Localisation en espace de la propriété de Feller avec application aux processus de type Lévy

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Mots clés : Processus de Markov, processus de Feller, processus de type Lévy, topologie de Skorokhod

Résumé : Dans cette thèse, nous donnons une localisation en espace de la théorie des processus de Feller. Un premier objectif est d'obtenir des résultats simples et précis sur la convergence de processus de Markov. Un second objectif est d'étudier le lien entre les notions de propriété de Feller, problème de martingales et topologie de Skorokhod.
Dans un premier temps nous donnons une version localisée en espace de la topologie de Skorokhod. Nous en étudions les notions de compacité et tension. Nous faisons le lien entre les topologies de Skorokhod localisée et non localisée, grâce à la notion de changement de temps.
Dans un second temps, à l'aide de la topologie de Skorokhod localisée et du changement de temps, nous étudions les problèmes de martingales. Nous montrons pour des processus l'équivalence entre, d'une part, être solution d'un problème de martingales bien posé, d'autre part, vérifier une version localisée de la propriété de Feller, et enfin, être markovien et continu en loi par rapport à sa condition initiale.
Nous caractérisons la convergence en loi pour les solutions de problèmes de martingale en terme de convergence des opérateurs associés et donnons un résultat similaire pour les approximations à temps discret.
Pour finir, nous appliquons la théorie des processus localement fellerien à deux exemples. Nous l'appliquons d'abord au processus de type Lévy et obtenons des résultats de convergence pour des processus à temps discret et continu, notamment des méthodes de simulation et schémas d'Euler. Nous appliquons ensuite cette même théorie aux diffusions unidimensionnelles dans des potentiels, nous obtenons des résultats de convergence de diffusions ou marches aléatoires vers des diffusions singulières. Comme conséquences, nous déduisons la convergence de marches aléatoires en milieux aléatoires vers des diffusions en potentiels aléatoires.

Title : Space localisation of the Feller property with application to Lévy-type processes

Keywords : Markov processes, Feller processes, Lévy-type processes, Skorokhod topology

Abstract: In this PhD thesis, we give a space localisation for the theory of Feller processes. A first objective is to obtain simple and precise results on the convergence of Markov processes. A second objective is to study the link between the notions of Feller property, martingale problem and Skorokhod topology.
First we give a localised in space version of the Skorokhod topology. We study the notions of compactness and tightness for this topology. We make the connexion between localised and unlocalised Skorokhod topologies, by using the notion of time change.
In a second step, using the localised Skorokhod topology and the time change, we study martingale problems. We show the equivalence between, on the one hand, to be solution of a well-posed martingale problem, on the other hand, to satisfy a localised version of the Feller property, and finally, to be a Markov process weakly continuous with respect to the initial condition.
We characterise the weak convergence for solutions of martingale problems in terms of convergence of associated operators and give a similar result for discrete time approximations.
Finally, we apply the theory of locally Feller process to some examples. We first apply it to the Lévy-type processes and obtain convergence results for discrete and continuous time processes, including simulation methods and Euler's schemes. We then apply the same theory to one-dimensional diffusions in a potential and we obtain convergence results of diffusions or random walks towards singular diffusions. As a consequence, we deduce the convergence of random walks in random environment towards diffusions in random potential.

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Chapitre 1

Introduction

1 Processus de Markov

Les principaux objets d'étude de ma thèse sont les processus de Markov, c'est-à-dire les processus sans mémoire. Formellement un processus de Markov sur un espace probabilisé $(\Omega, \mathcal{G}, \mathbb{P})$ est une famille de variables aléatoires $X_t : \Omega \to S$ indexée par $t \in \mathbb{R}_+$, à valeurs dans un espace mesurable d'états $(S, \mathcal{S})$, vérifiant

$$\forall s \leq t,\ \forall A \in \mathcal{S},\ \mathbb{P}\text{-p.s.,} \quad \mathbb{P}\left(X_t \in A \mid X_u,\ u \leq s\right) = \mathbb{P}\left(X_t \in A \mid X_s\right). \tag{1.1}$$

1.1 Équation de Chapman-Kolmogorov

Une première approche pour étudier ces processus est la suivante : en supposant que S est un espace de Borel[1], alors il existe une version régulière $\mu_{s,t}(X_s, \mathrm{d}b)$ de $\mathscr{L}(X_t \mid X_s)$, i.e. pour tout $s \leq t$

$$\forall a \in S, \quad \mu_{s,t}(a, \mathrm{d}b) \text{ est une mesure de probabilité sur } S, \tag{1.2}$$

$$\forall\ A \in \mathcal{S}, \quad \text{la fonction } a \in S \mapsto \mu_{s,t}(a, A) \text{ est mesurable,} \tag{1.3}$$

$$\forall\ A \in \mathcal{S},\ \mathbb{P}\text{-p.s.,} \quad \mathbb{P}\left(X_t \in A \mid X_u,\ u \leq s\right) = \mu_{s,t}(X_s, A), \tag{1.4}$$

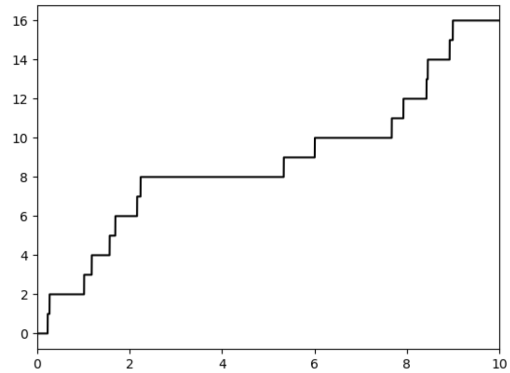
qui vérifie de plus l'équation de Chapman-Kolmogorov

$$\forall s \leq t \leq u,\ \forall a \in S,\ \forall A \in \mathcal{S}, \quad \mu_{s,u}(a, A) = \int \mu_{s,t}(a, \mathrm{d}b)\mu_{t,u}(b, A). \tag{1.5}$$

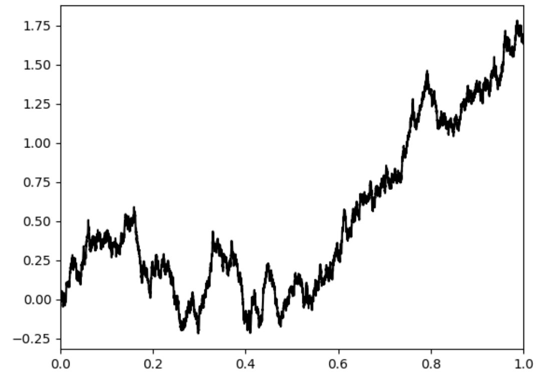
Ceci est démontré notamment dans [Kal02], Théorème 6.3 p. 107 et Corollaire 8.3 p. 142. De plus (voir [Kal02] Proposition 8.2 p. 142) les lois fini-dimensionnelles de X sont uniquement déterminées par $(\mu_{s,t}(\cdot, \mathrm{d}b))_{s \leq t}$ et par la loi de X_0. Réciproquement, par le théorème d'existence de Kolmogorov ([Kal02] Théorème 8.4 p. 143), pour toute famille $(\mu_{s,t}(\cdot, \mathrm{d}b))_{s \leq t}$ vérifiant (1.2), (1.3) et (1.5) et pour toute probabilité initiale ν_0 sur S, il existe un espace probabilisé et un processus de Markov X vérifiant (1.4) et tels que X_0 soit de loi ν_0.

Ceci permet la construction des processus de Markov classiques, tels que le processus de Poisson N sur $\mathbb{N}$, ainsi que le mouvement brownien B, le processus de Ornstein–Uhlenbeck

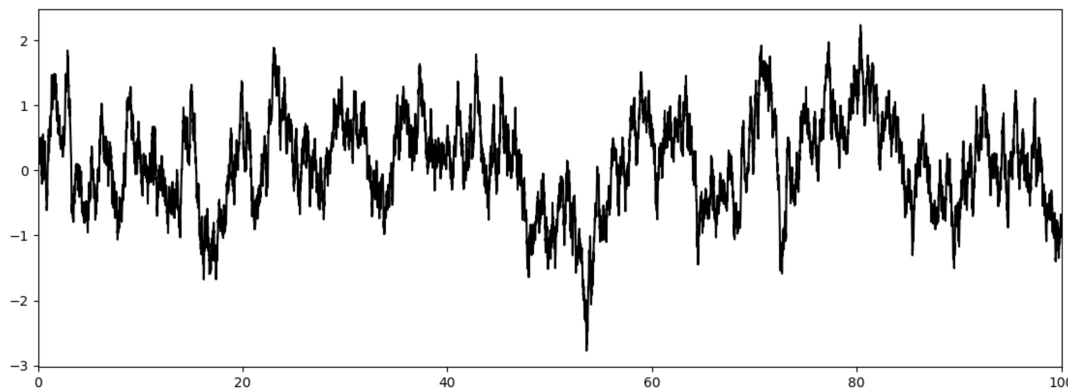
[1] Un espace mesurable est de Borel s'il est en bijection bimesurable avec un borélien d'un espace métrique séparable complet.



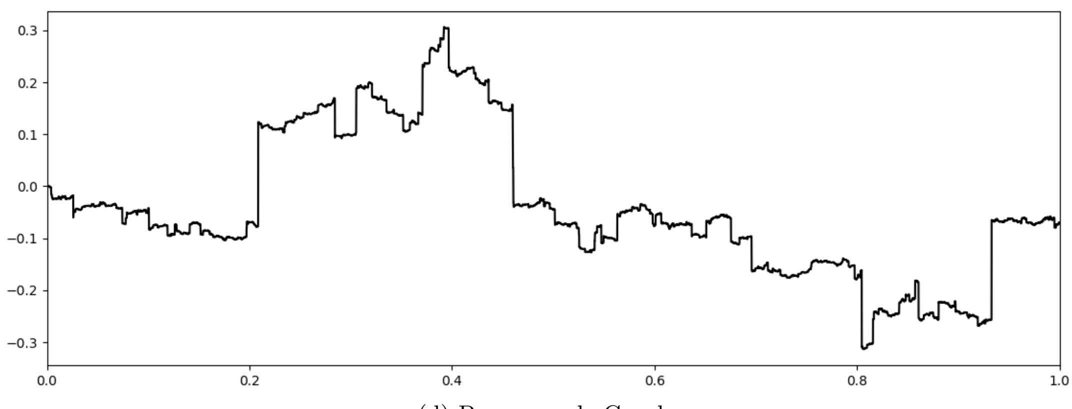
(a) Processus de Poisson



(b) Mouvement Brownien



(c) Processus d'Ornstein-Uhlenbeck



(d) Processus de Cauchy

Figure 1 – Exemples de processus ayant des probabilités de transitions explicites.

X^{OU} et le processus de Cauchy X^{C} sur $\mathbb{R}^d$ (voir Figure 1). On connaît, en effet, explicitement leurs probabilités de transition :

$$\begin{aligned}
&\mathscr{L}\left(N_t - N_s \mid N_s\right) = \mathcal{P}(t-s) = \mathrm{e}^{-(t-s)} \sum_{k\in\mathbb{N}} \frac{(t-s)^k}{k!} \boldsymbol{\delta}_k,\\
&\mathscr{L}\left(B_t - B_s \mid B_s\right) = \mathcal{N}(0, t-s) = (2\pi(t-s))^{-d/2} \exp\left(-\frac{\|b\|^2}{2(t-s)}\right) \mathrm{d}b,\\
&\mathscr{L}\left(X_t^{\mathrm{OU}} \mid X_s^{\mathrm{OU}}\right) = \mathcal{N}\left(\mathrm{e}^{-(t-s)} X_s^{\mathrm{OU}}, \frac{1-\mathrm{e}^{-2(t-s)}}{2}\right)\\
&\qquad\qquad = (\pi(1-\mathrm{e}^{-2(t-s)}))^{-d/2} \exp\left(-\frac{\|b - \mathrm{e}^{-(t-s)} X_s^{\mathrm{OU}}\|^2}{1-\mathrm{e}^{-2(t-s)}}\right) \mathrm{d}b,\\
&\mathscr{L}\left(X_t^{\mathrm{C}} - X_s^{\mathrm{C}} \mid X_s^{\mathrm{C}}\right) = \mathcal{C}(0, t-s) = \frac{\Gamma\left(\frac{d+1}{2}\right)(t-s)}{\Gamma\left(\frac{1}{2}\right)\pi^{d/2}} ((t-s)^2 - \|b\|^2)^{-(d+1)/2} \mathrm{d}b.
\end{aligned}$$

1.2 Exemple : les processus de Lévy

Dans cette section l'espace d'états est $\mathbb{R}^{d\Delta}$, qui est $\mathbb{R}^d$ auquel on a ajouté un point à l'infini Δ pour pouvoir prendre en compte une possible explosion instantanée. Les processus de Poisson, mouvements brownien et processus de Cauchy sont des exemples de processus de Lévy, c'est-à-dire à accroissements indépendants et identiquement distribués (voir par exemple [App09], [Ber96] ou [Sat13]). Cette propriété revient à être markovien et à posséder pour tout $s \leq t$ une version régulière $\mu_{s,t}(X_s, \mathrm{d}b)$ de $\mathscr{L}\left(X_t \mid X_s\right)$ qui en plus de vérifier (1.2)-(1.5), peut être factorisée en

$$\forall s \leq t,\ \forall a \in \mathbb{R}^d, \quad \mu_{s,t}(a, \mathrm{d}b) = \overline{\mu}_{t-s} * \boldsymbol{\delta}_a.$$

Ici pour $t \geq 0$, $\overline{\mu}_t$ est la loi de $X_t - X_0$ et $\boldsymbol{\delta}_a$ est la mesure de Dirac en a. L'équation de Chapman-Kolmogorov (1.5) devient alors l'équation de semi-groupe de convolution

$$\forall s, t \geq 0, \quad \overline{\mu}_{s+t} = \overline{\mu}_s * \overline{\mu}_t.$$

L'étude des lois infiniment divisibles permet de caractériser les semi-groupes de convolution et donc les processus de Lévy. La donnée de la loi d'un processus de Lévy est équivalente à la donnée d'un semi-groupe de convolution $(\overline{\mu}_t)_{t\geq 0}$ ce qui est encore équivalent à la donnée du triplet de Lévy (δ, γ, ν), où :

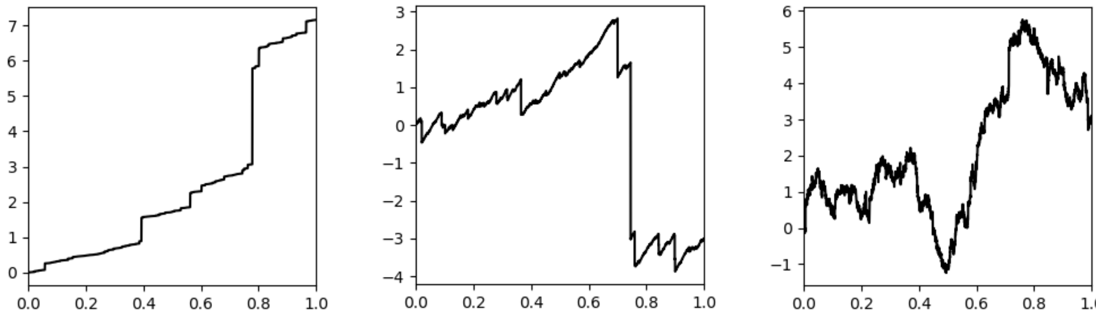
- une dérive $\delta \in \mathbb{R}^d$,
- une matrice de diffusion $\gamma \in \mathbb{R}^{d\times d}$ symétrique, semi-définie positive,
- une mesure (positive) de sauts ν sur $\mathbb{R}^{d\Delta} \setminus \{0\}$ satisfaisant
$$\int (1 \wedge \|b\|^2) \nu(db) < \infty.$$

L'équivalence passe par la description des fonctions caractéristiques :

$$\forall t \geq 0,\ \forall \alpha \in \mathbb{R}^d, \quad \mathbb{E}\left(\mathrm{e}^{\mathrm{i}(X_t - X_0)\cdot\alpha}\right) = \int \mathrm{e}^{\mathrm{i}b\cdot\alpha} \overline{\mu}_t(db) = \mathrm{e}^{-t\psi(\alpha)},$$

où l'exposant caractéristique ψ est de la forme

$$\psi(\alpha) = -\mathrm{i}\delta \cdot \alpha + \frac{1}{2}\alpha \cdot \gamma\alpha + \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - \mathrm{e}^{\mathrm{i}b\cdot\alpha} + \mathrm{i}\frac{b}{1+\|b\|^2} \cdot \alpha\right) \nu(\mathrm{d}b) + \nu(\{\Delta\}).$$



(a) Processus (4/5)-stable, à saut positif, sans compensation des sauts par dérive, avec taux de sauts $x^{-9/5}\mathbb{1}_{x>0}\mathrm{d}x$.

