Convergence and discrete schemes connected with some locally Feller processes

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Abstract: We characterise the convergence of a certain class of discrete time Markov processes toward locally Feller processes in terms of convergence of martingale problems. We apply our results of approximation to get convergence of some random walks to diffusions behaving into singular potentials. As a consequence we deduce the convergence of random walks in random medium to diffusions in random potential. The results on locally Feller processes are also applied to Lévy-type processes in order to get (or to improve) convergence results, simulation methods and Euler schemes.
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1 Introduction

A number of models for phenomena in several domains (physics, biology, ...) are based on a large class of stochastic processes, which are Feller processes. The present paper address some important questions focusing on two types of Feller processes : Brownian particle evolving in a some irregular potentials and Lévy-type processes. Recall that the Brownian particle evolving in a some irregular potentials is the solution of a SDE driven by a Brownian motion with the considered potential as a drift. Also, a Lévy-type process is a Markov process which, roughly speaking behaves locally like a Lévy process.

Our main goal is to study the convergence of sequences of such type of processes in the setting of these two examples. In particular we try to use approximating Markov sequences which could have continuous or discrete time parameter in order to cover scaling transformations or discrete schemes.

In the context of Feller processes there exist two corresponding results of convergence (see, for instance [Kal02], Theorems 19.25, p. 385 and 19.27, p. 387). For instance in the case of Lévy-type processes, when one needs to consider unbounded coefficients in the Lévy triplet technical difficulties could appear in the framework of Feller processes. On the other hand the cited results of convergence impose the knowledge of a core of the generator. This could not be the case in some probabilistic constructions. Detailed overviews on these topics and many other references on the subject can be found in [Jac05], [Hoh98], [Küh17] for the case of Lévy-type processes. Likely, for Brownian particles, it can be possible to consider potentials with very few constraints, in particular it could be singular, or random (see for instance [Man68],[Bro86] or [Car97]).

Our method to tackle these difficulties is to consider the context of the martingale local problems and of locally Feller processes, introduced in [GH17]. In this general framework we have already analysed the question of convergence of sequences of locally Feller processes. In the present paper we add the study of the convergence for processes indexed by a discrete time parameter toward processes indexed by a continuous time parameter. We obtain the characterisation of the convergence in terms of convergence of associated operators, by using the uniform convergence on compact sets, and hence operators with unbounded coefficients could be considered. Likewise, we do not impose that the operator is a generator, but we assume only the well-posed feature of the associated martingale local problem. Indeed, it could be more easy to verify the well-posed feature (see for instance, [Str75] for Lévy-type processes, [SV06] for diffusion processes, [Kur11] for Lévy-driven stochastic differential equations and forward equations...).

We apply our abstract results and we obtain sharp results of convergence in the context of the dynamic of a Brownian particle in a potential. It is often given by the solution of the one-dimensional stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{2}V'(X_t)\mathrm{d}t,$$

where $V : \mathbb{R} \to \mathbb{R}$. We prove the continuous dependence of the diffusion with respect to the potential. We point out that it can be possible to consider potentials with very few constraints. In particular we consider diffusions in random potentials as limits of random walks in random mediums, as an application of an approximation of the diffusion by random walks on $\mathbb{Z}$. An important example is the convergence of Sinai's random walk [Sin82] to the diffusion in a Poisson potential (recovering Theorem 2 from [Sei00], p. 296), to the diffusion in a Brownian potential, also called Brox's diffusion (improving Theorem 1 from [Sei00], p. 295) and, more generally, to the diffusion in a Lévy potential.

Using again our abstract results we obtain sharp results of convergence for discrete and continuous time sequences of processes toward Lévy-type process, in terms of Lévy triplet. We prefer the use of the Lévy triplet than the symbol associated to the operator, since the results are more precise in the situation of possibly instantaneous explosions. This is due essentially to the fact that the vague convergence of bounded measures cannot be characterised in terms of characteristic function. Our results can also be used to simulate Lévy-type processes and we improve Theorem 7.6 from [BSW13], p. 172, which is an approximation result of type Euler scheme. We state the results in terms of convergence of operators, but essentially one can deduce the convergence of the associated processes.

Let us describe the organisation of the paper. The next section contains notations and statements from our previous paper [GH17], which are useful for an easy reading of the present paper. In particular, we give the statements concerning the existence of solutions for martingale local problems and concerning the convergence of continuous time locally Feller processes. Section 3 is devoted to the limits of sequences of discrete time processes, while Section 4 contains the study of the diffusions evolving in a potential. Finally, two results of convergence toward general Lévy-type processes are studied in Section 5. The appendix contains the statements of several technical results already proved in [GH17].

2 Martingale local problem setting and related results

In the present section we recall some notations and results concerning the martingale local problems and locally Feller processes. Complete statements and proofs are contained in a entirely dedicated paper [GH17].

Let S be a locally compact Polish space. Take $\Delta \notin S$, and we will denote by $S^{\Delta} \supset S$ the one-point compactification of S, if S is not compact, or the topological sum $S \sqcup \{\Delta\}$, if S is compact (so Δ is an isolated point). The fact that a subset A is compactly embedded in an open subset $U \subset S^{\Delta}$ will be denoted by $A \Subset U$. If $x \in (S^{\Delta})^{\mathbb{R}_+}$ is a path on S we will denote its "explosion" time by

$$\xi(x) := \inf\{t \geq 0 \,|\, \{x_s\}_{s \leq t} \not\Subset S\}.$$

The set of exploding càdlàg paths is defined by

$$\mathbb{D}_{\mathrm{loc}}(S) := \left\{ x \in (S^{\Delta})^{\mathbb{R}_+} \;\middle|\; \begin{array}{l} \forall t \geq \xi(x),\ x_t = \Delta, \\ \forall t \geq 0,\ x_t = \lim_{s \downarrow t} x_s, \\ \forall t > 0 \text{ s.t. } \{x_s\}_{s<t} \Subset S,\ x_{t-} := \lim_{s \uparrow t} x_s \text{ exists} \end{array} \right\},$$

and it will be endowed with the local Skorokhod topology (see Theorem 2.4 from [GH18]) which also becomes Polish space. Recall also that a sequence $(x^k)_{k \in \mathbb{N}}$ in $\mathbb{D}_{\mathrm{loc}}(S)$ converges to x for the local Skorokhod topology if and only if there exists a sequence $(\lambda^k)_k$ of increasing homeomorphisms on $\mathbb{R}_+$ satisfying

$$\forall t \geq 0 \text{ s.t. } \{x_s\}_{s<t} \Subset S, \quad \lim_{k \to \infty} \sup_{s \leq t} d(x_s, x^k_{\lambda^k_s}) = 0 \quad \text{and} \quad \lim_{k \to \infty} \sup_{s \leq t} |\lambda^k_s - s| = 0.$$

The local Skorokhod topology does not depend on the (arbitrary) metric d on S^{Δ}, but only on the topology on S. We will always denote by X the canonical process on $\mathbb{D}_{\mathrm{loc}}(S)$. We endow $\mathbb{D}_{\mathrm{loc}}(S)$ with the Borel σ-algebra $\mathcal{F} := \sigma(X_s,\ 0 \leq s < \infty)$ and the filtration $\mathcal{F}_t := \sigma(X_s,\ 0 \leq s \leq t)$. For an open subset $U \Subset S$, τ^U denotes the stopping time given by

$$\tau^U := \inf\{t \geq 0 \mid X_t \notin U \text{ or } X_{t-} \notin U\}. \tag{2.1}$$

Denote by $\mathrm{C}(S) := \mathrm{C}(S, \mathbb{R})$, respectively by $\mathrm{C}(S^{\Delta}) := \mathrm{C}(S^{\Delta}, \mathbb{R})$, the set of real continuous functions on S, respectively on S^{Δ}, and by $\mathrm{C}_0(S)$ the set of functions f belonging to $\mathrm{C}(S)$ and vanishing at Δ. We endow the set $\mathrm{C}(S)$ with the topology of uniform convergence on compact sets, while $\mathrm{C}_0(S)$ with the topology of uniform convergence.

We proceed by recalling the notion of martingale local problem. An operator L from $\mathrm{C}_0(S)$ to $\mathrm{C}(S)$ will be denoted as a subset of $\mathrm{C}_0(S)\times\mathrm{C}(S)$. For $L\subset\mathrm{C}_0(S)\times\mathrm{C}_0(S)$ we define

$$L^{\Delta} := \mathrm{span}\left(L\cup\{(\mathbb{1}_{S^{\Delta}},0)\}\right)\subset\mathrm{C}(S^{\Delta})\times\mathrm{C}(S^{\Delta}). \tag{2.2}$$

The set $\mathcal{M}(L)$ of *solutions of the martingale local problem* associated to L is the set of probabilities $\mathbf{P}\in\mathcal{P}\left(\mathbb{D}_{\mathrm{loc}}(S)\right)$ such that, for all $(f,g)\in L$ and open subset $U\Subset S$,

$$f(X_{t\wedge\tau^U})-\int_0^{t\wedge\tau^U} g(X_s)\mathrm{d}s \text{ is a } \mathbf{P}\text{-martingale}$$

with respect to the filtration $(\mathcal{F}_t)_t$ or, equivalently, to the filtration $(\mathcal{F}_{t+})_t$.

In Theorem 3.10, p. 139 from [GH17], the following result of *existence of solutions for martingale local problem* is stated:

Theorem 2.1. *Let L be a linear subspace of $\mathrm{C}_0(S)\times\mathrm{C}(S)$ such that its domain $D(L):=\{f\in\mathrm{C}_0(S)\mid\exists g\in\mathrm{C}(S),\ (f,g)\in L\}$ is dense in $\mathrm{C}_0(S)$. Then, there is equivalence between*

i) existence of a solution for the martingale local problem: for any $a\in S$ there exists an element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathbf{P}(X_0=a)=1$;

ii) L satisfies the positive maximum principle: for all $(f,g)\in L$ and $a_0\in S$, if $f(a_0)=\sup_{a\in S}f(a)\geq 0$ then $g(a_0)\leq 0$.

A linear subspace $L\subset\mathrm{C}_0(S)\times\mathrm{C}(S)$ satisfying the positive maximum principle is univariate, so it can be equivalently considered as a linear operator $L:\mathrm{D}(L)\to\mathrm{C}(S)$.

The *martingale local problem* is said *well-posed* if there is existence and uniqueness of the solution, which means that for any $a\in S$ there exists an unique element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathbf{P}(X_0=a)=1$.

A family of probabilities $(\mathbf{P}_a)_a\in\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is called *locally Feller* if there exists $L\subset\mathrm{C}_0(S)\times\mathrm{C}(S)$ such that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$ and

$$\forall a\in S:\qquad \mathbf{P}\in\mathcal{M}(L)\ \text{ and }\ \mathbf{P}(X_0=a)=1\iff\mathbf{P}=\mathbf{P}_a.$$

The $\mathrm{C}_0\times\mathrm{C}$-*generator of a locally Feller family* $(\mathbf{P}_a)_a\in\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is the set of functions $(f,g)\in\mathrm{C}_0(S)\times\mathrm{C}(S)$ such that, for any $a\in S$ and any open subset $U\Subset S$,

$$f(X_{t\wedge\tau^U})-\int_0^{t\wedge\tau^U} g(X_s)\mathrm{d}s \text{ is a } \mathbf{P}_a\text{-martingale.}$$

It was noticed in Remark 4.15, p. 150 from [GH17], that if $\mathfrak{h}\in\mathrm{C}(S,\mathbb{R}_+^*)$ and if L is the $\mathrm{C}_0\times\mathrm{C}$-generator of a locally Feller family, then

$$\mathfrak{h}L:=\{(f,\mathfrak{h}g)\,|\,(f,g)\in L\}\ \text{ is the } \mathrm{C}_0\times\mathrm{C}\text{-generator of a locally Feller family.} \tag{2.3}$$

Remind that a Feller semi-group $(T_t)_{t\in\mathbb{R}_+}$ is a strongly continuous semi-group of positive linear contractions on $\mathrm{C}_0(S)$. A natural example of locally Feller family is the family of probability measures associated to a Feller semi-group (see Remark 4.6, p. 144 from [GH17]). The $\mathrm{C}_0 \times \mathrm{C}_0$-generator of the Feller semi-group is the set L_0 of $(f, g) \in \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ such that, for all $a \in S$

$$\lim_{t\to 0} \frac{1}{t}\big(T_t f(a) - f(a)\big) = g(a).$$

Thanks to Propositions 4.2 and 4.4, pp. 142-143 from [GH17], the martingale problem associated to L_0 admits a unique solution and, if L denotes the $\mathrm{C}_0(S) \times \mathrm{C}(S)$-generator of this solution then, taking the closure in $\mathrm{C}_0(S) \times \mathrm{C}(S)$, we have

$$L_0 = L \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_0(S)\big) \quad \text{and} \quad L = \overline{L_0}. \tag{2.4}$$

The following result of *convergence* is crucial for our further developments and it was stated in Theorem 4.17, p. 151, from [GH17]. We point out the fact that one does not need to know the generator of the limit family, but only the fact that a martingale local problem is well-posed. We denote the weakly convergence for the local Skorokhod topology by the symbol $\underset{n\to\infty}{\overset{\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))}{\longrightarrow}}$.

Theorem 2.2 (Convergence of locally Feller family). *For $n \in \mathbb{N} \cup \{\infty\}$, let $(\mathbf{P}^n_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be a locally Feller family and let L_n be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$. Suppose that for any $n \in \mathbb{N}$, $\overline{L_n}$ is the generator of $(\mathbf{P}^n_a)_a$. Furthermore assume that $\mathrm{D}(L_\infty)$ is dense in $\mathrm{C}_0(S)$ and that*

$$\forall a \in S: \qquad \mathbf{P} \in \mathcal{M}(L_\infty) \text{ and } \mathbf{P}(X_0 = a) = 1 \Longleftrightarrow \mathbf{P} = \mathbf{P}^\infty_a.$$

Then we have equivalence between:
i) the mapping

$$\begin{array}{ccc} \mathbb{N} \cup \{\infty\} \times \mathcal{P}(S^\Delta) & \to & \mathcal{P}\left(\mathbb{D}_{loc}(S)\right) \\ (n, \mu) & \mapsto & \mathbf{P}^n_\mu := \int \mathbf{P}_a \mu(\mathrm{d}a) \end{array}$$

is weakly continuous for the local Skorokhod topology, where $\mathbf{P}_\Delta(X_0 = \Delta) = 1$;
ii) for any $a_n, a \in S$ s.t. $a_n \underset{n\to\infty}{\longrightarrow} a$ we have $\mathbf{P}^n_{a_n} \underset{n\to\infty}{\overset{\mathcal{P}(\mathbb{D}_{loc}(S))}{\longrightarrow}} \mathbf{P}^\infty_a$;
iii) for any $f \in \mathrm{D}(L_\infty)$, there exist $f_n \in \mathrm{D}(L_n)$ for each n, such that we have $f_n \underset{n\to\infty}{\overset{\mathrm{C}_0(S)}{\longrightarrow}} f$ and $L_n f_n \underset{n\to\infty}{\overset{\mathrm{C}(S)}{\longrightarrow}} L_\infty f$.

The Appendix contains the statements of other technical results proved in [GH17] and which will be used only in some specific points of proofs of our results.

3 Convergence of discrete time locally Feller families

We start by introducing a discrete time version of the notion of locally Feller family.

