingale local problem was proved:

Theorem 2.1. *Let L be a linear subspace of $C_0(S) \times C(S)$ such that its domain $D(L) := \{f \in C_0(S) \mid \exists g \in C(S), (f, g) \in L\}$ is dense in $C_0(S)$. Then, there is equivalence between*

- i) existence of a solution for the martingale local problem: for any $a \in S$ there exists an element \mathbf{P} in $\mathcal{M}(L)$ such that $\mathbf{P}(X_0 = a) = 1$;*
- ii) L satisfies the positive maximum principle: for all $(f, g) \in L$ and $a_0 \in S$, if $f(a_0) = \sup_{a \in S} f(a) \geq 0$ then $g(a_0) \leq 0$.*

Let us note that a linear subspace $L \subset C_0(S) \times C(S)$ satisfying the positive maximum principle is univariate, so it can be equivalently considered as a linear operator

$$L : D(L) \rightarrow C(S).$$

The martingale local problem is said well-posed if there is existence and uniqueness of the solution, which means that for any $a \in S$ there exists a unique element \mathbf{P} in $\mathcal{M}(L)$ such that $\mathbf{P}(X_0 = a) = 1$.

A family of probabilities $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\text{loc}}(S))^S$ is called locally Feller if there exists $L \subset C_0(S) \times C(S)$ such that $D(L)$ is dense in $C_0(S)$ and

$$\forall a \in S : \quad \mathbf{P} \in \mathcal{M}(L) \text{ and } \mathbf{P}(X_0 = a) = 1 \iff \mathbf{P} = \mathbf{P}_a.$$

The $C_0 \times C$ -generator of a locally Feller family $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\text{loc}}(S))^S$ is the set of functions $(f, g) \in C_0(S) \times C(S)$ such that, for any $a \in S$ and any open subset $U \Subset S$,

$$f(X_{t \wedge \tau^U}) - \int_0^{t \wedge \tau^U} g(X_s) ds \text{ is a } \mathbf{P}_a\text{-martingale.}$$

It was noticed in Remark 4.11 in [GH17b] that if $h \in C(S, \mathbb{R}_+^*)$ and if L is the $C_0 \times C$ -generator of a locally Feller family, then

$$hL := \{(f, hg) \mid (f, g) \in L\} \text{ is the } C_0 \times C\text{-generator of a locally Feller family.} \quad (2.2)$$

A family of probability measures associated to a Feller semi-group constitutes a natural example of locally Feller family (see Theorem 4.8 from [GH17b]). We recall that a Feller semi-group $(T_t)_{t \in \mathbb{R}_+}$ is a strongly continuous semi-group of positive linear contractions on $C_0(S)$. Its $C_0 \times C_0$ -generator is the set L_0 of $(f, g) \in C_0(S) \times C_0(S)$ such that, for all $a \in S$

$$\lim_{t \rightarrow 0} \frac{1}{t} (T_t f(a) - f(a)) = g(a).$$

It was proved in Propositions 4.2 and 4.12 from [GH17b], that the martingale problem associated to L_0 admits a unique solution and, if L denotes its $C_0(S) \times C(S)$ -generator then, taking the closure in $C_0(S) \times C(S)$,

$$L_0 = L \cap C_0(S) \times C_0(S) \quad \text{and} \quad L = \overline{L_0}. \quad (2.3)$$

The following result of convergence is essential for our further development and it was proved in [GH17b]. As was already pointed out in the introduction, an improvement with respect to the classical result of convergence (for instance Theorem 19.25, p. 385, in [Kal02]), is that one does not need to know the generator of the limit family, but only the fact that a martingale local problem is well-posed.

Theorem 2.2 (Convergence of locally Feller family). *For $n \in \mathbb{N} \cup \{\infty\}$, let $(\mathbf{P}_a^n)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be a locally Feller family and let L_n be a subset of $C_0(S) \times C(S)$. Suppose that for any $n \in \mathbb{N}$, $\overline{L_n}$ is the generator of $(\mathbf{P}_a^n)_a$, suppose also that $D(L_\infty)$ is dense in $C_0(S)$ and*

$$\forall a \in S : \quad \mathbf{P} \in \mathcal{M}(L_\infty) \quad \text{and} \quad \mathbf{P}(X_0 = a) = 1 \iff \mathbf{P} = \mathbf{P}_a^\infty.$$

Then we have equivalence between:

a) the mapping

$$\begin{array}{ccc} \mathbb{N} \cup \{\infty\} \times \mathcal{P}(S^\Delta) & \rightarrow & \mathcal{P}(\mathbb{D}_{loc}(S)) \\ (n, \mu) & \mapsto & \mathbf{P}_\mu^n \end{array}$$

is weakly continuous for the local Skorokhod topology, where $\mathbf{P}_\mu := \int \mathbf{P}_a \mu(da)$ and $\mathbf{P}_\Delta(X_0 = \Delta) = 1$;

b) for any $a_n, a \in S$ such that $a_n \rightarrow a$, $\mathbf{P}_{a_n}^n$ converges weakly for the local Skorokhod topology to \mathbf{P}_a^∞ , as $n \rightarrow \infty$;

c) for any $f \in D(L_\infty)$, there exist for each n , $f_n \in D(L_n)$ such that $f_n \xrightarrow[n \rightarrow \infty]{C_0} f$,
 $L_n f_n \xrightarrow[n \rightarrow \infty]{C} L_\infty f$.

3 Convergence of families indexed by discrete time

We start our study by giving a discrete time version of the notion of locally Feller family.

Definition 3.1 (Discrete time locally Feller family). We denote by Y the discrete time canonical process on $(S^\Delta)^\mathbb{N}$ and we endow $(S^\Delta)^\mathbb{N}$ with the canonical σ -algebra. A family $(\mathbf{P}_a)_a \in \mathcal{P}((S^\Delta)^\mathbb{N})^S$ is said to be a discrete time locally Feller family if there exists an operator $T : C_0(S) \rightarrow C_b(S)$, called transition operator, such that for any $a \in S$: $\mathbf{P}_a(Y_0 = a) = 1$ and

$$\forall n \in \mathbb{N}, \forall f \in C_0(S), \quad \mathbf{E}_a(f(Y_{n+1}) \mid Y_0, \dots, Y_n) = \mathbf{1}_{\{Y_n \neq \Delta\}} T f(Y_n) \quad \mathbf{P}_a\text{-a.s.} \quad (3.1)$$

If we denote \mathbf{P}_Δ the probability defined by $\mathbf{P}_\Delta(\forall n \in \mathbb{N}, Y_n = \Delta) = 1$, then for $\mu \in \mathcal{P}(S^\Delta)$, $\mathbf{P}_\mu := \int \mathbf{P}_a \mu(da)$ satisfies also (3.1).

Now we can state the main result of this section which, similarly, is an improvement with respect to Theorem 19.27, p. 387, in [Kal02], in the sense that one does not need to know the generator of the limit family, but only the fact that a martingale local problem is well-posed.

Theorem 3.2 (Convergence). *Let L be a subset of $C_0(S) \times C(S)$ with $D(L)$ a dense subset of $C_0(S)$, such that the martingale local problem associated to L is well-posed, and let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be the associated continuous time locally Feller family. For each $n \in \mathbb{N}$ we introduce $(\mathbf{P}_a^n)_a \in \mathcal{P}((S^\Delta)^\mathbb{N})^S$ a discrete time locally Feller families having their transition operator T_n . We denote by L_n the operator $(T_n - \text{id})/\varepsilon_n$, where $(\varepsilon_n)_n$ is a sequence of positive constants converging to 0, as $n \rightarrow \infty$. There is equivalence between:*

a) for any $\mu_n, \mu \in \mathcal{P}(S^\Delta)$ such that $\mu_n \rightarrow \mu$ weakly, as $n \rightarrow \infty$,

$$\mathcal{L}_{\mathbf{P}_{\mu_n}^n} \left((Y_{\lfloor t/\varepsilon_n \rfloor})_t \right) \xrightarrow[n \rightarrow \infty]{\mathcal{P}(\mathbb{D}_{loc}(S))} \mathbf{P}_\mu;$$

b) for any $a_n, a \in S$ such that $a_n \rightarrow a$, as $n \rightarrow \infty$,

$$\mathcal{L}_{\mathbf{P}_{a_n}^n} \left((Y_{\lfloor t/\varepsilon_n \rfloor})_t \right) \xrightarrow[n \rightarrow \infty]{\mathcal{P}(\mathbb{D}_{loc}(S))} \mathbf{P}_a;$$

c) for any $f \in D(L)$, there exists $(f_n)_n \in C_0(S)^\mathbb{N}$ such that $f_n \xrightarrow[n \rightarrow \infty]{C_0(S)} f$, $L_n f_n \xrightarrow[n \rightarrow \infty]{C(S)} Lf$.

Here $\lfloor r \rfloor$ denotes the integer part of the real number r .

Proof. Let $\Omega := (S^\Delta)^\mathbb{N} \times \mathbb{R}_+^\mathbb{N}$ and $\mathcal{G} := \mathcal{B}(S^\Delta)^{\otimes \mathbb{N}} \otimes \mathcal{B}(\mathbb{R}_+)^{\otimes \mathbb{N}}$ be. For any $\mu \in \mathcal{P}(S^\Delta)$ and $n \in \mathbb{N}$, define $\mathbb{P}_\mu^n := \mathbf{P}_\mu^n \otimes \mathcal{E}(1)^{\otimes \mathbb{N}}$, where $\mathcal{E}(1)$ is the exponential distribution with expectation 1. Define

$$Y_n : \quad \Omega \quad \rightarrow \quad S \quad \text{and} \quad E_n : \quad \Omega \quad \rightarrow \quad \mathbb{R}_+ \\ ((y_k)_k, (s_k)_k) \mapsto y_n \quad \quad \quad ((y_k)_k, (s_k)_k) \mapsto s_n,$$

and introduce the standard Poisson process

$$\forall t \geq 0, \quad N_t := \inf \left\{ n \in \mathbb{N} \mid \sum_{k=1}^{n+1} E_k > t \right\}.$$

Step 1) For each $n \in \mathbb{N}$ define $Z_t^n := Y_{N_t/\varepsilon_n}$. Consider the modified assertions:

a') for any $\mu_n, \mu \in \mathcal{P}(S^\Delta)$ such that $\mu_n \rightarrow \mu$,

$$\mathcal{L}_{\mu_n}^{\mathbb{P}_n}(Z^n) \xrightarrow[n \rightarrow \infty]{\mathcal{P}(\mathbb{D}_{\text{loc}}(S))} \mathbf{P}_\mu;$$

b') for any $a_n, a \in S$ such that $a_n \rightarrow a$,

$$\mathcal{L}_{a_n}^{\mathbb{P}_n}(Z^n) \xrightarrow[n \rightarrow \infty]{\mathcal{P}(\mathbb{D}_{\text{loc}}(S))} \mathbf{P}_a,$$

We will verify that $a') \Leftrightarrow b') \Leftrightarrow c)$. We need to prove that for all $\mu \in \mathcal{P}(S^\Delta)$, $\mathcal{L}_{\mu}^{\mathbb{P}_n}(Z^n) \in \mathcal{M}(L_n)$. Taking $\mathcal{G}_t^n := \sigma(N_{s/\varepsilon_n}, Z_s^n, s \leq t)$, it is enough to prove that, for each $f \in C_0(S)$ and $0 \leq s \leq t$,

$$\mathbb{E}_\mu^n \left[f(Z_t^n) - f(Z_s^n) - \int_s^t L_n f(Z_u^n) du \mid \mathcal{G}_s^n \right] = 0.$$

Let us introduce the (\mathcal{G}_t^n) -stopping times $\tau_k^n := \inf \{ u \geq 0 \mid N_{u/\varepsilon_n} = k \}$. Then, for all $k \in \mathbb{N}$,

$$\begin{aligned} & \mathbb{E}_\mu^n \left[f(Z_{t \wedge (\tau_{k+1}^n \vee s)}^n) - f(Z_{t \wedge (\tau_k^n \vee s)}^n) \mid \mathcal{G}_{t \wedge (\tau_k^n \vee s)}^n \right] \\ &= \mathbf{1}_{\{t > \tau_k^n, s < \tau_{k+1}^n\}} \mathbb{E}_\mu^n \left[(f(Y_{k+1}) - f(Y_k)) \mathbf{1}_{\{\tau_{k+1}^n \leq t\}} \mid \mathcal{G}_{t \wedge (\tau_k^n \vee s)}^n \right] \\ &= \mathbf{1}_{\{t > \tau_k^n, s < \tau_{k+1}^n\}} \mathbb{E}_\mu^n \left[(f(Y_{k+1}) - f(Y_k)) \mathbf{1}_{\{\tau_{k+1}^n - \tau_k^n \vee s \leq t - \tau_k^n \vee s\}} \mid \mathcal{G}_{\tau_k^n \vee s}^n \right] \\ &= \mathbf{1}_{\{t > \tau_k^n, s < \tau_{k+1}^n\}} (T_n f(Y_k) - f(Y_k)) (1 - \exp(-(t - \tau_k^n \vee s)/\varepsilon_n)) \\ &= \mathbf{1}_{\{t > \tau_k^n, s < \tau_{k+1}^n\}} L_n f(Z_{\tau_k^n \vee s}^n) \varepsilon_n (1 - \exp(-(t - \tau_k^n \vee s)/\varepsilon_n)), \end{aligned}$$

where we used the fact that $(N_{u/\varepsilon_n})_u$ is a Poisson process. Similarly,

$$\begin{aligned} & \mathbb{E}_\mu^n \left[\int_{t \wedge (\tau_k^n \vee s)}^{t \wedge (\tau_{k+1}^n \vee s)} L_n f(Z_u^n) du \mid \mathcal{G}_{t \wedge (\tau_k^n \vee s)}^n \right] \\ &= \mathbf{1}_{\{t > \tau_k^n, s < \tau_{k+1}^n\}} L_n f(Z_{\tau_k^n \vee s}^n) \mathbb{E}_\mu^n \left[t \wedge \tau_{k+1}^n - \tau_k^n \vee s \mid \mathcal{G}_{t \wedge (\tau_k^n \vee s)}^n \right] \\ &= \mathbf{1}_{\{t > \tau_k^n, s < \tau_{k+1}^n\}} L_n f(Z_{\tau_k^n \vee s}^n) \mathbb{E}_\mu^n \left[(t - \tau_k^n \vee s) \wedge (\tau_{k+1}^n - \tau_k^n \vee s) \mid \mathcal{G}_{\tau_k^n \vee s}^n \right] \\ &= \mathbf{1}_{\{t > \tau_k^n, s < \tau_{k+1}^n\}} L_n f(Z_{\tau_k^n \vee s}^n) \int_0^\infty (1/\varepsilon_n) \exp(-u/\varepsilon_n) ((t - \tau_k^n \vee s) \wedge u) du \\ &= \mathbf{1}_{\{t > \tau_k^n, s < \tau_{k+1}^n\}} L_n f(Z_{\tau_k^n \vee s}^n) \varepsilon_n (1 - \exp(-(t - \tau_k^n \vee s)/\varepsilon_n)). \end{aligned}$$