(b) Processus (6/5)-stable, à saut négatif, avec compensation des sauts par dérive "infinie", avec taux de sauts $x^{-11/5}\mathbb{1}_{x<0}\mathrm{d}x$.

(c) Processus (9/5)-stable, symétrique, avec taux de sauts $x^{-14/5}\mathrm{d}x$.

FIGURE 2 – Exemples de processus de Lévy stable sans partie diffusion, voir aussi Figure 1, (a), (b) (2-stable) et (d) (1-stable).

1.3 Description infinitésimale

De même que les systèmes dynamiques déterministes sont souvent représentés par une EDO, une idée pour étudier les processus de Markov est de décrire leurs comportements lors d'un temps infinitésimal. On s'intéresse alors au comportement de $\mu_{t,t+h}(a)$, lorsque h tend vers 0+. Par exemple, dans le cas des diffusions on a que $\mu_{t,t+h}(a)$ ressemble à une gaussienne. Une façon de regarder ce comportement est de regarder X comme solution d'une EDS, par rapport à un mouvement brownien ([IW89] Chapitre IV), un processus de Lévy ([App09] Chapitre 6) ou encore un processus ponctuel de Poisson ([IW89] Chapitre IV Section 9).

Une idée commune aux processus de Feller (voir par exemple [EK86] Chapitre 4 Section 2, [Kal02] Chapitre 19 ou [RW94] Chapitre III Section 2) et aux problèmes de martingales (voir par exemple [EK86] Chapitre 4, [RW00] Chapitre V Section 4 ou [SV06]) est de regarder le comportement de $\int f(b)\mu_{t,t+h}(a, \mathrm{d}b)$, lorsque h tend vers 0+, et lorsque f est suffisamment régulière. Il s'agit de donner un sens, possiblement faible, à

$$L_t f(a) = \lim_{h\to 0+} \frac{\mathbb{E}\left(f(X_{t+h}) \mid X_t = a\right) - f(a)}{h} := \lim_{h\to 0+} \int \frac{f(b) - f(a)}{h} \mu_{t,t+h}(a, \mathrm{d}b),$$

pour une classe de fonctions f suffisamment large pour décrire la dynamique (aléatoire). Comme nous allons le voir dans les sections suivantes, la description de L_t est, la plupart du temps, assez simple, faisant intervenir des coefficients facilement interprétables. D'autres approches infinitésimales pour étudier les processus de Markov sont par exemple l'étude des formes de Dirichlet (voir par exemple [FOT11]) ou l'études des équations d'évolutions (voir par exemple [BSW13] Section 3.4).

Pour les EDO, on souhaite reconstruire les solutions à partir de l'équation. De même, une question fondamentale est de savoir dans quelle mesure $(L_t)_t$ caractérise $(\mu_{s,t})_{s\leq t}$ et donc les lois fini-dimensionnelles de notre processus. De même, quelles conditions imposer à L_t pour qu'il existe un processus de Markov associé. Ces deux questions sont très importantes, notamment en modélisation où le système est souvent décrit à l'aide des coefficients présents dans $(L_t)_t$. De manière générale les conditions sur L_t vont être du type : principe du maximum positif, domaine suffisamment grand, ellipticité et/ou régularité des coefficients ...

Une autre question, qui m'a beaucoup intéressé lors de ma thèse, est de savoir quel est le lien entre la convergence d'une suite de processus de Markov et la convergence des opérateurs associés, et donc des coefficients associés. Ces convergences permettent d'obtenir des limites d'échelles, des approximations, de la stabilité par rapport aux coefficients ...

1.4 Exemple : les processus à sauts

Les processus de Markov constants par morceaux sur un espace mesuré $(S, \mathcal{S})$ (voir par exemple [Kal02] Chapitre 12) fournissent des exemples simples de processus de Markov. Leurs dynamiques peuvent être décrites ainsi : pour tout $a \in S$ et $t \geq 0$ il existe une mesure de saut $\nu_t(a, \mathrm{d}b)$ sur $S \setminus \{a\}$, positive et finie, telle que pour tout mesurable $B \subset S \setminus \{a\}$,

$$\mathbb{P}\left(X \text{ saute de } a \text{ vers } B \text{ dans l'intervalle } [t, t+h] \mid X_t = a\right) = h\nu_t(a, B) + o(h). \quad (1.6)$$

On retrouve le processus de Poisson classique en prenant $S = \mathbb{N}$ et $\nu_t(n, \mathrm{d}b) = \delta_{n+1}$. On étend de manière naturelle $\nu_t(a, \mathrm{d}b)$ à S en définissant

$$\nu_t(a, \{a\}) := -\nu_t(a, S \setminus \{a\}).$$

Ainsi, on déduit de (1.6) que, pour h proche de $0+$, en notant encore $\boldsymbol{\delta}_a$ la mesure de Dirac en a,

$$\begin{aligned} \mathscr{L}\left(X_{t+h} \mid X_t = a\right) = \mu_{t,t+h}(a, \mathrm{d}b) &\simeq (1 - h\nu_t(a, S \setminus \{a\}))\boldsymbol{\delta}_a + h\mathbb{1}_{b\neq a}\nu_t(a, \mathrm{d}b) \\ &= \boldsymbol{\delta}_a + h\nu_t(a, \mathrm{d}b). \end{aligned}$$

Ainsi on a le comportement infinitésimal

$$\begin{aligned} L_t f(a) := \lim_{h\to 0+} \frac{\mathbb{E}\left(f(X_{t+h}) \mid X_t = a\right) - f(a)}{h} &= \int (f(b) - f(a))\nu_t(a, \mathrm{d}b) \\ &= \int_S f(b)\nu_t(a, \mathrm{d}b). \end{aligned}$$

Partant de $(\nu_t(\cdot, \mathrm{d}b))_t$, on a une construction explicite du processus associé : soit $(E_i)_{i\geq 0}$ une suite variables aléatoires i.i.d. de loi exponentielle de paramètre 1. Partant d'un point X_0 on définit les temps de sauts $(\tau_i)_i$ de X et les valeurs après les sauts $(X_{\tau_i})_i$ par

$$\begin{aligned} &\tau_0 = 0, \\ &\forall i \geq 0, \quad \tau_{i+1} := \inf\left\{t \;\middle|\; \int_{\tau_i}^t \nu_s(X_{\tau_i}, S \setminus \{X_{\tau_i}\})\mathrm{d}s \geq E_i\right\}, \\ &\forall i \geq 0, \quad \mathscr{L}\left(X_{\tau_{i+1}} \;\middle|\; \tau_{i+1},\ (X_t)_{t<\tau_{i+1}}\right) = \frac{\nu_{\tau_{i+1}}(X_{\tau_i}, \mathrm{d}b)_{|S\setminus\{X_{\tau_i}\}}}{\nu_{\tau_{i+1}}(X_{\tau_i}, S \setminus \{X_{\tau_i}\})}. \end{aligned} \quad (1.7)$$

Si $S = \mathbb{R}^d$, on peut ajouter une dérive pour obtenir des PDMP (Processus de Markov Déterministe par Morceaux), c'est-à-dire des processus qui sont déterministes entre des temps de sauts aléatoires sans accumulation (voir par exemple [Dav93] Chapitre 2). Dans les cas réguliers, on peut décrire leurs dérivées au temps t en a par $\delta_t(a) \in \mathbb{R}^d$, c'est-à-dire

$$\mathbb{P}\text{-p.s.}, \ \forall t \geq 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} X_{t+} = \delta_t(X_t).$$

On a encore le comportement infinitésimal

$$\mathscr{L}\left(f(X_{t+h}) \mid X_t = a\right) = \mu_{t,t+h}(a, \mathrm{d}b) \simeq \boldsymbol{\delta}_{a+h\delta_t(a)} + h\nu_t(a, \mathrm{d}b),$$

ou encore

$$L_t f(a) := \lim_{h\to 0+} \frac{\mathbb{E}\left(f(X_{t+h}) \mid X_t = a\right) - f(a)}{h} = \delta_t(a) \cdot \nabla f(a) + \int_S f(b)\nu_t(a, \mathrm{d}b).$$

De plus ces processus peuvent être construits de manière similaire à celle décrite en (1.7).

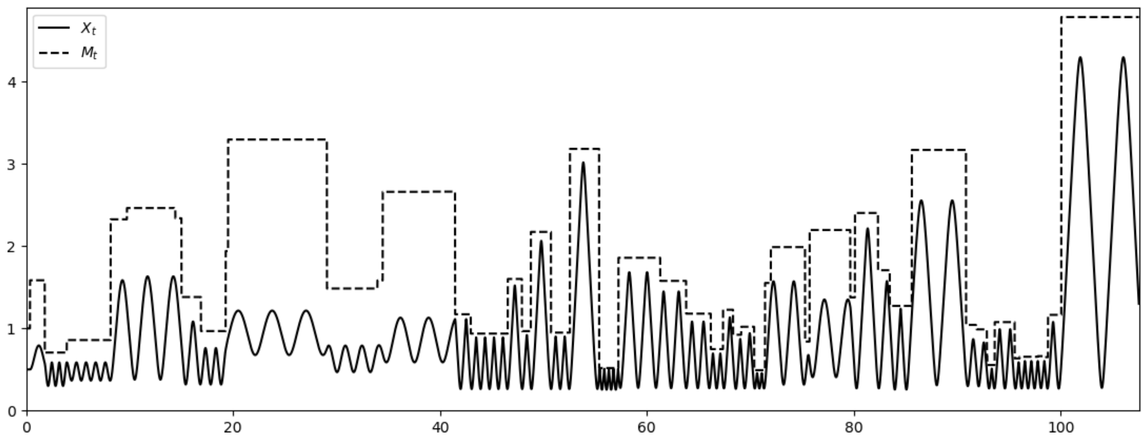


FIGURE 3 – Exemple de PDMP : $\mathrm{d}^2X_t/\mathrm{d}t^2 = 1/X_t^2 - C_t/(M_t - X_t)^2$ où (M_t, C_t) est constant par morceau et saute en $(m, (X_t - m)C_{t-}/(X_t - M_{t-}))$ avec intensité $\mathbb{1}_{m>X_t}\mathrm{e}^{-m}\mathrm{d}m$.

1.5 Exemple : les diffusions

Les diffusions sont des exemples fondamentaux de processus de Markov. On peut les voir comme des versions non homogènes en espace et en temps du mouvement brownien (voir par exemple [IW89], [RW00] Chapitre V ou [SV06]). Pour le mouvement brownien sur $\mathbb{R}^d$, on peut voir que

$$\lim_{h\to 0+} \frac{\mathbb{E}\left(f(B_{t+h}) \mid B_t = a\right) - f(a)}{h} = \frac{1}{2}\Delta f(a).$$

Formellement, un processus de Markov X est une diffusion dans $\mathbb{R}^d$ si pour tout $t \in \mathbb{R}_+$ et $a \in \mathbb{R}^d$, il existe une dérive $\delta_t(a) \in \mathbb{R}^d$ et une matrice de diffusion $\gamma_t(a) \in \mathbb{R}^{d\times d}$ symétrique semi-définie positive, telles que pour h petit,

$$\mathscr{L}\left(X_{t+h} \mid X_t = a\right) = \mu_{t,t+h}(a, \mathrm{d}b) \simeq \mathcal{N}(a + h\delta_t(a), h\gamma_t(a)). \tag{1.8}$$

Par exemple une solution de l'EDS

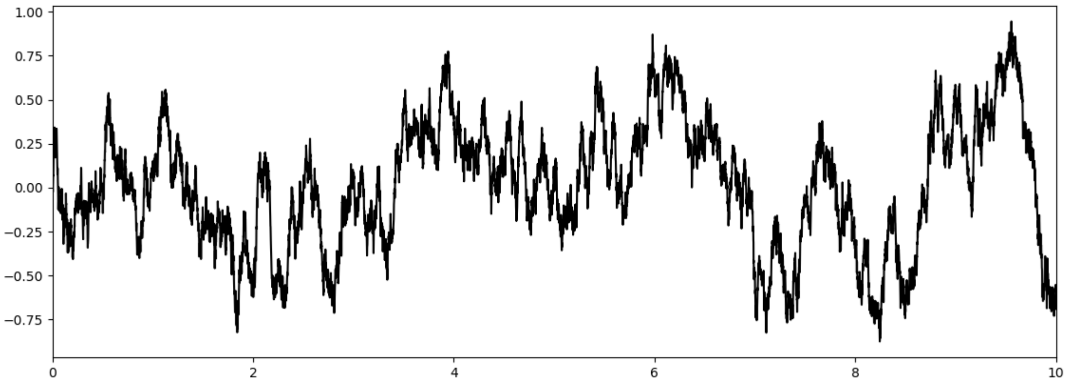
$$\mathrm{d}X_t = \delta_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}B_t$$

est une diffusion de dérive δ et matrice de diffusion $\sigma\sigma^{\mathrm{T}}$. On déduit de (1.8) que formellement, pour h qui tend vers $0+$, en notant $\nabla^2 f(a)$ la hessienne de f en a,

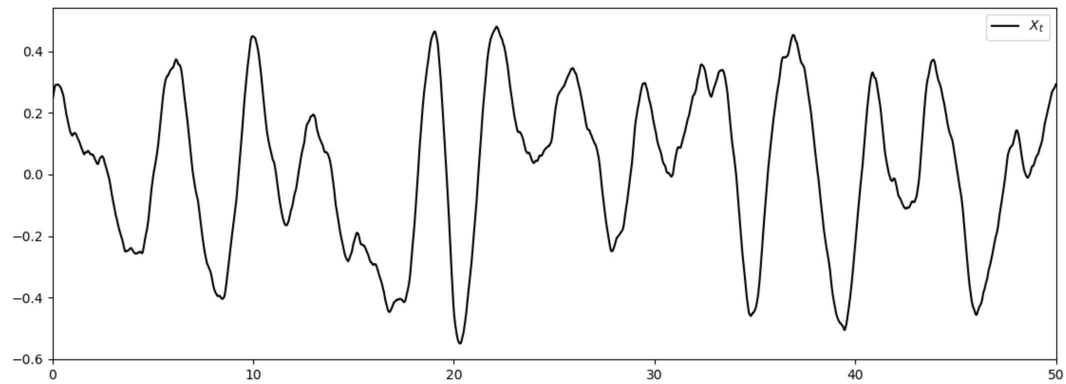
$$\begin{aligned}\mathbb{E}\left(f(X_{t+h}) \mid X_t = a\right) &= \mathbb{E}\Bigg(f(a) + (X_{t+h}-a)\cdot\nabla f(a) \\ &\qquad + \frac{1}{2}(X_{t+h}-a)\cdot\nabla^2 f(a)(X_{t+h}-a) + o(\|X_{t+h}-a\|^2)\ \Bigg|\ X_t = a\Bigg) \\ &= f(a) + h\delta_t(a)\cdot\nabla f(a) + \frac{1}{2}\mathrm{trace}\left(h\gamma_t(a)\nabla^2 f(a)\right) + o(h),\end{aligned}$$

d'où le comportement infinitésimal

$$L_t f(a) := \lim_{h\to 0+}\frac{\mathbb{E}\left(f(X_{t+h}) \mid X_t = a\right) - f(a)}{h} = \delta_t(a)\cdot\nabla f(a) + \frac{1}{2}\mathrm{trace}\left(\gamma_t(a)\nabla^2 f(a)\right). \tag{1.9}$$



(a) Elliptique : solutions de $\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}t/(1+X_t) - \mathrm{d}t/(1-X_t)$.



(b) Parabolique : solutions de $\mathrm{d}X_t = V_t\mathrm{d}t$ avec $\mathrm{d}V_t = \mathrm{d}B_t/2 + \mathrm{d}t/2(X_t+1)^2 - \mathrm{d}t/2(X_t-1)^2 - 2V_t\mathrm{d}t$.

FIGURE 4 – Deux exemples de diffusions sur $(-1,1)$.