Definition 3.1 (Discrete time locally Feller family)**.** Denote by Y the discrete time canonical process on $(S^\Delta)^{\mathbb{N}}$ and endow $(S^\Delta)^{\mathbb{N}}$ with the canonical σ-algebra. A family $(\mathbf{P}_a)_a \in \mathcal{P}\left((S^\Delta)^{\mathbb{N}}\right)^S$ is said to be a *discrete time locally Feller family* if there exists an operator $T : \mathrm{C}_0(S) \to \mathrm{C}_b(S)$, called *transition operator*, such that for any $a \in S$: $\mathbf{P}_a(Y_0 = a) = 1$ and

$$\forall n \in \mathbb{N},\ \forall f \in \mathrm{C}_0(S), \quad \mathbf{E}_a\left(f(Y_{n+1}) \mid Y_0, \ldots, Y_n\right) = \mathbb{1}_{\{Y_n \neq \Delta\}} Tf(Y_n) \quad \mathbf{P}_a\text{-a.s.} \tag{3.1}$$

If we denote by $\mathbf{P}_\Delta$ the probability defined by $\mathbf{P}_\Delta(\forall n \in \mathbb{N},\ Y_n = \Delta) = 1$, then for each $\mu \in \mathcal{P}(S^\Delta)$, $\mathbf{P}_\mu := \int \mathbf{P}_a \mu(\mathrm{d}a)$ satisfies also (3.1).

The following theorem contains our result of convergence of a discrete time locally Feller family to a continuous time locally Feller family. Once again, the main difference with respect to Theorem 19.27, p. 387 from [Kal02], is that one only needs to know the fact that a martingale local problem is well-posed. In what follows, as usual $\lfloor r \rfloor$ will denote the integer part of the real number r.

Theorem 3.2 (Convergence)**.** *Let $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ be an operator with $\mathrm{D}(L)$ a dense subset of $\mathrm{C}_0(S)$ and such that the martingale local problem associated to L is well-posed. Let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be the associated continuous time locally Feller family. For each $n \in \mathbb{N}$ we introduce $(\mathbf{P}^n_a)_a \in \mathcal{P}((S^\Delta)^{\mathbb{N}})^S$ a discrete time locally Feller family having its transition operator denoted by T_n. Set $L_n := (T_n - \mathrm{id})/\varepsilon_n$, where $(\varepsilon_n)_n$ is a sequence of positive constants, $\varepsilon_n \to 0$, as $n \to \infty$. There is equivalence between:*
a) for any $\mu_n, \mu \in \mathcal{P}(S^\Delta)$ s.t. $\mu_n \underset{n\to\infty}{\longrightarrow} \mu$ weakly, $\mathscr{L}_{\mathbf{P}^n_{\mu_n}}\left((Y_{\lfloor t/\varepsilon_n \rfloor})_t\right) \overset{\mathcal{P}(\mathbb{D}_{loc}(S))}{\underset{n\to\infty}{\longrightarrow}} \mathbf{P}_\mu$;
b) for any $a_n, a \in S$ s.t. $a_n \underset{n\to\infty}{\longrightarrow} a$, we have $\mathscr{L}_{\mathbf{P}^n_{a_n}}\left((Y_{\lfloor t/\varepsilon_n \rfloor})_t\right) \overset{\mathcal{P}(\mathbb{D}_{loc}(S))}{\underset{n\to\infty}{\longrightarrow}} \mathbf{P}_a$;
c) for any $f \in \mathrm{D}(L)$, there exists $(f_n)_n \in \mathrm{C}_0(S)^{\mathbb{N}}$ s.t. $f_n \overset{\mathrm{C}_0(S)}{\underset{n\to\infty}{\longrightarrow}} f$ and $L_n f_n \overset{\mathrm{C}(S)}{\underset{n\to\infty}{\longrightarrow}} Lf$.

Proof. Introduce $\Omega := (S^\Delta)^{\mathbb{N}} \times \mathbb{R}_+^{\mathbb{N}}$ and $\mathcal{G} := \mathcal{B}(S^\Delta)^{\otimes\mathbb{N}} \otimes \mathcal{B}(\mathbb{R}_+)^{\otimes\mathbb{N}}$. For any $\mu \in \mathcal{P}(S^\Delta)$ and $n \in \mathbb{N}$, we denote

$$\mathbb{P}^n_\mu := \mathbf{P}^n_\mu \otimes \mathcal{E}(1)^{\otimes\mathbb{N}}, \tag{3.2}$$

where $\mathcal{E}(1)$ is the exponential distribution with expectation 1. We also set

$$Y_n : \begin{array}{ccc} \Omega & \to & S \\ \left((y_k)_k, (s_k)_k\right) & \mapsto & y_n \end{array} \quad \text{and} \quad E_n : \begin{array}{ccc} \Omega & \to & \mathbb{R}_+ \\ \left((y_k)_k, (s_k)_k\right) & \mapsto & s_n \end{array} \tag{3.3}$$

and $N_t := \inf\{n \in \mathbb{N} \mid E_1 + \ldots + E_{n+1} > t\}$, $t \geq 0$, a standard Poisson process.
Step 1) For each $n \in \mathbb{N}$ we set

$$Z^n_t := Y_{N_{t/\varepsilon_n}}. \tag{3.4}$$

We will prove that $a') \Leftrightarrow b') \Leftrightarrow c)$, where $a')$ and $b')$ are the following assertions concerning processes Z^n:

$a')$ for any $\mu_n, \mu \in \mathcal{P}(S^\Delta)$ s.t. $\mu_n \underset{n\to\infty}{\longrightarrow} \mu$, $\mathscr{L}_{\mathbb{P}^n_{\mu_n}}\left(Z^n\right) \overset{\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))}{\underset{n\to\infty}{\longrightarrow}} \mathbf{P}_\mu$;

$b')$ for any $a_n, a \in S$ s.t. $a_n \underset{n\to\infty}{\longrightarrow} a$, $\mathscr{L}_{\mathbb{P}^n_{a_n}}(Z^n) \overset{\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))}{\underset{n\to\infty}{\longrightarrow}} \mathbf{P}_a$.

If we prove that for all $\mu \in \mathcal{P}(S^\Delta)$, $\mathscr{L}_{\mathbb{P}^n_\mu}(Z^n) \in \mathcal{M}(L_n)$, then invoking Theorem 2.2 applied to L_n and L, our claim $a') \Leftrightarrow b') \Leftrightarrow c)$ will be achieved. It is enough to prove that, for each $f \in \mathrm{C}_0(S)$ and $0 \le s \le t$,

$$\mathbb{E}^n_\mu\Big[f(Z^n_t) - f(Z^n_s) - \int_s^t L_n f(Z^n_u)\mathrm{d}u\Big|\mathcal{G}^n_s\Big] = 0, \tag{3.5}$$

where the filtration is given by $\mathcal{G}^n_t := \sigma(N_{s/\varepsilon_n}, Z^n_s,\ s \le t)$, Let us introduce the $(\mathcal{G}^n_t)_t$-stopping times $\tau^n_k := \inf\big\{u \ge 0 \,\big|\, N_{u/\varepsilon_n} = k\big\}$. Then, for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}^n_\mu\left[f(Z^n_{t\wedge(\tau^n_{k+1}\vee s)}) - f(Z^n_{t\wedge(\tau^n_k\vee s)}) \;\middle|\; \mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\right] \\
&\quad = \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}\mathbb{E}^n_\mu\left[(f(Y_{k+1}) - f(Y_k))\mathbb{1}_{\{\tau^n_{k+1}\le t\}} \;\middle|\; \mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\right] \\
&\quad = \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}\mathbb{E}^n_\mu\left[(f(Y_{k+1}) - f(Y_k))\mathbb{1}_{\{\tau^n_{k+1}-\tau^n_k\vee s\le t-\tau^n_k\vee s\}} \;\middle|\; \mathcal{G}^n_{\tau^n_k\vee s}\right].
\end{aligned}$$

Recalling that T_n is a transition operator and the fact that $(N_{u/\varepsilon_n})_u$ is a Poisson process, we get for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}^n_\mu\left[f(Z^n_{t\wedge(\tau^n_{k+1}\vee s)}) - f(Z^n_{t\wedge(\tau^n_k\vee s)}) \;\middle|\; \mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\right] \\
&\quad = \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}(T_n f(Y_k) - f(Y_k))\big(1 - \exp(-(t-\tau^n_k\vee s)/\varepsilon_n)\big) \\
&\quad = \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}L_n f(Z^n_{\tau^n_k\vee s})\varepsilon_n\big(1 - \exp(-(t-\tau^n_k\vee s)/\varepsilon_n)\big).
\end{aligned} \tag{3.6}$$

Similarly, we can compute, for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}^n_\mu\Big[\int_{t\wedge(\tau^n_k\vee s)}^{t\wedge(\tau^n_{k+1}\vee s)} L_n f(Z^n_u)\mathrm{d}u\Big|\mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\Big] \\
&\quad = \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}L_n f(Z^n_{\tau^n_k\vee s})\mathbb{E}^n_\mu\Big[t\wedge\tau^n_{k+1} - \tau^n_k\vee s\Big|\mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\Big] \\
&\quad = \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}L_n f(Z^n_{\tau^n_k\vee s})\mathbb{E}^n_\mu\Big[(t-\tau^n_k\vee s)\wedge(\tau^n_{k+1}-\tau^n_k\vee s)\Big|\mathcal{G}^n_{\tau^n_k\vee s}\Big]
\end{aligned}$$

Once again, since the distribution of $\tau^n_{k+1} - \tau^n_k$ is exponential we get, for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}^n_\mu\Big[\int_{t\wedge(\tau^n_k\vee s)}^{t\wedge(\tau^n_{k+1}\vee s)} L_n f(Z^n_u)\mathrm{d}u\Big|\mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\Big] \\
&\quad = \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}L_n f(Z^n_{\tau^n_k\vee s})\int_0^\infty (1/\varepsilon_n)\exp(-u/\varepsilon_n)((t-\tau^n_k\vee s)\wedge u)\mathrm{d}u \\
&\quad = \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}L_n f(Z^n_{\tau^n_k\vee s})\varepsilon_n\big(1 - \exp(-(t-\tau^n_k\vee s)/\varepsilon_n)\big).
\end{aligned} \tag{3.7}$$

Hence, substracting (3.7) from (3.6), we get, for all $k \in \mathbb{N}$,

$$\mathbb{E}^n_\mu\Big[f(Z^n_{t\wedge(\tau^n_{k+1}\vee s)}) - f(Z^n_{t\wedge(\tau^n_k\vee s)}) - \int_{t\wedge(\tau^n_k\vee s)}^{t\wedge(\tau^n_{k+1}\vee s)} L_n f(Z^n_u)\mathrm{d}u\Big|\mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\Big] = 0. \tag{3.8}$$

Recalling the definition of the stopping times τ_k^n and by summing on $k \in \mathbb{N}$, we also get

$$\begin{aligned}
&\mathbb{E}_\mu^n\Big[f(Z_t^n) - f(Z_s^n) - \int_s^t L_n f(Z_u^n)\mathrm{d}u\Big|\mathcal{G}_s^n\Big] \\
&= \mathbb{E}_\mu^n\Big[\sum_{k\geq 0}\Big((Z^n_{t\wedge(\tau^n_{k+1}\vee s)}) - f(Z^n_{t\wedge(\tau^n_k\vee s)}) - \int_{t\wedge(\tau^n_k\vee s)}^{t\wedge(\tau^n_{k+1}\vee s)} L_n f(Z_u^n)\mathrm{d}u\Big)\Big|\mathcal{G}_s^n\Big] \\
&= \sum_{k\geq 0}\mathbb{E}_\mu^n\Big[\mathbb{E}_\mu^n\Big[f(Z^n_{t\wedge(\tau^n_{k+1}\vee s)}) - f(Z^n_{t\wedge(\tau^n_k\vee s)}) - \int_{t\wedge(\tau^n_k\vee s)}^{t\wedge(\tau^n_{k+1}\vee s)} L_n f(Z_u^n)\mathrm{d}u\Big|\mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\Big]\Big|\mathcal{G}_s^n\Big].
\end{aligned}$$

Owing (3.8) we get (3.5) and we end up with $\mathscr{L}_{\mathbb{P}_\mu^n}(Z^n) \in \mathcal{M}(L_n)$, for each $n \in \mathbb{N}$.
Step 2. Set, for all $t \geq 0$ and $n \in \mathbb{N}$,

$$\Gamma_t^n := \varepsilon_n\Big(\sum_{k=1}^{\lfloor t/\varepsilon_n\rfloor} E_k + (t/\varepsilon_n - \lfloor t/\varepsilon_n\rfloor)E_{\lfloor t/\varepsilon_n\rfloor+1}\Big), \tag{3.9}$$

where the exponential independent random variables E_k has been introduced in (3.3). Thanks to (3.4), for any $t \geq 0$ and $n \in \mathbb{N}$, we have $Y_{\lfloor t/\varepsilon_n\rfloor} = Z^n_{\Gamma^n_t}$. We claim that

$$\forall t \geq 0,\ \forall \varepsilon > 0, \quad \sup_{\mu\in\mathcal{P}(S^\Delta)} \mathbb{P}_\mu^n\Big(\sup_{s\leq t}|\Gamma_s^n - s| \geq \varepsilon\Big) \underset{n\to\infty}{\longrightarrow} 0. \tag{3.10}$$

Fix $t \geq 0$, $\varepsilon > 0$, $n \in \mathbb{N}$ and $\mu \in \mathcal{P}(S^\Delta)$. Since Γ^n is a continuous piecewise affine function we have

$$\sup_{s\leq t}|\Gamma_s^n - s| \leq \sup_{\substack{k\in\mathbb{N}\\ k\leq\lceil t/\varepsilon_n\rceil}} |\Gamma^n_{k\varepsilon_n} - k\varepsilon_n| = \sup_{\substack{k\in\mathbb{N}\\ k\leq\lceil t/\varepsilon_n\rceil}} \Big|\varepsilon_n\sum_{i=1}^k E_i - k\varepsilon_n\Big| = \varepsilon_n \sup_{\substack{k\in\mathbb{N}\\ k\leq\lceil t/\varepsilon_n\rceil}} |M_k|\,,$$

where $M_k := \sum_{i=1}^k E_i - k$. Here $\lceil r\rceil$ denotes the smallest integer larger or equal than the real number r. Recalling again that E_i are independent random variables, with exponential distribution $\mathcal{E}(1)$, we have that the discrete martingale $(M_k)_k$ satisfies $\mathbb{E}_\mu^n[M_k^2] = k\mathbb{E}_\mu^n[(E_1-1)^2] = k$. Applying Markov's inequality and maximal Doob's inequality to the martingale M_k we get

$$\begin{aligned}
\mathbb{P}_\mu^n\Big(\sup_{s\leq t}|\Gamma_s^n - s| \geq \varepsilon\Big) &\leq \mathbb{P}_\mu^n\Big(\varepsilon_n \sup_{k\leq\lceil t/\varepsilon_n\rceil}|M_k| \geq \varepsilon\Big) \leq \frac{\mathbb{E}_\mu^n\Big[\sup_{k\leq\lceil t/\varepsilon_n\rceil} M_k^2\Big]\varepsilon_n^2}{\varepsilon^2} \\
&\leq \frac{4\mathbb{E}_\mu^n\Big[M^2_{\lceil t/\varepsilon_n\rceil}\Big]\varepsilon_n^2}{\varepsilon^2} = \frac{4\lceil t/\varepsilon_n\rceil\varepsilon_n^2}{\varepsilon^2} \leq \frac{4(t+\varepsilon_n)\varepsilon_n}{\varepsilon^2}.
\end{aligned}$$

The claim (3.10) is verified.
Step 3. To end the proof we need the following technical result

Lemma 3.3. *For $n \in \mathbb{N}$, let $(\Omega^n, \mathcal{G}^n, \mathbb{P}^n)$ be a probability space, let $Z^n : \Omega^n \to \mathbb{D}_{loc}(S)$ and $\Gamma^n : \Omega^n \to \mathrm{C}(\mathbb{R}_+, \mathbb{R}_+)$ be a increasing random bijection. Define $\widetilde{Z}^n := Z^n \circ \Gamma^n$. Suppose that for each $\varepsilon > 0$ and $t \in \mathbb{R}_+$*

$$\mathbb{P}^n\Big(\sup_{s\le t} |\Gamma^n_s - s| \ge \varepsilon\Big) \underset{n\to\infty}{\longrightarrow} 0. \tag{3.11}$$

Then for any $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{loc}(S))$,

$$\mathscr{L}_{\mathbb{P}^n}(Z^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P} \quad \Leftrightarrow \quad \mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}, \tag{3.12}$$

where the limits are for the weak topology associated to the local Skorokhod topology.