Hence

$$\mathbb{E}_\mu^n \left[f(Z_{t \wedge (\tau_{k+1}^n \vee s)}^n) - f(Z_{t \wedge (\tau_k^n \vee s)}^n) - \int_{t \wedge (\tau_k^n \vee s)}^{t \wedge (\tau_{k+1}^n \vee s)} L_n f(Z_u^n) du \mid \mathcal{G}_{t \wedge (\tau_k^n \vee s)}^n \right] = 0.$$

Hence

$$\begin{aligned}
& \mathbb{E}_\mu^n \left[f(Z_t^n) - f(Z_s^n) - \int_s^t L_n f(Z_u^n) du \mid \mathcal{G}_s^n \right] \\
&= \mathbb{E}_\mu^n \left[\sum_{k \geq 0} \left(f(Z_{t \wedge (\tau_{k+1}^n \vee s)}^n) - f(Z_{t \wedge (\tau_k^n \vee s)}^n) - \int_{t \wedge (\tau_k^n \vee s)}^{t \wedge (\tau_{k+1}^n \vee s)} L_n f(Z_u^n) du \right) \mid \mathcal{G}_s^n \right] \\
&= \sum_{k \geq 0} \mathbb{E}_\mu^n \left[\mathbb{E}_\mu^n \left[f(Z_{t \wedge (\tau_{k+1}^n \vee s)}^n) - f(Z_{t \wedge (\tau_k^n \vee s)}^n) - \int_{t \wedge (\tau_k^n \vee s)}^{t \wedge (\tau_{k+1}^n \vee s)} L_n f(Z_u^n) du \mid \mathcal{G}_{t \wedge (\tau_k^n \vee s)}^n \right] \mid \mathcal{G}_s^n \right] \\
&= 0,
\end{aligned}$$

so $\mathcal{L}_{\mathbb{P}_\mu^n}(Z^n) \in \mathcal{M}(L_n)$. Hence applying the convergence theorem 2.2 to L_n and L , we obtain the equivalences between a' , b') and c).

Step 2. To carry on with the proof we need the following technical result

Lemma 3.3. *For $n \in \mathbb{N}$, let $(\Omega^n, \mathcal{G}^n, \mathbb{P}^n)$ be a probability space, let $Z^n : \Omega^n \rightarrow \mathbb{D}_{loc}(S)$ and $\Gamma^n : \Omega^n \rightarrow \mathbb{C}(\mathbb{R}_+, \mathbb{R}_+)$ be a increasing random bijection. Define $\tilde{Z}^n := Z^n \circ \Gamma^n$. Suppose that for each $\varepsilon > 0$ and $t \in \mathbb{R}_+$*

$$\mathbb{P}^n \left(\sup_{s \leq t} |\Gamma_s^n - s| \geq \varepsilon \right) \xrightarrow{n \rightarrow \infty} 0.$$

Then for any $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{loc}(S))$,

$$\mathcal{L}_{\mathbb{P}^n}(Z^n) \xrightarrow{n \rightarrow \infty} \mathbf{P} \quad \Leftrightarrow \quad \mathcal{L}_{\mathbb{P}^n}(\tilde{Z}^n) \xrightarrow{n \rightarrow \infty} \mathbf{P},$$

where the limits are for the weak topology associated to the local Skorokhod topology.

We postpone the proof of this lemma and we finish the proof of the theorem. Let us note that for any $t \geq 0$ and $n \in \mathbb{N}$, $Y_{\lfloor t/\varepsilon_n \rfloor} = Z_{\Gamma_t^n}^n$ with

$$\Gamma_t^n := \varepsilon_n \left(\sum_{k=1}^{\lfloor t/\varepsilon_n \rfloor} E_k + (t/\varepsilon_n - \lfloor t/\varepsilon_n \rfloor) E_{\lfloor t/\varepsilon_n \rfloor + 1} \right).$$

Assuming that

$$\forall t \geq 0, \forall \varepsilon > 0, \sup_{\mu \in \mathcal{P}(S^\Delta)} \mathbb{P}_\mu^n \left(\sup_{s \leq t} |\Gamma_s^n - s| \geq \varepsilon \right) \xrightarrow{n \rightarrow \infty} 0, \quad (3.2)$$

then, by the latter lemma we get $a) \Leftrightarrow a')$ and $b \Leftrightarrow b')$, so $a) \Leftrightarrow b) \Leftrightarrow c)$.

Let us prove (3.2). Fix $t \geq 0$, $\varepsilon > 0$, $n \in \mathbb{N}$ and $\mu \in \mathcal{P}(S^\Delta)$, then since Γ^n is a continuous piecewise affine function, we have

$$\sup_{s \leq t} |\Gamma_s^n - s| \leq \sup_{\substack{k \in \mathbb{N} \\ k \leq \lfloor t/\varepsilon_n \rfloor}} |\Gamma_{k\varepsilon_n}^n - k\varepsilon_n| = \sup_{\substack{k \in \mathbb{N} \\ k \leq \lfloor t/\varepsilon_n \rfloor}} \left| \varepsilon_n \sum_{i=1}^k E_i - k\varepsilon_n \right| = \varepsilon_n \sup_{\substack{k \in \mathbb{N} \\ k \leq \lfloor t/\varepsilon_n \rfloor}} |M_k|$$

where $M_k := \sum_{i=1}^k E_i - k$ and $\lceil r \rceil$ denotes the smallest integer larger or equal than the real number r . Since the E_i are independent random variables with exponential distribution $\mathcal{E}(1)$ we have

$$\mathbb{E}_\mu^n[M_k^2] = k\mathbb{E}_\mu^n[(E_1 - 1)^2] = k.$$

By Markov's inequality and by the maximal Doob inequality applied to the discrete time martingale $(M_k)_k$ we can write

$$\begin{aligned} \mathbb{P}_\mu^n \left(\sup_{s \leq t} |\Gamma_s^n - s| \geq \varepsilon \right) &\leq \mathbb{P}_\mu^n \left(\varepsilon_n \sup_{k \leq \lceil t/\varepsilon_n \rceil} |M_k| \geq \varepsilon \right) \leq \frac{\mathbb{E}_\mu^n \left[\sup_{k \leq \lceil t/\varepsilon_n \rceil} M_k^2 \right] \varepsilon_n^2}{\varepsilon^2} \\ &\leq \frac{4\mathbb{E}_\mu^n \left[M_{\lceil t/\varepsilon_n \rceil}^2 \right] \varepsilon_n^2}{\varepsilon^2} = \frac{4\lceil t/\varepsilon_n \rceil \varepsilon_n^2}{\varepsilon^2} \leq \frac{4(t + \varepsilon_n)\varepsilon_n}{\varepsilon^2}. \end{aligned}$$

We deduce (3.2) and the proof of the theorem is complete except for the proof of Lemma 3.3. \square

Before giving the proof of Lemma 3.3 we state and prove a more general result:

Lemma 3.4. *Let E be a Polish topological space, for $n \in \mathbb{N}$, let $(\Omega^n, \mathcal{G}^n, \mathbb{P}^n)$ be a probability space and consider $Z^n, \tilde{Z}^n : \Omega^n \rightarrow E$ random variables. Suppose that for each compact subset $K \subset E$ and each open subset $U \subset E^2$ containing the diagonal $\{(z, z) \mid z \in E\}$,*

$$\mathbb{P}^n \left(Z^n \in K, (Z^n, \tilde{Z}^n) \notin U \right) \xrightarrow[n \rightarrow \infty]{} 0. \quad (3.3)$$

Then, for any $\mathbf{P} \in \mathcal{P}(E)$,

$$\mathcal{L}_{\mathbb{P}^n}(Z^n) \xrightarrow[n \rightarrow \infty]{} \mathbf{P} \quad \text{implies} \quad \mathcal{L}_{\mathbb{P}^n}(\tilde{Z}^n) \xrightarrow[n \rightarrow \infty]{} \mathbf{P},$$

where the limits are for the weak topology on $\mathcal{P}(E)$.

Proof. Suppose that $\mathcal{L}_{\mathbb{P}^n}(Z^n) \xrightarrow[n \rightarrow \infty]{} \mathbf{P}$ so for any bounded continuous function $f : E \rightarrow \mathbb{R}$, $\lim_{n \rightarrow \infty} \mathbb{E}^n[f(Z^n)] = \int f d\mathbf{P}$. Since E is a Polish space the sequence $(\mathcal{L}_{\mathbb{P}^n}(Z^n))_n$ is tight. Take an arbitrary $\varepsilon > 0$ and let K be a compact subset of E such that

$$\forall n \in \mathbb{N}, \quad \mathbb{P}^n(Z^n \notin K) \leq \varepsilon. \quad (3.4)$$

By (3.3) applied to K and $U := \{(z, \tilde{z}) \mid |f(\tilde{z}) - f(z)| < \varepsilon\}$, we have

$$\mathbb{P}^n \left(Z^n \in K, |f(\tilde{Z}^n) - f(Z^n)| \geq \varepsilon \right) \xrightarrow[n \rightarrow \infty]{} 0.$$

Hence by (3.4)

$$\begin{aligned} \left| \mathbb{E}^n[f(\tilde{Z}^n)] - \int f d\mathbf{P} \right| &\leq \left| \mathbb{E}^n[f(Z^n)] - \int f d\mathbf{P} \right| + \mathbb{E}^n |f(\tilde{Z}^n) - f(Z^n)| \\ &\leq \left| \mathbb{E}^n[f(Z^n)] - \int f d\mathbf{P} \right| + \mathbb{E}^n \left[|f(\tilde{Z}^n) - f(Z^n)| \mathbf{1}_{\{Z^n \in K, |f(\tilde{Z}^n) - f(Z^n)| \geq \varepsilon\}} \right] \\ &\quad + \mathbb{E}^n \left[|f(\tilde{Z}^n) - f(Z^n)| \mathbf{1}_{\{Z^n \in K, |f(\tilde{Z}^n) - f(Z^n)| < \varepsilon\}} \right] + \mathbb{E}^n \left[|f(\tilde{Z}^n) - f(Z^n)| \mathbf{1}_{\{Z^n \notin K\}} \right] \\ &\leq \left| \mathbb{E}^n[f(Z^n)] - \int f d\mathbf{P} \right| + 2\|f\| \mathbb{P}^n \left(Z^n \in K, |f(\tilde{Z}^n) - f(Z^n)| \geq \varepsilon \right) + \varepsilon(1 + 2\|f\|). \end{aligned}$$

Letting successively $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$, we deduce that

$$\mathbb{E}^n[f(\tilde{Z}^n)] \xrightarrow{n \rightarrow \infty} \int f d\mathbf{P},$$

hence, since f is an arbitrary bounded continuous function, we have $\mathcal{L}_{\mathbb{P}^n}(\tilde{Z}^n) \xrightarrow{n \rightarrow \infty} \mathbf{P}$. \square

Proof of Lemma 3.3. We denote by $\tilde{\Lambda}$ the space of increasing bijections λ from \mathbb{R}_+ to \mathbb{R}_+ , and for $t \in \mathbb{R}_+$ we denote $\|\lambda - \text{id}\|_t := \sup_{s \leq t} |\lambda_s - s|$. Since

$$\forall \lambda \in \tilde{\Lambda}, \forall t \in \mathbb{R}_+, \forall \varepsilon > 0, \quad \|\lambda - \text{id}\|_{t+\varepsilon} < \varepsilon \Rightarrow \|\lambda^{-1} - \text{id}\|_t < \varepsilon,$$

the hypotheses of Lemma 3.3 are symmetric with respect to Z and \tilde{Z} , so it suffices to prove only one implication. Hence we suppose $\mathcal{L}_{\mathbb{P}^n}(Z^n) \xrightarrow{n \rightarrow \infty} \mathbf{P}$ and, by applying Lemma 3.4, we prove $\mathcal{L}_{\mathbb{P}^n}(\tilde{Z}^n) \xrightarrow{n \rightarrow \infty} \mathbf{P}$. Let K be a compact subset of $\mathbb{D}_{\text{loc}}(S)$ and U be an open subset of $\mathbb{D}_{\text{loc}}(S)^2$ containing the diagonal $\{(z, z) \mid z \in \mathbb{D}_{\text{loc}}(S)\}$. We prove the assertion

$$\exists t \geq 0, \exists \varepsilon > 0, \forall z \in K, \forall \lambda \in \tilde{\Lambda}, \quad \|\lambda - \text{id}\|_t < \varepsilon \Rightarrow (z, z \circ \lambda) \in U. \quad (3.5)$$