1.6 Exemple : les processus de type Lévy

Soit X un processus de Lévy de triplet de Lévy (δ, γ, ν) et d'exposant caractéristique ψ, on peut déduire de $\mathbb{E}\left(\mathrm{e}^{\mathrm{i}(X_h - X_0)\cdot\alpha}\right) = \mathrm{e}^{-h\psi(\alpha)} \simeq 1 - h\psi(\alpha)$ la formule classique (voir par exemple [App09], [Ber96] ou [Sat13])

$$\forall t \geq 0,\ \forall a \in \mathbb{R}^d, \quad \lim_{h\to 0+} \frac{\mathbb{E}\left(f(X_{t+h}) \mid X_t = a\right) - f(a)}{h} = -\psi(\nabla)f(a),$$

où, pour f suffisamment régulière,

$$\begin{aligned} -\psi(\nabla)f(a) &:= \int_{\mathbb{R}^d} -\mathrm{e}^{\mathrm{i}a\cdot\alpha}\psi(\alpha)\widehat{f}(\alpha)\mathrm{d}\alpha \qquad \text{avec} \qquad \widehat{f}(\alpha) := (2\pi)^{-d}\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}a\cdot\alpha} f(b)\mathrm{d}b \\ &= \delta\cdot\nabla f(a) + \frac{1}{2}\mathrm{trace}\left(\gamma\nabla^2 f(a)\right) \\ &\quad + \int_{\mathbb{R}^d}\left(f(a+b) - f(a) - \frac{b}{1+\|b\|^2}\cdot\nabla f(a)\right)\nu(\mathrm{d}b). \end{aligned}$$

Compte tenu de la discussion des deux précédentes sections, ceci permet d'interpréter le processus de Lévy comme l'addition, d'une part d'une diffusion de matrice de diffusion γ et de dérive δ, et d'autre part d'un processus de saut d'intensité ν avec compensation des sauts sous forme de dérive. Il faut noter qu'il y a potentiellement une infinité de sauts en temps fini, non nécessairement sommables mais de carrés sommables, et que la compensation est potentiellement infinie.

Les processus de type Lévy sont des processus de Markov X qui ressemblent en temps petit à des processus de Lévy (voir par exemple [BSW13] ou [Küh17]). Précisément, pour tout $t \geq 0$, $a \in \mathbb{R}^d$ et h petit, la loi $\mathscr{L}\left(X_{t+h} \mid X_t = a\right)$ est proche de la loi d'un processus de Lévy au temps h, d'exposant caractéristique $q_t(a, \cdot)$ et partant de a, ou autrement

$$\mathbb{E}\left(\mathrm{e}^{\mathrm{i}(X_{t+h} - X_t)\cdot\alpha} \,\middle|\, X_t = a\right) \simeq \mathrm{e}^{-hq_t(a,\alpha)}.$$

On appelle q le symbole de X. Cette propriété est vérifiée pour les processus "suffisamment réguliers", comme on peut le voir, par exemple, dans de Théorème 2.21 p. 47 de [BSW13]. Une solution de l'EDS

$$\mathrm{d}X_t = \sigma_t(X_{t-})\mathrm{d}\ell_t$$

avec $(\ell_t)_t$ un processus de Lévy d'exposant caractéristique ψ, est un processus de type Lévy de symbole $(t, a, \alpha) \mapsto \psi\left(\sigma_t(a)^{\mathrm{T}}\alpha\right)$. De même, les solutions d'EDS par rapport à des processus ponctuels de Poisson sont des processus de type Lévy (voir [BSW13] Section 3.2). Pour un processus de type Lévy X de symbole q, on a, formellement, le comportement infinitésimal

$$\lim_{h\to 0+} \frac{\mathbb{E}\left(f(X_{t+h}) \mid X_t = a\right) - f(a)}{h} = -q_t(a, \nabla)f(a) = \int_{\mathbb{R}^d} -\mathrm{e}^{\mathrm{i}a\cdot\alpha} q_t(a,\alpha)\widehat{f}(\alpha)\mathrm{d}\alpha.$$

2 Processus de Feller

Une manière de reconstruire les probabilités de transition à partir de L est contenue dans la théorie des semi-groupes de Feller (voir par exemple [EK86] Chapitre 4 Section 2, [Kal02]

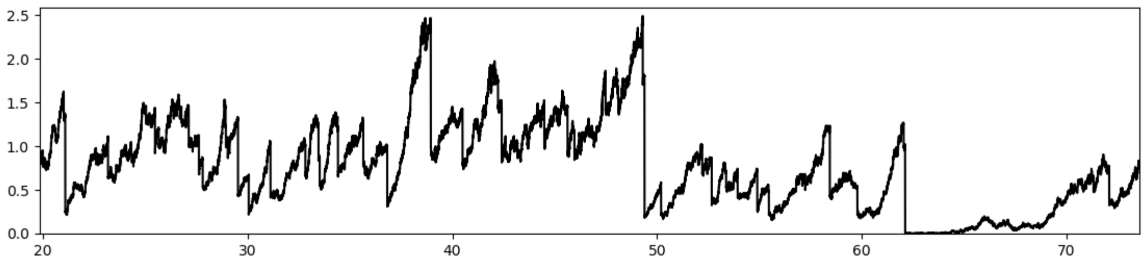


FIGURE 5 – Exemple de processus de type Lévy :
$Lf(a) = af'(a) + \frac{1}{32}af''(a) + \frac{1}{2}\int_0^a (f(a) - f(b))(1 - b/a)^{-3/2}\mathrm{d}b.$

Chapitre 19 ou [RW94] Chapitre III Section 2). Décrivons succinctement l'idée dans le cas où S est un espace localement compact à base dénombrable[2] muni de sa tribu borélienne. De manière à prendre en compte l'explosion possible des solutions, on ajoute un point à l'infini Δ à S : on assigne la valeur Δ à un processus qui a explosé.

On ne considérera que les processus de Markov homogènes en temps, c'est-à-dire ceux qui possèdent, pour tout $s \leq t$, une version régulière $\mu_{s,t}(X_s, \mathrm{d}b)$ de $\mathscr{L}(X_t \mid X_s)$ qui, en plus de vérifier (1.2)-(1.5), vérifie

$$\forall s \leq t,\ \forall a \in S, \quad \mu_{s,t}(a, \mathrm{d}b) = \mu_{0,t-s}(a, \mathrm{d}b).$$

Un processus de Markov est dit fellerien si pour tout $t \geq 0$ et pour toute fonction f appartenant à l'ensemble $\mathrm{C}_0(S)$ des fonctions continues réelles définies sur S qui tendent vers 0 en l'infini, la fonction

$$\begin{array}{rccl} T_t f : & S & \to & \mathbb{R} \\ & a & \mapsto & \mathbb{E}\left(f(X_t) \mid X_0 = a\right) := \int f(b)\mu_{0,t}(a, \mathrm{d}b) \end{array}$$

appartient également à $\mathrm{C}_0(S)$, où on a étendu f par $f(\Delta) = 0$. On obtient un opérateur borné T_t sur l'espace $\mathrm{C}_0(S)$ muni de la topologie de la convergence uniforme. De plus l'équation de Chapman-Kolmogorov (1.5) devient alors l'équation de semi-groupe (voir [Dav80])

$$\forall s, t \geq 0, \quad T_{s+t} = T_s T_t.$$

Une fonction $f \in \mathrm{C}_0(S)$ est dite dans le domaine $\mathrm{D}(L)$ du générateur infinitésimal L de $(T_t)_t$ si la limite

$$Lf(a) = \lim_{t \to 0+} \frac{\mathbb{E}\left(f(X_t) \mid X_0 = a\right) - f(0)}{t} := \lim_{t \to 0+} \frac{T_t f(a) - f(a)}{t}$$

existe pour tout a et si $Lf \in \mathrm{C}_0(S)$. Comme on peut le voir dans les exemples des sections précédentes, L décrit le comportement infinitésimal de la dynamique. Hille et Yosida ont montré ([Kal02] Lemme 19.5 p. 371) que la loi de X, ou de manière équivalente le semi-groupe $(T_t)_t$, est uniquement déterminé par L, ce qui justifie l'écriture

$$\forall t \geq 0, \qquad T_t = \mathrm{e}^{tL}.$$

[2]Cette propriété topologique est équivalente à être séparable et métrisable par une métrique à boules compactes, donc S est polonais.

Ils ont de plus montré ([Kal02] Théorème 19.11 p. 375) que la fermeture d'un opérateur quelconque L est le générateur d'un processus de Feller si et seulement si, pour $\lambda > 0$ quelconque,

1. le domaine $\mathrm{D}(L)$ est dense dans $\mathrm{C}_0(S)$,
2. l'image de $\lambda - L$ est dense dans $\mathrm{C}_0(S)$,
3. l'opérateur L vérifie le principe du maximum positif : pour tout $f \in \mathrm{D}(L)$ et $a \in S$ tels que $f(a) = \sup_{b \in S} f(b) \geq 0$, $Lf(a) \leq 0$.

Ce résultat permet la construction de processus de Feller, les première et troisième conditions sont en pratique faciles à obtenir, toute la difficulté est dans la deuxième condition. Celle-ci peut être vérifiée à l'aide de méthodes analytiques, considérons par exemple un opérateur de diffusion

$$L : f \in \mathrm{C}_\mathrm{c}^\infty \mapsto \delta \cdot \nabla f + \frac{1}{2}\mathrm{trace}\left(\gamma \nabla^2 f\right).$$

Sous les conditions que γ et δ soient globalement lipschitziennes et que γ soit uniformément définie positive, la fermeture de L est le générateur d'un processus fellerien (voir [EK86] Théorème 1.6 p. 370). De même, si δ est globalement lipschitzienne et si γ est C^2 bornée et que ses dérivées jusqu'à l'ordre deux sont bornées, alors la fermeture de L est le générateur d'un processus fellerien (voir [EK86] Théorème 2.5 p. 373).
Des résultats similaires existent pour des diffusions avec conditions de bord (voir [EK86] Théorème 1.4 p. 368 et Théorème 1.5 p. 369) et des processus de type Lévy (voir par exemple [BSW13] Section 3.1).

3 L'espace des fonctions cadlag et la topologie de Skorokhod

Pour étudier un processus de Markov il apparaît nécessaire d'imposer de la régularité sur les trajectoires, pour au moins deux raisons. Une première est l'utilisation des temps d'arrêts : en effet rien n'indique a priori que les temps aléatoires classiques, tel que le temps d'atteinte d'un borélien, soit mesurable. Si τ est un temps d'arrêt, rien n'indique non plus que X_τ soit mesurable et on ne peut donc pas donner de sens à la propriété de Markov forte, i.e. l'équation (1.1) écrite en remplaçant s et t par des temps d'arrêt.

Une deuxième motivation est l'obtention de théorèmes limites. On veut alors une topologie forte sur l'espace des trajectoires avec des critères de tension utilisables en pratique.

Si S est un espace topologique, l'ensemble des trajectoires cadlag, c'est-à-dire continues à droite avec limites à gauche est tout à fait adéquat, l'apparition des sauts permet de modéliser une grande classe de processus. Cet espace apparaît naturellement de la théorie des martingales, notamment avec les résultats de Doob sur la régularisation ([Kal02] Théorème 7.27 p. 134) et sur l'arrêt optimal ([Kal02] Théorème 7.29 p. 135).

Grâce à ces résultats sur les martingales, Kinney a pu montrer ([Kal02] Théorème 19.15 p. 379) que les processus de Feller admettent une version cadlag et Dynkin, Yushkevich et Blumenthal ont montré ([Kal02] Théorème 19.17 p. 380) que ces versions vérifient la propriété de Markov forte pour la filtration augmentée.

De plus, le "théorème du début" ([DM78] Théorème IV.50 p. 116) permet de voir que les temps d'atteinte de boréliens sont des temps d'arrêt pour les tribus augmentées. Dans mes travaux j'ai décidé de ne pas augmenter les tribus, car ceci n'est pas nécessaire. Les temps d'atteintes que j'utilise sont

$$\text{pour tout } U \text{ ouvert de } S, \quad \tau^U := \inf\{t \geq 0 \mid X_{t-} \notin U \text{ ou } X_t \notin U\}.$$

Ce sont bien des temps d'arrêts pour la filtration canonique de X si X est cadlag.

Une première notion de convergence pour des trajectoires cadlag serait de prendre la convergence uniforme. Cependant des suites de trajectoires dont on souhaiterait la convergence, tel que $\mathbb{1}_{[0,1+n^{-1})}$ vers $\mathbb{1}_{[0,1)}$ ne convergent pas uniformément. De plus la topologie induite n'est pas séparable, ce qui complique son utilisation. Skorokhod a eu l'idée de modifier la convergence uniforme en permettant de décaler temporellement les trajectoires pour permettre de synchroniser les sauts. On peut trouver dans [Bil99] Chapitre 3 une description précise de cette topologie.
Si S est un espace localement compact à base dénombrable, les trajectoires cadlag $(x_t^n)_{t\geq 0}$ convergent vers $(x_t)_{t\geq 0}$ au sens de Skorokhod s'il existe une suite $(\lambda^k)_k$ de bijections continues croissantes de $\mathbb{R}_+$ telles que

$$\forall t \geq 0, \quad \lim_{k\to\infty} \sup_{s\leq t} d(x_s, x^k_{\lambda^k_s}) = 0 \quad \text{et} \quad \lim_{k\to\infty} \sup_{s\leq t} |\lambda^k_s - s| = 0,$$

où d est une métrique quelconque de S. Cette notion de convergence définit une topologie polonaise et, de plus, on possède une caractérisation de la compacité similaire au critère d'Ascoli. De cette caractérisation on peut déduire des critères de tension, dont le critère d'Aldous, qui est une version du critère d'Ascoli le long des temps d'arrêts.

La topologie de Skorokhod et notamment le critère d'Aldous a permis à Trotter, Sova, Kurtz et Mackevičius d'obtenir une caractérisation de la convergence en loi des processus de Feller pour la topologie de Skorokhod en termes de semi-groupes ou en termes de générateurs (voir par exemple [Kal02], Théorème 19.25 p. 385). Précisément, on dit qu'une suite de générateurs L_n converge vers L si

$$\forall f \in \mathrm{D}(L),\ \exists (f_n)_n, \quad f_n \to f,\ f_n \in \mathrm{D}(L_n),\ L_n f_n \to Lf. \tag{3.1}$$

Cette convergence est plus facile à obtenir en pratique (lorsqu'on connaît $\mathrm{D}(L)$), et se déduit souvent de la convergence des paramètres des processus. Il existe un critère similaire pour la convergence des processus de Markov à temps discret (voir par exemple [Kal02], Théorème 19.28 p. 387) et on peut alors obtenir des résultats de convergence de type Donsker.

4 Problèmes de Martingales

La théorie des processus de Feller permet d'obtenir une description simple des processus de Markov à l'aide du générateur et permet de ramener l'étude des convergences des processus à l'étude, plus simple, des convergences de générateurs. La principale difficulté est alors de montrer qu'un opérateur L, ou sa fermeture, est générateur d'un processus de Feller.

Entre autres pour palier à cette difficulté, Itô a développé le calcul stochastique. On peut construire des processus de Markov, en résolvant l'EDS

$$\mathrm{d}X_t = \delta(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t.$$

Dans le cas où $\delta : \mathbb{R}^d \to \mathbb{R}^d$ et $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ sont localement lipschitziennes et sous-linéaires, on peut résoudre cette EDS ([IW89] Théorème 3.1 pp. 164-165) et voir que la solution est fellerienne. Cependant, on ne connait pas, en général, le domaine (ou un core) du générateur, on sait simplement qu'il s'agit d'une extension de l'opérateur $f \in \mathrm{C}_\mathrm{c}^\infty \mapsto \delta \cdot \nabla f + \frac{1}{2}\mathrm{trace}\left(\sigma\sigma^\mathrm{T}\nabla^2 f\right)$. Des résultats similaires sont connus pour les EDS avec condition de bord ([IW89] Chapitre IV Section 7) par rapport à des processus Lévy ([App09] Chapitre 6) et/ou des processus ponctuels de Poisson ([IW89] Chapitre IV Section 9).

Par ailleurs, Stroock et Varadhan ont donné un autre cadre pour étudier les processus de Markov, celui du problème de martingales (voir par exemple [EK86] Chapitre 4, [RW00] Chapitre V Section 4 ou [SV06]). Partant d'un opérateur L, un processus X est solution du problème de martingales associé si

$$\forall f \in \mathrm{D}(L), \qquad f(X_t) - \int_0^t Lf(X_s)\mathrm{d}s \quad \text{est une martingale en la variable } t.$$

Ceci est une formalisation de l'équation heuristique $\mathbb{E}\left(f(X_{t+h}) \mid X_t\right) \simeq f(X_t) + hLf(X_t)$. Sous de bonnes hypothèses sur L, notamment le principe du maximum positif, il y a toujours existence de solutions au problème de martingales, et de plus toute solution admet une version cadlag et quasi-continue.

Le problème de martingales est dit bien posé si, partant de n'importe quel point initial, il existe une unique (en loi) solution du problème de martingales. Alors L définit de manière unique (en loi et à loi au temps initial fixée) un processus, qui vérifie la propriété de Markov forte.