Thanks to Lemma 3.3 we get $a) \Leftrightarrow a')$ and $b \Leftrightarrow b')$. By *Step 1* we end up with $a) \Leftrightarrow b) \Leftrightarrow c)$. The proof is complete except for the proof of Lemma 3.3. ☐

In fact we will state and prove a more general result:

Lemma 3.4. *Let E be a Polish topological space, for $n \in \mathbb{N}$, let $(\Omega^n, \mathcal{G}^n, \mathbb{P}^n)$ be a probability space and consider random variables $Z^n, \widetilde{Z}^n : \Omega^n \to E$. Suppose that for each compact subset $\mathcal{K} \subset E$ and each open subset $\mathcal{U} \subset E^2$ containing the diagonal $\{(z,z) \,|\, z \in E\}$,*

$$\mathbb{P}^n\left(Z^n \in \mathcal{K},\ (Z^n, \widetilde{Z}^n) \notin \mathcal{U}\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{3.13}$$

Then, for any $\mathbf{P} \in \mathcal{P}(E)$,

$$\mathscr{L}_{\mathbb{P}^n}(Z^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P} \quad \textit{implies} \quad \mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}, \tag{3.14}$$

where the limits are for the weak topology on $\mathcal{P}(E)$.

Proof of Lemma 3.4. Assume that $\mathscr{L}_{\mathbb{P}^n}(Z^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$. This means that for any bounded continuous function $f : E \to \mathbb{R}$, $\lim_{n\to\infty} \mathbb{E}^n[f(Z^n)] = \int f \mathrm{d}\mathbf{P}$. Since E is a Polish space the sequence $(\mathscr{L}_{\mathbb{P}^n}(Z^n))_n$ is tight. Take an arbitrary $\varepsilon > 0$ and let $\mathcal{K}$ be a compact subset of E such that

$$\forall n \in \mathbb{N}, \quad \mathbb{P}^n(Z^n \notin \mathcal{K}) \le \varepsilon. \tag{3.15}$$

By (3.13) applied to $\mathcal{K}$ and $\mathcal{U} := \{(z, \widetilde{z}) \mid |f(\widetilde{z}) - f(z)| < \varepsilon\}$, we have

$$\mathbb{P}^n\left(Z^n \in \mathcal{K},\ |f(\widetilde{Z}^n) - f(Z^n)| \ge \varepsilon\right) \underset{n\to\infty}{\longrightarrow} 0.$$

We decompose

$$\left|\mathbb{E}^n[f(\widetilde{Z}^n)] - \int f \mathrm{d}\mathbf{P}\right| \le \left|\mathbb{E}^n[f(Z^n)] - \int f \mathrm{d}\mathbf{P}\right| + \mathbb{E}^n\left|f(\widetilde{Z}^n) - f(Z^n)\right|$$

and also we split the second term on the right hand side of the above inequality

$$\mathbb{E}^n\big|f(\widetilde{Z}^n)-f(Z^n)\big| = \mathbb{E}^n\Big[\big|f(\widetilde{Z}^n)-f(Z^n)\big|\mathbb{1}_{\{Z^n\in\mathcal{K},|f(\widetilde{Z}^n)-f(Z^n)|\geq\varepsilon\}}\Big] \\ +\mathbb{E}^n\Big[\big|f(\widetilde{Z}^n)-f(Z^n)\big|\mathbb{1}_{\{Z^n\in\mathcal{K},|f(\widetilde{Z}^n)-f(Z^n)|<\varepsilon\}}\Big]+\mathbb{E}^n\Big[\big|f(\widetilde{Z}^n)-f(Z^n)\big|\mathbb{1}_{\{Z^n\notin\mathcal{K}\}}\Big].$$

Hence by (3.15)

$$\Big|\mathbb{E}^n[f(\widetilde{Z}^n)]-\int f\mathrm{d}\mathbf{P}\Big| \leq \Big|\mathbb{E}^n[f(Z^n)]-\int f\mathrm{d}\mathbf{P}\Big| \\ +2\|f\|\,\mathbb{P}^n\left(Z^n\in\mathcal{K},\ |f(\widetilde{Z}^n)-f(Z^n)|\geq\varepsilon\right)+\varepsilon(1+2\|f\|).$$

Letting successively $n\to\infty$ and then $\varepsilon\to 0$, we deduce that $\mathbb{E}^n[f(\widetilde{Z}^n)] \underset{n\to\infty}{\longrightarrow} \int f\mathrm{d}\mathbf{P}$. Hence, since f is an arbitrary bounded continuous function, we get $\mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$. □

We can now give the

Proof of Lemma 3.3. We denote by $\widetilde{\Lambda}$ the space of increasing bijections λ from $\mathbb{R}_+$ to $\mathbb{R}_+$, and for $t\in\mathbb{R}_+$ we denote $\|\lambda-\mathrm{id}\|_t := \sup_{s\leq t}|\lambda_s - s|$. Since

$$\forall\lambda\in\widetilde{\Lambda},\ \forall t\in\mathbb{R}_+,\ \forall\varepsilon>0,\quad \|\lambda-\mathrm{id}\|_{t+\varepsilon}<\varepsilon\Rightarrow\|\lambda^{-1}-\mathrm{id}\|_t<\varepsilon,$$

the hypotheses of Lemma 3.3 are symmetric with respect to Z and $\widetilde{Z}$, so it suffices to prove only one implication. Hence we suppose $\mathscr{L}_{\mathbb{P}^n}(Z^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$ and we will use Lemma 3.4 to get $\mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$.

Let $\mathcal{K}$ be a compact subset of $\mathbb{D}_{\mathrm{loc}}(S)$ and $\mathcal{U}$ be an open subset of $\mathbb{D}_{\mathrm{loc}}(S)^2$ containing the diagonal $\{(z,z)\,|\,z\in\mathbb{D}_{\mathrm{loc}}(S)\}$. We prove the assertion

$$\exists t\geq 0,\ \exists\varepsilon>0,\ \forall z\in\mathcal{K},\ \forall\lambda\in\widetilde{\Lambda},\quad \|\lambda-\mathrm{id}\|_t<\varepsilon\Rightarrow(z,z\circ\lambda)\in\mathcal{U}. \tag{3.16}$$

If we suppose that (3.16) is false, then we can find two sequences $(z^n)_n\in\mathcal{K}^{\mathbb{N}}$ and $(\lambda^n)_n\in\widetilde{\Lambda}^{\mathbb{N}}$ such that, for all $n\in\mathbb{N}$, $(z^n,z^n\circ\lambda^n)\notin\mathcal{U}$ and for all $t\geq 0$, $\|\lambda_n-\mathrm{id}\|_t\to 0$, as $n\to\infty$. Since $\mathcal{K}$ is compact, possibly by taking a subsequence, we may suppose the existence of $z\in\mathcal{K}$ such that $z^n\to z$ as $n\to\infty$. Then, it is straightforward to obtain

$$\mathcal{U}\not\ni(z^n,z^n\circ\lambda^n) \underset{n\to\infty}{\longrightarrow} (z,z)\in\mathcal{U}.$$

This is a contradiction with the fact that $\mathcal{U}$ is open, so (3.16) is proved. Take t and ε given by (3.16), then

$$\mathbb{P}^n\left(Z^n\in\mathcal{K},\ (Z^n,\widetilde{Z}^n)\notin\mathcal{U}\right)\leq\mathbb{P}^n\left(\|\Gamma^n-\mathrm{id}\|_t\geq\varepsilon\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Hence, thanks to Lemma 3.4, $\mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$. □

4 Approximate diffusions evolving in measurable potential

As usual we denote by $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ the space of locally Lebesgue integrable functions. A real continuous function f is called locally absolutely continuous if its distributional derivative f' belongs to $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$. We introduce the set of potential functions

$$\mathscr{V} := \left\{ V : \mathbb{R} \to \mathbb{R} \text{ measurable} \;\middle|\; \mathrm{e}^{|V|} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}) \right\}.$$

It is straightforward to prove that there exists a unique Polish topology on $\mathscr{V}$ such that a sequence $(V_n)_n$ in $\mathscr{V}$ converges to $V \in \mathscr{V}$ if and only if

$$\forall M \in \mathbb{R}_+, \quad \lim_{n\to\infty} \int_{-M}^{M} |\mathrm{e}^{V(a)} - \mathrm{e}^{V_n(a)}| \vee |\mathrm{e}^{-V(a)} - \mathrm{e}^{-V_n(a)}| \,\mathrm{d}a = 0.$$

For a potential $V \in \mathscr{V}$, the operator

$$L^V := \frac{1}{2}\mathrm{e}^V \frac{\mathrm{d}}{\mathrm{d}a}\mathrm{e}^{-V}\frac{\mathrm{d}}{\mathrm{d}a} \tag{4.1}$$

is the set of couples $(f,g) \in \mathrm{C}_0(\mathbb{R}) \times \mathrm{C}(\mathbb{R})$ such that f and $\mathrm{e}^{-V}f'$ are locally absolutely continuous and $g = \frac{1}{2}\mathrm{e}^V(\mathrm{e}^{-V}f')'$. Notice that it is a particular case of the operator $\boldsymbol{D}_m\boldsymbol{D}_p^+$ described in [Man68], pp. 21-22. Heuristically, the solutions of the martingale local problem associated to L^V are solutions of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{2}V'(X_t)\mathrm{d}t,$$

where B is a standard Brownian motion. Our first main result of this section are applications of Theorems 2.1 and 2.2 (or Theorems 3.10 and 4.17 in [GH17]). In particular we will say that the sequence of operators L_n converges to L_∞ in the sense of Theorem 2.2 if *iii)* of that theorem holds.

Theorem 4.1 (Diffusions in a potential)**.**

1. *For any potential $V \in \mathscr{V}$, the operator L^V is the generator of a locally Feller family.*
2. *For any sequence of potentials $(V_n)_n$ in $\mathscr{V}$ converging to $V \in \mathscr{V}$ for the topology of $\mathscr{V}$, the sequence of operators L^{V_n} converges to L^V, in the sense of Theorem 2.2.*

The proof of this theorem involves the use of a technical lemma which is stated below. Its proof is essentially an application of the second chapter of [Man68] and it will postponed at the end of this section.

Lemma 4.2. *Let V be a potential in $\mathscr{V}$ and let $\mathfrak{h} \in \mathrm{C}(\mathbb{R}, \mathbb{R}_+^*)$ be a function such that, for all $n \in \mathbb{N}$,*

$$\inf_{n\le|a|\le n+1} \mathfrak{h}(a) \le \frac{1}{n}\Big[\int_n^{n+1}\int_0^a \mathrm{e}^{V(b)-V(a)}\mathrm{d}b\,\mathrm{d}a \wedge \int_{n+1}^{n+2}\int_n^{n+1} \mathrm{e}^{V(a)-V(b)}\mathrm{d}b\,\mathrm{d}a \wedge \int_{-n-1}^{-n}\int_a^0 \mathrm{e}^{V(b)-V(a)}\mathrm{d}b\,\mathrm{d}a \wedge \int_{-n-2}^{-n-1}\int_{-n-1}^{-n} \mathrm{e}^{V(a)-V(b)}\mathrm{d}b\,\mathrm{d}a\Big]. \tag{4.2}$$

Then the operator $(\mathfrak{h}L^V)\cap\big(\mathrm{C}_0(\mathbb{R})\times\mathrm{C}_0(\mathbb{R})\big)$ *is the* $\mathrm{C}_0\times\mathrm{C}_0$*-generator of a Feller semigroup, with* $\mathfrak{h}L^V$ *introduced in* (2.3).

Remark 4.3. Consider $a_1,a_2\in\mathbb{R}$ and let $V:[a_1\wedge a_2,a_1\vee a_2]\to\mathbb{R}$ be a measurable function such that $\mathrm{e}^{|V|}\in\mathrm{L}^1([a_1\wedge a_2,a_1\vee a_2])$. For any absolutely continuous function $f\in\mathrm{C}([a_1\wedge a_2,a_1\vee a_2],\mathbb{R})$ such that $\mathrm{e}^{-V}f'$ is absolutely continuous and $g:=\frac{1}{2}\mathrm{e}^V(\mathrm{e}^{-V}f')'$ is continuous, we have

$$f(a_2)=f(a_1)+\int_{a_1}^{a_2}f'(b)\mathrm{d}b=f(a_1)+\int_{a_1}^{a_2}\mathrm{e}^{V(b)}\Big((\mathrm{e}^{-V}f')(a_1)+\int_{a_1}^{b}(\mathrm{e}^{-V}f')'(c)\mathrm{d}c\Big)\mathrm{d}b.$$

Hence

$$f(a_2)=f(a_1)+\int_{a_1}^{a_2}\mathrm{e}^{V(b)}\Big((\mathrm{e}^{-V}f')(a_1)+2\int_{a_1}^{b}\mathrm{e}^{-V(c)}g(c)\mathrm{d}c\Big)\mathrm{d}b\,, \tag{4.3}$$

and furthermore

$$\begin{aligned}f(a_2)=f(a_1)+(\mathrm{e}^{-V}f')(a_1)\int_{a_1}^{a_2}\mathrm{e}^{V(b)}\mathrm{d}b+2g(a_1)\int_{a_1}^{a_2}\int_{a_1}^{b}\mathrm{e}^{V(b)-V(c)}\mathrm{d}c\,\mathrm{d}b\\+2\int_{a_1}^{a_2}\int_{a_1}^{b}\mathrm{e}^{V(b)-V(c)}(g(c)-g(a_1))\mathrm{d}c\,\mathrm{d}b.\end{aligned} \tag{4.4}$$

◊

Proof of Theorem 4.1. The first part of theorem is an application of Theorem 2.1. Thanks to Lemma 4.2 and using (2.3)-(2.4) we deduce that the operator

$$\widetilde{L}:=\frac{1}{\mathfrak{h}}\overline{(\mathfrak{h}L^V)\cap\big(\mathrm{C}_0(\mathbb{R})\times\mathrm{C}_0(\mathbb{R})\big)}$$

is the generator of a locally Feller family. Here the closure is taken in $\mathrm{C}_0(\mathbb{R})\times\mathrm{C}(\mathbb{R})$, and it is clear that $\widetilde{L}\subset\overline{L^V}$. Secondly, thanks to the representation (4.3) it is straightforward to obtain $L^V=\overline{L^V}$ and thanks to (4.4) it is straightforward to obtain that L^V satisfies the positive maximum principle. Finally, using Theorem 2.1 we deduce the existence for the martingale local problem associated to L^V. Hence $L^V=\widetilde{L}$ is the generator of a locally Feller family.

We proceed with the proof of the second part of the theorem. Let us denote by $(\mathbf{P}^n_a)_a$ and $(\mathbf{P}^\infty_a)_a$ the locally Feller families associated, respectively, to L^{V_n} and L^V. Owing Theorem 2.2 it is enough to prove that for each sequence of real numbers $(a_n)_n$ converging to $a_\infty\in\mathbb{R}$, $\mathbf{P}^n_{a_n}$ converges weakly to $\mathbf{P}^\infty_{a_\infty}$ for the local Skorokhod topology.