If we suppose that (3.5) is false, then we can find two sequences $(z^n)_n \in K^{\mathbb{N}}$ and $(\lambda^n)_n \in \tilde{\Lambda}^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $(z^n, z^n \circ \lambda^n) \notin U$ and for all $t \geq 0$, $\|\lambda^n - \text{id}\|_t \rightarrow 0$, as $n \rightarrow \infty$. By compactness of K , possibly by taking a subsequence, we may suppose the existence of $z \in K$ such that $z^n \rightarrow z$ as $n \rightarrow \infty$. Then, it is straightforward to obtain

$$U \not\ni (z^n, z^n \circ \lambda^n) \xrightarrow{n \rightarrow \infty} (z, z) \in U.$$

This is a contradiction with the fact that U is open, so we have proved (3.5). Take t and ε given by (3.5), then

$$\mathbb{P}^n(Z^n \in K, (Z^n, \tilde{Z}^n) \notin U) \leq \mathbb{P}^n(\|\Gamma^n - \text{id}\|_t \geq \varepsilon) \xrightarrow{n \rightarrow \infty} 0.$$

Hence by Lemma 3.4, $\mathcal{L}_{\mathbb{P}^n}(\tilde{Z}^n) \xrightarrow{n \rightarrow \infty} \mathbf{P}$. \square

4 Lévy-type processes: convergence and discrete scheme

In this section we take $d \in \mathbb{N}^*$, we denote by $|\cdot|$ the Euclidean norm and by $\mathbb{R}^{d\Delta}$ the one point compactification of \mathbb{R}^d . Let also $C_c^\infty(\mathbb{R}^d)$ be the set of compactly supported infinitely differentiable functions from \mathbb{R}^d to \mathbb{R} . We are interested in the dynamics which locally looks like as Lévy's dynamics. To simplify notations all along of the present section, let us introduce a linear functional on $C_c^\infty(\mathbb{R}^d)$ which describes a dynamic in a neighbourhood of a point $a \in \mathbb{R}^d$: for $f \in C_c^\infty(\mathbb{R}^d)$,

$$T_{\chi, a}(\delta, \gamma, \nu)f := \frac{1}{2} \sum_{i, j=1}^d \gamma_{ij} \partial_{ij}^2 f(a) + \delta \cdot \nabla f(a) + \int_{\mathbb{R}^{d\Delta}} (f(b) - f(a) - \chi(a, b) \cdot \nabla f(a)) \nu(db),$$

where

– the compensation function $\chi : \mathbb{R}^d \times \mathbb{R}^{d\Delta} \rightarrow \mathbb{R}^d$ is a bounded measurable function satisfying, for any compact subset $K \subset \mathbb{R}^d$,

$$\sup_{b,c \in K, b \neq c} \frac{|\chi(b,c) - (c-b)|}{|c-b|^2} < \infty; \quad (\text{H1})$$

– the drift vector is $\delta \in \mathbb{R}^d$, the diffusion matrix $\gamma \in \mathbb{R}^{d \times d}$ is symmetric positive semi-definite and the jump measure ν is a measure on $\mathbb{R}^{d\Delta}$ satisfying $\nu(\{a\}) = 0$ and

$$\int_{\mathbb{R}^{d\Delta}} (1 \wedge |b-a|^2) \nu(db) < \infty.$$

Usually we take for compensation function

$$\chi_1(a,b) := (b-a)/(1+|b-a|^2) \quad \text{or} \quad \chi_2(a,b) := (b-a)\mathbb{1}_{|b-a|<1}. \quad (4.1)$$

It is well known (see for instance Theorem 2.12 pp. 21-22 from [Hoh98]) that for any linear operator $L : C_c^\infty(\mathbb{R}^d) \rightarrow C(\mathbb{R}^d)$ satisfying the positive maximum principle and for any χ satisfying (H1): for each $a \in \mathbb{R}^d$ there exist $\delta(a)$, $\gamma(a)$ and $\nu(a)$ satisfying (H2(a)) such that

$$\forall f \in C_c^\infty(\mathbb{R}^d), \forall a \in \mathbb{R}^d, \quad Lf(a) = T_{\chi,a}(\delta(a), \gamma(a), \nu(a))f.$$

In the following we will call a such expression of L a *Lévy-type operator*.

In order to obtain a converse sentence and to get the convergence of sequences of Lévy-type operators, we have to made a more restrictive hypothesis on the couple (χ, ν) : for $a \in \mathbb{R}^d$

– the compensation function $\chi : \mathbb{R}^d \times \mathbb{R}^{d\Delta} \rightarrow \mathbb{R}^d$ is a bounded measurable function satisfying, for any compact subset $K \subset \mathbb{R}^d$,

$$\sup_{b,c \in K, 0 < |c-b| \leq \varepsilon} \frac{|\chi(b,c) - (c-b)|}{|c-b|^2} \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (\text{H3(a)})$$

and $\nu(\{b \in \mathbb{R}^{d\Delta} \mid \chi \text{ is not continuous at } (a,b)\}) = 0$.

For example, χ_1 given in (4.1) satisfies (H3(a)) for any ν and $\chi_2(a,b)$ satisfies (H3(a)) whenever $\nu(\{b \in \mathbb{R}^d \mid |b-a| = 1\}) = 0$.

The following theorem contains a necessary and sufficient condition for the convergence of sequences Lévy-type operators (and processes) in terms of their Lévy triplet. Before we introduce some notations.

- Let $\chi : \mathbb{R}^d \times \mathbb{R}^{d\Delta} \rightarrow \mathbb{R}^d$ be a compensation function. For each $a \in \mathbb{R}^d$ let $(\delta(a), \gamma(a), \nu(a))$ and $(\chi, \nu(a))$ be satisfying respectively (H2(a)) and (H3(a)). Set

$$Lf(a) := T_{\chi,a}(\delta(a), \gamma(a), \nu(a))f, \quad \text{for any } f \in C_c^\infty(\mathbb{R}^d). \quad (4.2)$$

- For each $n \in \mathbb{N}$ and $a \in \mathbb{R}^d$ let $(\delta_n(a), \gamma_n(a), \nu_n(a))$ be satisfying (H2(a)). Set

$$L_n f(a) := T_{\chi, a}(\delta_n(a), \gamma_n(a), \nu_n(a))f, \quad \text{for any } f \in C_c^\infty(\mathbb{R}^d). \quad (4.3)$$

- For each $n \in \mathbb{N}$ and $a \in \mathbb{R}^d$ let $\mu_n(a)$ be a probability measure on $\mathbb{R}^{d\Delta}$ and let $\varepsilon_n > 0$ be a sequence converging to 0. Set

$$T_n f(a) := \int f(b) \mu_n(a, db), \quad \text{for any } f \in C(\mathbb{R}^{d\Delta}). \quad (4.4)$$

Theorem 4.1 (Characterisation of the convergence toward Lévy-type operators).

1) The function Lf is continuous for any $f \in C_c^\infty(\mathbb{R}^d)$ if and only if

- $a \mapsto \delta(a)$ is continuous on \mathbb{R}^d ,
- $a \mapsto \int f(b) \nu(a, db)$ is continuous on the interior of $\{f = 0\} \cap \mathbb{R}^d$, for any $f \in C(\mathbb{R}^{d\Delta})$,
- $a \mapsto \gamma_{ij}(a) + \int \chi_i(a, b) \chi_j(a, b) \nu(a, db)$ is continuous on \mathbb{R}^d , for any $1 \leq i, j \leq d$.

2) Assume that Lf is continuous for any $f \in C_c^\infty(\mathbb{R}^d)$. The uniform convergence on compact sets, $L_n f \rightarrow Lf$, as $n \rightarrow \infty$, holds for all $f \in C_c^\infty(\mathbb{R}^d)$ if and only if

- $\delta_n(a) \rightarrow \delta(a)$, uniformly for a varying in compact subsets of \mathbb{R}^d ,
- $\int f(b) \nu_n(a, db) \rightarrow \int f(b) \nu(a, db)$, uniformly for a varying in compact subsets of the interior of $\{f = 0\} \cap \mathbb{R}^d$, for any $f \in C(\mathbb{R}^{d\Delta})$,
- $\gamma_{n,ij}(a) + \int (\chi_i \chi_j)(a, b) \nu_n(a, db) \rightarrow \gamma_{ij}(a) + \int (\chi_i \chi_j)(a, b) \nu(a, db)$, uniformly for a varying in compact subsets of \mathbb{R}^d , for any $1 \leq i, j \leq d$.

3) Assume that Lf is continuous for any $f \in C_c^\infty(\mathbb{R}^d)$. The uniform convergence on compact sets, $\varepsilon_n^{-1}(T_n f - f) \rightarrow Lf$, as $n \rightarrow \infty$, holds for all $f \in C_c^\infty(\mathbb{R}^d)$ if and only if

- $\varepsilon_n^{-1} \int_{\mathbb{R}^{d\Delta} \setminus \{a\}} \chi(a, b) \mu_n(a, db) \rightarrow \delta(a)$, uniformly for a varying in compact subsets of \mathbb{R}^d ,
- $\varepsilon_n^{-1} \int f(b) \mu_n(a, db) \rightarrow \int f(b) \nu(a, db)$, uniformly for a varying in the compact subsets of the interior of $\{f = 0\} \cap \mathbb{R}^d$, for any $f \in C(\mathbb{R}^{d\Delta})$,
- $\varepsilon_n^{-1} \int_{\mathbb{R}^{d\Delta} \setminus \{a\}} (\chi_i \chi_j)(a, b) \mu_n(a, db) \rightarrow \gamma_{ij}(a) + \int (\chi_i \chi_j)(a, b) \nu(a, db)$, uniformly for a varying in the compact subsets of \mathbb{R}^d , for any $1 \leq i, j \leq d$.

Remark 4.2. Thanks to Theorems 2.2 and 3.2 we can deduce from Theorem 4.1 sharp results of convergence for the processes associated to L_n , T_n and L . In particular, the third part of Theorem 4.1 one could be seen as an improvement of the classical Donsker theorem, and, for instance allows us to simulate Lévy-type processes. We illustrate this fact by the following example. \diamond

Example 4.3 (Symmetric stable type operator). Let $c \in C(\mathbb{R}^d, \mathbb{R}_+)$ and $\alpha \in C(\mathbb{R}^d, (0, 2))$ be and denote, for $f \in C_0(\mathbb{R}^d)$ and $a \in \mathbb{R}^d$,

$$Lf(a) := \int_{\mathbb{R}^d} (f(b) - f(a) - (b-a) \cdot \nabla f(a) \mathbb{1}_{|b-a| \leq 1}) c(a) |b-a|^{-d-\alpha(a)} db.$$

As a consequence of the first part of Theorem 4.1, L maps $C_0(\mathbb{R}^d)$ to $C(\mathbb{R}^d)$. For $a \in \mathbb{R}^d$ and $n \in \mathbb{N}^*$, define the probability measure

$$\mu_n(a, db) := \frac{c(a)}{n} |b-a|^{-d-\alpha(a)} \mathbb{1}_{|b-a| \geq \varepsilon_n(a)} db, \quad \text{with} \quad \varepsilon_n(a) := \left(\frac{c(a) S_{d-1}}{n \alpha(a)} \right)^{1/\alpha(a)}$$

and where $S_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the measure of the unity sphere in \mathbb{R}^d . Thanks to the third part of Theorem 4.1, for any $f \in C_c^\infty(\mathbb{R}^d)$,

$$\lim_{n \rightarrow \infty} n \left(\int f(b) \mu_n(a, db) - f(a) \right) = Lf(a), \quad \text{uniformly for } a \text{ in compact subsets of } \mathbb{R}^d.$$

To go further, it is straightforward that for any $a \in \mathbb{R}^d$ and $n \in \mathbb{N}^*$, $\mu_n(a)$ is the distribution of the random variable

$$a + Q \left(\frac{c(a) S_{d-1}}{n \alpha(a) U} \right)^{1/\alpha(a)}, \quad \text{with independent } Q \sim \mathcal{U}(\mathbb{S}^{d-1}), U \sim \mathcal{U}([0, 1]),$$

where $\mathcal{U}(\mathbb{S}^{d-1})$ and $\mathcal{U}([0, 1])$ are the uniform distributions, respectively on the unity sphere of \mathbb{R}^d and on $[0, 1]$. To simulate the discrete time locally Feller processes associated to $(\mu_n(a))_a$ we can proceed as follows. Let $(Q_k, U_k)_k$ be an i.i.d. sequence of random variables with distributions $\mathcal{U}(\mathbb{S}^{d-1}) \otimes \mathcal{U}([0, 1])$ and define, for $n \in \mathbb{N}^*$ and $k \in \mathbb{N}$,

$$Z_{k+1}^n := Z_k^n + Q_k \left(\frac{c(Z_k^n) S_{d-1}}{n \alpha(Z_k^n) U_k} \right)^{1/\alpha(Z_k^n)}.$$

Hence thanks to Theorem 3.2, if the martingale local problem associated to L is well-posed, then $(Z_{[nt]}^n)_t$ converges in distribution to the solution of the martingale local problem. \diamond

Remark 4.4. This example is adaptable when we want to simulate more general Lévy-type processes. The heuristics is as follows: first we approximate the Lévy measure by finite measures, we renormalise them and then we convolute with a Gaussian measure having well chosen parameters. \diamond

Before proceeding to the proof Theorem 4.1, we give a second approximation result inspired from [BSW13], Theorem 7.6 p. 172. Let $L : C_c^\infty(\mathbb{R}^d) \rightarrow C(\mathbb{R}^d)$ be an operator satisfying the positive maximum principle. Let the translation of f by $h \in \mathbb{R}^d$ be the mapping $\tau_h f(a) = f(a+h)$. For $a_0 \in \mathbb{R}^d$, we introduce the operator

$$L(a_0) : C_c^\infty(\mathbb{R}^d) \rightarrow C_0(\mathbb{R}^d) \quad \text{by} \quad L(a_0)f(a) := L(\tau_{a-a_0}f)(a_0). \quad (4.5)$$

Clearly $Lf(a) = L(a)f(a)$. Since $L(a_0)$ is invariant with respect to the translation and satisfies the positive maximum principle then its closure in $C_0(\mathbb{R}^d) \times C_0(\mathbb{R}^d)$ is the $C_0 \times C_0$ -generator of a Lévy family (see for instance, Section 2.1 pp. 32-41 from [BSW13]). We denote by $(T_t(a_0))_{t \geq 0}$ its Feller semi-group and we state:

Theorem 4.5 (Approximation with Lévy increments).