Cette théorie permet la construction de processus de Markov sous des conditions plus faibles. Initialement, Stroock et Varadhan ont montré ([SV06] Théorème 7.2.1 p. 187) que le problème de martingales associé à $f \in \mathrm{C}_\mathrm{c}^\infty \mapsto \delta \cdot \nabla f + \frac{1}{2}\mathrm{trace}\left(\gamma\nabla^2 f\right)$ était bien posé sous les conditions que δ soit mesurable bornée et γ soit continue définie positive bornée. De nombreux résultats sur le caractère bien posé des problèmes de martingales associés aux processus de type Lévy ont été obtenus (voir par exemple [BSW13] Section 3.5).

Il s'avère que le point de vue problème de martingales est plutôt universel, en effet Kurtz ([Kur11]) a montré l'équivalence entre le caractère bien posé d'une EDS, le caractère bien posé d'une équation d'évolution et le caractère bien posé d'un problème de martingales (correspondant à un opérateur pseudo-différentiel de la forme $-q(\cdot, \nabla)$ et de domaine $\mathrm{C}_\mathrm{c}^\infty$).

De même Van Casteren a montré (voir [vC92]) que dans le cas où S est compact, être solution d'un problème de martingales bien posé est équivalent à être un processus de Feller. Cependant il faut noter que les points de vue restent différents car on peut connaître le caractère bien posé d'un problème de martingales sans connaître le générateur de la solution.

5 Localisation en espace

Un des objectifs de ma thèse a été d'étudier finement les problèmes de martingales, notamment de généraliser le résultat de Van Casteren au cas non compact et d'obtenir des résultats généraux de convergence pour les solutions de problèmes de martingales.

Mon travail mêle les trois notions que sont la topologie de Skorokhod, les processus de Feller et les problèmes de martingales. Cependant, dans le cadre localement compact, il va être fructueux de localiser en espace ces notions, d'une part pour enlever les hypothèses

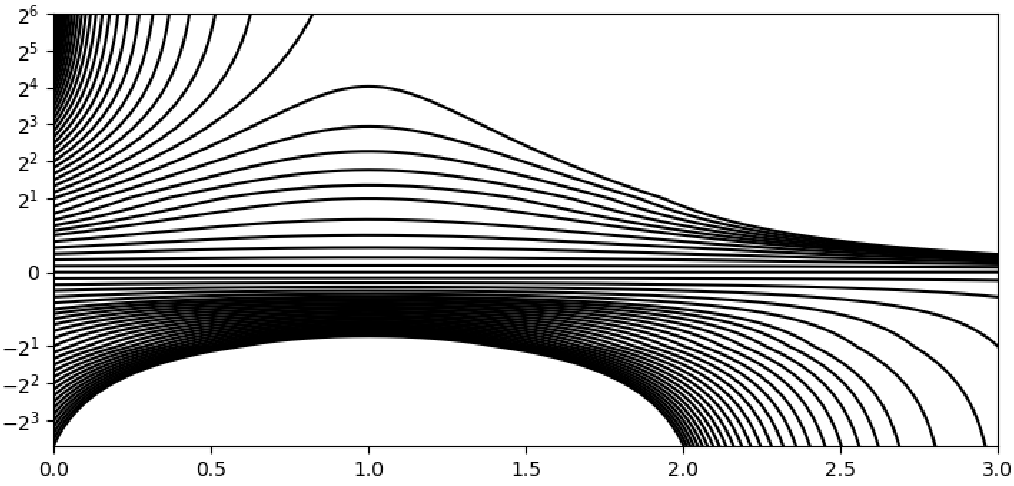


FIGURE 6 – Solutions de $\dot{x}_t = (1-t)x_t^2$.

sur le comportement à l'infini des coefficients des dynamiques et d'autre part pour obtenir des résultats généraux.

Pour illustrer les raisons de la localisation, considérons l'EDO

$$\dot{x}_t = (1-t)x_t^2, \quad t > 0, \quad x_0 \in \mathbb{R}.$$

Celle-ci est très régulière et admet, pour toute condition initiale x_0, une unique solution maximale

$$x_t = \left(\frac{t^2}{2} - t + \frac{1}{x_0}\right)^{-1} \text{ avant explosion en } t_{\max} = \begin{cases} \infty, & \text{pour } x_0 \in [0,2), \\ 1 - \sqrt{1-2/x_0}, & \text{pour } x_0 \geq 2, \\ 1 + \sqrt{1-2/x_0}, & \text{pour } x_0 < 0. \end{cases}$$

Cependant même en compactifiant $\mathbb{R}$, les trajectoires ne sont pas continues par rapport à la donnée initiale x_0, pour la convergence uniforme sur les intervalles bornés, la discontinuité apparaissant à la condition initiale 2 (voir Figure 6). Pour obtenir la continuité par rapport à la condition initiale l'idée est de demander, pour tout compact de l'espace d'états, la convergence uniforme uniquement lorsque la trajectoire reste dans ce compact.

Partant de cette idée on peut construire la topologie de Skorokhod locale : des trajectoires cadlag, avec possible explosion, $(x_t^n)_t$ convergent vers $(x_t)_t$ au sens de Skorokhod locale s'il existe une suite $(\lambda^k)_k$ de bijections continues croissantes de $\mathbb{R}_+$ telles que

$$\forall t \geq 0 \text{ t.q. } \{x_s\}_{s<t} \text{ soit compact,} \quad \lim_{k\to\infty} \sup_{s\leq t} d(x_s, x^k_{\lambda^k_s}) = 0 \quad \text{et} \quad \lim_{k\to\infty} \sup_{s\leq t} |\lambda^k_s - s| = 0,$$

où d est une métrique quelconque de S. Cette notion de convergence définit une topologie polonaise et de plus on possède une caractérisation de la compacité, similaire au critère d'Ascoli. De cette caractérisation on peut déduire une amélioration du critère d'Aldous de tension, qui devient une équivalence.

La version localisée du problème de martingales consiste seulement à considérer des opérateurs à valeurs dans l'ensemble des fonctions continues, non-nécessairement bornées, en utilisant une notion voisine à celle de martingale locale. Ceci permet essentiellement de considérer des processus qui ont des sauts très grands. Un processus sera dit localement fellerien si, pour tout compact, il coïncide avec un processus de Feller tant qu'il reste dans ce compact.

Nous avons montré qu'un processus est localement fellerien si et seulement s'il est solution d'un problème local de martingales bien posé et si et seulement s'il est markovien et continu en loi par rapport à la donnée initiale, pour la topologie de Skorokhod locale. De plus il vérifie la propriété de Markov forte pour la filtration augmentée.

De plus nous avons montré que des solutions de problèmes locaux de martingales bien posés convergent si et seulement si les opérateurs associés convergent, au sens de (3.1). La principale amélioration par rapport au résultat similaire pour les processus de Feller est qu'ici on n'impose pas que l'opérateur limite soit un générateur, mais uniquement que le problème local de martingales associé soit bien posé. Ainsi ce résultat est applicable dans bien plus de cas. Nous avons aussi donné un résultat du même type concernant la convergence des processus de Markov à temps discrets.

Enfin, comme exemples, nous avons appliqué ces résultats aux processus de type Lévy et aux diffusions unidimensionnelles évoluant en potentiels aléatoires.

6 Notations et principaux résultats obtenus

Dans toute la suite S désigne un espace topologique localement compact à base dénombrable. Cette propriété topologique est équivalente à être séparable et métrisable par une métrique à boules compactes, donc S est polonais. Soit $\Delta \notin S$, nous notons $S^\Delta \supset S$ le compactifié de S par le point Δ si S est non compact, et la somme topologique $S \sqcup \{\Delta\}$ si S est compact. Ainsi on a la caractérisation de la convergence vers Δ :

$$S \ni a_n \underset{n\to\infty}{\longrightarrow} \Delta \quad \text{s.s.i.} \quad \forall K \subset S \text{ compact}, \exists n_0 \in \mathbb{N},\ \forall n \geq n_0, \quad a_n \notin K.$$

Nous notons $\mathrm{C}(S) := \mathrm{C}(S, \mathbb{R})$ l'ensemble des fonctions continues réelles définies sur S. Nous notons $\mathrm{C}_0(S)$ l'ensemble des fonctions $f \in \mathrm{C}(S)$ s'annulant en Δ. L'espace $\mathrm{C}(S)$ est muni de la topologie de la convergence uniforme sur les compacts et $\mathrm{C}_0(S)$ de la topologie de la convergence uniforme.

La propriété qu'un sous-ensemble $A \subset S$ soit relativement compact dans un ouvert $U \subset S$, c'est-à-dire que la fermeture $\overline{A}$ soit un compact inclus dans U, est notée $A \Subset U$.

Nous noterons X le processus canonique sur $(S^\Delta)^{\mathbb{R}_+}$ ou, sans danger de confusion, sur n'importe lequel de ses sous-espaces. Nous munissons $(S^\Delta)^{\mathbb{R}_+}$, et ses sous-espaces de la tribu $\mathcal{F} := \sigma(X_s,\ 0 \leq s < \infty)$ et de la filtration $\mathcal{F}_t := \sigma(X_s,\ 0 \leq s \leq t)$. Enfin nous notons le temps d'explosion

$$\xi := \inf\{t \geq 0 \,|\, \{X_s\}_{s \leq t} \not\Subset S\}.$$

6.1 Topologie de Skorokhod locale

Nous exposons dans cette section les résultats obtenus dans [GH17b], que l'on peut retrouver dans le chapitre 2. Un espace de trajectoires utilisé pour étudier les processus est l'ensemble des trajectoires cadlag à valeurs dans S^Δ :

$$\mathbb{D}(S^\Delta) := \left\{ x \in (S^\Delta)^{\mathbb{R}_+} \;\middle|\; \begin{array}{l} \forall t \geq 0,\ x_t = \lim_{s \downarrow t} x_s, \quad \text{et} \\ \forall t > 0,\ x_{t-} := \lim_{s \uparrow t} x_s \quad \text{existe dans } S^\Delta \end{array} \right\},$$

muni de la topologie de Skorokhod (voir, par exemple, Chap. 3 de [EK86], pp. 116-147), que nous appelons globale et qui est polonaise. Une suite $(x^k)_k$ de $\mathbb{D}(S^\Delta)$ converge vers x pour cette topologie si et seulement s'il existe une suite $(\lambda^k)_k$ de bijections continues croissantes de $\mathbb{R}_+$ telles que

$$\forall t \geq 0, \quad \lim_{k\to\infty} \sup_{s\leq t} d(x_s, x^k_{\lambda^k_s}) = 0 \quad \text{et} \quad \lim_{k\to\infty} \sup_{s\leq t} |\lambda^k_s - s| = 0.$$

La topologie de Skorokhod globale ne dépend pas de la métrique arbitraire d sur S^Δ et sa tribu borélienne coïncide avec $\mathcal{F}$.

Nous avons souhaité localiser cette topologie en espace. Pour une trajectoire $x \in (S^\Delta)^{\mathbb{R}_+}$ nous rappelons que son temps d'explosion est défini par

$$\xi(x) := \inf\{t \geq 0 \,|\, \{x_s\}_{s\leq t} \not\Subset S\}.$$

Nous avons introduit un espace de trajectoires cadlag à valeurs dans S avec explosion :

$$\mathbb{D}_{\text{loc}}(S) := \left\{ x \in (S^\Delta)^{\mathbb{R}_+} \;\middle|\; \begin{array}{l} \forall t \geq \xi(x),\ x_t = \Delta, \\ \forall t \geq 0,\ x_t = \lim_{s\downarrow t} x_s, \\ \forall t > 0 \text{ t.q. } \{x_s\}_{s<t} \Subset S,\ x_{t-} := \lim_{s\uparrow t} x_s \text{ existe} \end{array} \right\}.$$

Nous disons qu'une suite $(x^k)_{k\in\mathbb{N}}$ de $\mathbb{D}_{\text{loc}}(S)$ converge vers x au sens de Skorokhod locale s'il existe une suite $(\lambda^k)_k$ de bijections continues croissantes de $\mathbb{R}_+$ telles que

$$\forall t \geq 0 \text{ t.q. } \{x_s\}_{s<t} \Subset S, \quad \lim_{k\to\infty} \sup_{s\leq t} d(x_s, x^k_{\lambda^k_s}) = 0 \quad \text{et} \quad \lim_{k\to\infty} \sup_{s\leq t} |\lambda^k_s - s| = 0.$$

Dans notre Théorème 2.4, nous avons montré que cette convergence définit une topologie polonaise sur $\mathbb{D}_{\text{loc}}(S)$, qui ne dépend pas de la métrique arbitraire d sur S^Δ, et que sa tribu borélienne coïncide avec $\mathcal{F}$.

Nous nous sommes ensuite intéressés à la compacité pour la topologie de Skorokhod locale, l'objectif étant d'avoir une caractérisation similaire au cas de la topologie de Skorokhod globale (voir par exemple [Bil99], Théorème 16.5 p. 172). Pour cela nous avons besoin de modules de continuité modifiés : pour $x \in \mathbb{D}_{\text{loc}}(S)$, $t \geq 0$, $K \subset S$ compact et $\delta > 0$, nous avons défini

$$\omega'_{t,K,x}(\delta) := \inf \left\{ \sup_{\substack{0\leq i<N \\ t_i\leq s_1,s_2<t_{i+1}}} d(x_{s_1}, x_{s_2}) \;\middle|\; \begin{array}{l} N \in \mathbb{N},\ 0 = t_0 < \cdots < t_N \leq \xi(x) \\ (t_N, x_{t_N}) \notin [0,t] \times K \\ \forall 0 \leq i < N :\ t_{i+1} - t_i > \delta \end{array} \right\}.$$

Nous avons alors montré, dans notre Théorème 2.8, qu'un sous-ensemble $D \subset \mathbb{D}_{\text{loc}}(S)$ est relativement compact si et seulement si

$$\forall t \geq 0,\ K \subset S \text{ compact}, \quad \sup_{x\in D} \omega'_{t,K,x}(\delta) \underset{\delta\to 0}{\longrightarrow} 0.$$

À partir de ce résultat nous avons pu améliorer le critère d'Aldous (voir par exemple [Bil99], Théorème 16.10 pp. 178-179) pour obtenir une caractérisation de la tension dans l'espace $\mathcal{P}(\mathbb{D}_{\text{loc}}(S))$ des probabilités sur $\mathbb{D}_{\text{loc}}(S)$, du type "uniformément cadlag le long des temps d'arrêts". Pour U ouvert de S, on définit le temps d'arrêt

$$\tau^U := \inf\{t \geq 0 \mid X_{t-} \notin U \text{ ou } X_t \notin U\} \wedge \xi. \tag{6.1}$$

Dans notre Théorème 2.11, nous avons montré que pour un sous-ensemble $\mathcal{P} \subset \mathcal{P}(\mathbb{D}_{\text{loc}}(S))$, nous avons équivalence entre :

1. $\mathcal{P}$ est tendu,

2. pour tout $t \geq 0$, $\varepsilon > 0$ et tout compact K de S,

$$\sup_{\mathbf{P}\in\mathcal{P}} \mathbf{P}\left(\omega'_{t,K,X}(\delta) \geq \varepsilon\right) \underset{\delta\to 0}{\longrightarrow} 0,$$

3. pour tout $t \geq 0$, $\varepsilon > 0$ et tout ouvert $U \Subset S$,

$$\sup_{\mathbf{P}\in\mathcal{P}} \sup_{\substack{\tau_1\leq\tau_2\leq\tau_3 \\ \tau_3\leq(\tau_1+\delta)\wedge t\wedge\tau^U}} \mathbf{P}(R \geq \varepsilon) \underset{\delta\to 0}{\longrightarrow} 0,$$

où le supremum est pris pour des temps d'arrêts τ_i et où

$$R := \begin{cases} d(X_{\tau_1}, X_{\tau_2}) \wedge d(X_{\tau_2}, X_{\tau_3}) & \text{si } 0 < \tau_1 < \tau_2, \\ d(X_{\tau_2-}, X_{\tau_2}) \wedge d(X_{\tau_2}, X_{\tau_3}) & \text{si } 0 < \tau_1 = \tau_2, \\ d(X_{\tau_1}, X_{\tau_2}) & \text{si } 0 = \tau_1. \end{cases}$$

Ayant pour but d'avoir un lien entre les topologies de Skorokhod locale et globale nous avons introduit un changement de temps. Soit $g \in \mathrm{C}(S, \mathbb{R}_+)$, pour $t \in \mathbb{R}_+$ nous notons le temps d'arrêt

$$\tau_t^g := \inf\left\{ s \geq 0 \;\middle|\; s \geq \tau^{\{g\neq 0\}} \text{ ou } \int_0^s \frac{\mathrm{d}u}{g(X_u)} \geq t \right\}.$$

Nous introduisons alors le changement de temps, $\mathcal{F}$-mesurable,

$$\begin{array}{rccc} g\cdot X : & \mathbb{D}_{\mathrm{loc}}(S) & \to & \mathbb{D}_{\mathrm{loc}}(S) \\ & x & \mapsto & g\cdot x \end{array},$$

défini par : pour $t \in \mathbb{R}_+$,

$$(g\cdot X)_t := \begin{cases} X_{\tau^{\{g\neq 0\}}-} & \text{si } \tau_t^g = \tau^{\{g\neq 0\}},\ X_{\tau^{\{g\neq 0\}}-} \text{ existe dans } \{g = 0\}, \\ X_{\tau_t^g} & \text{sinon.} \end{cases}$$

La notation vient du fait que

$$\forall g_1, g_2 \in \mathrm{C}(S, \mathbb{R}_+), \quad g_1\cdot(g_2\cdot X) = (g_1 g_2)\cdot X.$$

Pour $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$, notons également $g\cdot\mathbf{P}$ la mesure image de $\mathbf{P}$ par $g\cdot X$. Nous avons étudié l'ensemble de continuité du changement de temps dans notre Théorème 3.5. Dans notre Proposition 3.9 nous explorons un lien entre les topologies locale et globale. Soit E un espace localement compact à base dénombrable arbitraire et soit U un ouvert de S, pour toute application

$$\begin{array}{rccc} \mathbf{P} : & E & \to & \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S)) \\ & a & \mapsto & \mathbf{P}_a \end{array},$$

l'assertion i) implique l'assertion ii), où

i) l'application $\mathbf{P}$ est étroitement continue pour la topologie de Skorokhod locale,

ii) il existe $g \in \mathrm{C}(S, \mathbb{R}_+)$ telle que $\{g \neq 0\} = U$, telle que pour tout $a \in E$

$$g \cdot \mathbf{P}_a \left(0 < \xi < \infty \Rightarrow X_{\xi-} \text{ existe dans } U\right) = 1,$$

et telle que l'application

$$g \cdot \mathbf{P} : \begin{array}{ccc} E & \to & \mathcal{P}(\{0 < \xi < \infty \Rightarrow X_{\xi-} \text{ existe dans } U\}) \\ a & \mapsto & g \cdot \mathbf{P}_a \end{array}$$

soit étroitement continue pour la topologie de Skorokhod globale.