At this level we need to employ one of the results in the Apendix : thanks to Lemma A.1, for $M\in\mathbb{N}^*$, there exists $\mathfrak{h}_M\in\mathrm{C}(\mathbb{R},[0,1])$ such that

$$\{\mathfrak{h}_M\neq 0\}=(-2M,2M),\quad \{\mathfrak{h}_M=1\}=[-M,M],$$

and, for all $n\in\mathbb{N}$, the martingale local problems associated to $\mathfrak{h}_ML^V$ and to $\mathfrak{h}_ML^{V_n}$ are well-posed. For $n\in\mathbb{N}$ and $M\in\mathbb{N}^*$, denote by $(\mathbf{P}^{n,M}_a)_a$ and $(\mathbf{P}^{\infty,M}_a)_a$ the locally

Feller families associated, respectively with $\mathfrak{h}_M L^{V_n}$ and $\mathfrak{h}_M L^V$. For $n \in \mathbb{N}$, define the extension of $h_M L^{V_n}$:

$$\widetilde{L_{n,M}} := \Big\{(f,g) \in \mathrm{C}_0(\mathbb{R}) \times \mathrm{C}(\mathbb{R}) \,\Big|\, g = \frac{1}{2}\mathfrak{h}_M \mathrm{e}^{V_n}(\mathrm{e}^{-V_n} f')' \mathbb{1}_{(-2M,2M)}\Big\},$$

where f and $\mathrm{e}^{-V_n} f'$ are supposed locally absolutely continuous only on $(-2M, 2M)$. Thanks to (4.4) it is straightforward to obtain that $\widetilde{L_{n,M}}$ satisfies the positive maximum principle, so using again Theorem 2.1, we get that $\widetilde{L_{n,M}}$ is a linear subspace of the generator of the family $(\mathbf{P}_a^{n,M})_a$. We will prove that the sequence of operators $\widetilde{L_{n,M}}$ converges to the operator $\mathfrak{h}_M L^V$ in the sense of Theorem 2.2. Let $f \in \mathrm{D}(L)$ be and define $f_n \in \mathrm{C}_0(\mathbb{R})$ by

$$f_n(a) := \begin{cases} f(a), \quad a \notin (-2M - n^{-1}, 2M + n^{-1}) \\[2ex] f(0) + \displaystyle\int_0^a \mathrm{e}^{V_n(b)}\Big[(\mathrm{e}^{-V} f')(0) + 2\int_0^b \mathrm{e}^{-V_n(c)} L^V f(c)\mathrm{d}c\Big]\mathrm{d}b, \quad a \in [-2M, 2M], \end{cases}$$

with f_n affine on $[-2M - n^{-1}, -2M]$ and on $[2M, 2M + n^{-1}]$. Hence $f_n \in \mathrm{D}(\widetilde{L_{n,M}})$ and $\widetilde{L_{n,M}} f_n = \mathfrak{h}_M L^V f$. We have

$$\|f_n - f\| \le \sup_{a \in [-2M,2M]} |f_n(a) - f(a)| + \sup_{\substack{2M \le |a_1|,|a_2| \le 2M+n^{-1} \\ 0 \le a_1 a_2}} |f(a_2) - f(a_1)|.$$

Since f is continuous, the second supremum in the latter equation tends to 0. It is straightforward to deduce from (4.3), by using the expression of f_n and the convergence $V_n \to V$, that

$$\sup_{a \in [-2M,2M]} |f_n(a) - f(a)| \underset{n\to\infty}{\longrightarrow} 0.$$

Hence $\|f_n - f\| \to 0$ as $n \to \infty$, so the by Theorem 2.2:

$$\mathbf{P}_{a_n}^{n,M} \underset{n\to\infty}{\longrightarrow} \mathbf{P}_{a_\infty}^{\infty,M}. \tag{4.5}$$

Again, we need to use two results stated in the Appendix : thanks to Lemma A.2, for all $M \in \mathbb{N}^*$ and $n \in \mathbb{N} \cup \{\infty\}$,

$$\mathscr{L}_{\mathbf{P}_{a_n}^{n,M}}\left(X^{\tau^{(-M,M)}}\right) = \mathscr{L}_{\mathbf{P}_{a_n}^{n,M}}\left(X^{\tau^{(-M,M)}}\right). \tag{4.6}$$

Finally, we employ the result of localisation of the continuity contained in Lemma A.3. Gathering (4.5) and (4.6) and letting $M \to \infty$, we end up with $\mathbf{P}_{a_n}^{n} \underset{n\to\infty}{\longrightarrow} \mathbf{P}_{a_\infty}^{\infty}$. □

The second main result of this section gives an approximation result of a diffusion in a potential by using a sequence of random walks. Its proof is based on the result Theorem 3.2 in the preceding section.

Theorem 4.4 (Approximation by random walks on $\mathbb{Z}$)**.** *For $(n,k) \in \mathbb{N} \times \mathbb{Z}$, choose real numbers $q_{n,k}$ and strictly positive numbers ε_n. For all $n \in \mathbb{N}$, in accordance with Definition 3.1, let $(\mathbf{P}^n_k)_k \in \mathcal{P}(\mathbb{Z}^{\mathbb{N}})^{\mathbb{Z}}$ be the unique discrete time locally Feller family such that*

$$\mathbf{P}^n_k(Y_1 = k+1) = 1 - \mathbf{P}^n_k(Y_1 = k-1) = \frac{1}{\mathrm{e}^{q_{n,k}} + 1}.$$

We introduce the sequence of potentials in $\mathscr{V}$ given by

$$V_n(a) := \sum_{k=1}^{\lfloor a/\varepsilon_n \rfloor} q_{n,k} \mathbb{1}_{a \geq \varepsilon_n} - \sum_{k=0}^{-\lfloor a/\varepsilon_n \rfloor - 1} q_{n,-k} \mathbb{1}_{a<0}\,,$$

such that V_n converges for the topology of $\mathscr{V}$ to a potential of $\mathscr{V}$, say V. Let $(\mathbf{P}_a)_a$ be the locally Feller family associated with L^V. If the sequence $\varepsilon_n \to 0$, then, for any sequence $\mu_n \in \mathcal{P}(\mathbb{Z})$ such that their pushforwards with respect to the mappings $k \mapsto \varepsilon_n k$ converge to a probability measure $\mu \in \mathcal{P}(\mathbb{R})$, we have

$$\mathscr{L}_{\mathbf{P}^n_{\mu_n}}\left((\varepsilon_n Y_{\lfloor t/\varepsilon_n^2 \rfloor})_t\right) \underset{n\to\infty}{\overset{\mathcal{P}(\mathbb{D}_{loc}(S))}{\longrightarrow}} \mathbf{P}_\mu.$$

Before proving this theorem, we give an important consequence concerning a random walk and a diffusion in random environment. Then we will discuss some examples.

Corollary 4.5. *For each $n \in \mathbb{N}$, let $(\Omega^n, \mathcal{G}^n, \mathbb{P}^n)$ be a probability space and consider the random variables*

$$(q_{n,k})_k : \Omega^n \to \mathbb{R}^{\mathbb{Z}}, \quad (Z^n_k)_k : \Omega^n \to \mathbb{Z}^{\mathbb{N}} \quad \text{and} \quad \varepsilon_n : \Omega^n \to \mathbb{R}^*_+\,.$$

Suppose that for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, $\mathbb{P}^n$-almost surely,

$$\mathbb{P}^n\left(Z^n_{k+1} = Z^n_k + 1 \;\middle|\; \varepsilon_n, (q_{n,\ell})_{\ell\in\mathbb{Z}}, (Z^n_\ell)_{0\leq\ell\leq k}\right) = \frac{1}{\mathrm{e}^{q_{n,Z_k}} + 1}$$
$$\mathbb{P}^n\left(Z^n_{k+1} = Z^n_k - 1 \;\middle|\; \varepsilon_n, (q_{n,\ell})_{\ell\in\mathbb{Z}}, (Z^n_\ell)_{0\leq\ell\leq k}\right) = \frac{1}{\mathrm{e}^{-q_{n,Z_k}} + 1} = 1 - \frac{1}{\mathrm{e}^{q_{n,Z_k}} + 1}.$$

For any $n \in \mathbb{N}$ and $a \in \mathbb{R}$, denote the random potential in $\mathscr{V}$ by

$$W_n(a) := \sum_{k=1}^{\lfloor a/\varepsilon_n \rfloor} q_{n,k} \mathbb{1}_{a \geq \varepsilon_n} - \sum_{k=0}^{-\lfloor a/\varepsilon_n \rfloor - 1} q_{n,-k} \mathbb{1}_{a<0}\,. \tag{4.7}$$

Furthermore on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ consider the random variables $W : \Omega \to \mathscr{V}$ and $Z : \Omega \to \mathbb{D}_{loc}(\mathbb{R})$, such that the conditional distribution of Z with respect to W satisfies, $\mathbb{P}$-a.s.

$$\mathscr{L}_{\mathbb{P}}(Z \mid W) \in \mathcal{M}(L^W).$$

If ε_n converges in distribution to 0, if $\varepsilon_n Z^n_0$ converges in distribution to Z_0 and if W_n converges in distribution to W for the topology of $\mathscr{V}$, then $(\varepsilon_n Z^n_{\lfloor t/\varepsilon_n^2 \rfloor})_t$ converges in distribution to Z for the local Skorokhod topology.

Proof of Corollary 4.5. For any $a \in \mathbb{R}$, $V \in \mathscr{V}$ and $\varepsilon > 0$, let $\mathbf{P}^{a,V,\varepsilon} \in \mathcal{P}(\mathbb{Z}^{\mathbb{N}})$ be the unique probability such that $\mathbf{P}^{a,V,\varepsilon}(Y_0 = \lfloor a/\varepsilon \rfloor) = 1$ and, $\mathbf{P}^{a,V,\varepsilon}$-almost surely, for all $k \in \mathbb{N}$,

$$\begin{aligned}\mathbf{P}^{a,V,\varepsilon}\left(Y_{k+1} = Y_k + 1 \mid Y_0, \ldots, Y_k\right) &= 1 - \mathbf{P}^{a,V,\varepsilon}\left(Y_{k+1} = Y_k - 1 \mid Y_0, \ldots, Y_k\right) \\ &= \int_{\varepsilon Y_k - \varepsilon}^{\varepsilon Y_k} \mathrm{e}^{V(a)} \mathrm{d}a \Big/ \int_{\varepsilon Y_k - \varepsilon}^{\varepsilon Y_k + \varepsilon} \mathrm{e}^{V(a)} \mathrm{d}a.\end{aligned}$$

Let $\mathbf{P}^{a,V,0} \in \mathcal{P}(\mathbb{D}_{\text{loc}}(\mathbb{R}))$ be the unique element belonging to $\mathcal{M}(L^V)$ and starting from a. Consider F a bounded continuous function from $\mathbb{D}_{\text{loc}}(\mathbb{R})$ to $\mathbb{R}$ and define the bounded mapping $G : \mathbb{R} \times \mathscr{V} \times \mathbb{R}_+ \to \mathbb{R}$ as follows:

$$G(a, V, \varepsilon) := \mathbf{E}^{a,V,\varepsilon}\left[F\left((\varepsilon Y_{\lfloor t/\varepsilon^2 \rfloor})_t\right)\right] \quad \text{and} \quad G(a, V, 0) := \mathbf{E}^{a,V,0}\left[F(X)\right].$$

Thanks to Theorem 4.4, the mapping G is continuous at every point of $\mathbb{R} \times \mathscr{V} \times \{0\}$. Therefore we have

$$\mathbb{E}^n\left[G(\varepsilon_n Z_0^n, W_n, \varepsilon_n)\right] \underset{n \to \infty}{\longrightarrow} \mathbb{E}\left[G(Z_0, W, 0)\right].$$

Hence

$$\begin{aligned}\mathbb{E}^n\Big[F\big((\varepsilon_n Z^n_{\lfloor t/\varepsilon_n^2 \rfloor})_t\big)\Big] &= \mathbb{E}^n\Big[\mathbb{E}^n\Big[F\big((\varepsilon_n Z^n_{\lfloor t/\varepsilon_n^2 \rfloor})_t\big)\Big| \varepsilon_n,\ Z_0^n,\ (q_{n,\ell})_{\ell \in \mathbb{Z}}\Big]\Big] \\ &= \mathbb{E}^n\big[G(\varepsilon_n Z_0^n, W_n, \varepsilon_n)\big] \underset{n \to \infty}{\longrightarrow} \mathbb{E}\big[G(Z_0, W, 0)\big] = \mathbb{E}\Big[\mathbb{E}\big[F(Z)\big| Z_0,\ W\big]\Big] = \mathbb{E}\big[F(Z)\big].\end{aligned}$$

We conclude that $(\varepsilon_n Z^n_{\lfloor t/\varepsilon_n^2 \rfloor})_t$ converges in distribution to Z. □

Example 4.6. 1) Let $(q_k)_k$ be a sequence of centred real i.i.d random variables with finite variance σ^2 and suppose that $q_{n,k} = \sqrt{\varepsilon_n} q_k$, where ε_n are strictly positive numbers. Suppose also that W is a Brownian motion with variance σ^2. Clearly, by Donsker's theorem, W_n given by (4.7) converges in distribution to W. Therefore we can apply Corollary 4.5 to deduce the convergence of Sinai's random walk in a random i.i.d. medium (introduced in [Sin82]) to the diffusion in a Brownian potential (introduced in [Bro86]). Hence, we recover Theorem 1 from [Sei00], p. 295, without assuming the hypothesis that the distribution of q_0 is compactly supported.

2) Fix deterministic $q \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+^*$. Suppose that, for each $n \in \mathbb{N}$, $(q_{n,k})_k$ is a sequence of real i.i.d random variables such that $\mathbb{P}^n(q_{n,k} = q) = 1 - \mathbb{P}^n(q_{n,k} = 0) = \lambda \varepsilon_n$, where again ε_n are strictly positive numbers.. Suppose also that $W(a) = qN_{\lambda a}$, where N is a standard Poisson process on $\mathbb{R}$. Then, it is classical (see for instance [Car97]), that W_n given by (4.7) converges in distribution to W, so we can apply again Corollary 4.5. We deduce the convergence of Sinai's random walk to the diffusion in a Poisson potential. so we recover Theorem 2 from [Sei00], p. 296.