Let $(\varepsilon_n)_n$ be a sequence of positive numbers such that $\varepsilon_n \rightarrow 0$ and define the transition operators T_n by

$$T_n f(a) := T_{\varepsilon_n}(a)f(a), \quad \text{for } f \in C_0(\mathbb{R}^d).$$

Then, for any $f \in C_c^\infty(\mathbb{R}^d)$,

$$\frac{1}{\varepsilon_n}(T_n f - f) \xrightarrow{n \rightarrow \infty} Lf, \quad \text{uniformly on compact sets.}$$

Remark 4.6. If the martingale local problem associated to L is well-posed, by Theorem 3.2, one deduces the convergence of the associated probability families. \diamond

Excepting the fact that the present convergence is for the local Skorokhod topology, Theorem 4.5 is an improvement of Theorem 7.6 p. 172 from [BSW13]. More precisely, we do not need that the closure of L is a generator of a Feller semi-group, but we only suppose that the martingale local problem is well-posed. We postpone the proof of the latter theorem to the end of this section.

The proof of Theorem 4.1 is obtained as a straightforward application of the following proposition whose result is somehow similar to Theorem 8.7, pp.41-42 of [Sat13] (see also part ii) of Lemma 4.8 below).

Proposition 4.7. For each $n \in \mathbb{N} \cup \{\infty\}$ take $a_n \in \mathbb{R}^d$ such that $a_n \rightarrow a_\infty$ and consider $(\delta_n, \gamma_n, \nu_n)$ satisfying (H2(a_n)). Let also χ be such that the couple (χ, ν_∞) satisfies (H3(a_∞)). Then

$$\forall f \in C_c^\infty(\mathbb{R}^d), \quad T_{\chi, a_n}(\delta_n, \gamma_n, \nu_n)f \xrightarrow{n \rightarrow \infty} T_{\chi, a_\infty}(\delta_\infty, \gamma_\infty, \nu_\infty)f, \quad (4.6)$$

if and only if the following three conditions hold

$$\left\{ \begin{array}{l} \delta_n \xrightarrow{n \rightarrow \infty} \delta_\infty, \\ \forall f \in C(\mathbb{R}^{d\Delta}) \text{ vanishing in a neighbourhood of } a_\infty, \int f(b)\nu_n(db) \xrightarrow{n \rightarrow \infty} \int f(b)\nu_\infty(db), \\ \left(\gamma_{n,ij} + \int (\chi_i \chi_j)(a_n, b)\nu_n(db) \right)_{i,j} \xrightarrow{n \rightarrow \infty} \left(\gamma_{\infty,ij} + \int (\chi_i \chi_j)(a_\infty, b)\nu_\infty(db) \right)_{i,j}. \end{array} \right. \quad (4.7)$$

Proof of Theorem 4.1. Parts 1) and 2) are direct consequences of the latter proposition. To verify Part 3) we remark that for any $f \in C_c^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$ and $a \in \mathbb{R}^d$, we have

$$(T_n f(a) - f(a))/\varepsilon_n = T_{\chi, a}(\delta_n(a), 0, \nu_n(a))f$$

with

$$\delta_n(a) := \varepsilon_n^{-1} \int_{\mathbb{R}^{d\Delta} \setminus \{a\}} \chi(a, b) \mu_n(a, db) \quad \text{and} \quad \nu_n(a, db) := \varepsilon_n^{-1} \mathbb{1}_{\mathbb{R}^{d\Delta} \setminus \{a\}}(b) \mu_n(a, db),$$

hence we can apply again Proposition 4.7. \square

In order, to prove Proposition 4.7 we need the following lemma on the convergence of measures:

Lemma 4.8. *For $n \in \mathbb{N} \cup \{\infty\}$ let $a_n \in \mathbb{R}^d$ be such that $a_n \rightarrow a_\infty$ and let ν_n be Radon measures on $\mathbb{R}^{d\Delta} \setminus \{a_n\}$. Suppose that, for any $f \in C(\mathbb{R}^{d\Delta})$ such that f vanishes in a neighbourhood of a_∞ , is constant in a neighbourhood of Δ and is infinitely differentiable in \mathbb{R}^d , we have*

$$\int f(b) \nu_n(db) \xrightarrow{n \rightarrow \infty} \int f(b) \nu_\infty(db).$$

i) *Then, for any sequence $(f_n)_{n \in \mathbb{N} \cup \{\infty\}}$ of measurable uniformly bounded functions from $\mathbb{R}^{d\Delta}$ to \mathbb{R} such that the f_n vanish in the same neighbourhood of a_∞ , for $n \in \mathbb{N} \cup \{\infty\}$, and such that*

$$\nu_\infty\left(\mathbb{R}^{d\Delta} \setminus \left\{b_0 \in \mathbb{R}^{d\Delta} \mid \lim_{n \rightarrow \infty, b \rightarrow b_0} f_n(b) = f_\infty(b_0)\right\}\right) = 0, \quad (4.8)$$

we have

$$\int f_n(b) \nu_n(db) \xrightarrow{n \rightarrow \infty} \int f_\infty(b) \nu_\infty(db).$$

ii) *Assume, moreover, that there exists $\eta > 0$ such that*

$$\sup_{n \in \mathbb{N} \cup \{\infty\}} \int |b - a_n|^2 \mathbb{1}_{|b - a_n| \leq \eta} \nu_n(db) < \infty.$$

Then, for any sequence $(f_n)_{n \in \mathbb{N} \cup \{\infty\}}$ of measurable uniformly bounded functions from $\mathbb{R}^{d\Delta}$ to \mathbb{R} satisfying $f_n(a_n) = 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{0 < |h| \leq \delta} \frac{f_n(a_n + h)}{|h|^2} = 0, \quad (4.9)$$

and (4.8), we have

$$\int f_n(b) \nu_n(db) \xrightarrow{n \rightarrow \infty} \int f_\infty(b) \nu_\infty(db).$$

Proof. Consider a sequence of functions $(f_n)_{n \in \mathbb{N} \cup \{\infty\}}$ as in the first part of lemma. Let U_1 be an open subset such that $U_1 \Subset \mathbb{R}^{d\Delta} \setminus \{a_\infty\}$ and

$$U_1 \supset \bigcup_{n \in \mathbb{N} \cup \{\infty\}} \{f_n \neq 0\} \supset \mathbb{R}^{d\Delta} \setminus \left\{b_0 \in \mathbb{R}^{d\Delta} \mid \lim_{n \rightarrow \infty, b \rightarrow b_0} f_n(b) = f_\infty(b_0)\right\}.$$

Let $\varphi_1 \in C(\mathbb{R}^{d\Delta})$ be such that φ_1 vanishes in a neighbourhood of a_∞ and is constant in a neighbourhood of Δ , φ_1 is infinitely differentiable in \mathbb{R}^d and such that $\varphi_1 \geq \mathbb{1}_{U_1}$. Then

$$\int \varphi_1(b) \nu_n(db) \xrightarrow{n \rightarrow \infty} \int \varphi_1(b) \nu_\infty(db),$$

so

$$\sup_{n \in \mathbb{N} \cup \{\infty\}} \nu_n(U_1) \leq \sup_{n \in \mathbb{N} \cup \{\infty\}} \int \varphi_1(b) \nu_n(db) < \infty.$$

Since $\mathbb{R}^{d\Delta} \setminus \{a_\infty\}$ is a Polish space, the measure ν_∞ is inner regular on this set. Hence, if $\varepsilon > 0$ is chosen arbitrary, there exists a compact subset $K_\varepsilon \subset U_1$ satisfying

$$K_\varepsilon \subset \left\{ b_0 \in \mathbb{R}^{d\Delta} \mid \lim_{n \rightarrow \infty, b \rightarrow b_0} f_n(b) = f_\infty(b_0) \right\} \quad \text{and} \quad \nu_\infty(K_\varepsilon) \geq \nu_\infty(U_1) - \varepsilon. \quad (4.10)$$

Hence f_∞ is continuous on K_ε and f_n converges uniformly to f_∞ on K_ε . There exists a function $\varphi_2 \in C(\mathbb{R}^{d\Delta})$ such that φ_2 is constant in a neighbourhood of Δ , is infinitely differentiable in \mathbb{R}^d and such that $\{\varphi_2 \neq 0\} \subset U_1$, $\|\varphi_2\| \leq \|f_\infty\|$ and $\|\varphi_2 - f_\infty\|_{K_\varepsilon} \leq \varepsilon$. Since (4.10), by compactness there exists an open subset $U_2 \subset U_1$ such that

$$K_\varepsilon \subset U_2 \subset \left\{ b_0 \in \mathbb{R}^{d\Delta} \mid \limsup_{n \rightarrow \infty, b \rightarrow b_0} |f_n(b) - \varphi_2(b_0)| \leq 2\varepsilon \right\}.$$

By dominated convergence there exists a function $\varphi_3 \in C(\mathbb{R}^{d\Delta})$ such that φ_3 vanishes in a neighbourhood of a_∞ , is constant in a neighbourhood of Δ , is infinitely differentiable in \mathbb{R}^d , and such that $\mathbb{1}_{U_2} \geq \varphi_3$ and $\int \varphi_3(b) \nu_\infty(db) \geq \nu_\infty(U_2) - \varepsilon$. Hence

$$\begin{aligned} \liminf_{n \rightarrow \infty} \nu_n(U_2) &\geq \liminf_{n \rightarrow \infty} \int \varphi_3(b) \nu_n(db) = \int \varphi_3(b) \nu_\infty(db) \geq \nu_\infty(U_2) - \varepsilon \\ &\geq \nu_\infty(K_\varepsilon) - \varepsilon \geq \nu_\infty(U_1) - 2\varepsilon. \end{aligned}$$

Therefore we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left| \int f_n(b) \nu_n(db) - \int f_\infty(b) \nu_\infty(db) \right| &\leq \limsup_{n \rightarrow \infty} \left| \int \varphi_2(b) \nu_n(db) - \int \varphi_2(b) \nu_\infty(db) \right| \\ &\quad + \limsup_{n \rightarrow \infty} \left| \int_{U_2} (f_n(b) - \varphi_2(b)) \nu_n(db) \right| + \limsup_{n \rightarrow \infty} \left| \int_{U_1 \setminus U_2} (f_n(b) - \varphi_2(b)) \nu_n(db) \right| \\ &\quad + \limsup_{n \rightarrow \infty} \left| \int_{K_\varepsilon} (f_\infty(b) - \varphi_2(b)) \nu_\infty(db) \right| + \limsup_{n \rightarrow \infty} \left| \int_{U_1 \setminus K_\varepsilon} (f_\infty(b) - \varphi_2(b)) \nu_\infty(db) \right| \\ &\leq 0 + 2\varepsilon \sup_{n \in \mathbb{N}} \nu_n(U_1) + 4\varepsilon \sup_{n \in \mathbb{N} \cup \{\infty\}} \|f_n\| + \varepsilon \nu_\infty(U_1) + 2\varepsilon \|f_\infty\| \\ &\leq 3\varepsilon \left(\sup_{n \in \mathbb{N} \cup \{\infty\}} \nu_n(U_1) + 2 \sup_{n \in \mathbb{N} \cup \{\infty\}} \|f_n\| \right). \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we obtain that

$$\int f_n(b) \nu_n(db) \xrightarrow{n \rightarrow \infty} \int f_\infty(b) \nu_\infty(db).$$

We proceed with the proof of the part ii) of lemma. Fix $\eta > 0$ as in the statement and choose an arbitrary $\varepsilon > 0$. By (4.9), there exists $0 < \delta < \eta/2$ such that

$$\limsup_{n \rightarrow \infty} \sup_{0 < |h| \leq 2\delta} \frac{f_n(a_n + h)}{|h|^2} \leq \frac{\varepsilon}{1 \vee \sup_{n \in \mathbb{N} \cup \{\infty\}} \int |b - a_n|^2 \mathbb{1}_{|b - a_n| \leq \eta} \nu_n(db)}.$$