De plus, si $U = S$ les assertions i) et ii) sont équivalentes.

6.2 Processus localement fellerien et problèmes locaux de martingales

Nous exposons dans cette section les résultats obtenus dans [GH17c], que l'on peut retrouver dans le chapitre 3. Pour un opérateur

$$L : \mathrm{D}(L) \subset \mathrm{C}_0(S) \to \mathrm{C}(S)$$

de domaine $\mathrm{D}(L)$ un sous-espace vectoriel dense de $\mathrm{C}_0(S)$, l'ensemble $\mathcal{M}(L)$ des solutions du problème local de martingales associé à L est l'ensemble des probabilités $\mathbf{P} \in \mathcal{P}\left(\mathbb{D}_{\mathrm{loc}}(S)\right)$ telle que pour tout $f \in \mathrm{D}(L)$ et tout ouvert $U \Subset S$:

$$f(X_{t\wedge\tau^U}) - \int_0^{t\wedge\tau^U} Lf(X_s)\mathrm{d}s \text{ est une } \mathbf{P}\text{-martingale}$$

par rapport à la filtration $(\mathcal{F}_t)_t$. Nous rappelons que τ^U est défini par (6.1). En se ramenant au Théorème 5.4 p. 199 de [EK86], nous avons obtenu, dans notre Théorème 3.9, l'équivalence entre les deux assertions suivantes

i) existence au problème local de martingales : pour tout $a \in S$ il existe un élément $\mathbf{P}$ de $\mathcal{M}(L)$ tel que $\mathbf{P}(X_0 = a) = 1$,

ii) L satisfait le principe du maximum positif : pour tout $f \in \mathrm{D}(L)$ et $a_0 \in S$, si $f(a_0) = \sup_{a\in S} f(a) \geq 0$ alors $Lf(a_0) \leq 0$.

Le problème local de martingales est dit bien posé si pour tout $a \in S$ il existe un unique élément $\mathbf{P}$ de $\mathcal{M}(L)$ tel que $\mathbf{P}(X_0 = a) = 1$.

Nous avons exploré le lien entre les notions de problème de martingales, topologie de Skorokhod et processus de Feller. Dans notre Théorème 4.5 nous obtenons que pour une famille de probabilités $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$, les assertions suivantes sont équivalentes :

1. (continuité) la famille $(\mathbf{P}_a)_a$ est $\mathcal{F}_t$-markovienne et $a \mapsto \mathbf{P}_a$ est étroitement continue pour la topologie de Skorokhod locale ;

2. (changement de temps) il existe $g \in \mathrm{C}(S, \mathbb{R}_+^*)$ telle que $(g \cdot \mathbf{P}_a)_a$ est une famille fellerienne ;

3. (martingales) il existe un opérateur $L : \mathrm{D}(L) \to \mathrm{C}(S)$ de domaine dense dans $\mathrm{C}_0(S)$ tel que

$$\forall a \in S, \quad \mathbf{P} \in \mathcal{M}(L) \text{ et } \mathbf{P}(X_0 = a) = 1 \Longleftrightarrow \mathbf{P} = \mathbf{P}_a; \tag{6.2}$$

4. (localisation) pour tout ouvert $U \Subset S$ il existe une famille de probabilités felleriennes $(\widetilde{\mathbf{P}}_a)_a$ telles que, pour tout $a \in S$,

$$\mathscr{L}_{\mathbf{P}_a}\left(X^{\tau^U}\right) = \mathscr{L}_{\widetilde{\mathbf{P}}_a}\left(X^{\tau^U}\right),$$

où X^{τ^U} est le processus arrêté à la sortie de U.

La dernière assertion nous autorise à appeler de tels processus localement felleriens. Nous obtenons de plus que les processus localement felleriens sont $(\mathcal{F}_{t+})_t$-fortement markoviens et quasi-continus. Pour une famille localement fellerienne $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$, on appelle générateur l'opérateur L maximal vérifiant (6.2).

Dans notre Théorème 4.8 nous avons montré qu'une famille $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ est fellerienne si et seulement si elle est localement fellerienne et

$$\forall t \geq 0,\ \forall K \subset S \text{ compact}, \quad \mathbf{P}_a(X_t \in K) \underset{a \to \Delta}{\longrightarrow} 0.$$

Nous avons ensuite étudié la convergence des solutions de problèmes de martingales. Soit, pour tout $n \in \mathbb{N}$, $(\mathbf{P}^n_a)_a$ une famille de probabilités localement felleriennes de générateurs L_n et soit $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ une famille localement fellerienne solution d'un problème local de martingales bien posé d'opérateur L. Il y a équivalence entre (voir notre Théorème 4.13)

1. pour tout $a_n, a \in S$ tels que $a_n \to a$, $\mathbf{P}^n_{a_n}$ converge étroitement pour la topologie de Skorokhod locale vers $\mathbf{P}_a$,

2. pour tout $f \in \mathrm{D}(L)$, il existe une suite d'éléments $f_n \in \mathrm{D}(L_n)$ telle que

$$f_n \underset{n\to\infty}{\xrightarrow{\mathrm{C}_0}} f \quad \text{et} \quad Lf_n \underset{n\to\infty}{\xrightarrow{\mathrm{C}}} Lf.$$

On obtient ainsi une caractérisation de la convergence, similaire au résultat classique sur les processus felleriens (voir par exemple le Théorème 19.25, p. 385, dans [Kal02]). L'amélioration principale est encore que nous n'imposons plus que L soit le générateur, mais uniquement que le problème de martingales associé soit bien posé. Ainsi notre résultat s'applique à plus de constructions de processus markoviens.

Enfin on peut utiliser les bonnes propriétés de localisation des problèmes de martingales, voir par exemple la Section 6.6 pp. 161-165 de [SV06]. Nous avons ainsi obtenu dans notre Théorème 4.16 que pour un opérateur $L : \mathrm{D}(L) \to \mathrm{C}(S)$ de domaine dense dans $\mathrm{C}_0(S)$, si pour tout $a \in S$ il existe un voisinage V de a et un opérateur $\widetilde{L} : \mathrm{D}(L) \to \mathrm{C}(S)$ de même domaine pour lequel le problème local de martingales est bien posé et tel que

$$\forall f \in \mathrm{D}(L), \quad Lf_{|V} = \widetilde{L}f_{|V},$$

alors le problème local de martingales associé à L est bien posé. Dans notre Théorème 4.18 nous obtenons un résultat similaire pour le générateur.

6.3 Processus de type Lévy : convergences et schémas discrets

Nous exposons dans cette section les résultats obtenus dans [GH17a], que l'on peut retrouver dans le chapitre 4. Dans le but d'étudier les approximations discrètes de processus localement felleriens, nous avons montré dans notre Théorème 3.2 qu'une suite de processus de Markov à temps discret $\{k\varepsilon_n \mid k \in \mathbb{N}\}$, de pas de temps $\varepsilon_n \to 0+$, et d'opérateurs de transitions T_n, converge vers un processus localement fellerien, solution d'un problème local de martingales bien posé associé à un opérateur $L : \mathrm{D}(L) \subset \mathrm{C}_0(S) \to \mathrm{C}(S)$ si et seulement si

$$\forall f \in \mathrm{D}(L),\ \exists (f_n)_n \in \mathrm{C}_0(S)^{\mathbb{N}}, \quad \text{tel que } f_n \xrightarrow[n\to\infty]{\mathrm{C}_0} f \quad \text{et} \quad \frac{T_n f_n - f_n}{\varepsilon_n} \xrightarrow[n\to\infty]{\mathrm{C}} Lf.$$

On obtient ainsi une caractérisation de la convergence, similaire au résultat classique sur les processus felleriens (voir par exemple le Théorème 19.27, p. 387, dans [Kal02]). L'amélioration principale est encore que nous n'imposons pas que L soit le générateur, mais uniquement que le problème local de martingales associé soit bien posé.

Nous avons voulu appliquer les résultats obtenus sur les processus localement felleriens aux processus de type Lévy sur $\mathbb{R}^d$. Un symbole est une fonction $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ telle que, pour tout $a \in \mathbb{R}^d$, $q(a, \cdot)$ est l'exposant caractéristique d'un processus de Lévy avec possible explosion (voir Section 1.2). On sait (voir par exemple [BSW13], Proposition 2.17 p. 43) que

$$\forall a \in \mathbb{R}^d,\ \forall \alpha \in \mathbb{R}^d, \quad |q(a,\alpha)| \leq 2(1+|\alpha|^2) \sup_{|\beta|\leq 1} |q(a,\beta)|.$$

Ceci permet de définir l'opérateur pseudo-différentiel de symbole q :

$$\forall f \in \mathrm{C}_\mathrm{c}^\infty, \quad -q(\cdot,\nabla)f(a) = -q(a,\nabla)f(a) := \int_{\mathbb{R}^d} -\mathrm{e}^{\mathrm{i}a\cdot\alpha} q(a,\alpha)\widehat{f}(\alpha)\mathrm{d}\alpha$$
$$\text{avec} \quad \widehat{f}(\alpha) := (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}a\cdot\alpha} f(b)\mathrm{d}b.$$

Pour appliquer les résultats sur les processus localement felleriens, on souhaite que l'opérateur $q(\cdot,\nabla)$ envoie $\mathrm{C}_\mathrm{c}^\infty$ dans $\mathrm{C}(\mathbb{R}^d)$. Par convergence dominée on voit qu'il suffit que

$$\forall \alpha \in \mathbb{R}^d, \quad a \mapsto q(a,\alpha) \text{ soit continue.}$$

De même, pour une suite de symboles q_n, dans le but d'appliquer les théorèmes de convergence, on voudrait savoir sous quelles conditions, pour tout $f \in \mathrm{C}_\mathrm{c}^\infty$, $q_n(\cdot,\nabla)f$ converge vers $q(\cdot,\nabla)f$. Encore par convergence dominée, on voit qu'il suffit que

$$\forall \alpha \in \mathbb{R}^d, \quad q_n(\cdot,\alpha) \underset{n\to\infty}{\longrightarrow} q(\cdot,\alpha) \text{ uniformément sur les compacts de } \mathbb{R}^d.$$

Dans notre Théorème 4.1 on obtient des conditions nécessaires et suffisantes, en termes de triplet de Lévy, pour ces deux problèmes et pour le problème similaire associé aux schémas discrets.

Dans notre Théorème 4.5 nous obtenons la convergence d'un schéma d'aproximation d'Euler pour les processus de type Lévy. On améliore ainsi le Théorème 7.6 p. 172 de [BSW13]. En effet on ne demande plus que la fermeture de $-q(\cdot,\nabla)$ soit le générateur d'un processus de Feller, mais uniquement que le problème local de martingales associé soit bien posé.

Enfin, dans nos Proposition 5.1 et Proposition 5.2, nous nous sommes intéressés aux diffusions unidimensionnelles évoluant dans des potentiels cadlag. En effet, en utilisant les résultats de [Man68] on peut voir que pour $V : \mathbb{R} \to \mathbb{R}$ cadlag, l'opérateur

$$L^V := \frac{1}{2}\mathrm{e}^V \frac{\mathrm{d}}{\mathrm{d}a}\mathrm{e}^{-V} \frac{\mathrm{d}}{\mathrm{d}a}$$

associé à l'EDS formelle

$$\mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{2}V'(X_t)\mathrm{d}t$$

est le générateur d'un processus localement fellerien. De plus, en utilisant nos résultats de convergence, on a montré que si V_n converge vers V pour la topologie de Skorokhod, alors la suite de générateurs L^{V_n} converge vers L^V et donc les processus associés convergent aussi.

Nous avons aussi obtenu un résultat de convergence de marches aléatoires sur $\mathbb{Z}$ vers des diffusions dans des potentiels. On a pu l'appliquer aux marches aléatoires de Sinai (introduites dans [Sin82]), c'est-à-dire les marches aléatoires sur $\mathbb{Z}$ en environnements aléatoires i.i.d. On a montré des convergences vers des diffusions dans des potentiels aléatoires de Lévy (introduits dans [Bro86] pour le cas brownien et dans [Car97] pour le cas général), améliorant ainsi les résultats de [Sei00].

Chapter 2

Local Skorokhod topology on the space of cadlag processes

Abstract: We modify the global Skorokhod topology, on the space of cadlag paths, by localising with respect to space variable, in order to include the eventual explosions. The tightness of families of probability measures on the paths space endowed with this local Skorokhod topology is studied and a characterization of Aldous type is obtained. The local and global Skorokhod topologies are compared by using a time change transformation.
Key words: cadlag processes, explosion time, local Skorokhod topology, Aldous tightness criterion, time change transformation
MSC2010 Subject Classification: Primary 60B10; Secondary 60J75, 54E70, 54A10

1 Introduction

The study of cadlag Lévy-type processes has been an important challenge during the last twenty years. This was due to the fact that phenomena like jumps and unbounded coefficients of characteristic exponent (or symbol) should be taken in consideration in order to get more realistic models.

To perform a systematic study of this kind of trajectories one needs, on one hand, to consider the space of cadlag paths with some appropriate topologies, e.g. Skorokhod's topologies. On the other hand, it was a very useful observation that a unified manner to tackle a lot of questions about large classes of processes is the martingale problem approach. Identifying tightness is an important step when studying sequences of distributions of processes solving associated martingale problems and the Aldous criterion is one of the most employed.

The martingale problem approach was used for several situations: diffusion processes, stochastic differential equations driven by Brownian motion, Lévy processes, Poisson random measures (see, for instance, Stroock [Str75], Stroock and Varadhan [SV06], Kurtz [Kur11]...). Several technical hypotheses (for instance, entire knowledge of the generator, bounded coefficients hypothesis, assumptions concerning explosions...) provide some limitation on the conclusions of certain results, in particular, on convergence results.

The present paper constitutes our first step in studying Markov processes with explosion and, in particular in the martingale problem setting. It contains the study of the so-called local Skorokhod topology and of a time change transformation of cadlag paths. The detailed study of the martingale problem, of Lévy-type processes and several applications will be presented elsewhere (see [GH17c] and [GH17a]).

One of our motivations is that we wonder whether the solution of a well-posed martingale problem is continuous with respect to the initial distribution. The classical approach when one needs to take into consideration the explosion of the solution is to compactify the state space by one point, say Δ, and to endow the cadlag paths space by the Skorokhod topology (see for instance Ethier and Kurtz [EK86], Kallenberg [Kal02]). Unfortunately, this usual topology is not appropriate when we relax hypotheses on the martingale problem setting.