3) More generally, suppose that for each $n \in \mathbb{N}$, $(q_{n,k})_k$ is an i.i.d sequence of random variables. Likewise, suppose that W_n given again by (4.7), converges in distribution to a Lévy process W. We can apply Corollary 4.5 to deduce the convergence of Sinai's random walk to the diffusion in a Lévy potential and introduced in [Car97]. ◊

Proof of Theorem 4.4. For $n \in \mathbb{N}$, define the continuous function $\varphi_n : \mathbb{R}\times\mathbb{R} \to \mathbb{R}_+$ given by

$$\varphi_n(a,h) := 2\int_a^{a+h}\int_a^b \mathrm{e}^{V_n(b)-V_n(c)}\mathrm{d}c\,\mathrm{d}b.$$

For each $a \in \mathbb{R}$, it is clear that $\varphi_n(a,\cdot)$ is strictly increasing on $\mathbb{R}_+$ and $\varphi_n(a,0) = 0$. Furthermore, since V_n is constant on the interval $\big[\varepsilon_n\lceil a/\varepsilon_n\rceil, \varepsilon_n(\lceil a/\varepsilon_n\rceil + 1)\big)$,

$$\varphi_n(a,2\varepsilon_n) \geq 2\int_{\varepsilon_n\lceil a/\varepsilon_n\rceil}^{\varepsilon_n(\lceil a/\varepsilon_n\rceil+1)}\int_{\varepsilon_n\lceil a/\varepsilon_n\rceil}^b \mathrm{e}^{V_n(b)-V_n(c)}\mathrm{d}c\,\mathrm{d}b = \varepsilon_n^2.$$

Hence, there exists a unique $\psi_{1,n}(a) \in (0,2\varepsilon_n]$ such that

$$\varphi_n(a,\psi_{1,n}(a)) = \varepsilon_n^2. \tag{4.8}$$

Using the continuity of φ_n and the compactness of $[0,2\varepsilon_n]$, it is straightforward to verify that $\psi_{1,n}$ is continuous. In the same manner, we may prove that, for each $a \in \mathbb{R}$, there exists a unique $\psi_{2,n}(a) \in (0,2\varepsilon_n]$ such that

$$\varphi_n(a,-\psi_{2,n}(a)) = \varepsilon_n^2, \tag{4.9}$$

and also that $\psi_{2,n}$ is continuous. Introduce the continuous function $p_n : \mathbb{R} \to (0,1)$ given by

$$p_n(a) := \int_{a-\psi_{2,n}(a)}^a \mathrm{e}^{V_n(b)}\mathrm{d}b \Big/ \int_{a-\psi_{2,n}(a)}^{a+\psi_{1,n}(a)} \mathrm{e}^{V_n(b)}\mathrm{d}b, \tag{4.10}$$

and define a transition operator $T_n : \mathrm{C}_0(\mathbb{R}) \to \mathrm{C}_0(\mathbb{R})$ by

$$T_nf(a) := p_n(a)f(a+\psi_{1,n}(a)) + (1-p_n(a))f(a-\psi_{2,n}(a)).$$

According to Definition 3.1, let $(\widetilde{\mathbf{P}}^n_a)_a \in \mathcal{P}\left(\mathbb{R}^{\mathbb{N}}\right)^{\mathbb{R}}$ be the discrete time locally Feller family with transition operator T_n. Since V_n is constant on $[\varepsilon_nk, \varepsilon_n(k+1))$ and on $[\varepsilon_n(k-1), \varepsilon_nk)$, for any $k \in \mathbb{Z}$, we have

$$\varphi_n(\varepsilon_nk, \pm\varepsilon_n) = 2\int_{\varepsilon_nk}^{\varepsilon_n(k\pm1)}\int_{\varepsilon_nk}^b \mathrm{d}c\,\mathrm{d}b = \varepsilon_n^2.$$

Therefore we get $\psi_{1,n}(\varepsilon_nk) = \psi_{2,n}(\varepsilon_nk) = \varepsilon_n$. Furthermore

$$p_n(\varepsilon_nk) = \frac{\int_{\varepsilon_n(k-1)}^{\varepsilon_nk}\mathrm{e}^{V_n(b)}\mathrm{d}b}{\int_{\varepsilon_n(k-1)}^{\varepsilon_n(k+1)}\mathrm{e}^{V_n(b)}\mathrm{d}b} = \frac{\varepsilon_n\mathrm{e}^{V_n(\varepsilon_n(k-1))}}{\varepsilon_n\mathrm{e}^{V_n(\varepsilon_n(k-1))} + \varepsilon_n\mathrm{e}^{V_n(\varepsilon_nk)}} = \frac{1}{1+\mathrm{e}^{q_{n,k}}}.$$

Reporting in the definition of the transition operator, for any $f \in \mathrm{C}_0(\mathbb{R})$, we obtain

$$T_nf(\varepsilon_nk) = \frac{1}{1+\mathrm{e}^{q_{n,k}}}f(\varepsilon_n(k+1)) + \frac{1}{1+\mathrm{e}^{-q_{n,k}}}f(\varepsilon_n(k-1)).$$

We deduce that for any $\mu \in \mathcal{P}(\mathbb{Z})$ and $n \in \mathbb{N}$, $\mathscr{L}_{\mathbf{P}^n_\mu}(\varepsilon_n Y) = \widetilde{\mathbf{P}}^n_{\widetilde{\mu}}$, where $\widetilde{\mu}$ is the pushforward measure of μ with respect to the mapping $k \mapsto \varepsilon_n k$.

We will employ Theorem 3.2 of convergence of discrete time Markov families. If $f \in \mathrm{D}(L^V)$, we need to prove that there exists a sequence of continuous functions $f_n \in \mathrm{C}_0(\mathbb{R})$ converging to f such that $(T_n f_n - f_n)/\varepsilon_n^2$ converges to $L^V f$. Thanks to the second part of Theorem 4.1, there exists a sequence of continuous functions $f_n \in \mathrm{D}(L^{V_n})$ such that f_n converges to f and $L^{V_n} f_n$ converges to $L^V f$. Applying (4.4) to f_n and V_n and invoking (4.8) and (4.9), we have for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned} f(a+\psi_{1,n}(a)) =& f(a) + (\mathrm{e}^{-V} f')(a) \int_a^{a+\psi_{1,n}(a)} \mathrm{e}^{V(b)} \mathrm{d}b + \varepsilon_n^2 L^{V_n} f_n(a) \\ &+ 2 \int_a^{a+\psi_{1,n}(a)} \int_a^b \mathrm{e}^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) \mathrm{d}c \, \mathrm{d}b, \end{aligned}$$

and

$$\begin{aligned} f(a-\psi_{2,n}(a)) =& f(a) - (\mathrm{e}^{-V} f')(a) \int_{a-\psi_{2,n}(a)}^{a} \mathrm{e}^{V(b)} \mathrm{d}b + \varepsilon_n^2 L^{V_n} f_n(a) \\ &+ 2 \int_a^{a-\psi_{2,n}(a)} \int_a^b \mathrm{e}^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) \mathrm{d}c \, \mathrm{d}b. \end{aligned}$$

Employing once again the definition of the transition operator we can bound, for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned} &\Big| \frac{T_n f_n(a) - f_n(a)}{\varepsilon_n^2} - L^{V_n} f_n(a) \Big| \\ &\quad \le \frac{2 p_n(a)}{\varepsilon_n^2} \Big| \int_a^{a+\psi_{1,n}(a)} \int_a^b \mathrm{e}^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) \mathrm{d}c \, \mathrm{d}b \Big| \\ &\qquad + \frac{2(1-p_n(a))}{\varepsilon_n^2} \Big| \int_a^{a-\psi_{2,n}(a)} \int_a^b \mathrm{e}^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) \mathrm{d}c \, \mathrm{d}b \Big|. \end{aligned}$$

It is then straightforward to deduce that, for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\Big| \frac{T_n f_n(a) - f_n(a)}{\varepsilon_n^2} - L^{V_n} f_n(a) \Big| \le \sup_{|h| \le 2\varepsilon_n} |L^{V_n} f_n(a+h) - L^{V_n} f_n(a)|.$$

Then it is not difficult to conclude that $(T_n f_n - f_n)/\varepsilon_n^2$ converges to $L^V f$. Finally, we will use Theorem 3.2 of convergence of discrete time Markov families. For $\mu_n \in \mathcal{P}(\mathbb{Z})$ we denote by $\widetilde{\mu}_n$ the push-forward of μ_n with respect to the mappings $k \mapsto \varepsilon_n k$. Then for any sequence $\mu_n \in \mathcal{P}(\mathbb{Z})$ such that $\widetilde{\mu}_n$ converges to a probability measure $\mu \in \mathcal{P}(\mathbb{R})$, we deduce that

$$\mathscr{L}_{\mathbf{P}^n_{\mu_n}} \big((\varepsilon_n Y_{\lfloor t/\varepsilon_n^2 \rfloor})_t \big) = \mathscr{L}_{\widetilde{\mathbf{P}}^n_{\widetilde{\mu}_n}} \big((Y_{\lfloor t/\varepsilon_n^2 \rfloor})_t \big) \overset{\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))}{\underset{n\to\infty}{\longrightarrow}} \mathbf{P}_\mu.$$

□

Proof of Lemma 4.2. As was already announced this proof is essentially an application of the second chapter of [Man68]. For the sake of completeness we give here some details.

The operator $\mathfrak{h}L^V$ coincides on $\mathrm{C}_0(\mathbb{R})\times\mathrm{C}_0(\mathbb{R})$ with the operator described in [Man68], pp. 21-22, $D_mD_p^+ \subset \mathrm{C}(\overline{\mathbb{R}}) \times \mathrm{C}(\overline{\mathbb{R}})$ on the extended real line $\overline{\mathbb{R}}$. This operator involves the measures

$$\mathrm{d}m(a) := \frac{2\mathrm{e}^{-V(a)}}{\mathfrak{h}(a)}\mathrm{d}a \quad \text{and} \quad \mathrm{d}p(a) := \mathrm{e}^{V(a)}\mathrm{d}a.$$

Owing the hypothesis (4.2), we have

$$\begin{aligned}
\int_0^\infty \int_0^a \mathrm{d}m(b)\mathrm{d}p(a) &\geq \limsup_{n\to\infty} \int_{n+1}^{n+2}\int_n^{n+1} \mathrm{d}m(b)\mathrm{d}p(a) \geq \limsup_{n\to\infty} 2n = \infty,\\
\int_0^\infty \int_0^a \mathrm{d}p(b)\mathrm{d}m(a) &\geq \limsup_{n\to\infty} \int_n^{n+1}\int_0^a \mathrm{d}p(b)\mathrm{d}m(a) \geq \limsup_{n\to\infty} 2n = \infty,\\
\int_{-\infty}^0 \int_a^0 \mathrm{d}m(b)\mathrm{d}p(a) &\geq \limsup_{n\to\infty} \int_{-n-2}^{-n-1}\int_{-n-1}^{-n} \mathrm{d}m(b)\mathrm{d}p(a) \geq \limsup_{n\to\infty} 2n = \infty,\\
\int_{-\infty}^0 \int_a^0 \mathrm{d}p(b)\mathrm{d}m(a) &\geq \limsup_{n\to\infty} \int_{-n-1}^{-n}\int_a^0 \mathrm{d}p(b)\mathrm{d}m(a) \geq \limsup_{n\to\infty} 2n = \infty.
\end{aligned}$$

Thus the boundary points $-\infty$ and $+\infty$ are natural, according to the definition given in [Man68], pp. 24-25. Thanks to Theorem 1 and Remark 2 p. 38 of [Man68], $D_mD_p^+$ is the generator of a conservative Feller semi-group on $\mathrm{C}(\overline{\mathbb{R}})$. Furthermore

$$D_mD_p^+f(-\infty) = D_mD_p^+f(+\infty) = 0, \quad \forall f \in \mathrm{D}(D_mD_p^+),$$

invoking Steps 7 and 8 in [Man68], pp. 31-32. Therefore, the operator

$$(\mathfrak{h}L^V) \cap \big(\mathrm{C}_0(\mathbb{R}) \times \mathrm{C}_0(\mathbb{R})\big) = D_mD_p^+ \cap \big(\mathrm{C}_0(\mathbb{R}) \times \mathrm{C}_0(\mathbb{R})\big)$$

is the $\mathrm{C}_0 \times \mathrm{C}_0$-generator of a Feller semi-group. □

5 Convergence toward some Lévy-type processes

In this section d denotes a strictly positive integer, $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$, and $\mathbb{R}^{d\Delta}$ denotes the one point compactification of $\mathbb{R}^d$. Let also $\mathrm{C}_c^\infty(\mathbb{R}^d)$ be the set of compactly supported infinitely differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$. We are interested in the dynamics which locally looks like as Lévy processes dynamic.

All along of the present section we will use a linear functional on $\mathrm{C}_c^\infty(\mathbb{R}^d)$ which describes dynamic in a neighbourhood of a point $a \in \mathbb{R}^d$: for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$T_{\chi,a}(\delta,\gamma,\nu)f := \frac{1}{2}\sum_{i,j=1}^d \gamma_{ij}\partial_{ij}^2 f(a) + \delta\cdot\nabla f(a) + \int_{\mathbb{R}^{d\Delta}} \big(f(b) - f(a) - \chi(a,b)\cdot\nabla f(a)\big)\nu(\mathrm{d}b), \quad (5.1)$$

where

– *the compensation function* $\chi : \mathbb{R}^d \times \mathbb{R}^{d\Delta} \to \mathbb{R}^d$ *is a bounded measurable function satisfying for any compact subset* $K \subset \mathbb{R}^d$*,*

$$\sup_{b,c\in K,\, b\neq c} \frac{|\chi(b,c)-(c-b)|}{|c-b|^2} < \infty; \tag{H1}$$

– *the drift vector is* $\delta \in \mathbb{R}^d$*, the diffusion matrix* $\gamma \in \mathbb{R}^{d\times d}$ *is symmetric positive semi-definite and the jump measure* ν *is a measure on* $\mathbb{R}^{d\Delta}$ *satisfying* $\nu(\{a\}) = 0$ *and* (H2(a))

$$\nu(\{a\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^{d\Delta}} (1\wedge |b-a|^2)\nu(db) < \infty.$$

Usually, the compensation function is

$$\chi_1(a,b) := (b-a)/(1+|b-a|^2) \quad \text{or} \quad \chi_2(a,b) := (b-a)\mathbb{1}_{|b-a|<1}. \tag{5.2}$$

It is well known (see for instance Theorem 2.12 pp. 21-22 from [Hoh98], see also [Cou66], [Jac05], [JS01]) that for any linear operator $L : \mathrm{C}_c^\infty(\mathbb{R}^d) \to \mathrm{C}(\mathbb{R}^d)$ satisfying the positive maximum principle and for any χ satisfying (H1): for each $a \in \mathbb{R}^d$ there exist $\delta(a)$, $\gamma(a)$ and $\nu(a)$ satisfying (H2(a)) such that

$$\forall f \in \mathrm{C}_c^\infty(\mathbb{R}^d),\ \forall a \in \mathbb{R}^d, \quad Lf(a) = T_{\chi,a}(\delta(a),\gamma(a),\nu(a))f.$$

In the following we will call a such expression of L a *Lévy-type operator*.

In order to obtain a converse sentence and to get the convergence of sequences of Lévy-type operators, we need to impose a more restrictive hypothesis on the couple (χ,ν): for $a \in \mathbb{R}^d$

– *the compensation function* $\chi : \mathbb{R}^d \times \mathbb{R}^{d\Delta} \to \mathbb{R}^d$ *is a bounded measurable function satisfying, for any compact subset* $K \subset \mathbb{R}^d$*,*

$$\sup_{b,c\in K,\, 0<|c-b|\leq\varepsilon} \frac{|\chi(b,c)-(c-b)|}{|c-b|^2} \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{H3(a)}$$

and $\nu\left(\{b \in \mathbb{R}^{d\Delta} \mid \chi \text{ is not continuous at } (a,b)\}\right) = 0.$

For example, χ_1 given in (5.2) satisfies (H3(a)) for any ν and $\chi_2(a,b)$ satisfies (H3(a)) whenever $\nu\big(\{b \in \mathbb{R}^d : |b-a| = 1\}\big) = 0$.

The main result of this section is stated below. It has some similarities with Theorem 8.7, pp. 41-42 from [Sat13].