Consider a function $\varphi \in C(\mathbb{R}^{d\Delta}, [0, 1])$ which vanishes in a neighbourhood of a_∞ and such that $\varphi(a) = 1$ for any a satisfying $|a - a_\infty| \geq \delta$. Then, by the first part i),

$$\int \varphi(b) f_n(b) \nu_n(db) \xrightarrow{n \rightarrow \infty} \int \varphi(b) f_\infty(b) \nu_\infty(db).$$

For $n \in \mathbb{N}$ large enough, $|a - a_n| \leq \delta$, hence

$$\left| \int (1 - \varphi(b)) f_n(b) \nu_n(db) \right| \leq \int |b - a_n|^2 \mathbb{1}_{|b - a_n| \leq \eta} \nu_n(db) \sup_{0 < |h| \leq 2\delta} \frac{f_n(a_n + h)}{|h|^2},$$

so $\limsup_{n \rightarrow \infty} \left| \int (1 - \varphi(b)) f_n(b) \nu_n(db) \right| \leq \varepsilon$. We also have

$$\begin{aligned} & \left| \int (1 - \varphi(b)) f_\infty(b) \nu_\infty(db) \right| \\ & \leq \int |b - a_\infty|^2 \mathbb{1}_{|b - a_\infty| \leq \eta} \nu_\infty(db) \limsup_{n \rightarrow \infty} \sup_{0 < |h| \leq 2\delta} \frac{f_n(a_n + h)}{|h|^2} \leq \varepsilon, \end{aligned}$$

so

$$\limsup_{n \rightarrow \infty} \left| \int f_n(b) \nu_n(db) - \int f_\infty(b) \nu_\infty(db) \right| \leq 2\varepsilon.$$

Letting $\varepsilon \rightarrow 0$ we can conclude. \square

Proof of Proposition 4.7. Suppose first (4.6). Let $f \in C(\mathbb{R}^{d\Delta})$ be such that f vanishes in a neighbourhood of a_∞ , is constant in a neighbourhood of Δ and is infinitely differentiable in \mathbb{R}^d . Hence $f - f(\Delta) \in C_c^\infty(\mathbb{R}^d)$, and

$$T_{\chi, a_\infty}(\delta_\infty, \gamma_\infty, \nu_\infty)(f - f(\Delta)) = \int f(b) \nu_\infty(db),$$

while, for n large enough,

$$T_{\chi, a_n}(\delta_n, \gamma_n, \nu_n)(f - f(\Delta)) = \int f(b) \nu_n(db).$$

We deduce that

$$\int f(b) \nu_n(db) \xrightarrow{n \rightarrow \infty} \int f(b) \nu_\infty(db).$$

Therefore we can apply the first part of Lemma 4.8 and in particular, for any $f \in C(\mathbb{R}^{d\Delta})$ vanishing in a neighbourhood of a ,

$$\int f(b) \nu_n(db) \xrightarrow{n \rightarrow \infty} \int f(b) \nu_\infty(db).$$

Consider $\theta \in C(\mathbb{R}_+, [0, 1])$ such that $\theta(r) = 1$ for $r \leq 1$ and $\theta(r) = 0$ for $r \geq 2$. For $(a, b) \in \mathbb{R}^d \times \mathbb{R}^{d\Delta}$ and $n \in \mathbb{N} \cup \{\infty\}$, define

$$\tilde{\chi}(a, b) := \theta(|b - a|)(b - a)\mathbb{1}_{b \neq \Delta} \quad \text{and} \quad \tilde{\delta}_n := \delta_n + \int (\tilde{\chi}(a_n, b) - \chi(a_n, b))\nu_n(db). \quad (4.11)$$

Therefore, for all $f \in C_c^\infty(\mathbb{R}^d)$ and all $n \in \mathbb{N} \cup \{\infty\}$

$$T_{\chi, a_n}(\delta_n, \gamma_n, \nu_n)f = T_{\tilde{\chi}, a_n}(\tilde{\delta}_n, \gamma_n, \nu_n)f.$$

Let ϕ be an arbitrary linear form on \mathbb{R}^d and consider $f \in C_c^\infty(\mathbb{R}^d)$ such that $f(b) = \phi \cdot (b - a_\infty)$ in a neighbourhood of a_∞ . Then

$$T_{\tilde{\chi}, a_\infty}(\tilde{\delta}_\infty, \gamma_\infty, \nu_\infty)f = \phi \cdot \tilde{\delta}_\infty + \int (f(b) - \phi \cdot \tilde{\chi}(a_\infty, b))\nu_\infty(db)$$

and for n large enough

$$T_{\tilde{\chi}, a_n}(\tilde{\delta}_n, \gamma_n, \nu_n)f = \phi \cdot \tilde{\delta}_n + \int (f(b) - f(a_n) - \phi \cdot \tilde{\chi}(a_n, b))\nu_n(db).$$

Thanks to the first part of Lemma 4.8

$$\int (f(b) - f(a_n) - \phi \cdot \tilde{\chi}(a_n, b))\nu_n(db) \xrightarrow{n \rightarrow \infty} \int (f(b) - \phi \cdot \tilde{\chi}(a_\infty, b))\nu_\infty(db),$$

so $\phi \cdot \tilde{\delta}_n \xrightarrow{n \rightarrow \infty} \phi \cdot \tilde{\delta}_\infty$ and since ϕ was chosen arbitrary, $\tilde{\delta}_n \xrightarrow{n \rightarrow \infty} \tilde{\delta}_\infty$.

Let Φ be an arbitrary symmetric bilinear form on \mathbb{R}^d and if (e_1, \dots, e_d) is the canonical basis of \mathbb{R}^d , we denote $\Phi_{ij} = \Phi(e_i, e_j)$, $i, j = 1, \dots, d$. Consider $f \in C_c^\infty(\mathbb{R}^d)$ such that $f(b) = \Phi(b - a_\infty, b - a_\infty)$ in a neighbourhood of a_∞ . Then, for n large enough,

$$\begin{aligned} & T_{\tilde{\chi}, a_n}(\tilde{\delta}_n, \gamma_n, \nu_n)f \\ &= \sum_{i,j=1}^d \Phi_{ij}\gamma_{n,ij} + 2\Phi(a_n - a_\infty, \tilde{\delta}_n) + \int (f(b) - f(a_n) - 2\Phi(a_n - a_\infty, \tilde{\chi}(a_n, b)))\nu_n(db) \\ &= \sum_{i,j=1}^d \Phi_{ij} \left(\gamma_{n,ij} + \int (\tilde{\chi}_i \tilde{\chi}_j)(a_n, b)\nu_n(db) \right) + 2\Phi(a_n - a_\infty, \tilde{\delta}_n) \\ & \quad + \int (f(b) - f(a_n) - 2\Phi(a_n - a_\infty, \tilde{\chi}(a_n, b)) - \sum_{i,j=1}^d \Phi_{ij} (\tilde{\chi}_i \tilde{\chi}_j)(a_n, b))\nu_n(db). \end{aligned}$$

A similar equality holds with the index n replaced by ∞ :

$$\begin{aligned} & T_{\tilde{\chi}, a_\infty}(\tilde{\delta}_\infty, \gamma_\infty, \nu_\infty)f = \sum_{i,j=1}^d \Phi_{ij}\gamma_{\infty,ij} + \int f(b)\nu_\infty(db) \\ &= \sum_{i,j=1}^d \Phi_{ij} \left(\gamma_{\infty,ij} + \int (\tilde{\chi}_i \tilde{\chi}_j)(a_\infty, b)\nu_\infty(db) \right) + \int (f(b) - \sum_{i,j=1}^d \Phi_{ij} (\tilde{\chi}_i \tilde{\chi}_j)(a_\infty, b))\nu_\infty(db). \end{aligned}$$

Thanks to the first part of Lemma 4.8

$$\begin{aligned} \int \left(f(b) - f(a_n) - 2\Phi(a_n - a_\infty, \tilde{\chi}(a_n, b)) - \sum_{i,j=1}^d \Phi_{ij}(\tilde{\chi}_i \tilde{\chi}_j)(a_n, b) \right) \nu_n(db) \\ \xrightarrow{n \rightarrow \infty} \int \left(f(b) - \sum_{i,j=1}^d \Phi_{ij}(\tilde{\chi}_i \tilde{\chi}_j)(a_\infty, b) \right) \nu_\infty(db), \end{aligned}$$

so

$$\sum_{i,j=1}^d \Phi_{ij} \left(\gamma_{n,ij} + \int (\tilde{\chi}_i \tilde{\chi}_j)(a_n, b) \nu_n(db) \right) \xrightarrow{n \rightarrow \infty} \sum_{i,j=1}^d \Phi_{ij} \left(\gamma_{\infty,ij} + \int (\tilde{\chi}_i \tilde{\chi}_j)(a_\infty, b) \nu_\infty(db) \right).$$

Since Φ was chosen arbitrary, for all $1 \leq i, j \leq d$

$$\gamma_{n,ij} + \int (\tilde{\chi}_i \tilde{\chi}_j)(a_n, b) \nu_n(db) \xrightarrow{n \rightarrow \infty} \gamma_{\infty,ij} + \int (\tilde{\chi}_i \tilde{\chi}_j)(a_\infty, b) \nu_\infty(db).$$

So we can apply the second part of Lemma 4.8 and in particular

$$\lim_{n \rightarrow \infty} \int (\tilde{\chi}(a_n, b) - \chi(a_n, b)) \nu_n(db) = \int (\tilde{\chi}(a_\infty, b) - \chi(a_\infty, b)) \nu_\infty(db),$$

so by (4.11), $\delta_n \xrightarrow{n \rightarrow \infty} \delta_\infty$. By the second part of Lemma 4.8 we also have, for all $1 \leq i, j \leq d$,

$$\int ((\tilde{\chi}_i \tilde{\chi}_j)(a_n, b) - (\chi_i \chi_j)(a_n, b)) \nu_n(db) \xrightarrow{n \rightarrow \infty} \int ((\tilde{\chi}_i \tilde{\chi}_j)(a_\infty, b) - (\chi_i \chi_j)(a_\infty, b)) \nu_\infty(db),$$

so we deduce

$$\gamma_{n,ij} + \int (\chi_i \chi_j)(a_n, b) \nu_n(db) \xrightarrow{n \rightarrow \infty} \gamma_{\infty,ij} + \int (\chi_i \chi_j)(a_\infty, b) \nu_\infty(db).$$

We prove the converse, by supposing (4.7) and applying the second part of Lemma 4.8. Let $f \in C_c^\infty(\mathbb{R}^d)$ be, for all $n \in \mathbb{N} \cup \{\infty\}$,

$$\begin{aligned} T_{\chi, a_n}(\delta_n, \gamma_n, \nu_n)f &:= \frac{1}{2} \sum_{i,j=1}^d \gamma_{n,ij} \partial_{ij}^2 f(a_n) + \delta_n \cdot \nabla f(a_n) \\ &\quad + \int (f(b) - f(a_n) - \chi(a_n, b) \cdot \nabla f(a_n)) \nu_n(db) \\ &= \frac{1}{2} \sum_{i,j=1}^d \left(\gamma_{n,ij} + \int (\chi_i \chi_j)(a_n, b) \nu_n(db) \right) \partial_{ij}^2 f(a_n) + \delta_n \cdot \nabla f(a_n) \\ &\quad + \int \left(f(b) - f(a_n) - \chi(a_n, b) \cdot \nabla f(a_n) - \sum_{i,j=1}^d (\chi_i \chi_j)(a_n, b) \partial_{ij}^2 f(a_n) \right) \nu_n(db). \end{aligned}$$

Applying the second part of Lemma 4.8 to the last term of the previous equation we deduce

$$T_{\chi, a_n}(\delta_n, \gamma_n, \nu_n)f \xrightarrow{n \rightarrow \infty} T_{\chi, a_\infty}(\delta_\infty, \gamma_\infty, \nu_\infty)f. \quad \square$$

We finish the section with the proof of Theorem 4.5. Recall that $\chi_1(a, b)$ is given by (4.1). Thanks to Theorem 2.12 pp. 21-22 from [Hoh98], for each $a \in \mathbb{R}^d$ there exists a triplet $(\delta(a), \gamma(a), \nu(a))$ satisfying (H2(a)) such that, $Lf(a) := T_{\chi_1, a}(\delta(a), \gamma(a), \nu(a))f$, for all $f \in C_c^\infty(\mathbb{R}^d)$. It is clear that for any $a_0, a \in \mathbb{R}^d$ and $f \in C_c^\infty(\mathbb{R}^d)$, the Lévy operator $L(a_0)$ defined by (4.5) satisfies also

$$L(a_0)f(a) = T_{\chi_1, a}(\delta(a_0), \gamma(a_0), \nu_a(a_0)).$$

Here and elsewhere $\nu_a(a_0)$ is the pushforward measure of $\nu(a_0)$ with respect to the translation $b \mapsto b - a_0 + a$.