The simplest example is provided by the differential equation

$$\dot{x}_t = b(t, x_t), \quad t > 0, \quad \text{starting from } x_0 \in \mathbb{R}^d,$$

where $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz function. The unique maximal solution exists by setting $x_t = \Delta$, after the explosion time. In general, for some $t > 0$, the mapping $x_0 \mapsto x_t$ is not continuous, and in particular $x_0 \mapsto x_\bullet$ is not continuous for the usual (global) Skorokhod topology. As an illustration, let us consider

$$\dot{x}_t = (1 - t)x_t^2, \quad t > 0, \quad x_0 \in \mathbb{R}.$$

For any initial condition x_0, the unique maximal solution is given by

$$x_t = \left(\frac{t^2}{2} - t + \frac{1}{x_0}\right)^{-1} \quad \text{before} \quad t_{\max} = \begin{cases} \infty, & \text{if } x_0 \in [0, 2), \\ 1 - \sqrt{1 - 2/x_0}, & \text{if } x_0 \geq 2, \\ 1 + \sqrt{1 - 2/x_0}, & \text{if } x_0 < 0, \end{cases}$$

and $x_t := \Delta$, after $t_{\max}$. Indeed, this trajectory is not continuous with respect to the initial condition in the neighbourhood of $x_0 = 2$. To achieve the continuity of the mapping $x_0 \mapsto x_\bullet$ our idea will be to localise the topology on the paths space, not only with respect to the time variable but also with respect to the space variable. More precisely, we need to consider uniform convergence until the exit time from some compact subset of $\mathbb{R}_+ \times \mathbb{R}^d$.

We adapt this idea to cadlag paths by following a similar approach as in Billingsley [Bil99] and we get the local Skorokhod topology which is weaker than the usual (global) Skorokhod topology. Then, we describe the compactness and the tightness in connection with this topology. Furthermore, we state and prove a version of the Aldous criterion, which is an equivalence, as in [Reb79].

As in Ethier and Kurtz [EK86], pp. 306-311, we introduce a time change transformation of the cadlag path x by the positive continuous function g. Roughly speaking, it is defined by $(g \cdot x)_t := x_{\tau_t}$ with τ_t the unique solution starting from 0 of the equation $\dot{\tau}_t = g(x_{\tau_t})$. Another novelty of our paper is the employ of the time change transformation to compare the local Skorokhod topology with the usual (global) Skorokhod topology.

Our paper is organised as follows: the following section is mainly devoted to the study of the local Skorokhod topology on spaces of cadlag paths: the main result is a tightness criterion. Properties of the time change mapping, in particular the continuity, and the connection between the local and global Skorokhod topologies are described in Section 3. The last section contains technical proofs, based on local Skorokhod metrics, of results stated in Section 2.

2 Paths spaces

2.1 Local spaces of cadlag paths

Let S be a locally compact Hausdorff space with countable base. This topological feature is equivalent with the fact that S is separable and can be endowed by a metric for which the unit balls are compact, so, S is a Polish space. Take $\Delta \notin S$, and we will denote by $S^\Delta \supset S$ the one-point compactification of S, if S is not compact, or the topological sum $S \sqcup \{\Delta\}$, if S is compact (so, Δ is an isolated point). Clearly, S^Δ is a compact Hausdorff space with countable base which could be also endowed with a metric. This latter metric will be used to construct various useful functions, compact and open subsets.

For any topological space A and any subset $B \subset \mathbb{R}$, we will denote by $\mathrm{C}(A, B)$ the set of continuous functions from A to B, and by $\mathrm{C}_b(A, B)$ its subset of bounded continuous functions. We will abbreviate $\mathrm{C}(A) := \mathrm{C}(A, \mathbb{R})$ and $\mathrm{C}_b(A) := \mathrm{C}_b(A, \mathbb{R})$. All along the paper we will denote $C \Subset A$ for a subset C which is compactly embedded in A. Similarly, $C \not\Subset A$ means either that C is not a subset of A or C is not compactly embedded in A.

We start with the definition of our spaces of trajectories:

Definition 2.1 (Spaces of cadlag paths)**.** Define the space of exploding cadlag paths

$$\mathbb{D}_{\mathrm{exp}}(S) := \left\{ x : [0, \xi(x)) \to S \;\middle|\; \begin{array}{ll} 0 \le \xi(x) \le \infty, & \\ \forall t_0 \in [0, \xi(x)) & x_{t_0} = \lim\limits_{t \downarrow t_0} x_t, \\ \forall t_0 \in (0, \xi(x)) & x_{t_0-} := \lim\limits_{t \uparrow t_0} x_t \text{ exists in } S \end{array} \right\}.$$

When $\xi(x) = 0$, we assume that x is the empty subset of $[0, 0) \times S$. For a path x from $\mathbb{D}_{\mathrm{exp}}(S)$, $\xi(x)$ is its lifetime or explosion time. We identify $\mathbb{D}_{\mathrm{exp}}(S)$ with a subset of $(S^\Delta)^{\mathbb{R}_+}$ by using the mapping

$$\begin{array}{ccc} \mathbb{D}_{\mathrm{exp}}(S) & \hookrightarrow & (S^\Delta)^{\mathbb{R}_+} \\ x & \mapsto & (x_t)_{t \ge 0} \end{array} \qquad \text{with} \quad x_t := \Delta \quad \text{if} \quad t \ge \xi(x).$$

We define the local cadlag space as the subspace

$$\mathbb{D}_{\mathrm{loc}}(S) := \big\{ x \in \mathbb{D}_{\mathrm{exp}}(S) \;\big|\; \xi(x) \in (0, \infty) \text{ and } \{x_s\}_{s < \xi(x)} \Subset S \text{ imply } x_{\xi(x)-} \text{ exists} \big\}. \quad (2.1)$$

We also introduce the global cadlag space as the subspace of $\mathbb{D}_{\mathrm{loc}}(S)$

$$\mathbb{D}(S) := \{ x \in \mathbb{D}_{\mathrm{loc}}(S) \mid \xi(x) = \infty \} \subset S^{\mathbb{R}_+}.$$

We will always denote by X the canonical process on $\mathbb{D}_{\mathrm{exp}}(S)$, $\mathbb{D}_{\mathrm{loc}}(S)$ and $\mathbb{D}(S)$ without danger of confusion. We endow each of $\mathbb{D}_{\mathrm{exp}}(S)$, $\mathbb{D}_{\mathrm{loc}}(S)$ and $\mathbb{D}(S)$ with a σ-algebra $\mathcal{F} := \sigma(X_s,\ 0 \le s < \infty)$ and a filtration $\mathcal{F}_t := \sigma(X_s,\ 0 \le s \le t)$. We will always skip the argument X for the explosion time $\xi(X)$ of the canonical process.

The following result provides a useful class of measurable mappings:

Proposition 2.2. *For $t_0 \in \mathbb{R}_+$, the mapping*

$$\begin{array}{ccc} \mathbb{D}_{exp}(S) \times [0, t_0] & \to & S^\Delta \\ (x, t) & \mapsto & x_t \end{array}$$

is $\mathcal{F}_{t_0} \otimes \mathcal{B}([0,t_0])$*-measurable. For* $t_0 \in \mathbb{R}_+^*$*, the set*

$$A := \left\{(x,t) \in \mathbb{D}_{exp}(S) \times (0,t_0] \;\middle|\; x_{t-} \text{ exists in } S^\Delta\right\}$$

belongs to $\mathcal{F}_{t_0-} \otimes \mathcal{B}((0,t_0])$ *and the mapping*

$$\begin{array}{rcl} A & \to & S^\Delta \\ (x,t) & \mapsto & x_{t-} \end{array}$$

is $\mathcal{F}_{t_0-} \otimes \mathcal{B}((0,t_0])$*-measurable. For* U *an open subset of* S *and for* $t_0 \in \mathbb{R}_+$*, the set*

$$B := \left\{(x,s,t) \in \mathbb{D}_{exp}(S) \times [0,t_0]^2 \;\middle|\; \{x_u\}_{s\wedge t \le u < s \vee t} \Subset U\right\}$$

belongs to $\mathcal{F}_{t_0-} \otimes \mathcal{B}([0,t_0])^{\otimes 2}$ *and the mapping*

$$\begin{array}{rcl} B \times \mathrm{C}(U) & \to & \mathbb{R} \\ (x,s,t,h) & \mapsto & \displaystyle\int_s^t h(x_u)\mathrm{d}u \end{array}$$

is $\mathcal{F}_{t_0-} \otimes \mathcal{B}([0,t_0])^{\otimes 2} \otimes \mathcal{B}(\mathrm{C}(U))$*-measurable.*

Before proving this proposition we state a corollary which give a useful class of stopping times:

Corollary 2.3. *For any* $(\mathcal{F}_t)$*-stopping time* τ_0*,* $\mathcal{U}$ *an open subset of* S^2*,* $h \in \mathrm{C}(\mathcal{U}, \mathbb{R}_+)$ *a continuous function and* $M : \mathbb{D}_{exp}(S) \to [0,\infty]$ *a* $\mathcal{F}_{\tau_0}$*-measurable map, the mapping*

$$\tau := \inf\left\{t \ge \tau_0 \,|\, \{(X_{\tau_0}, X_s)\}_{\tau_0 \le s \le t} \not\Subset \mathcal{U} \text{ or } \int_{\tau_0}^t h(X_{\tau_0}, X_s)\mathrm{d}s \ge M\right\}$$

is a $(\mathcal{F}_t)$*-stopping time. In particular,* ξ *is a stopping time. Furthermore, if* $U \subset S$ *is an open subset,*

$$\tau^U := \inf\{t \ge 0 \mid X_{t-} \notin U \text{ or } X_t \notin U\} \le \xi \tag{2.2}$$

is a stopping time.

Proof of Corollary 2.3. For each $t \ge 0$, using Proposition 2.2 it is straightforward to obtain that

$$Y := \begin{cases} -1 & \text{if } \tau_0 > t, \\ \int_{\tau_0}^t h(X_{\tau_0}, X_s)\mathrm{d}s & \text{if } \tau_0 \le t \text{ and } \{(X_{\tau_0}, X_s)\}_{\tau_0 \le s \le t} \Subset \mathcal{U}, \\ \infty & \text{otherwise.} \end{cases}$$

is $\mathcal{F}_t$-measurable. Hence

$$\{\tau \le t\} = \{Y \ge M\} = \{Y \ge M\} \cap \{\tau_0 \le t\} \in \mathcal{F}_t,$$

so, τ is a $(\mathcal{F}_t)$-stopping time. □

Proof of Proposition 2.2. Let d be a complete metric for the topology of S, note that

$$A = \bigcap_{\varepsilon \in \mathbb{Q}_+^*} \bigcup_{\delta \in \mathbb{Q}_+^*} \bigcap_{q_1, q_2 \in \mathbb{Q}_+ \cap [0,t_0)} \{q_1, q_2 \in [t-\delta, t) \Rightarrow d(x_{q_1}, x_{q_2}) \le \varepsilon\}$$

so, A belongs to $\mathcal{F}_{t_0-} \otimes \mathcal{B}((0,t_0])$. It is clear that for each $n \in \mathbb{N}$

$$\begin{array}{rcl} A & \to & S^\Delta \\ (x,t) & \mapsto & x_{\frac{t_0}{n+1}\left\lfloor \frac{nt}{t_0} \right\rfloor} \end{array}$$

is $\mathcal{F}_{t_0-} \otimes \mathcal{B}((0,t_0])$-measurable, where $\lfloor r \rfloor$ denotes the integer part of the real number r. Letting $n \to \infty$ we obtain that $(x,t) \mapsto x_{t-}$ is $\mathcal{F}_{t_0-} \otimes \mathcal{B}((0,t_0])$-measurable. The proof is similar for $(x,t) \mapsto x_t$. To prove that B is measurable, let $(K_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact subsets of U such that $U = \bigcup_n K_n$. Then

$$\begin{aligned} B &= \bigcup_{n\in\mathbb{N}} \left\{ (x,s,t) \in \mathbb{D}_{\exp}(S) \times [0,t_0]^2 \;\middle|\; \{x_u\}_{s\wedge t \le u < s\vee t} \subset K_n \right\} \\ &= \bigcup_{n\in\mathbb{N}} \bigcap_{\substack{q\in\mathbb{Q}_+ \\ q<t_0}} \left\{ (x,s,t) \in \mathbb{D}_{\exp}(S) \times [0,t_0]^2 \;\middle|\; s\wedge t \le q < s\vee t \Rightarrow x_q \in K_n \right\}, \end{aligned}$$

so, $B \in \mathcal{F}_{t_0-} \otimes \mathcal{B}([0,t_0])^{\otimes 2}$. To verify the last part, let us note that for $n \in \mathbb{N}^*$ the mapping from $B \times \mathrm{C}(U)$

$$(x,s,t,h) \mapsto \frac{\mathrm{sign}(t-s)}{n} \sum_{i=0}^{n-1} h(x_{\frac{it_0}{n}}) \mathbb{1}_{s\wedge t \le \frac{it_0}{n} < s\vee t}$$

is $\mathcal{F}_{t_0-} \otimes \mathcal{B}([0,t_0])^{\otimes 2} \otimes \mathcal{B}(\mathrm{C}(U))$-measurable. Here and elsewhere we denote by $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ the set of positive entire numbers. Letting $n \to \infty$, the same thing is true for the mapping

$$\begin{array}{rcl} B \times \mathrm{C}(U) & \to & \mathbb{R} \\ (x,s,t,h) & \mapsto & \displaystyle\int_s^t h(x_u)\mathrm{d}u. \end{array}$$

□

2.2 Local Skorokhod topology

To simplify some statements, in this section we will consider a metric d on S. To describe the convergence of a sequence $(x^k)_{k\in\mathbb{N}} \subset \mathbb{D}_{\mathrm{loc}}(S)$ for our topology on $\mathbb{D}_{\mathrm{loc}}(S)$, we need to introduce the following two spaces: we denote by Λ the space of increasing bijections from $\mathbb{R}_+$ to $\mathbb{R}_+$, and by $\widetilde{\Lambda} \subset \Lambda$ the space of increasing bijections λ with λ and λ^{-1} locally Lipschitz. For $\lambda \in \Lambda$ and $t \in \mathbb{R}_+$ we denote

$$\|\lambda - \mathrm{id}\|_t := \sup_{0\le s\le t} |\lambda_s - s| = \|\lambda^{-1} - \mathrm{id}\|_{\lambda_t}. \tag{2.3}$$

For $\lambda \in \widetilde{\Lambda}$, let $\dot{\lambda} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathrm{d}s)$ be the density of $\mathrm{d}\lambda$ with respect to the Lebesgue measure. This density is non-negative and locally bounded below, and for $t \in \mathbb{R}_+$ denote

$$\|\log \dot{\lambda}\|_t := \mathrm{esssup}_{0\le s\le t} \|\log \dot{\lambda}_s\| = \sup_{0\le s_1<s_2\le t} \left| \log \frac{\lambda_{s_2} - \lambda_{s_1}}{s_2 - s_1} \right| = \left\| \log\left(\frac{\mathrm{d}\lambda_s^{-1}}{\mathrm{d}s} \right) \right\|_{\lambda_t}.$$

The proofs of the following theorems use the strategy developed in §12, pp. 121-137 from [Bil99], and are postponed to Section 4.

Theorem 2.4 (Local Skorokhod topology)**.** *There exists a unique Polish topology on $\mathbb{D}_{loc}(S)$, such that a sequence $(x^k)_{k\in\mathbb{N}}$ converges to x for this topology if and only if there exists a sequence $(\lambda^k)_{k\in\mathbb{N}}$ in Λ such that*

- *either $\xi(x) < \infty$ and $\{x_s\}_{s<\xi(x)} \Subset S$: $\lambda^k_{\xi(x)} \leq \xi(x^k)$ for k large enough and*

$$\sup_{s<\xi(x)} d(x_s, x^k_{\lambda^k_s}) \longrightarrow 0, \qquad x^k_{\lambda^k_{\xi(x)}} \longrightarrow \Delta, \qquad \|\lambda^k - \mathrm{id}\|_{\xi(x)} \longrightarrow 0, \quad \text{as } k \to \infty,$$

- *or $\xi(x) = \infty$ or $\{x_s\}_{s<\xi(x)} \not\Subset S$: for all $t < \xi(x)$, for k large enough $\lambda^k_t < \xi(x^k)$ and*

$$\sup_{s\leq t} d(x_s, x^k_{\lambda^k_s}) \longrightarrow 0, \qquad \|\lambda^k - \mathrm{id}\|_t \longrightarrow 0, \quad \text{as } k \to \infty.$$

This topology is also described by a similar characterisation with $\lambda^k \in \Lambda$ and $\|\lambda^k - \mathrm{id}\|$ replaced, respectively, by $\lambda^k \in \widetilde{\Lambda}$ and $\|\log \dot{\lambda}^k\|$. Moreover, the Borel σ-algebra $\mathcal{B}(\mathbb{D}_{loc}(S))$ coincides with the σ-algebra $\mathcal{F}$.