Theorem 5.1. *For each* $n \in \mathbb{N}\cup\{\infty\}$ *take* $a_n \in \mathbb{R}^d$ *such that* $a_n \to a_\infty$ *and consider* $(\delta_n,\gamma_n,\nu_n)$ *satisfying* (H2(a_n))*. Let also* χ *be such that the couple* (χ,ν_∞) *satisfies* (H3(a_∞))*. Then, there is equivalence between*

$$\forall f \in \mathrm{C}_c^\infty(\mathbb{R}^d), \quad T_{\chi,a_n}(\delta_n,\gamma_n,\nu_n)f \underset{n\to\infty}{\longrightarrow} T_{\chi,a_\infty}(\delta_\infty,\gamma_\infty,\nu_\infty)f, \tag{5.3}$$

and the following three conditions

$$\begin{cases} \delta_n \underset{n\to\infty}{\longrightarrow} \delta_\infty, \\ \forall f \in \mathrm{C}(\mathbb{R}^{d\Delta}) \text{ vanishing in a neighbourhood of } a_\infty, \int f(b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int f(b)\nu_\infty(\mathrm{d}b), \\ \Big(\gamma_{n,ij} + \int (\chi_i\chi_j)(a_n, b)\nu_n(\mathrm{d}b)\Big)_{i,j} \underset{n\to\infty}{\longrightarrow} \Big(\gamma_{\infty,ij} + \int (\chi_i\chi_j)(a_\infty, b)\nu_\infty(\mathrm{d}b)\Big)_{i,j}. \end{cases} \tag{5.4}$$

Remark 5.2. Let us point out that this theorem is not contradictory with the statement of Theorem 8.7 in Sato's book. We will see that there is an equivalence with a condition with double limit as in Sato's result, as a consequence of the part ii) of our Lemma 5.9 below. ♢

Before proving Theorem 5.1, let us first look to some of its consequences. We get necessary and sufficient conditions for the continuity of the limit function in (5.3) or for the convergence of sequences of Lévy-type operators (and processes) in terms of their Lévy triplets. We start by introducing other notations to simplify our statements.

- Let $\chi : \mathbb{R}^d \times \mathbb{R}^{d\Delta} \to \mathbb{R}^d$ be a compensation function. For each $a \in \mathbb{R}^d$ consider $(\delta(a), \gamma(a), \nu(a))$ and $(\chi, \nu(a))$ satisfying respectively (H2(a)) and (H3(a)). We denote

$$Lf(a) := T_{\chi,a}(\delta(a), \gamma(a), \nu(a))f \qquad \text{for any } f \in \mathrm{C}_c^\infty(\mathbb{R}^d). \tag{5.5}$$

- For each $n \in \mathbb{N}$ and $a \in \mathbb{R}^d$ consider $(\delta_n(a), \gamma_n(a), \nu_n(a))$ satisfying (H2(a)). We denote

$$L_nf(a) := T_{\chi,a}(\delta_n(a), \gamma_n(a), \nu_n(a))f \qquad \text{for any } f \in \mathrm{C}_c^\infty(\mathbb{R}^d). \tag{5.6}$$

- For each $n \in \mathbb{N}$ and $a \in \mathbb{R}^d$ let $\mu_n(a)$ be a probability measure on $\mathbb{R}^{d\Delta}$. We denote

$$T_nf(a) := \int f(b)\mu_n(a, \mathrm{d}b), \qquad \text{for any } f \in \mathrm{C}(\mathbb{R}^{d\Delta}). \tag{5.7}$$

Corollary 5.3 (Continuity feature)**.** *The function Lf given by* (5.5) *is continuous for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ if and only if the following three conditions hold*

- $a \mapsto \delta(a)$ *is continuous on* $\mathbb{R}^d$,
- $a \mapsto \int f(b)\nu(a, \mathrm{d}b)$ *is continuous on the interior of* $\{f = 0\} \cap \mathbb{R}^d$, *for any* $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$,
- $a \mapsto \gamma_{ij}(a) + \int \chi_i(a, b)\chi_j(a, b)\nu(a, \mathrm{d}b)$ *is continuous on* $\mathbb{R}^d$, *for any* $1 \le i, j \le d$.

Example 5.4 (Neveu's counterexample)**.** In [BCP68], pp. 423-424 one describes the following example due to Neveu (see also [Nev58]). Let φ be an arbitrary function in $\mathrm{C}(\mathbb{R})$. Considers the operator

$$Lf(x) := \begin{cases} [f(x + \varphi(x)) + f(x - \varphi(x)) - 2f(x)]/[\varphi(x)^2], & \text{on } \{\varphi \neq 0\} \\ 2 \cdot \frac{1}{2} f''(x), & \text{on } \{\varphi = 0\} \end{cases}.$$

The jump measure associated to this operator is

$$\nu(x) = \frac{1}{\varphi(x)^2}\big(\delta_{x+\varphi(x)} + \delta_{x-\varphi(x)}\big)\mathbb{1}_{\varphi(x)\neq 0}$$

and its diffusion coefficient is $\gamma(x) = 2 \cdot \mathbb{1}_{\varphi(x)=0}$. A consequence of Corollary 5.3 is the fact that, for the present case, Lf is a continuous function, for any $f \in \mathrm{C}_c^\infty(\mathbb{R})$. Indeed, considering the compensation function χ_1 given in (5.2), the third condition is clearly verified since

$$\gamma(x) + \int \frac{(y-x)^2}{(1+(y-x)^2)^2}\,\nu(x,\mathrm{d}y) = \frac{2}{(1+\varphi(x)^2)^2}.$$

$\diamondsuit$

Corollary 5.5 (Convergence towards Lévy-type operators)**.** *Assume that Lf given by (5.5) is continuous for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. The uniform convergence on compact sets, $L_n f \to Lf$, as $n \to \infty$, holds for all $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ if and only if the following three conditions hold*

- $\delta_n(a) \to \delta(a)$, *uniformly for a varying in compact subsets of* $\mathbb{R}^d$,
- $\int f(b)\nu_n(a,\mathrm{d}b) \to \int f(b)\nu(a,\mathrm{d}b)$, *uniformly for a varying in compact subsets of the interior of* $\{f = 0\} \cap \mathbb{R}^d$, *for any* $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$,
- $\gamma_{n,ij}(a) + \int(\chi_i\chi_j)(a,b)\nu_n(a,\mathrm{d}b) \to \gamma_{ij}(a) + \int(\chi_i\chi_j)(a,b)\nu(a,\mathrm{d}b)$, *uniformly for a varying in compact subsets of* $\mathbb{R}^d$, *for any* $1 \le i,j \le d$.

Corollaries 5.3 and 5.5 are straightforward consequences of Theorem 5.1.

Corollary 5.6 (Convergence towards Lévy-type operators - discrete context)**.** *Assume that Lf given by (5.5) is continuous for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. The uniform convergence on compact sets, $(T_n f - f)/\varepsilon_n \to Lf$, as $n \to \infty$, holds for all $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ if and only if the following three conditions hold*

- $\frac{1}{\varepsilon_n}\int_{\mathbb{R}^{d\Delta}\setminus\{a\}} \chi(a,b)\mu_n(a,\mathrm{d}b) \to \delta(a)$, *uniformly for a in compact subsets of* $\mathbb{R}^d$,
- $\frac{1}{\varepsilon_n}\int f(b)\mu_n(a,\mathrm{d}b) \to \int f(b)\nu(a,\mathrm{d}b)$, *uniformly for a in compact subsets of the interior of* $\{f = 0\} \cap \mathbb{R}^d$, *for any* $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$,
- $\frac{1}{\varepsilon_n}\int_{\mathbb{R}^{d\Delta}\setminus\{a\}}(\chi_i\chi_j)(a,b)\mu_n(a,\mathrm{d}b) \to \gamma_{ij}(a) + \int(\chi_i\chi_j)(a,b)\nu(a,\mathrm{d}b)$, *uniformly for a in compact subsets of* $\mathbb{R}^d$, *for any* $1 \le i,j \le d$.

Proof of Corollary 5.6. Notice that for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$ and $a \in \mathbb{R}^d$, we have

$$(T_n f(a) - f(a))/\varepsilon_n = T_{\chi,a}(\delta_n(a), 0, \nu_n(a))f,$$

with

$$\delta_n(a) := \varepsilon_n^{-1}\int_{\mathbb{R}^{d\Delta}\setminus\{a\}} \chi(a,b)\mu_n(a,\mathrm{d}b) \quad \text{and} \quad \nu_n(a,\mathrm{d}b) := \varepsilon_n^{-1}\mathbb{1}_{\mathbb{R}^{d\Delta}\setminus\{a\}}(b)\mu_n(a,\mathrm{d}b).$$

We conclude by applying again Theorem 5.1. $\square$

Remark 5.7. Combining Theorems 2.2, 3.2 and Corollaries 5.5, 5.6 we can deduce some sharp results of convergence for the processes associated to L_n, T_n and L. In particular, Corollary 5.6 could be think as an improvement of the classical Donsker theorem, and for instance, allows us to simulate Lévy-type processes. We illustrate this fact in the following example. $\Diamond$

Example 5.8 (Symmetric stable type operator)**.** Consider two continuous functions $c \in \mathrm{C}(\mathbb{R}^d, \mathbb{R}_+)$ and $\alpha \in \mathrm{C}(\mathbb{R}^d, (0,2))$ and denote, for $f \in \mathrm{C}_0(\mathbb{R}^d)$ and $a \in \mathbb{R}^d$,

$$Lf(a) := \int_{\mathbb{R}^d} (f(b) - f(a) - (b-a)\cdot\nabla f(a)\mathbb{1}_{|b-a|\leq 1})c(a)|b-a|^{-d-\alpha(a)}\mathrm{d}b.$$

As a consequence of Corollary 5.3, L maps $\mathrm{C}_0(\mathbb{R}^d)$ to $\mathrm{C}(\mathbb{R}^d)$. For $a \in \mathbb{R}^d$ and $n \in \mathbb{N}^*$, define the probability measure

$$\mu_n(a, \mathrm{d}b) := \frac{c(a)}{n}|b-a|^{-d-\alpha(a)}\mathbb{1}_{|b-a|\geq\varepsilon_n(a)}\mathrm{d}b, \quad \text{with} \quad \varepsilon_n(a) := \left(\frac{c(a)S_{d-1}}{n\,\alpha(a)}\right)^{1/\alpha(a)}.$$

Here $S_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the measure of the unit sphere in $\mathbb{R}^d$. Thanks to Corollary 5.6, for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$\lim_{n\to\infty} n\Big(\int f(b)\mu_n(a,\mathrm{d}b) - f(a)\Big) = Lf(a), \text{ uniformly for } a \text{ in compact subsets of } \mathbb{R}^d.$$

It is straightforward that for any $a \in \mathbb{R}^d$ and $n \in \mathbb{N}^*$, $\mu_n(a)$ is the distribution of the random variable

$$a + Q\left(\frac{c(a)S_{d-1}}{n\,\alpha(a)U}\right)^{1/\alpha(a)}, \text{ with independent r.v. } Q \sim \mathcal{U}(\mathbb{S}^{d-1}),\ U \sim \mathcal{U}([0,1]).$$

Here $\mathcal{U}(\mathbb{S}^{d-1})$ and $\mathcal{U}([0,1])$ denote the uniform distribution, respectively on the unity sphere of $\mathbb{R}^d$ and on $[0,1]$. To simulate a discrete time locally Feller processes associated to $(\mu_n(a))_a$ we can proceed as follows. Let $(Q_k, U_k)_k$ be a sequence of i.i.d. random variables with distributions $\mathcal{U}(\mathbb{S}^{d-1})\otimes\mathcal{U}([0,1])$ and define, for $n \in \mathbb{N}^*$ and $k \in \mathbb{N}$,

$$Z_{k+1}^n := Z_k^n + Q_k\left(\frac{c(Z_k^n)S_{d-1}}{n\,\alpha(Z_k^n)U_k}\right)^{1/\alpha(Z_k^n)}.$$

Thanks to Theorem 3.2, provided that the martingale local problem associated to L is well-posed, the sequence of processes $(Z_{\lfloor nt\rfloor}^n)_t$ converges in distribution to the solution of the martingale local problem.

Let us note that it is possible to adapt this example when we try to simulate more general Lévy-type processes. The heuristics is as follows: first we approximate the Lévy measure by finite measures, we renormalise them, and then we convolute with a Gaussian measure having well chosen parameters. $\Diamond$

The proof of Theorem 5.1 requires to use a technical lemma concerning the convergence of measures.

Lemma 5.9. *For $n \in \mathbb{N} \cup \{\infty\}$ let $a_n \in \mathbb{R}^d$ be such that $a_n \to a_\infty$. Consider also ν_n a sequence of Radon measures on $\mathbb{R}^{d\Delta} \setminus \{a_n\}$. Suppose that, for any $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$ such that f vanishes on a neighbourhood of a_∞, it is constant on a neighbourhood of Δ and it is infinitely differentiable in $\mathbb{R}^d$, we have*

$$\int f(b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int f(b)\nu_\infty(\mathrm{d}b). \tag{5.8}$$

i) Let $(f_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a sequence of measurable uniformly bounded functions from $\mathbb{R}^{d\Delta}$ to $\mathbb{R}$ such that, all f_n, with $n \in \mathbb{N} \cup \{\infty\}$, vanish on the same neighbourhood of a_∞. In addition, suppose that

$$\nu_\infty\Big(\mathbb{R}^{d\Delta} \setminus \big\{b_0 \in \mathbb{R}^{d\Delta} \mid \lim_{n\to\infty, b\to b_0} f_n(b) = f_\infty(b_0)\big\}\Big) = 0. \tag{5.9}$$

Then we have

$$\int f_n(b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int f_\infty(b)\nu_\infty(\mathrm{d}b). \tag{5.10}$$

ii) Assume, furthermore, that there exists $\eta > 0$ such that

$$\sup_{n\in\mathbb{N}\cup\{\infty\}} \int |b - a_n|^2 \mathbb{1}_{|b-a_n|\le\eta}\, \nu_n(\mathrm{d}b) < \infty. \tag{5.11}$$

Then, for any sequence $(f_n)_{n\in\mathbb{N}\cup\{\infty\}}$ of measurable uniformly bounded functions from $\mathbb{R}^{d\Delta}$ to $\mathbb{R}$ satisfying (5.9), $f_n(a_n) = 0$ and that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{0<|h|\le\delta} \frac{f_n(a_n + h)}{|h|^2} = 0, \tag{5.12}$$

we have the same conclusion, that is (5.10).

Proof of Theorem 5.1. Suppose first (5.3). Let $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$ be such that f vanishes on a neighbourhood of a_∞, it is constant on a neighbourhood of Δ and it is infinitely differentiable in $\mathbb{R}^d$. Hence $f - f(\Delta) \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, and

$$T_{\chi,a_\infty}(\delta_\infty, \gamma_\infty, \nu_\infty)(f - f(\Delta)) = \int f(b)\nu_\infty(\mathrm{d}b),$$

while, for n large enough,

$$T_{\chi,a_n}(\delta_n, \gamma_n, \nu_n)(f - f(\Delta)) = \int f(b)\nu_n(\mathrm{d}b).$$

We deduce that

$$\int f(b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int f(b)\nu_\infty(\mathrm{d}b).$$

Therefore we can apply the first part of Lemma 5.9 and in particular, for any $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$ vanishing on a neighbourhood of a, we get the second statement in (5.4).