Proof of Theorem 4.5. It suffices to prove that for any function $f_0 \in C_c^\infty(\mathbb{R}^d)$ and any sequence $a_n \in \mathbb{R}^d$ converging to $a_\infty \in \mathbb{R}^d$, the sequence $(T_n f_0(a_n) - f_0(a_n))/\varepsilon_n$ converges to $Lf_0(a_\infty)$. Thanks to Proposition 4.7 we have, $\delta(a_n) \xrightarrow[n \rightarrow \infty]{} \delta(a_\infty)$,

$$\forall f \in C(\mathbb{R}^{d\Delta}) \text{ vanishing in a neighbourhood of } a_\infty, \int f(b)\nu(a_n, db) \xrightarrow[n \rightarrow \infty]{} \int f(b)\nu(a_\infty, db),$$

and for all $1 \leq i, j \leq d$

$$\gamma_{ij}(a_n) + \int (\chi_i \chi_j)(a_n, b)\nu(a_n, db) \xrightarrow[n \rightarrow \infty]{} \gamma_{ij}(a_\infty) + \int (\chi_i \chi_j)(a_\infty, b)\nu(a_\infty, db).$$

It is not difficult to deduce that, there exists $C \in \mathbb{R}_+$ such that, for all $n \in \mathbb{N} \cup \{\infty\}$ and $f \in C_c^\infty(\mathbb{R}^d)$,

$$\|L(a_n)f\| \leq C\|f\| \vee \max_{1 \leq i \leq d} \|\partial_i f\| \vee \max_{1 \leq i, j \leq d} \|\partial_{ij}^2 f\|.$$

Hence $\sup_{n \in \mathbb{N} \cup \{\infty\}} \|L(a_n)f_0\| < \infty$. Consider $b_\infty \in \mathbb{R}^d$, a sequence $b_n \rightarrow b_\infty$ and a function $f \in C(\mathbb{R}^{d\Delta})$ vanishing in a neighbourhood of b_∞ . By using the first part of Lemma 4.8,

$$\begin{aligned} \int f(b)\nu_{b_n}(a_n, db) &= \int f(b - a_n + b_n)\nu(a_n, db) \\ &\xrightarrow[n \rightarrow \infty]{} \int f(b - a_\infty + b_\infty)\nu(a_\infty, db) = \int f(b)\nu_{b_\infty}(a_\infty, db). \end{aligned}$$

Hence by the second part of Theorem 4.1, $L(a_n)f$ converges uniformly on compact sets toward $L(a_\infty)f$, for all $f \in C_c^\infty(\mathbb{R}^d)$. In particular, for each $\varepsilon > 0$ there exists an open neighbourhood U of a_∞ and $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \forall a \in U, \quad |L(a_n)f_0(a) - L(a_\infty)f_0(a_\infty)| \leq \varepsilon.$$

Let \mathbf{P}_n be the unique element of $\mathcal{M}(L(a_n))$ such that $\mathbf{P}_n(X_0 = a_n) = 1$. Then, for all

$n \geq n_0$

$$\begin{aligned}
& \left| \frac{T_n f_0(a_n) - f_0(a_n)}{\varepsilon_n} - L(a_\infty) f_0(a_\infty) \right| = \left| \frac{\mathbf{E}_n[f_0(X_{\varepsilon_n})] - f_0(a_n)}{\varepsilon_n} - L(a_\infty) f_0(a_\infty) \right| \\
&= \left| \mathbf{E}_n \frac{1}{\varepsilon_n} \int_0^{\varepsilon_n} (L(a_n) f_0(X_s) - L(a_\infty) f_0(a_\infty)) ds \right| \\
&\leq \mathbf{E}_n \left[\mathbf{1}_{\{\tau^U < \varepsilon_n\}} \frac{1}{\varepsilon_n} \int_0^{\varepsilon_n} |L(a_n) f_0(X_s) - L(a_\infty) f_0(a_\infty)| ds \right] \\
&\quad + \mathbf{E}_n \left[\mathbf{1}_{\{\tau^U \geq \varepsilon_n\}} \frac{1}{\varepsilon_n} \int_0^{\varepsilon_n} |L(a_n) f_0(X_s) - L(a_\infty) f_0(a_\infty)| ds \right] \\
&\leq 2\mathbf{P}_n(\tau^U < \varepsilon_n) \sup_{m \in \mathbb{N} \cup \{\infty\}} \|L(a_m) f_0\| + \varepsilon.
\end{aligned}$$

We apply Lemma A.1 of uniform continuity along stopping times with a compact neighbourhood $\mathcal{K} \subset U$ of a_∞ and with $\mathcal{U} := \mathbb{R}^d \times U \cup (\mathbb{R}^d \setminus \mathcal{K}) \times \mathbb{R}^d$. We deduce that

$$\lim_{n \rightarrow \infty} \mathbf{P}_n(\tau^U < \varepsilon_n) = 0.$$

Hence

$$\limsup_{n \rightarrow \infty} \left| \frac{T_n f_0(a_n) - f_0(a_n)}{\varepsilon_n} - L(a_\infty) f_0(a_\infty) \right| \leq \varepsilon,$$

and we conclude by letting $\varepsilon \rightarrow 0$. □

5 Diffusion in a potential

We recall that $L_{\text{loc}}^1(\mathbb{R})$ denotes the space of locally Lebesgue integrable functions. A real continuous function f is called locally absolutely continuous if its distributional derivative f' belongs to $L_{\text{loc}}^1(\mathbb{R})$. We introduce the set of potential functions

$$\mathcal{V} := \left\{ V : \mathbb{R} \rightarrow \mathbb{R} \text{ measurable} \mid e^{|V|} \in L_{\text{loc}}^1(\mathbb{R}) \right\}.$$

It is straightforward to prove that there exists a unique Polish topology on \mathcal{V} such that a sequence $(V_n)_n$ in \mathcal{V} converges to $V \in \mathcal{V}$ if and only if

$$\forall M \in \mathbb{R}_+, \quad \lim_{n \rightarrow \infty} \int_{-M}^M |e^{V(a)} - e^{V_n(a)}| \vee |e^{-V(a)} - e^{-V_n(a)}| da = 0.$$

For a potential $V \in \mathcal{V}$, the operator

$$L^V := \frac{1}{2} e^V \frac{d}{da} e^{-V} \frac{d}{da} \tag{5.1}$$

is the set of couples $(f, g) \in C_0(\mathbb{R}) \times C(\mathbb{R})$ such that f and $e^{-V} f'$ are locally absolutely continuous and $g = \frac{1}{2} e^V (e^{-V} f)'$. Let us notice that it is a particular case of the operator

$D_m D_p^+$ described in [Man68], pp. 21-22. Heuristically, the solutions of the martingale local problem associated to L^V are solutions of the stochastic differential equation

$$dX_t = dB_t - \frac{1}{2}V'(X_t)dt,$$

where B is a standard Brownian motion.

Proposition 5.1 (Diffusions on potential and random walks on \mathbb{Z}).

1. For any potential $V \in \mathcal{V}$, the operator L^V is the generator of a locally Feller family.
2. For any sequence of potentials $(V_n)_n$ in \mathcal{V} converging to $V \in \mathcal{V}$ for the topology of \mathcal{V} , the sequence of operators L^{V_n} converges to L^V , in the sense of the third statement of the convergence Theorem 2.2.
3. For $(n, k) \in \mathbb{N} \times \mathbb{Z}$, let $q_{n,k} \in \mathbb{R}$ and $\varepsilon_n > 0$ be. For all $n \in \mathbb{N}$, in accordance with Definition 3.1, let $(\mathbf{P}_k^n)_k \in \mathcal{P}(\mathbb{Z}^{\mathbb{N}})^{\mathbb{Z}}$ be the unique discrete time locally Feller family such that

$$\mathbf{P}_k^n(Y_1 = k + 1) = 1 - \mathbf{P}_k^n(Y_1 = k - 1) = \frac{1}{e^{q_{n,k}} + 1}.$$

Denote the sequence of potential in \mathcal{V} by

$$V_n(a) := \sum_{k=1}^{\lfloor a/\varepsilon_n \rfloor} q_{n,k} \mathbb{1}_{a \geq \varepsilon_n} - \sum_{k=0}^{-\lfloor a/\varepsilon_n \rfloor - 1} q_{n,-k} \mathbb{1}_{a < 0}.$$

Let V be a potential in \mathcal{V} and let $(\mathbf{P}_a)_a$ be the locally Feller family associated with L^V . Assume that V_n converges to V for the topology of \mathcal{V} , and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Then, for any sequence $\mu_n \in \mathcal{P}(\mathbb{Z})$ such that their pushforwards with respect to the mappings $k \mapsto \varepsilon_n k$ converge to a probability measure $\mu \in \mathcal{P}(\mathbb{R})$, we have

$$\mathcal{L}_{\mu_n}^{\mathbf{P}_n} \left((\varepsilon_n Y_{\lfloor t/\varepsilon_n^2 \rfloor})_t \right) \xrightarrow[n \rightarrow \infty]{\mathcal{P}(\mathbb{D}_{loc}(S))} \mathbf{P}_\mu.$$

Before proving this proposition, we give an important consequence concerning a random walk and a diffusion in random environment. For $n \in \mathbb{N}$, let $(\Omega^n, \mathcal{G}^n, \mathbb{P}^n)$ be a probability space and consider the random variables

$$(q_{n,k})_k : \Omega^n \rightarrow \mathbb{R}^{\mathbb{Z}}, \quad (Z_k^n)_k : \Omega^n \rightarrow \mathbb{Z}^{\mathbb{N}} \quad \text{and} \quad \varepsilon_n : \Omega^n \rightarrow \mathbb{R}_+^*.$$

Suppose that for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, \mathbb{P}^n -almost surely,

$$\begin{aligned} \mathbb{P}^n(Z_{k+1}^n = Z_k^n + 1 \mid \varepsilon_n, (q_{n,\ell})_{\ell \in \mathbb{Z}}, (Z_\ell^n)_{0 \leq \ell \leq k}) &= \frac{1}{e^{q_{n,Z_k^n}} + 1} \\ \mathbb{P}^n(Z_{k+1}^n = Z_k^n - 1 \mid \varepsilon_n, (q_{n,\ell})_{\ell \in \mathbb{Z}}, (Z_\ell^n)_{0 \leq \ell \leq k}) &= \frac{1}{e^{-q_{n,Z_k^n}} + 1} = 1 - \frac{1}{e^{q_{n,Z_k^n}} + 1}. \end{aligned}$$

For any $n \in \mathbb{N}$ and $a \in \mathbb{R}$, denote the random potential in \mathcal{V} by

$$W_n(a) := \sum_{k=1}^{\lfloor a/\varepsilon_n \rfloor} q_{n,k} \mathbb{1}_{a \geq \varepsilon_n} - \sum_{k=0}^{-\lfloor a/\varepsilon_n \rfloor - 1} q_{n,-k} \mathbb{1}_{a < 0}. \quad (5.2)$$

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space and consider random variables

$$W : \Omega \rightarrow \mathcal{V} \quad \text{and} \quad Z : \Omega \rightarrow \mathbb{D}_{\text{loc}}(\mathbb{R}).$$

Suppose that the conditional distribution of Z with respect to W satisfies, \mathbb{P} -a.s.

$$\mathcal{L}_{\mathbb{P}}(Z | W) \in \mathcal{M}(L^W).$$

Proposition 5.2. *Suppose that ε_n converges in distribution to 0, that $\varepsilon_n Z_0^n$ converges in distribution to Z_0 and that W_n converges in distribution to W for the topology of \mathcal{V} . Then $(\varepsilon_n Z_{\lfloor t/\varepsilon_n^2 \rfloor}^n)_t$ converges in distribution to Z for the local Skorokhod topology.*

Example 5.3. 1) Let $(q_k)_k$ be an i.i.d sequence of centred real random variables with finite variance σ^2 and suppose that $q_{n,k} = \sqrt{\varepsilon_n} q_k$. Suppose also that W is a Brownian motion with variance σ^2 . Then, by Donsker's theorem, W_n given by (5.2) converges in distribution to W , so we can apply Proposition 5.2 to deduce the convergence of the random walk in a random i.i.d. medium (introduced by Sinai in [Sin82]) to the diffusion in a Brownian potential (introduced in [Bro86]). Hence we recover Theorem 1 from [Sei00], p. 295, without the hypothesis that the distribution of q_0 is compactly supported.

2) Fix deterministic $q \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+^*$. Suppose that for each $n \in \mathbb{N}$, $(q_{n,k})_k$ is an i.i.d sequence of random variables such that $\mathbb{P}^n(q_{n,k} = q) = 1 - \mathbb{P}^n(q_{n,k} = 0) = \lambda \varepsilon_n$. Suppose also that $W(a) = qN_{\lambda a}$, where N is a standard Poisson process on \mathbb{R} . Then, it is classical (see for instance [Car97]), that W_n given by (5.2) converges in distribution to W , so we can apply Proposition 5.2 to deduce the convergence of Sinai's random walk to the diffusion in a Poisson potential. Hence we recover Theorem 2 from [Sei00], p. 296.