Definition 2.5. The topology on $\mathbb{D}_{\mathrm{loc}}(S)$ whose existence is stated in the latter theorem will be called the local Skorokhod topology. The trace topology from $\mathbb{D}_{\mathrm{loc}}(S)$ to $\mathbb{D}(S)$ will be called the global Skorokhod topology.

Remark 2.6. 1) We point out that these topologies do not depend on the metric d of S and this is a consequence of the fact that two metrics on a compact set are uniformly equivalent (cf. Lemma 4.3 below).
2) The convergence conditions of Theorem 2.4 may be summarised as: a sequence $(x_k)_k$ converges to x for the local Skorokhod topology if and only if there exists a sequence $(\lambda^k)_k$ in Λ satisfying that for any $t \in \mathbb{R}_+$ such that $\{x_s\}_{s<t} \Subset S$, for k large enough $\lambda^k_t \leq \xi(x^k)$ and

$$\sup_{s<t} d(x_s, x^k_{\lambda^k_s}) \longrightarrow 0, \quad x^k_{\lambda^k_t} \longrightarrow x_t, \quad \|\lambda^k - \mathrm{id}\|_t \longrightarrow 0, \quad \text{as } k \to \infty.$$

3) A sequence $(x^k)_k$ from $\mathbb{D}(S)$ converges to $x \in \mathbb{D}(S)$ for the global Skorokhod topology if and only if there exists a sequence $(\lambda^k)_k$ in Λ such that for all $t \geq 0$

$$\sup_{s\leq t} d(x_s, x^k_{\lambda^k_s}) \longrightarrow 0, \qquad \|\lambda^k - \mathrm{id}\|_t \longrightarrow 0, \quad \text{as } k \to \infty.$$

In fact, we recover the usual Skorokhod topology on $\mathbb{D}(S)$, which is described, for instance, in §16 pp. 166-179 from [Bil99]. $\Diamond$

We are now interested to characterise the sets of $\mathbb{D}_{\mathrm{loc}}(S)$ which are compact and also to obtain a criterion for the tightness of a subset of probability measures in $\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$. For $x \in \mathbb{D}_{\mathrm{exp}}(S)$, $t \geq 0$, $K \subset S$ compact and $\delta > 0$, define

$$\omega'_{t,K,x}(\delta) := \inf \left\{ \sup_{\substack{0\leq i<N \\ t_i\leq s_1,s_2<t_{i+1}}} d(x_{s_1}, x_{s_2}) \;\middle|\; \begin{array}{l} N \in \mathbb{N},\ 0 = t_0 < \cdots < t_N \leq \xi(x) \\ (t_N, x_{t_N}) \notin [0,t] \times K \\ \forall 0 \leq i < N :\ t_{i+1} - t_i > \delta \end{array} \right\}. \tag{2.4}$$

We give some properties of ω'.

Proposition 2.7.

i) Consider $x \in \mathbb{D}_{exp}(S)$. Then x belongs to $\mathbb{D}_{loc}(S)$ if and only if

$$\forall t \geq 0,\ \forall K \subset S \text{ compact}, \quad \omega'_{t,K,x}(\delta) \underset{\delta \to 0}{\longrightarrow} 0.$$

ii) For all $t \geq 0$, $K \subset S$ compact and $\delta > 0$, the mapping

$$\begin{array}{ccc} \mathbb{D}_{loc}(S) & \to & [0, +\infty] \\ x & \mapsto & \omega'_{t,K,x}(\delta) \end{array}$$

is upper semi-continuous.

Proof. Suppose that $x \in \mathbb{D}_{\mathrm{loc}}(S)$ and let $t \geq 0$ be and consider a compact set $K \subset S$. There exists $T \leq \xi(x)$ such that $(T, x_T) \notin [0,t] \times K$ and the limit x_{T-} exists in S. Let $\varepsilon > 0$ be arbitrary and consider I the set of times $s \leq T$ for which there exists a subdivision

$$0 = t_0 < \cdots < t_N = s$$

such that

$$\sup_{\substack{0 \leq i < N \\ t_i \leq s_1, s_2 < t_{i+1}}} d(x_{s_1}, x_{s_2}) \leq \varepsilon.$$

It is clear that I is an interval of $[0,T]$ containing 0: set $t^* := \sup I$. Since there is existence of the limit x_{t^*-}, then $t^* \in I$, and, since x is right-continuous, $t^* = T$. Hence, $T \in I$ and there exists $\delta > 0$ such that $\omega'_{t,K,x}(\delta) \leq \varepsilon$.

Conversely, let's take $x \in \mathbb{D}_{\mathrm{exp}}(S)$ such that $\xi(x) < \infty$, $\{x_s\}_{s<\xi(x)} \Subset S$ and

$$\forall t \geq 0,\ \forall K \subset S \text{ compact}, \quad \omega'_{t,K,x}(\delta) \underset{\delta \to 0}{\longrightarrow} 0.$$

We need to prove that the limit $x_{\xi(x)-}$ exists in S. Let y_1, y_2 be any two limits points of x_s, as $s \to \xi(x)$. We will prove that $y_1 = y_2$. Let $\varepsilon > 0$ be arbitrary. By taking $t = \xi(x)$ and $K = \overline{\{x_s\}}_{s<\xi(x)}$ in (2.4) there exists a subdivision,

$$0 = t_0 < \cdots < t_N = \xi(x),$$

such that

$$\sup_{\substack{0 \leq i < N \\ t_i \leq s_1, s_2 < t_{i+1}}} d(x_{s_1}, x_{s_2}) \leq \varepsilon.$$

Replacing in the latter inequality the two sub-sequences tending toward y_1, y_2, we can deduce that $d(y_1, y_2) \leq \varepsilon$, and letting $\varepsilon \to 0$ we get $y_1 = y_2$.

We proceed with the proof of part *ii)*. Let $(x^k)_k \subset \mathbb{D}_{\mathrm{loc}}(S)$ be such that x^k converges to $x \in \mathbb{D}_{\mathrm{loc}}(S)$ and let $(\lambda^k)_k \subset \widetilde{\Lambda}$ be such in Theorem 2.4. We need to prove that,

$$\limsup_{k \to \infty} \omega'_{t,K,x^k}(\delta) \leq \omega'_{t,K,x}(\delta).$$

We can suppose that $\omega'_{t,K,x}(\delta) < \infty$. Let $\varepsilon > 0$ be arbitrary and consider a subdivision $0 = t_0 < \cdots < t_N \leq \xi(x)$ such that

$$\sup_{\substack{0 \leq i < N \\ t_i \leq s_1, s_2 < t_{i+1}}} d(x_{s_1}, x_{s_2}) \leq \omega'_{t,K,x}(\delta) + \varepsilon,$$

$t_{i+1} > t_i + \delta$ and $(t_N, x_{t_N}) \notin [0,t] \times K$. If $t_N = \xi(x)$ and $\{x_s\}_{s<\xi(x)} \not\Subset S$, then we can find $\widetilde{t}_N$ such that $t_{N-1} + \delta < \widetilde{t}_N < \xi(x)$ and $x_{\widetilde{t}_N} \notin K$. We can suppose, possibly by replacing t_N by $\widetilde{t}_N$, that

$$t_N = \xi(x) \text{ implies } \{x_s\}_{s<\xi(x)} \Subset S.$$

Hence, for k large enough, $\lambda^k_{t_N} \leq \xi(x^k)$ and

$$\sup_{s<t_N} d(x_s, x^k_{\lambda^k_s}) \longrightarrow 0, \qquad x^k_{\lambda^k_{t_N}} \longrightarrow x_{t_N}, \qquad \|\lambda^k - \mathrm{id}\|_{t_N} \longrightarrow 0, \quad \text{ as } k \to \infty.$$

We deduce that, for k large enough, we have $0 = \lambda^k_{t_0} < \cdots < \lambda^k_{t_N} \leq \xi(x^k)$, $\lambda^k_{t_{i+1}} > \lambda^k_{t_i} + \delta$, $(\lambda^k_{t_N}, x^k_{\lambda^k_{t_N}}) \notin [0,t] \times K$, and moreover,

$$\begin{aligned}
\sup_{\substack{0\leq i<N \\ \lambda^k_{t_i} \leq s_1, s_2 < \lambda^k_{t_{i+1}}}} d(x^k_{s_1}, x^k_{s_2}) &\leq \sup_{\substack{0\leq i<N \\ t_i \leq s_1, s_2 < t_{i+1}}} d(x_{s_1}, x_{s_2}) + 2 \sup_{s<t_N} d(x_s, x^k_{\lambda^k_s}) \\
&\leq \omega'_{t,K,x}(\delta) + \varepsilon + 2 \sup_{s<t_N} d(x_s, x^k_{\lambda^k_s}) \underset{k\to\infty}{\longrightarrow} \omega'_{t,K,x}(\delta) + \varepsilon.
\end{aligned}$$

Therefore,

$$\limsup_{k\to\infty} \omega'_{t,K,x^k}(\delta) \leq \omega'_{t,K,x}(\delta) + \varepsilon,$$

and we conclude by letting $\varepsilon \to 0$. $\square$

We can give now a characterisation of the relative compactness for the local Skorokhod topology:

Theorem 2.8 (Compact sets of $\mathbb{D}_{\mathrm{loc}}(S)$)**.** *For any subset $D \subset \mathbb{D}_{loc}(S)$, D is relatively compact if and only if*

$$\forall t \geq 0,\ K \subset S \textit{ compact}, \quad \sup_{x\in D} \omega'_{t,K,x}(\delta) \underset{\delta\to 0}{\longrightarrow} 0. \tag{2.5}$$

The proof follows the strategy developed in §12 pp. 121-137 from [Bil99] and it is postponed to Section 4.

We conclude this section with a version of the Aldous criterion of tightness:

Proposition 2.9 (Aldous criterion)**.** *Let $\mathcal{P}$ be a subset of $\mathcal{P}(\mathbb{D}_{loc}(S))$. If for all $t \geq 0$, $\varepsilon > 0$, and an open subset $U \Subset S$, we have:*

$$\inf_{F\subset\mathcal{P}} \sup_{\mathbf{P}\in\mathcal{P}\setminus F} \sup_{\substack{\tau_1\leq\tau_2 \\ \tau_2 \leq (\tau_1+\delta)\wedge t\wedge\tau^U}} \mathbf{P}\big(\tau_1 < \tau_2 = \xi \textit{ or } d(X_{\tau_1}, X_{\tau_2}) \mathbb{1}_{\{\tau_1<\xi\}} \geq \varepsilon\big) \underset{\delta\to 0}{\longrightarrow} 0, \tag{2.6}$$

then $\mathcal{P}$ is tight. Here the infimum is taken on all finite subsets $F \subset \mathcal{P}$ and the supremum is taken on all stopping times τ_i.

Remark 2.10. As in [Bil99], Theorem 16.9, p. 177, an equivalent condition for tightness can be obtained by replacing (2.6) by

$$\inf_{F\subset\mathcal{P}} \sup_{\mathbf{P}\in\mathcal{P}\setminus F} \sup_{\tau\leq t\wedge\tau^U} \mathbf{P}\big(\tau < (\tau+\delta)\wedge\tau^U = \xi \text{ or } d(X_\tau, X_{(\tau+\delta)\wedge\tau^U}) \mathbb{1}_{\{\tau<\xi\}} \geq \varepsilon\big) \underset{\delta\to 0}{\longrightarrow} 0,$$

by taking infimum on all finite subsets $F \subset \mathcal{P}$ and the supremum on all stopping times τ. $\lozenge$

In fact we will state and prove a version of the of the Aldous criterion, which is a necessary and sufficient condition, similarly as in [Reb79]:

Theorem 2.11 (Tightness for $\mathbb{D}_{\mathrm{loc}}(S)$)**.** *For any subset $\mathcal{P} \subset \mathcal{P}(\mathbb{D}_{loc}(S))$, the following assertions are equivalent:*

1. *$\mathcal{P}$ is tight,*
2. *for all $t \geq 0$, $\varepsilon > 0$ and K a compact set we have*
$$\sup_{\mathbf{P} \in \mathcal{P}} \mathbf{P}\left(\omega'_{t,K,X}(\delta) \geq \varepsilon\right) \underset{\delta \to 0}{\longrightarrow} 0,$$
3. *for all $t \geq 0$, $\varepsilon > 0$, and open subset $U \Subset S$ we have:*
$$\alpha(\varepsilon, t, U, \delta) := \sup_{\mathbf{P} \in \mathcal{P}} \sup_{\substack{\tau_1 \leq \tau_2 \leq \tau_3 \\ \tau_3 \leq (\tau_1 + \delta) \wedge t \wedge \tau^U}} \mathbf{P}(R \geq \varepsilon) \underset{\delta \to 0}{\longrightarrow} 0,$$
where the supremum is taken on τ_i stopping times and with
$$R := \begin{cases} d(X_{\tau_1}, X_{\tau_2}) \wedge d(X_{\tau_2}, X_{\tau_3}) & \text{if } 0 < \tau_1 < \tau_2 < \tau_3 < \xi, \\ d(X_{\tau_2 -}, X_{\tau_2}) \wedge d(X_{\tau_2}, X_{\tau_3}) & \text{if } 0 < \tau_1 = \tau_2 < \tau_3 < \xi, \\ d(X_{\tau_1}, X_{\tau_2}) & \text{if } 0 = \tau_1 \leq \tau_2 < \xi \text{ or } 0 < \tau_1 < \tau_2 < \tau_3 = \xi, \\ d(X_{\tau_2 -}, X_{\tau_2}) & \text{if } 0 < \tau_1 = \tau_2 < \tau_3 = \xi, \\ 0 & \text{if } \tau_1 = \xi \text{ or } 0 < \tau_1 \leq \tau_2 = \tau_3, \\ \infty & \text{if } 0 = \tau_1 < \tau_2 = \xi. \end{cases}$$

Remark 2.12. 1) If d is obtained from a metric on S^Δ, then if $\varepsilon < d(\Delta, U)$ the expression of $\alpha(\varepsilon, t, U, \delta)$ may be simplified as follows:
$$\alpha(\varepsilon, t, U, \delta) = \sup_{\mathbf{P} \in \mathcal{P}} \sup_{\substack{\tau_1 \leq \tau_2 \leq \tau_3 \\ \tau_3 \leq (\tau_1 + \delta) \wedge t \wedge \tau^U}} \mathbf{P}(\widetilde{R} \geq \varepsilon) \underset{\delta \to 0}{\longrightarrow} 0,$$
where the supremum is taken on τ_i stopping times and with
$$\widetilde{R} := \begin{cases} d(X_{\tau_1}, X_{\tau_2}) \wedge d(X_{\tau_2}, X_{\tau_3}) & \text{if } 0 < \tau_1 < \tau_2, \\ d(X_{\tau_2 -}, X_{\tau_2}) \wedge d(X_{\tau_2}, X_{\tau_3}) & \text{if } 0 < \tau_1 = \tau_2, \\ d(X_{\tau_1}, X_{\tau_2}) & \text{if } 0 = \tau_1. \end{cases}$$
2) It is straightforward to verify that a subset $D \subset \mathbb{D}(S)$ is relatively compact for $\mathbb{D}(S)$ if and only if D is relatively compact for $\mathbb{D}_{\mathrm{loc}}(S)$ and
$$\forall t \geq 0, \quad \{x_s \mid x \in D,\ s \leq t\} \Subset S.$$
Hence we may recover the classical characterisation of compact sets of $\mathbb{D}(S)$ and the classical Aldous criterion. Moreover, we may obtain a version of Theorem 2.11 for $\mathbb{D}(S)$.
3) The difficult part of Theorem 2.11 is the implication 3⇒2, and its proof is adapted from the proof of Theorem 16.10 pp. 178-179 from [Bil99]. Roughly speaking the assertion 3 uses
$$\omega''_x(\delta) := \sup_{s_1 \leq s_2 \leq s_3 \leq s_1 + \delta} d(x_{s_1}, x_{s_2}) \wedge d(x_{s_2}, x_{s_3}),$$

while the Aldous criterion uses

$$\omega_x(\delta) := \sup_{s_1 \le s_2 \le s_1+\delta} d(x_{s_1}, x_{s_2}).$$

The term $d(X_{\tau_2-}, X_{\tau_2})$ appears because, in contrary to the deterministic case, some stopping time may not be approximate by the left. We refer to the proof of Theorem 12.4 pp. 132-133 from [Bil99] for the relation between ω'' and ω'. $\Diamond$