Consider $\theta \in \mathrm{C}(\mathbb{R}_+, [0,1])$ such that $\theta(r) = 1$ for $r \leq 1$ and $\theta(r) = 0$ for $r \geq 2$. For $(a, b) \in \mathbb{R}^d \times \mathbb{R}^{d\Delta}$ and $n \in \mathbb{N} \cup \{\infty\}$, define

$$\widetilde{\chi}(a,b) := \theta(|b-a|)(b-a)\mathbb{1}_{b\neq\Delta} \quad \text{and} \quad \widetilde{\delta}_n := \delta_n + \int (\widetilde{\chi}(a_n, b) - \chi(a_n, b))\nu_n(\mathrm{d}b). \quad (5.13)$$

Therefore, for all $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ and all $n \in \mathbb{N} \cup \{\infty\}$, we recast

$$T_{\chi,a_n}(\delta_n, \gamma_n, \nu_n)f = T_{\widetilde{\chi},a_n}(\widetilde{\delta}_n, \gamma_n, \nu_n)f.$$

Let ϕ be an arbitrary linear form on $\mathbb{R}^d$ and consider $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ such that $f(b) = (b - a_\infty)\phi$ on a neighbourhood of a_∞. Then we have

$$T_{\widetilde{\chi},a_\infty}(\widetilde{\delta}_\infty, \gamma_\infty, \nu_\infty)f = \widetilde{\delta}_\infty\phi + \int (f(b) - \widetilde{\chi}(a_\infty, b)\phi)\nu_\infty(\mathrm{d}b)$$

and for n large enough

$$T_{\widetilde{\chi},a_n}(\widetilde{\delta}_n, \gamma_n, \nu_n)f = \widetilde{\delta}_n\phi + \int (f(b) - f(a_n) - \widetilde{\chi}(a_n, b)\phi)\nu_n(\mathrm{d}b).$$

Thanks to the first part of Lemma 5.9 we deduce

$$\int (f(b) - f(a_n) - \widetilde{\chi}(a_n, b)\phi)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int (f(b) - \widetilde{\chi}(a_\infty, b)\phi)\nu_\infty(\mathrm{d}b).$$

We conclude that $\widetilde{\delta}_n\phi \underset{n\to\infty}{\longrightarrow} \widetilde{\delta}_\infty\phi$, and since ϕ was chosen arbitrary, $\widetilde{\delta}_n \underset{n\to\infty}{\longrightarrow} \widetilde{\delta}_\infty$.
Let Φ be an arbitrary symmetric bilinear form on $\mathbb{R}^d$ and if $(e_1, \dots, e_d)$ is the canonical basis of $\mathbb{R}^d$, we denote $\Phi_{ij} = \Phi(e_i, e_j)$, $i, j = 1, \dots, d$. Consider $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ such that $f(b) = \Phi(b - a_\infty, b - a_\infty)$ on a neighbourhood of a_∞. Then, for n large enough, we can write

$$\begin{aligned} &T_{\widetilde{\chi},a_n}(\widetilde{\delta}_n, \gamma_n, \nu_n)f \\ &= \sum_{i,j=1}^d \Phi_{ij}\gamma_{n,ij} + 2\Phi(a_n - a_\infty, \widetilde{\delta}_n) + \int \big(f(b) - f(a_n) - 2\Phi(a_n - a_\infty, \widetilde{\chi}(a_n, b))\big)\nu_n(\mathrm{d}b), \end{aligned}$$

or equivalently,

$$\begin{aligned} T_{\widetilde{\chi},a_n}(\widetilde{\delta}_n, \gamma_n, \nu_n)f = &\sum_{i,j=1}^d \Phi_{ij} \left(\gamma_{n,ij} + \int (\widetilde{\chi}_i\widetilde{\chi}_j)(a_n, b)\nu_n(\mathrm{d}b)\right) + 2\Phi(a_n - a_\infty, \widetilde{\delta}_n) \\ &+ \int \left(f(b) - f(a_n) - 2\Phi(a_n - a_\infty, \widetilde{\chi}(a_n, b)) - \sum_{i,j=1}^d \Phi_{ij}\,(\widetilde{\chi}_i\widetilde{\chi}_j)(a_n, b)\right)\nu_n(\mathrm{d}b). \end{aligned}$$

A similar equality holds with the index n replaced by ∞:

$$T_{\widetilde{\chi},a_\infty}(\widetilde{\delta}_\infty,\gamma_\infty,\nu_\infty)f=\sum_{i,j=1}^d\Phi_{ij}\gamma_{\infty,ij}+\int f(b)\nu_\infty(\mathrm{d}b)$$
$$=\sum_{i,j=1}^d\Phi_{ij}\Big(\gamma_{\infty,ij}+\int(\widetilde{\chi}_i\widetilde{\chi}_j)(a_\infty,b)\nu_\infty(\mathrm{d}b)\Big)\Big)+\int\Big(f(b)-\sum_{i,j=1}^d\Phi_{ij}\,(\widetilde{\chi}_i\widetilde{\chi}_j)(a_\infty,b)\Big)\nu_\infty(\mathrm{d}b).$$

Invoking again the first part of Lemma 5.9 we can write

$$\int\Big(f(b)-f(a_n)-2\Phi(a_n-a_\infty,\widetilde{\chi}(a_n,b))-\sum_{i,j=1}^d\Phi_{ij}\,(\widetilde{\chi}_i\widetilde{\chi}_j)(a_n,b)\Big)\nu_n(\mathrm{d}b)$$
$$\underset{n\to\infty}{\longrightarrow}\int\Big(f(b)-\sum_{i,j=1}^d\Phi_{ij}\,(\widetilde{\chi}_i\widetilde{\chi}_j)(a_\infty,b)\Big)\nu_\infty(\mathrm{d}b).$$

Hence we get

$$\sum_{i,j=1}^d\Phi_{ij}\Big(\gamma_{n,ij}+\int(\widetilde{\chi}_i\widetilde{\chi}_j)(a_n,b)\nu_n(\mathrm{d}b))\Big)\underset{n\to\infty}{\longrightarrow}\sum_{i,j=1}^d\Phi_{ij}\Big(\gamma_{\infty,ij}+\int(\widetilde{\chi}_i\widetilde{\chi}_j)(a_\infty,b)\nu_\infty(\mathrm{d}b))\Big).$$

Since Φ was chosen arbitrary, for all $1\le i,j\le d$ we get

$$\gamma_{n,ij}+\int(\widetilde{\chi}_i\widetilde{\chi}_j)(a_n,b)\nu_n(\mathrm{d}b))\underset{n\to\infty}{\longrightarrow}\gamma_{\infty,ij}+\int(\widetilde{\chi}_i\widetilde{\chi}_j)(a_\infty,b)\nu_\infty(\mathrm{d}b)).$$

Due to the second part of Lemma 5.9 we deduce in particular

$$\lim_{n\to\infty}\int(\widetilde{\chi}(a_n,b)-\chi(a_n,b))\nu_n(\mathrm{d}b)=\int(\widetilde{\chi}(a_\infty,b)-\chi(a_\infty,b))\nu_\infty(\mathrm{d}b).$$

Owing (5.13) we ends up with $\delta_n\underset{n\to\infty}{\longrightarrow}\delta_\infty$, which is the first sentence in (5.4). Invoking again the second part of Lemma 5.9 we also have, for all $1\le i,j\le d$,

$$\int((\widetilde{\chi}_i\widetilde{\chi}_j)(a_n,b)-(\chi_i\chi_j)(a_n,b))\nu_n(\mathrm{d}b)\underset{n\to\infty}{\longrightarrow}\int((\widetilde{\chi}_i\widetilde{\chi}_j)(a_\infty,b)-(\chi_i\chi_j)(a_\infty,b))\nu_\infty(\mathrm{d}b),$$

so we deduce the third sentence in (5.4).

We prove the converse, so we suppose that (5.4) holds. Let $f\in \mathrm{C}_c^\infty(\mathbb{R}^d)$ be. For each $n\in\mathbb{N}\cup\{\infty\}$,

$$T_{\chi,a_n}(\delta_n,\gamma_n,\nu_n)f=\frac{1}{2}\sum_{i,j=1}^d\gamma_{n,ij}\partial^2_{ij}f(a_n)+\delta_n\cdot\nabla f(a_n)$$
$$+\int(f(b)-f(a_n)-\chi(a_n,b)\cdot\nabla f(a_n))\nu_n(\mathrm{d}b),$$

or, equivalently,

$$T_{\chi,a_n}(\delta_n,\gamma_n,\nu_n)f = \frac{1}{2}\sum_{i,j=1}^{d}\Big(\gamma_{n,ij} + \int (\chi_i\chi_j)(a_n,b)\nu_n(\mathrm{d}b)\Big)\,\partial^2_{ij}f(a_n) + \delta_n\cdot\nabla f(a_n)$$
$$+ \int \Big(f(b) - f(a_n) - \chi(a_n,b)\cdot\nabla f(a_n) - \sum_{i,j=1}^{d}(\chi_i\chi_j)(a_n,b)\partial^2_{ij}f(a_n)\Big)\nu_n(\mathrm{d}b).$$

Applying the second part of Lemma 5.9 to the last term of the previous equation we deduce

$$T_{\chi,a_n}(\delta_n,\gamma_n,\nu_n)f \underset{n\to\infty}{\longrightarrow} T_{\chi,a_\infty}(\delta_\infty,\gamma_\infty,\nu_\infty)f.$$

The proof is complete except for the proof of Lemma 5.9. □

Proof of Lemma 5.9. Consider a sequence of functions $(f_n)_{n\in\mathbb{N}\cup\{\infty\}}$ as in the first part of lemma. Let U_1 be an open subset such that $U_1 \Subset \mathbb{R}^{d\Delta}\setminus\{a_\infty\}$ and

$$U_1 \supset \bigcup_{n\in\mathbb{N}\cup\{\infty\}}\Big\{f_n\neq 0\Big\} \supset \mathbb{R}^{d\Delta}\setminus\Big\{b_0\in\mathbb{R}^{d\Delta}\;\Big|\;\lim_{n\to\infty,b\to b_0} f_n(b) = f_\infty(b_0)\Big\}.$$

Let $\varphi_1 \in \mathrm{C}(\mathbb{R}^{d\Delta})$ be such that $\varphi_1 \geq \mathbb{1}_{U_1}$, φ_1 is infinitely differentiable in $\mathbb{R}^d$, it vanishes in a neighbourhood of a_∞ and is constant in a neighbourhood of Δ, φ_1. Then we have

$$\int \varphi_1(b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int \varphi_1(b)\nu_\infty(\mathrm{d}b).$$

Therefore

$$\sup_{n\in\mathbb{N}\cup\{\infty\}}\nu_n(U_1) \leq \sup_{n\in\mathbb{N}\cup\{\infty\}}\int\varphi_1(b)\nu_n(\mathrm{d}b) < \infty.$$

Since $\mathbb{R}^{d\Delta}\setminus\{a_\infty\}$ is a Polish space, the measure ν_∞ is inner regular on this set. Hence, if $\varepsilon>0$ is chosen arbitrary, there exists a compact subset $K_\varepsilon\subset U_1$ satisfying

$$K_\varepsilon \subset \Big\{b_0\in\mathbb{R}^{d\Delta}\mid \lim_{n\to\infty,b\to b_0} f_n(b) = f_\infty(b_0)\Big\} \quad\text{and}\quad \nu_\infty(K_\varepsilon)\geq\nu_\infty(U_1)-\varepsilon. \tag{5.14}$$

Hence f_∞ is continuous on K_ε and f_n converges uniformly to f_∞ on K_ε. There exists a function $\varphi_2\in\mathrm{C}(\mathbb{R}^{d\Delta})$ such that $\{\varphi_2\neq 0\}\subset U_1$, $\|\varphi_2\|\leq\|f_\infty\|$ and $\|\varphi_2 - f_\infty\|_{K_\varepsilon}\leq\varepsilon$, φ_2 is infinitely differentiable in $\mathbb{R}^d$ and is constant in a neighbourhood of Δ. Combining (5.14) and a compactness argument, we deduce that there exists an open subset $U_2\subset U_1$ such that

$$K_\varepsilon\subset U_2\subset\Big\{b_0\in\mathbb{R}^{d\Delta}\mid \limsup_{n\to\infty,b\to b_0}\big|f_n(b)-\varphi_2(b_0)\big|\leq 2\varepsilon\Big\}.$$

Arguing by dominated convergence, there exists a function $\varphi_3\in\mathrm{C}(\mathbb{R}^{d\Delta})$ such that $\mathbb{1}_{U_2}\geq\varphi_3$ and $\int\varphi_3(b)\nu_\infty(\mathrm{d}b)\geq\nu_\infty(U_2)-\varepsilon$, φ_3 is infinitely differentiable in $\mathbb{R}^d$, it vanishes in a neighbourhood of a_∞ and is constant in a neighbourhood of Δ. Hence

$$\begin{aligned}\liminf_{n\to\infty}\nu_n(U_2) &\geq \liminf_{n\to\infty}\int\varphi_3(b)\nu_n(\mathrm{d}b) = \int\varphi_3(b)\nu_\infty(\mathrm{d}b) \geq \nu_\infty(U_2)-\varepsilon\\ &\geq \nu_\infty(K_\varepsilon)-\varepsilon \geq \nu_\infty(U_1)-2\varepsilon.\end{aligned}$$

Therefore we have

$$\begin{aligned}\limsup_{n\to\infty}\Big|\int f_n(b)\nu_n(\mathrm{d}b)-\int f_\infty(b)\nu_\infty(\mathrm{d}b)\Big| &\leq \limsup_{n\to\infty}\Big|\int\varphi_2(b)\nu_n(\mathrm{d}b)-\int\varphi_2(b)\nu_\infty(\mathrm{d}b)\Big|\\ &+\limsup_{n\to\infty}\Big|\int_{U_2}(f_n(b)-\varphi_2(b))\nu_n(\mathrm{d}b)\Big|+\limsup_{n\to\infty}\Big|\int_{U_1\setminus U_2}(f_n(b)-\varphi_2(b))\nu_n(\mathrm{d}b)\Big|\\ &+\limsup_{n\to\infty}\Big|\int_{K_\varepsilon}(f_\infty(b)-\varphi_2(b))\nu_\infty(\mathrm{d}b)\Big|+\limsup_{n\to\infty}\Big|\int_{U_1\setminus K_\varepsilon}(f_\infty(b)-\varphi_2(b))\nu_\infty(\mathrm{d}b)\Big|,\end{aligned}$$

and we deduce

$$\begin{aligned}&\limsup_{n\to\infty}\Big|\int f_n(b)\nu_n(\mathrm{d}b)-\int f_\infty(b)\nu_\infty(\mathrm{d}b)\Big|\\ &\quad\leq 0+2\varepsilon\sup_{n\in\mathbb{N}}\nu_n(U_1)+4\varepsilon\sup_{n\in\mathbb{N}\cup\{\infty\}}\|f_n\|+\varepsilon\nu_\infty(U_1)+2\varepsilon\|f_\infty\|\\ &\quad\leq 3\varepsilon\Big(\sup_{n\in\mathbb{N}\cup\{\infty\}}\nu_n(U_1)+2\sup_{n\in\mathbb{N}\cup\{\infty\}}\|f_n\|\Big).\end{aligned}$$

Letting $\varepsilon\to 0$ we obtain that

$$\int f_n(b)\nu_n(\mathrm{d}b)\underset{n\to\infty}{\longrightarrow}\int f_\infty(b)\nu_\infty(\mathrm{d}b).$$

We proceed with the proof of the part *ii)* of lemma. Fix $\eta>0$ as in the statement and choose an arbitrary $\varepsilon>0$. Thanks to (5.12), there exists $0<\delta<\eta/2$ such that

$$\limsup_{n\to\infty}\sup_{0<|h|\leq 2\delta}\frac{f_n(a_n+h)}{|h|^2}\leq\frac{\varepsilon}{1\vee\sup_{n\in\mathbb{N}\cup\{\infty\}}\int|b-a_n|^2\mathbb{1}_{|b-a_n|\leq\eta}\nu_n(\mathrm{d}b)}.$$