3) More generally, suppose that for each $n \in \mathbb{N}$, $(q_{n,k})_k$ is an i.i.d sequence of random variables. Likewise, suppose that W_n given again by (5.2), converges in distribution to a Lévy process W . We can apply Proposition 5.2 to deduce the convergence of Sinai's random walk to the diffusion in a Lévy potential (introduced in [Car97]). \diamond

Proof of Proposition 5.2. Let F be a bounded continuous function from $\mathbb{D}_{\text{loc}}(\mathbb{R})$ to \mathbb{R} . Define the bounded mapping

$$G : \mathbb{R} \times \mathcal{V} \times \mathbb{R}_+ \rightarrow \mathbb{R}$$

as follows: for any $a \in \mathbb{R}$, $V \in \mathcal{V}$ and $\varepsilon \in \mathbb{R}_+^*$, let $\mathbf{P}^{a,V,\varepsilon} \in \mathcal{P}(\mathbb{Z}^{\mathbb{N}})$ be the unique probability such that $\mathbf{P}^{a,V,\varepsilon}(Y_0 = \lfloor a/\varepsilon \rfloor) = 1$ and, $\mathbf{P}^{a,V,\varepsilon}$ -almost surely, for all $k \in \mathbb{N}$,

$$\begin{aligned} \mathbf{P}^{a,V,\varepsilon}(Y_{k+1} = Y_k + 1 \mid Y_0, \dots, Y_k) &= 1 - \mathbf{P}^{a,V,\varepsilon}(Y_{k+1} = Y_k - 1 \mid Y_0, \dots, Y_k) \\ &= \int_{\varepsilon Y_k - \varepsilon}^{\varepsilon Y_k} e^{V(a)} da \Big/ \int_{\varepsilon Y_k - \varepsilon}^{\varepsilon Y_k + \varepsilon} e^{V(a)} da. \end{aligned}$$

Let $\mathbf{P}^{a,V,0} \in \mathcal{P}(\mathbb{D}_{\text{loc}}(\mathbb{R}))$ be the unique element belonging to $\mathcal{M}(L^V)$ and starting from a . We set

$$G(a, V, \varepsilon) := \mathbf{E}^{a,V,\varepsilon} [F((\varepsilon Y_{[t/\varepsilon^2]})_t)] \quad \text{and} \quad G(a, V, 0) := \mathbf{E}^{a,V,0} [F(X)].$$

By Proposition 5.1, the mapping G is continuous at every point of $\mathbb{R} \times \mathcal{V} \times \{0\}$. Thus,

$$\mathbb{E}^n [G(\varepsilon_n Z_0^n, W_n, \varepsilon_n)] \xrightarrow{n \rightarrow \infty} \mathbb{E} [G(Z_0, W, 0)].$$

Hence

$$\begin{aligned} \mathbb{E}^n [F((\varepsilon_n Z_{[t/\varepsilon_n^2]}^n)_t)] &= \mathbb{E}^n [\mathbb{E}^n [F((\varepsilon_n Z_{[t/\varepsilon_n^2]}^n)_t) \mid \varepsilon_n, Z_0^n, (q_{n,\ell})_{\ell \in \mathbb{Z}}]] \\ &= \mathbb{E}^n [G(\varepsilon_n Z_0^n, W_n, \varepsilon_n)] \xrightarrow{n \rightarrow \infty} \mathbb{E} [G(Z_0, W, 0)] = \mathbb{E} [\mathbb{E} [F(Z) \mid Z_0, W]] = \mathbb{E} [F(Z)]. \end{aligned}$$

Then $(\varepsilon_n Z_{[t/\varepsilon_n^2]}^n)_t$ converges in distribution to Z . \square

Before starting the proof of Proposition 5.1, let us give a preliminary computation. Let $a_1, a_2 \in \mathbb{R}$ be and let $V : [a_1 \wedge a_2, a_1 \vee a_2] \rightarrow \mathbb{R}$ be a measurable function such that $e^{|V|} \in L^1([a_1 \wedge a_2, a_1 \vee a_2])$. For any absolutely continuous function $f \in C([a_1 \wedge a_2, a_1 \vee a_2], \mathbb{R})$ such that $e^{-V} f'$ is absolutely continuous and $g := \frac{1}{2} e^V (e^{-V} f')'$ is continuous, we have

$$\begin{aligned} f(a_2) &= f(a_1) + \int_{a_1}^{a_2} f'(b) db = f(a_1) + \int_{a_1}^{a_2} e^{V(b)} \left((e^{-V} f')(a_1) + \int_{a_1}^b (e^{-V} f')'(c) dc \right) db \\ &= f(a_1) + \int_{a_1}^{a_2} e^{V(b)} \left((e^{-V} f')(a_1) + 2 \int_{a_1}^b e^{-V(c)} g(c) dc \right) db \end{aligned} \quad (5.3)$$

$$\begin{aligned} &= f(a_1) + (e^{-V} f')(a_1) \int_{a_1}^{a_2} e^{V(b)} db + 2g(a_1) \int_{a_1}^{a_2} \int_{a_1}^b e^{V(b)-V(c)} dc db \\ &\quad + 2 \int_{a_1}^{a_2} \int_{a_1}^b e^{V(b)-V(c)} (g(c) - g(a_1)) dc db. \end{aligned} \quad (5.4)$$

Proof of Proposition 5.1. Proof of 1. This part is essentially an application of the second chapter of [Man68]. For the sake of completeness we give here some details. Let $h \in C(\mathbb{R}, \mathbb{R}_+^*)$ be such that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \inf_{n \leq |a| \leq n+1} h(a) &\leq \frac{1}{n} \left[\int_n^{n+1} \int_0^a e^{V(b)-V(a)} db da \wedge \int_{n+1}^{n+2} \int_n^{n+1} e^{V(a)-V(b)} db da \right. \\ &\quad \left. \wedge \int_{-n-1}^{-n} \int_a^0 e^{V(b)-V(a)} db da \wedge \int_{-n-2}^{-n-1} \int_{-n-1}^{-n} e^{V(a)-V(b)} db da \right] \end{aligned} \quad (5.5)$$

The operator hL^V coincides on $C_0(\mathbb{R}) \times C_0(\mathbb{R})$ with the operator $D_m D_p^+ \subset C(\overline{\mathbb{R}}) \times C(\overline{\mathbb{R}})$ on the extended real line $\overline{\mathbb{R}}$, described in [Man68], pp. 21-22, where

$$dm(a) := \frac{2e^{-V(a)}}{h(a)} da \quad \text{and} \quad dp(a) := e^{V(a)} da.$$

Applying (5.5) we have

$$\begin{aligned}
\int_0^\infty \int_0^a dm(b)dp(a) &\geq \limsup_{n \rightarrow \infty} \int_{n+1}^{n+2} \int_n^{n+1} dm(b)dp(a) \geq \limsup_{n \rightarrow \infty} 2n = \infty, \\
\int_0^\infty \int_0^a dp(b)dm(a) &\geq \limsup_{n \rightarrow \infty} \int_n^{n+1} \int_0^a dp(b)dm(a) \geq \limsup_{n \rightarrow \infty} 2n = \infty, \\
\int_{-\infty}^0 \int_a^0 dm(b)dp(a) &\geq \limsup_{n \rightarrow \infty} \int_{-n-2}^{-n-1} \int_{-n-1}^{-n} dm(b)dp(a) \geq \limsup_{n \rightarrow \infty} 2n = \infty, \\
\int_{-\infty}^0 \int_a^0 dp(b)dm(a) &\geq \limsup_{n \rightarrow \infty} \int_{-n-1}^{-n} \int_a^0 dp(b)dm(a) \geq \limsup_{n \rightarrow \infty} 2n = \infty.
\end{aligned}$$

Thus according to the definition given in [Man68], pp. 24-25, the boundary points $-\infty$ and $+\infty$ are natural. Thanks to Theorem 1 and Remark 2 p. 38 of [Man68], $D_m D_p^+$ is the generator of a conservative Feller semi-group on $C(\overline{\mathbb{R}})$. Furthermore by steps 7 and 8 in [Man68], pp. 31-32,

$$D_m D_p^+ f(-\infty) = D_m D_p^+ f(+\infty) = 0, \quad \forall f \in D(D_m D_p^+),$$

so that the operator

$$D_m D_p^+ \cap C_0(\mathbb{R}) \times C_0(\mathbb{R}) = (hL^V) \cap C_0(\mathbb{R}) \times C_0(\mathbb{R})$$

is the $C_0 \times C_0$ -generator of a Feller semi-group. Hence, by (2.3) and (2.2) we deduce that the operator

$$\tilde{L} := \frac{1}{h} \overline{(hL^V) \cap C_0(\mathbb{R}) \times C_0(\mathbb{R})}$$

is the generator of a locally Feller family. Here the closure is taken in $C_0(\mathbb{R}) \times C(\overline{\mathbb{R}})$, and it is clear that $\tilde{L} \subset \overline{L^V}$. Thanks to (5.3) it is straightforward to obtain $L^V = \overline{L^V}$ and thanks to (5.4) it is straightforward to obtain that L^V satisfies the positive maximum principle. Thanks to Theorem 2.1 we deduce the existence for the martingale local problem associated to L^V . Hence $L^V = \tilde{L}$ is the generator of a locally Feller family.

Proof of 2. Denote by $(\mathbf{P}_a^n)_a$ and $(\mathbf{P}_a^\infty)_a$ the locally Feller families associated, respectively, to L^{V_n} and L^V . By Theorem 2.2 it is enough to prove that for each sequence $a_n \in \mathbb{R}$ converging to $a_\infty \in \mathbb{R}$, $\mathbf{P}_{a_n}^n$ converges weakly to $\mathbf{P}_{a_\infty}^\infty$ for the local Skorokhod topology. Thanks to Lemma A.3, for $M \in \mathbb{N}^*$, there exists $h_M \in C(\mathbb{R}, [0, 1])$ such that

$$\{h_M \neq 0\} = (-2M, 2M), \quad \{h_M = 1\} = [-M, M]$$

and, for all $n \in \mathbb{N}$, the martingale local problems associated to $h_M L^{V_n}$ and to $h_M L^V$ are well-posed. For $n \in \mathbb{N}$ and $M \in \mathbb{N}^*$, denote by $(\mathbf{P}_a^{n,M})_a$ and $(\mathbf{P}_a^{\infty,M})_a$ the locally Feller families associated, respectively with $h_M L^{V_n}$ and $h_M L^V$. For $n \in \mathbb{N}$, define the extension of $h_M L^{V_n}$:

$$\widetilde{L}_{n,M} := \left\{ (f, g) \in C_0(\mathbb{R}) \times C(\mathbb{R}) \mid g = \frac{1}{2} h_M e^{V_n} (e^{-V_n} f')' \mathbf{1}_{(-2M, 2M)} \right\},$$

where f and $e^{-V_n} f'$ are supposed locally absolutely continuous only on $(-2M, 2M)$. By (5.4) it is straightforward to obtain that $\widetilde{L_{n,M}}$ satisfies the positive maximum principle, so by Theorem 2.1, $\widetilde{L_{n,M}}$ is a linear subspace of the generator of the family $(\mathbf{P}_a^{n,M})_a$. We will prove that the sequence of operators $\widetilde{L_{n,M}}$ converges to the operator $h_M L^V$ in the sense of the third statement of Theorem 2.2. Let $f \in D(L)$ be and define $f_n \in C_0(\mathbb{R})$ by

$$f_n(a) := \begin{cases} f(a), & a \notin (-2M - n^{-1}, 2M + n^{-1}) \\ f(0) + \int_0^a e^{V_n(b)} \left[(e^{-V} f')(0) + 2 \int_0^b e^{-V_n(c)} L^V f(c) dc \right] db, & a \in [-2M, 2M], \end{cases}$$

with f_n affine on $[-2M - n^{-1}, -2M]$ and on $[2M, 2M + n^{-1}]$. Hence $f_n \in D(\widetilde{L_{n,M}})$ and $\widetilde{L_{n,M}} f_n = h_M L^V f$. We have

$$\|f_n - f\| \leq \sup_{a \in [-2M, 2M]} |f_n(a) - f(a)| + \sup_{\substack{2M \leq |a_1|, |a_2| \leq 2M + n^{-1} \\ 0 \leq a_1 a_2}} |f(a_2) - f(a_1)|.$$

Since f is continuous, the second supremum in the latter equation tends to 0. It is straightforward to deduce from (5.3), by using the expression of f_n and the convergence $V_n \rightarrow V$, that

$$\sup_{a \in [-2M, 2M]} |f_n(a) - f(a)| \xrightarrow{n \rightarrow \infty} 0.$$

Hence $\|f_n - f\| \rightarrow 0$ as $n \rightarrow \infty$, so the by Theorem 2.2:

$$\mathbf{P}_{a_n}^{n,M} \xrightarrow{n \rightarrow \infty} \mathbf{P}_{a_\infty}^{\infty,M}. \quad (5.6)$$

By using Lemma A.2, for all $M \in \mathbb{N}^*$ and $n \in \mathbb{N} \cup \{\infty\}$,

$$\mathcal{L}_{\mathbf{P}_{a_n}^{n,M}} \left(X^{\tau^{(-M,M)}} \right) = \mathcal{L}_{\mathbf{P}_{a_n}^n} \left(X^{\tau^{(-M,M)}} \right). \quad (5.7)$$

At this level we use a result of localisation of the continuity contained in Lemma A.4. Therefore, from (5.6) and (5.7), letting $M \rightarrow \infty$, we deduce

$$\mathbf{P}_{a_n}^n \xrightarrow{n \rightarrow \infty} \mathbf{P}_{a_\infty}^\infty.$$

Proof of 3. For $n \in \mathbb{N}$, define the continuous function $\varphi_n : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+$ given by

$$\varphi_n(a, h) := 2 \int_a^{a+h} \int_a^b e^{V_n(b) - V_n(c)} dc db.$$

For each $a \in \mathbb{R}$, it is clear that $\varphi_n(a, \cdot)$ is strictly increasing on \mathbb{R}_+ and $\varphi_n(a, 0) = 0$. Furthermore, since V_n is constant on the interval $[\varepsilon_n \lceil a/\varepsilon_n \rceil, \varepsilon_n (\lceil a/\varepsilon_n \rceil + 1))$,

$$\varphi_n(a, 2\varepsilon_n) \geq 2 \int_{\varepsilon_n \lceil a/\varepsilon_n \rceil}^{\varepsilon_n (\lceil a/\varepsilon_n \rceil + 1)} \int_{\varepsilon_n \lceil a/\varepsilon_n \rceil}^b e^{V_n(b) - V_n(c)} dc db = \varepsilon_n^2.$$