Consider a function $\varphi\in\mathrm{C}(\mathbb{R}^{d\Delta},[0,1])$ which vanishes in a neighbourhood of a_∞ and such that $\varphi(a)=1$ for any a satisfying $|a-a_\infty|\geq\delta$. Then, using the first part *i)*,

$$\int\varphi(b)f_n(b)\nu_n(\mathrm{d}b)\underset{n\to\infty}{\longrightarrow}\int\varphi(b)f_\infty(b)\nu_\infty(\mathrm{d}b).$$

Clearly for $n\in\mathbb{N}$ large enough, $|a-a_n|\leq\delta$, hence

$$\Big|\int(1-\varphi(b))f_n(b)\nu_n(\mathrm{d}b)\Big|\leq\int|b-a_n|^2\mathbb{1}_{|b-a_n|\leq\eta}\,\nu_n(\mathrm{d}b)\cdot\sup_{0<|h|\leq 2\delta}\frac{f_n(a_n+h)}{|h|^2}.$$

We deduce that $\limsup_{n\to\infty} \Big| \int (1-\varphi(b)) f_n(b) \nu_n(\mathrm{d}b) \Big| \leq \varepsilon$. Similarly,

$$\Big| \int (1-\varphi(b)) f_\infty(b) \nu_\infty(\mathrm{d}b) \Big| \leq \int |b-a_\infty|^2 \mathbb{1}_{|b-a_\infty|\leq \eta}\, \nu_n(\mathrm{d}b) \cdot \limsup_{n\to\infty} \sup_{0<|h|\leq 2\delta} \frac{f_n(a_n+h)}{|h|^2} \leq \varepsilon.$$

Therefore

$$\limsup_{n\to\infty} \Big| \int f_n(b)\nu_n(\mathrm{d}b) - \int f_\infty(b)\nu_\infty(\mathrm{d}b) \Big| \leq 2\varepsilon.$$

Letting $\varepsilon \to 0$ we can conclude the proof of *ii)*. □

To conclude this section let us give another consequence of Theorem 5.1. It is an approximation result inspired from [BSW13], Theorem 7.6 p. 172. Let $L : \mathrm{C}_c^\infty(\mathbb{R}^d) \to \mathrm{C}(\mathbb{R}^d)$ be an operator satisfying the positive maximum principle. We will denote by $\tau_h f(a) = f(a+h)$ the translation of f by $h \in \mathbb{R}^d$. For $a_0 \in \mathbb{R}^d$, we introduce the operator

$$L(a_0) : \mathrm{C}_c^\infty(\mathbb{R}^d) \to \mathrm{C}_0(\mathbb{R}^d) \quad \text{given by} \quad L(a_0)f(a) := L(\tau_{a-a_0} f)(a_0). \tag{5.15}$$

Clearly $Lf(a) = L(a)f(a)$. Since $L(a_0)$ is invariant with respect to the translation and satisfies the positive maximum principle then its closure in $\mathrm{C}_0(\mathbb{R}^d) \times \mathrm{C}_0(\mathbb{R}^d)$ is the $\mathrm{C}_0 \times \mathrm{C}_0$-generator of a Lévy family (see for instance, Section 2.1 pp. 32-41 from [BSW13]). We denote by $(P_t(a_0))_{t\geq 0}$ its Feller semi-group.

Corollary 5.10 (Approximation with Lévy increments)**.** *Let $(\varepsilon_n)_n$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ and define the transition operators P_n by*

$$P_n f(a) := P_{\varepsilon_n}(a) f(a), \quad \textit{for } f \in \mathrm{C}_0(\mathbb{R}^d).$$

Then, for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$\frac{1}{\varepsilon_n}(P_n f - f) \underset{n\to\infty}{\longrightarrow} Lf, \quad \textit{uniformly on compact sets.}$$

Remark 5.11. 1) If the martingale local problem associated to L is well-posed, by Theorem 3.2, one then deduces the convergence of the associated probability families.
2) Excepting the fact that the present convergence is for the local Skorokhod topology, Corollary 5.10 is an improvement of Theorem 7.6 p. 172 from [BSW13]. More precisely, we do not ask that the closure of L should be a generator of a Feller semi-group, but only suppose that the martingale local problem is well-posed. ♢

Proof of Corollary 5.10. Recall that $\chi_1(a,b)$ is given by (5.2). Thanks to Theorem 2.12 pp. 21-22 from [Hoh98], for each $a \in \mathbb{R}^d$ there exists a triplet $(\delta(a), \gamma(a), \nu(a))$ satisfying (H2(a)) such that, $Lf(a) := T_{\chi_1,a}(\delta(a),\gamma(a),\nu(a))f$, for all $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. It is clear that for any $a_0, a \in \mathbb{R}^d$ and $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, the Lévy operator $L(a_0)$ defined by (5.15) satisfies also

$$L(a_0)f(a) = T_{\chi 1,a}(\delta(a_0), \gamma(a_0), \nu_a(a_0)).$$

Here and elsewhere $\nu_a(a_0)$ is the pushforward measure of $\nu(a_0)$ with respect to the translation $b \mapsto b - a_0 + a$.

To get the result of the corollary it suffices to prove that for any function $f_0 \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ and any sequence $a_n \in \mathbb{R}^d$ converging to $a_\infty \in \mathbb{R}^d$, we have

$$\frac{1}{\varepsilon_n}\big(P_n f_0(a_n) - f_0(a_n)\big) \underset{n\to\infty}{\longrightarrow} Lf_0(a_\infty). \tag{5.16}$$

Thanks to Theorem 5.1 we have,

$$\delta(a_n) \underset{n\to\infty}{\longrightarrow} \delta(a_\infty),$$

$$\forall f \in \mathrm{C}(\mathbb{R}^{d\Delta}) \text{ vanishing in a neighbourhood of } a_\infty, \int f(b)\nu(a_n, \mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int f(b)\nu(a_\infty, \mathrm{d}b),$$

and for all $1 \le i, j \le d$

$$\gamma_{ij}(a_n) + \int (\chi_i\chi_j)(a_n, b)\nu(a_n, \mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \gamma_{ij}(a_\infty) + \int (\chi_i\chi_j)(a_\infty, b)\nu(a_\infty, \mathrm{d}b).$$

It is not difficult to deduce that, there exists $C \in \mathbb{R}_+$ such that, for all $n \in \mathbb{N} \cup \{\infty\}$ and $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$\|L(a_n)f\| \le C\|f\| \vee \max_{1\le i\le d} \|\partial_i f\| \vee \max_{1\le i,j\le d} \|\partial_{ij}^2 f\|.$$

Hence $\sup_{n\in\mathbb{N}\cup\{\infty\}} \|L(a_n)f_0\| < \infty$. Consider $b_\infty \in \mathbb{R}^d$, a sequence $b_n \to b_\infty$ and a function $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$ vanishing on a neighbourhood of b_∞. Owing the first part of Lemma 5.9 we deduce that

$$\begin{aligned}\int f(b)\nu_{b_n}(a_n, \mathrm{d}b) &= \int f(b - a_n + b_n)\nu(a_n, \mathrm{d}b)\\ &\underset{n\to\infty}{\longrightarrow} \int f(b - a_\infty + b_\infty)\nu(a_\infty, \mathrm{d}b) = \int f(b)\nu_{b_\infty}(a_\infty, \mathrm{d}b).\end{aligned}$$

Thanks to Corollary 5.5, $L(a_n)f$ converges uniformly on compact sets toward $L(a_\infty)f$, for all $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. In particular, for each $\varepsilon > 0$ there exists an open neighbourhood U of a_∞ and $n_0 \in \mathbb{N}$ such that

$$\forall n \ge n_0,\ \forall a \in U, \quad |L(a_n)f_0(a) - L(a_\infty)f_0(a_\infty)| \le \varepsilon. \tag{5.17}$$

Let $\mathbf{P}_n$ be the unique element of $\mathcal{M}(L(a_n))$ such that $\mathbf{P}_n(X_0 = a_n) = 1$. Then,

$$\begin{aligned}\Big|\frac{1}{\varepsilon_n}\big(P_n f_0(a_n) - f_0(a_n)\big) - L(a_\infty)f_0(a_\infty)\Big| &= \Big|\frac{1}{\varepsilon_n}\big(\mathbf{E}_n[f_0(X_{\varepsilon_n})] - f_0(a_n)\big) - L(a_\infty)f_0(a_\infty)\Big|\\ &= \Big|\mathbf{E}_n \frac{1}{\varepsilon_n}\int_0^{\varepsilon_n} \big(L(a_n)f_0(X_s) - L(a_\infty)f_0(a_\infty)\big)\mathrm{d}s\Big|\end{aligned}$$

On the right hand side we split the expectation into two terms, by using the position of τ^U introduced in (2.1) with respect to ε_n, and we get

$$
\begin{aligned}
\Big|\frac{1}{\varepsilon_n}\big(P_n f_0(a_n) - f_0(a_n)\big) - L(a_\infty)f_0(a_\infty)\Big| \\
&\le \mathbf{E}_n\Big[\mathbb{1}_{\{\tau^U<\varepsilon_n\}}\frac{1}{\varepsilon_n}\int_0^{\varepsilon_n}\big|L(a_n)f_0(X_s) - L(a_\infty)f_0(a_\infty)\big|\mathrm{d}s\Big] \\
&\quad + \mathbf{E}_n\Big[\mathbb{1}_{\{\tau^U\ge\varepsilon_n\}}\frac{1}{\varepsilon_n}\int_0^{\varepsilon_n}\big|L(a_n)f_0(X_s) - L(a_\infty)f_0(a_\infty)\big|\mathrm{d}s\Big].
\end{aligned}
$$

Plugging (5.17) into the second term we obtain, for all $n \ge n_0$

$$\Big|\frac{1}{\varepsilon_n}\big(P_n f_0(a_n) - f_0(a_n)\big) - L(a_\infty)f_0(a_\infty)\Big| \le 2\mathbf{P}_n(\tau^U < \varepsilon_n)\sup_{m\in\mathbb{N}\cup\{\infty\}}\|L(a_m)f_0\| + \varepsilon.$$

At this level we apply Lemma A.4 result stated in the Apendix, concerning the uniform continuity along stopping times with a compact neighbourhood $K \subset U$ of a_∞ and with $\mathcal{U} := \big(\mathbb{R}^d \times U\big) \cup \big((\mathbb{R}^d \setminus K) \times \mathbb{R}^d\big)$. We deduce that

$$\lim_{n\to\infty}\mathbf{P}_n(\tau^U < \varepsilon_n) = 0.$$

Hence

$$\limsup_{n\to\infty}\Big|\frac{1}{\varepsilon_n}\big(P_n f_0(a_n) - f_0(a_n)\big) - L(a_\infty)f_0(a_\infty)\Big| \le \varepsilon,$$

and we end up with (5.16) by letting $\varepsilon \to 0$. □

Appendix

We collect in this appendix several results proved in [GH17] and used in some proofs in Sections 4 and 5. These results have technical statements and we use the notations introduced in Section 2. We refer the reader to the article [GH17] for the introductory context and for the proofs of each of the following lemmas.

Lemma A.1 (cf. Lemma 4.22, p. 154, in [GH17]). *Let U be an open subset of S and L be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ such that $\mathrm{D}(L)$ is dense in $\mathrm{C}(S)$. Assume that the martingale local problem associated to L is well-posed. Then there exist a subset L_0 of L and a function $\mathfrak{h}_0$ of $\mathrm{C}(S,\mathbb{R}_+)$ with $\{\mathfrak{h}_0 \neq 0\} = U$, such that the following properties hold.*

i) $\overline{L} = \overline{L_0}$ *and* $\mathfrak{h}_0 L_0 \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$.

ii) If $\mathfrak{h} \in \mathrm{C}(S,\mathbb{R}_+)$ is an arbitrary function such that $\{\mathfrak{h} \neq 0\} = U$ and $\sup_{a\in U}\frac{\mathfrak{h}}{\mathfrak{h}_0}(a) < \infty$, then the (classical) martingale problem associated to $(\mathfrak{h}L_0)^\Delta$, obtained as in (2.2), is well-posed in the space of càdlàg paths having values in S^Δ.

Lemma A.2 (cf. Proposition 4.20, p. 153, in [GH17]). *Let L_1, L_2 be two subsets of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ such that $\mathrm{D}(L_1) = \mathrm{D}(L_2)$ is a dense subset in $\mathrm{C}_0(S)$ and take an open subset $U \subset S$. Assume that the martingale local problems associated to L_1 and L_2 are well-posed and let $\mathbf{P}^1 \in \mathcal{M}(L_1)$ and $\mathbf{P}^2 \in \mathcal{M}(L_2)$ be two solutions of these problems having the same initial distribution. If for all $f \in D(L_1)$, $(L_2 f)_{|U} = (L_1 f)_{|U}$, then*

$$\mathscr{L}_{\mathbf{P}^2}\left(X^{\tau^U}\right) = \mathscr{L}_{\mathbf{P}^1}\left(X^{\tau^U}\right),$$

where τ^U is given by (2.1)*.*

Lemma A.3 (cf. Lemma A.1, p. 159, in [GH17]). *Consider $U_n \subset S$ an increasing sequence of open subsets such that $S = \bigcup_n U_n$. For $(n, a) \in \mathbb{N} \times \mathbb{R}$, let $\mathbf{P}^n_a \in \mathcal{P}(\mathbb{D}_{loc}(S))$ be, such that*

i) for each $n \leq m$ and $a \in \mathbb{R}$, $\mathscr{L}_{\mathbf{P}^m_a}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}^n_a}\left(X^{\tau^{U_n}}\right)$;

ii) for each $n \in \mathbb{N}$, $a \mapsto \mathbf{P}^n_a$ is weakly continuous for the local Skorokhod topology.

Then we have

j) for $a \in \mathbb{R}$, there exists a unique $\mathbf{P}^\infty_a \in \mathcal{P}(\mathbb{D}_{loc}(S))$, such that for any $(n, a) \in \mathbb{N} \times \mathbb{R}$, $\mathscr{L}_{\mathbf{P}^\infty_a}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}^n_a}\left(X^{\tau^{U_n}}\right)$;

jj) the mapping $(a, n) \mapsto \mathbf{P}^n_a$ on $\big(\mathbb{N} \cup \{\infty\}\big) \times \mathbb{R}$ with values in $\mathcal{P}(\mathbb{D}_{loc}(S))$ is weakly continuous for the local Skorokhod topology.

Lemma A.4 (cf. Lemma 3.8, p. 139, in [GH17]). *Let $L_1, \ldots, L_n, \ldots$ and L_∞ be subsets of $\mathrm{C}_0(S) \times \mathrm{C}(S)$. Assume that $\mathrm{D}(L_\infty)$ is dense in $\mathrm{C}_0(S)$ and that, the sequence of operators L_n converges to L_∞, in the sense of Theorem 2.2. Consider K a compact subset of S and $\mathcal{U}$ an open subset of $S \times S$ containing $\{(a, a) \,|\, a \in S\}$. Then for each $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$, $\delta > 0$ such that, for any $\tau_1 \leq \tau_2$ two $(\mathcal{F}_{t+})_t$-stopping times, for any $n \geq n_0$, and for any $\mathbf{P} \in \mathcal{M}(L_n)$ satisfying $\mathbf{E}[(\tau_2 - \tau_1)\mathbb{1}_{\{X_{\tau_1} \in K\}}] \leq \delta$, we have*

$$\mathbf{P}(X_{\tau_1} \in K,\ \tau(\tau_1) \leq \tau_2) \leq \varepsilon,$$

with the convention $X_\infty := \Delta$ and where $\tau(\tau_1)$ denotes the $(\mathcal{F}_{t+})_t$-stopping time

$$\tau(\tau_1) := \inf\left\{t \geq \tau_1 \;|\; \{(X_{\tau_1}, X_s)\}_{\tau_1 \leq s \leq t} \notin \mathcal{U}\right\}.$$

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