Hence, there exists a unique $\psi_{1,n}(a) \in (0, 2\varepsilon_n]$ such that

$$\varphi_n(a, \psi_{1,n}(a)) = \varepsilon_n^2. \quad (5.8)$$

Using the continuity of φ_n and the compactness of $[0, 2\varepsilon_n]$, it is straightforward to obtain that $\psi_{1,n}$ is continuous. In the same manner, we may prove that, for each $a \in \mathbb{R}$, there exists a unique $\psi_{2,n}(a) \in (0, 2\varepsilon_n]$ such that

$$\varphi_n(a, -\psi_{2,n}(a)) = \varepsilon_n^2, \quad (5.9)$$

and that $\psi_{2,n}$ is continuous. Introduce the continuous function $p_n : \mathbb{R} \rightarrow (0, 1)$ given by

$$p_n(a) := \int_{a-\psi_{2,n}(a)}^a e^{V_n(b)} db \Big/ \int_{a-\psi_{2,n}(a)}^{a+\psi_{1,n}(a)} e^{V_n(b)} db, \quad (5.10)$$

and define a transition operator $T_n : C_0(\mathbb{R}) \rightarrow C_0(\mathbb{R})$ by

$$T_n f(a) := p_n(a) f(a + \psi_{1,n}(a)) + (1 - p_n(a)) f(a - \psi_{2,n}(a)).$$

According to Definition 3.1, let $(\tilde{\mathbf{P}}_a^n)_{a \in \mathbb{R}} \in \mathcal{P}(\mathbb{R}^{\mathbb{N}})^{\mathbb{R}}$ be the discrete time locally Feller family with transition operator T_n . For any $k \in \mathbb{Z}$, since V_n is constant on $[\varepsilon_n k, \varepsilon_n(k+1))$ and on $[\varepsilon_n(k-1), \varepsilon_n k)$, we have

$$\varphi_n(\varepsilon_n k, \pm \varepsilon_n) = 2 \int_{\varepsilon_n k}^{\varepsilon_n(k \pm 1)} \int_{\varepsilon_n k}^b dc db = \varepsilon_n^2$$

and therefore $\psi_{1,n}(\varepsilon_n k) = \psi_{2,n}(\varepsilon_n k) = \varepsilon_n$. Furthermore

$$p_n(\varepsilon_n k) := \frac{\int_{\varepsilon_n(k-1)}^{\varepsilon_n k} e^{V_n(b)} db}{\int_{\varepsilon_n(k-1)}^{\varepsilon_n(k+1)} e^{V_n(b)} db} = \frac{\varepsilon_n e^{V_n(\varepsilon_n(k-1))}}{\varepsilon_n e^{V_n(\varepsilon_n(k-1))} + \varepsilon_n e^{V_n(\varepsilon_n k)}} = \frac{1}{1 + e^{q_{n,k}}},$$

hence for any $f \in C_0(\mathbb{R})$,

$$T_n f(\varepsilon_n k) := \frac{1}{1 + e^{q_{n,k}}} f(\varepsilon_n(k+1)) + \frac{1}{1 + e^{-q_{n,k}}} f(\varepsilon_n(k-1)).$$

We deduce that for any $\mu \in \mathcal{P}(\mathbb{Z})$ and $n \in \mathbb{N}$, $\mathcal{L}_{\tilde{\mathbf{P}}_\mu^n}(\varepsilon_n Y) = \tilde{\mathbf{P}}_\mu^n$, where $\tilde{\mu}$ is the pushforward measure of μ with respect to the mapping $k \mapsto \varepsilon_n k$.

We shall use Theorem 3.2 of convergence of discrete time Markov families. If $f \in D(L^V)$, we need to prove that there exists a sequence of continuous functions $f_n \in C_0(\mathbb{R})$ converging to f such that $(T_n f_n - f_n)/\varepsilon_n^2$ converges to $L^V f$. By the second part of Proposition 5.1, there exists a sequence of continuous functions $f_n \in D(L^{V_n})$ such that f_n converges to f and $L^{V_n} f_n$ converges to $L^V f$. Applying (5.4) to f_n and V_n and recalling (5.8) and (5.9), we have for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned} f(a + \psi_{1,n}(a)) &= f(a) + (e^{-V} f')(a) \int_a^{a+\psi_{1,n}(a)} e^{V(b)} db + \varepsilon_n^2 L^{V_n} f_n(a) \\ &\quad + 2 \int_a^{a+\psi_{1,n}(a)} \int_a^b e^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) dc db, \end{aligned}$$

$$\begin{aligned}
f(a - \psi_{2,n}(a)) &= f(a) - (e^{-V} f')(a) \int_{a-\psi_{2,n}(a)}^a e^{V(b)} db + \varepsilon_n^2 L^{V_n} f_n(a) \\
&\quad + 2 \int_a^{a-\psi_{2,n}(a)} \int_a^b e^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) dc db.
\end{aligned}$$

Hence by (5.10), for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
&\left| \frac{T_n f_n(a) - f_n(a)}{\varepsilon_n^2} - L^{V_n} f_n(a) \right| \\
&\leq \frac{2p_n(a)}{\varepsilon_n^2} \left| \int_a^{a+\psi_{1,n}(a)} \int_a^b e^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) dc db \right| \\
&\quad + \frac{2(1-p_n(a))}{\varepsilon_n^2} \left| \int_a^{a-\psi_{2,n}(a)} \int_a^b e^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) dc db \right| \\
&\leq \sup_{|h| \leq 2\varepsilon_n} |L^{V_n} f_n(a+h) - L^{V_n} f_n(a)|.
\end{aligned}$$

It is not difficult to deduce that $(T_n f_n - f_n)/\varepsilon_n^2$ converges to $L^V f$. Finally, by the Theorem 3.2 of convergence of discrete time Markov families, for any sequence $\mu_n \in \mathcal{P}(\mathbb{Z})$ such that $\tilde{\mu}_n$ converges to a probability measure $\mu \in \mathcal{P}(\mathbb{R})$, we have

$$\mathcal{L}_{\mathbf{P}_{\mu_n}}^n \left((\varepsilon_n Y_{\lfloor t/\varepsilon_n^2 \rfloor})_t \right) = \mathcal{L}_{\tilde{\mu}_n}^n \left((Y_{\lfloor t/\varepsilon_n^2 \rfloor})_t \right) \xrightarrow[n \rightarrow \infty]{\mathcal{P}(\mathbb{D}_{\text{loc}}^1(S))} \mathbf{P}_\mu,$$

where $\tilde{\mu}_n$ are the pushforwards of μ_n with respect to the mappings $k \mapsto \varepsilon_n k$. \square

Appendix

We collect in this appendix several results already proved in [GH17b] having somehow technical statements and used in the previous sections. We refer the interested reader to the paper [GH17b] for the introductory contexts and proofs of each lemma.

Lemma A.1 (cf. Lemma 3.6 in [GH17b]). *Let $L_n, L_\infty \subset C_0(S) \times C(S)$ be such that $D(L_\infty)$ is dense in $C_0(S)$ and assume that, for any $f \in D(L_\infty)$, there exist, for each n , $f_n \in D(L_n)$ such that $f_n \xrightarrow[n \rightarrow \infty]{C_0} f$, $L_n f_n \xrightarrow[n \rightarrow \infty]{C} L_\infty f$. Consider \mathcal{K} a compact subset of S and \mathcal{U} an open subset of $S \times S$ containing $\{(a, a) \mid a \in S\}$. For an arbitrary $(\mathcal{F}_{t+})_t$ -stopping time τ_1 we denote the $(\mathcal{F}_{t+})_t$ -stopping time*

$$\tau(\tau_1) := \inf \{t \geq \tau_1 \mid \{(X_{\tau_1}, X_s)\}_{\tau_1 \leq s \leq t} \notin \mathcal{U}\}.$$

Then for each $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that: for any $n \geq n_0$, for any $\tau_1 \leq \tau_2$, $(\mathcal{F}_{t+})_t$ -stopping times, and for any $\mathbf{P} \in \mathcal{M}(L_n)$ satisfying $\mathbf{E}[(\tau_2 - \tau_1) \mathbf{1}_{\{X_{\tau_1} \in \mathcal{K}\}}] \leq \delta$, we have

$$\mathbf{P}(X_{\tau_1} \in \mathcal{K}, \tau(\tau_1) \leq \tau_2) \leq \varepsilon,$$

with the convention $X_\infty := \Delta$.

Lemma A.2 (cf. Proposition 4.15 in [GH17b]). *Let $L_1, L_2 \subset C_0(S) \times C(S)$ be such that $D(L_1) = D(L_2)$ is dense in $C_0(S)$ and assume that the martingale local problems associated to L_1 and L_2 are well-posed. Let $\mathbf{P}^1 \in \mathcal{M}(L_1)$ and $\mathbf{P}^2 \in \mathcal{M}(L_2)$ be two solutions of these problems having the same initial distribution and let $U \subset S$ be an open subset. If*

$$\forall f \in D(L_1), (L_2 f)|_U = (L_1 f)|_U, \quad \text{then} \quad \mathcal{L}_{\mathbf{P}^2}(X^{\tau^U}) = \mathcal{L}_{\mathbf{P}^1}(X^{\tau^U}).$$

Lemma A.3 (cf. Lemma 4.17 in [GH17b]). *Let U be an open subset of S and L be a subset of $C_0(S) \times C(S)$ with $D(L)$ is dense in $C(S)$. Assume that the martingale local problem associated to L is well-posed. Then there exists a function $h_0 \in C(S, \mathbb{R}_+)$ satisfying $\{h_0 \neq 0\} = U$, such that for all $h \in C(S, \mathbb{R}_+)$ with $\{h \neq 0\} = U$ and $\sup_{a \in U} (h/h_0)(a) < \infty$, the martingale local problem associated to hL is well-posed.*

Lemma A.4 (cf. Lemma A.2 in [GH17b]). *Let $(U_m)_{m \in \mathbb{N}}$ be an increasing sequence of open subsets such that $S = \bigcup_m U_m$. For $n, m \in \mathbb{N} \cup \{\infty\}$, let $\mathbf{P}^{n,m} \in \mathcal{P}(\mathbb{D}_{loc}(S))$ be such that*

i) *for each $m \in \mathbb{N}$, $\mathbf{P}^{n,m} \xrightarrow[n \rightarrow \infty]{} \mathbf{P}^{\infty,m}$, weakly for the local Skorokhod topology,*

ii) *for each $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$, $\mathcal{L}_{\mathbf{P}^{n,m}}(X^{\tau^{U_m}}) = \mathcal{L}_{\mathbf{P}^{n,\infty}}(X^{\tau^{U_m}})$.*

Then $\mathbf{P}^{n,\infty} \xrightarrow[n \rightarrow \infty]{} \mathbf{P}^{\infty,\infty}$, weakly for the local Skorokhod topology.

References

- [Bro86] Th. Brox. A one-dimensional diffusion process in a Wiener medium. *Ann. Probab.*, 14(4):1206–1218, 1986.
- [BSW13] Björn Böttcher, René Schilling, and Jian Wang. *Lévy matters. III*, volume 2099 of *Lecture Notes in Mathematics*. Springer, Cham, 2013. Lévy-type processes: construction, approximation and sample path properties, With a short biography of Paul Lévy by Jean Jacod, Lévy Matters.
- [Car97] Philippe Carmona. The mean velocity of a Brownian motion in a random Lévy potential. *Ann. Probab.*, 25(4):1774–1788, 1997.
- [GH17a] Mihai Gradinaru and Tristan Haugomat. Local Skorokhod topology on the space of cadlag processes. *arXiv: 1706.03669*, June 2017.
- [GH17b] Mihai Gradinaru and Tristan Haugomat. Locally Feller processes and martingale local problems. *arXiv: 1706.04880*, June 2017.
- [Hoh98] Walter Hoh. *Pseudo-Differential Operators Generating Markov Processes*. Habilitationsschrift Universität Bielefeld, Bielefeld, 1998.

- [Jac05] N. Jacob. *Pseudo differential operators and Markov processes. Vol. III*. Imperial College Press, London, 2005. Markov processes and applications.
- [Kal02] Olav Kallenberg. *Foundations of modern probability*. Probability and its Applications. Springer-Verlag, New York, second edition, 2002.
- [Küh17] Franziska Kühn. *Lévy matters. VI*, volume 2187 of *Lecture Notes in Mathematics*. Springer, Cham, 2017. Lévy-type processes: moments, construction and heat kernel estimates, Lévy Matters.
- [Kur11] Thomas G. Kurtz. Equivalence of stochastic equations and martingale problems. In *Stochastic analysis 2010*, pages 113–130. Springer-Verlag, Berlin, Heidelberg, 2011.
- [Man68] Petr Mandl. *Analytical treatment of one-dimensional Markov processes*. Die Grundlehren der mathematischen Wissenschaften, Band 151. Springer-Verlag New York Inc., New York, 1968.
- [Sat13] Ken-iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013.
- [Sei00] Paul Seignourel. Discrete schemes for processes in random media. *Probab. Theory Related Fields*, 118(3):293–322, 2000.
- [Sin82] Ya. G. Sinaĭ. The limit behavior of a one-dimensional random walk in a random environment. *Teor. Veroyatnost. i Primenen.*, 27(2):247–258, 1982.
- [Str75] Daniel W. Stroock. Diffusion processes associated with Lévy generators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 32(3):209–244, 1975.
- [SV06] Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.