Proof of Theorem 2.11.
2⇒1 Consider $(t_n)_{n\ge1}$ a sequence of times tending to infinity and $(K_n)_{n\ge1}$ an increasing sequence of compact subsets of S such that $S = \bigcup_n K_n$. Let $\eta > 0$ be a real number, and for $n \ge 1$ define δ_n such that

$$\sup_{\mathbf{P}\in\mathcal{P}} \mathbf{P}\left(\omega'_{t_n,K_n,X}(\delta_n) \ge n^{-1}\right) \le 2^{-n}\eta\,.$$

Set

$$D := \{\forall n \in \mathbb{N}^*,\ \omega'_{t_n,K_n,X}(\delta_n) < n^{-1}\}.$$

By Theorem 2.8, D is relatively compact and moreover

$$\sup_{\mathbf{P}\in\mathcal{P}} \mathbf{P}\left(D^c\right) \le \sum_{n\ge1} 2^{-n}\eta = \eta$$

so, $\mathcal{P}$ is tight.
1⇒3 Consider ε, η two arbitrary positive real numbers. There exists a compact set $D \subset \mathbb{D}_{\mathrm{loc}}(S)$ such that

$$\sup_{\mathbf{P}\in\mathcal{P}} \mathbf{P}(D^c) \le \eta.$$

By Theorem 2.8, there exists $\delta > 0$ such that

$$D \subset \{\omega'_{t,\overline{U},X}(\delta) < \varepsilon\}.$$

Since for all $\tau_1 \le \tau_2 \le \tau_3 \le (\tau_1+\delta)\wedge t \wedge \tau^U$ we have

$$\{\omega'_{t,\overline{U},X}(\delta) < \varepsilon\} \subset \{R < \varepsilon\},$$

we conclude that

$$\sup_{\mathbf{P}\in\mathcal{P}} \sup_{\substack{\tau_1\le\tau_2\le\tau_3 \\ \tau_3 \le (\tau_1+\delta)\wedge t\wedge\tau^U}} \mathbf{P}(R \ge \varepsilon) \le \eta\,.$$

3⇒2 For all $\varepsilon > 0$, $t \ge 0$ and open subset $U \Subset S$, up to consider $\widetilde{\tau}_i := \tau_i \wedge (\tau_1+\delta)\wedge t\wedge\tau^U$ we have a new expression of $\alpha(\varepsilon, t, U, \delta)$:

$$\alpha(\varepsilon,t,U,\delta) = \sup_{\mathbf{P}\in\mathcal{P}} \sup_{\tau_1\le\tau_2\le\tau_3\le\xi} \mathbf{P}(R \ge \varepsilon,\ \tau_3 \le (\tau_1+\delta)\wedge t\wedge\tau^U) \underset{\delta\to0}{\longrightarrow} 0. \tag{2.7}$$

Consider $\varepsilon_0 > 0$, $t \ge 0$ and K a compact subset of S. We need to prove that

$$\inf_{\mathbf{P}\in\mathcal{P}} \mathbf{P}(\omega'_{t,K,X}(\delta) < \varepsilon_0) \underset{\delta\to0}{\longrightarrow} 1.$$

Choose $0 < \varepsilon_1 < \varepsilon_0/4$ such that

$$U := \{y \in S \mid d(y, K) < \varepsilon_1\} \Subset S.$$

For $n \in \mathbb{N}$ and $\varepsilon > 0$, define inductively the stopping times (see Corollary 2.3)

$$\begin{aligned}
&\tau_0 := 0,\\
&\tau_n^\varepsilon := \inf\left\{s > \tau_n \,\middle|\, d(X_{\tau_n}, X_s) \vee d(X_{\tau_n}, X_{s-}) \geq \varepsilon\right\} \wedge (t+2) \wedge \tau^U,\\
&\tau_{n+1} := \tau_n^{\varepsilon_1},
\end{aligned}$$

It is clear that τ_n^ε increases to τ_{n+1} when ε increases to ε_1. If we choose $0 < \varepsilon_2 < \varepsilon_1$, then for all $\mathbf{P} \in \mathcal{P}$,

$$\begin{aligned}
&\limsup_{\substack{\varepsilon \to \varepsilon_1\\ \varepsilon < \varepsilon_1}} \mathbf{P}(X_{\tau_n} \in K,\ \tau_n^\varepsilon < \xi,\ d(X_{\tau_n}, X_{\tau_n^\varepsilon}) \leq \varepsilon_2,\ \tau_n^\varepsilon \leq t+1) \qquad (2.8)\\
&\qquad \leq \mathbf{P}\left(\limsup_{\substack{\varepsilon \to \varepsilon_1\\ \varepsilon < \varepsilon_1}} \{X_{\tau_n} \in K,\ \tau_n^\varepsilon < \xi,\ d(X_{\tau_n}, X_{\tau_n^\varepsilon}) \leq \varepsilon_2,\ \tau_n^\varepsilon \leq t+1\}\right) = \mathbf{P}(\emptyset) = 0.
\end{aligned}$$

For all $\mathbf{P} \in \mathcal{P}$, $\delta \leq 1$ and $0 < \varepsilon < \varepsilon_1$ we have using the expression (2.7) with stopping times $0 \leq \tau_0^\varepsilon \leq \tau_0^{\varepsilon_1} = \tau_1$

$$\begin{aligned}
\mathbf{P}(X_0 \in K,\ \tau_0^\varepsilon \leq \delta) &= \mathbf{P}(X_0 \in K,\ X_{\tau_0^\varepsilon} \notin B(X_0, \varepsilon_2),\ \tau_0^\varepsilon \leq \delta)\\
&\quad + \mathbf{P}(X_0 \in K,\ \tau_0^\varepsilon < \xi,\ d(X_0, X_{\tau_0^\varepsilon}) < \varepsilon_2,\ \tau_0^\varepsilon \leq \delta)\\
&\leq \alpha(\varepsilon_2, t+2, U, \delta)\\
&\quad + \mathbf{P}(X_0 \in K,\ \tau_0^\varepsilon < \xi,\ d(X_0, X_{\tau_0^\varepsilon}) < \varepsilon_2,\ \tau_0^\varepsilon \leq t+1)
\end{aligned}$$

so, letting $\varepsilon \to \varepsilon_1$, since $\tau_0^\varepsilon \uparrow \tau_1$, by (2.8) we obtain

$$\mathbf{P}(X_0 \in K,\ \tau_1 \leq \delta) \leq \alpha(\varepsilon_2, t+2, U, \delta). \tag{2.9}$$

For all $\mathbf{P} \in \mathcal{P}$, $\delta \leq 1$, $n \in \mathbb{N}$ and $0 < \varepsilon < \varepsilon_1$ we have also using the expression (2.7) with stopping times $\tau_n \leq \tau_n^\varepsilon \leq \tau_{n+1}$ and $\tau_n \leq \tau_n^\varepsilon \leq \tau_{n+1}^\varepsilon$

$$\begin{aligned}
&\mathbf{P}(\tau_{n+1} \leq t,\ X_{\tau_n}, X_{\tau_{n+1}} \in K,\ \tau_{n+1}^\varepsilon - \tau_n \leq \delta)\\
&\quad \leq \mathbf{P}(X_{\tau_n} \in K,\ \tau_n^\varepsilon < \xi,\ d(X_{\tau_n}, X_{\tau_n^\varepsilon}) < \varepsilon_2,\ \tau_n^\varepsilon \leq t+1)\\
&\qquad + \mathbf{P}(X_{\tau_{n+1}} \in K,\ \tau_{n+1}^\varepsilon < \xi,\ d(X_{\tau_{n+1}}, X_{\tau_{n+1}^\varepsilon}) < \varepsilon_2,\ \tau_{n+1}^\varepsilon \leq t+1)\\
&\qquad + \mathbf{P}\Big(\tau_{n+1} \leq t,\ X_{\tau_n}, X_{\tau_{n+1}} \in K,\ d(X_{\tau_n}, X_{\tau_n^\varepsilon}) \geq \varepsilon_2,\\
&\qquad\qquad\qquad d(X_{\tau_n^\varepsilon}, X_{\tau_{n+1}}) \geq \frac{\varepsilon_2}{2}, \tau_{n+1} - \tau_n \leq \delta\Big)\\
&\qquad + \mathbf{P}\Big(\tau_{n+1} \leq t,\ X_{\tau_n}, X_{\tau_{n+1}} \in K,\ d(X_{\tau_n}, X_{\tau_n^\varepsilon}) \geq \varepsilon_2,\ X_{\tau_{n+1}^\varepsilon} \notin B(X_{\tau_n^\varepsilon}, \frac{\varepsilon_2}{2}),\\
&\qquad\qquad\qquad \tau_{n+1}^\varepsilon - \tau_n \leq \delta\Big)\\
&\quad \leq \mathbf{P}(X_{\tau_n} \in K,\ \tau_n^\varepsilon < \xi,\ d(X_{\tau_n}, X_{\tau_n^\varepsilon}) \leq \varepsilon_2,\ \tau_n^\varepsilon \leq t+1)\\
&\qquad + \mathbf{P}(X_{\tau_{n+1}} \in K,\ \tau_{n+1}^\varepsilon < \xi,\ d(X_{\tau_{n+1}}, X_{\tau_{n+1}^\varepsilon}) \leq \varepsilon_2,\ \tau_{n+1}^\varepsilon \leq t+1)\\
&\qquad + 2\alpha\left(\frac{\varepsilon_2}{2}, t+2, U, \delta\right)
\end{aligned}$$

so, letting $\varepsilon \to \varepsilon_1$, since $\tau_{n+1}^\varepsilon \uparrow \tau_{n+2}$, by (2.8) we obtain

$$\mathbf{P}(\tau_{n+1} \le t,\ X_{\tau_n}, X_{\tau_{n+1}} \in K,\ \tau_{n+2} - \tau_n \le \delta) \le 2\alpha\left(\frac{\varepsilon_2}{2}, t+2, U, \delta\right). \tag{2.10}$$

For all $\mathbf{P} \in \mathcal{P}$, $\delta \le 1$, $n \in \mathbb{N}^*$ and $0 < \varepsilon < \varepsilon_1$ we can write using the expression (2.7) with stopping times $\tau_n \le \tau_n \le \tau_n^\varepsilon$

$$\begin{aligned}
&\mathbf{P}(\tau_n \le t,\ X_{\tau_n} \in K,\ d(X_{\tau_n -}, X_{\tau_n}) \ge \varepsilon_2,\ \tau_n^\varepsilon - \tau_n \le \delta) \\
&\qquad \le \mathbf{P}(X_{\tau_n} \in K,\ \tau_n^\varepsilon < \xi,\ d(X_{\tau_n}, X_{\tau_n^\varepsilon}) < \varepsilon_2,\ \tau_n^\varepsilon \le t+1) \\
&\qquad\quad + \mathbf{P}(\tau_n \le t,\ X_{\tau_n} \in K,\ d(X_{\tau_n -}, X_{\tau_n}) \ge \varepsilon_2,\ X_{\tau_n^\varepsilon} \notin B(X_{\tau_n}, \varepsilon_2),\ \tau_n^\varepsilon - \tau_n \le \delta) \\
&\qquad \le \mathbf{P}(X_{\tau_n} \in K,\ \tau_n^\varepsilon < \xi,\ d(X_{\tau_n}, X_{\tau_n^\varepsilon}) \le \varepsilon_2,\ \tau_n^\varepsilon \le t+1) + \alpha(\varepsilon_2, t+2, U, \delta)
\end{aligned}$$

so, letting $\varepsilon \to \varepsilon_1$, since $\tau_n^\varepsilon \uparrow \tau_{n+1}$, by (2.8) we obtain

$$\mathbf{P}(\tau_n \le t,\ X_{\tau_n} \in K,\ d(X_{\tau_n -}, X_{\tau_n}) \ge \varepsilon_2,\ \tau_{n+1} - \tau_n \le \delta) \le \alpha(\varepsilon_2, t+2, U, \delta). \tag{2.11}$$

Let $m \in 2\mathbb{N}$ and $0 < \delta' \le 1$ be such that $m > 2t/\delta'$ and denote the event

$$A := \{\tau_m \le t \text{ and } \forall n \le m,\ X_{\tau_n} \in K\}.$$

Then for all $0 \le i < m$, thanks to (2.10)

$$\mathbf{E}[\tau_{i+2} - \tau_i \mid A] \ge \delta' \mathbf{P}\left(\tau_{i+2} - \tau_i \ge \delta' \mid A\right) \ge \delta' \left(1 - \frac{2\alpha\left(\frac{\varepsilon_2}{2}, t+2, U, \delta'\right)}{\mathbf{P}(A)}\right).$$

Hence

$$t \ge \mathbf{E}[\tau_m \mid A] = \sum_{i=0}^{(m-2)/2} \mathbf{E}[\tau_{2i+2} - \tau_{2i} \mid A] \ge \frac{m\delta'}{2}\left(1 - \frac{2\alpha\left(\frac{\varepsilon_2}{2}, t+2, U, \delta'\right)}{\mathbf{P}(A)}\right)$$

so,

$$\mathbf{P}(A) \le \frac{2\alpha\left(\frac{\varepsilon_2}{2}, t+2, U, \delta'\right)}{1 - 2t/(m\delta')}. \tag{2.12}$$

Taking $0 < \delta \le 1$ and setting

$$B_{m,\delta} := \left\{ \begin{array}{l} (\tau_m, X_{\tau_0}, \dots, X_{\tau_m}) \notin [0,t] \times K^{m+1}, \\ X_0 \in K \Rightarrow \tau_1 > \delta, \\ \forall 0 \le n \le m-2,\ \tau_{n+1} \le t \text{ and } X_{\tau_n}, X_{\tau_{n+1}} \in K \Rightarrow \tau_{n+2} - \tau_n > \delta, \\ \forall 0 \le n < m,\ \tau_n \le t,\ X_{\tau_n} \in K,\ d(X_{\tau_n -}, X_{\tau_n}) \ge \varepsilon_2 \Rightarrow \tau_{n+1} - \tau_n > \delta \end{array} \right\},$$

by (2.9), (2.10), (2.11) and (2.12) we obtain that

$$\begin{aligned}
\inf_{\mathbf{P} \in \mathcal{P}} \mathbf{P}(B_{m,\delta}) \ge 1 &- \frac{2\alpha\left(\frac{\varepsilon_2}{2}, t+2, U, \delta'\right)}{1 - 2t/(m\delta')} - \alpha(\varepsilon_2, t+2, U, \delta) \\
&- 2(m-1)\alpha\left(\frac{\varepsilon_2}{2}, t+2, U, \delta\right) - m\alpha(\varepsilon_2, t+2, U, \delta).
\end{aligned}$$

Hence

$$\sup_{m\in\mathbb{N}} \inf_{\mathbf{P}\in\mathcal{P}} \mathbf{P}(B_{m,\delta}) \underset{\delta\to 0}{\longrightarrow} 1.$$

Recalling that $\varepsilon_1 < 4\varepsilon_0$, a straightforward computation gives

$$B_{m,\delta} \subset \left\{\omega'_{t,K,X}(\delta) < \varepsilon_0\right\}.$$

We conclude that

$$\inf_{\mathbf{P}\in\mathcal{P}} \mathbf{P}(\omega'_{t,K,X}(\delta) < \varepsilon_0) \underset{\delta\to 0}{\longrightarrow} 1.$$

□

3 Time change and Skorokhod topologies

3.1 Definition and properties of time change

First we give the definition of the time change mapping (see also §6.1 pp. 306-311 from [EK86], §V.26 pp. 175-177 from [RW00]).

Definition 3.1 (Time Change)**.** Let us introduce

$$\mathrm{C}^{\neq 0}(S,\mathbb{R}_+) := \{g : S \to \mathbb{R}_+ \mid \{g = 0\} \text{ is closed and } g \text{ is continuous on } \{g \neq 0\}\},$$

and for $g \in \mathrm{C}^{\neq 0}(S,\mathbb{R}_+)$, $x \in \mathbb{D}_{\exp}(S)$ and $t \in [0,\infty]$ we denote

$$\tau^g_t(x) := \inf\{s \geq 0 \mid A^g_s(x) \geq t\}, \quad \text{where} \quad A^g_t(x) := \begin{cases} \int_0^t \frac{\mathrm{d}u}{g(x_u)}, & \text{if } t \in [0,\tau^{\{g\neq 0\}}(x)], \\ \infty & \text{otherwise.} \end{cases} \tag{3.1}$$

For $g \in \mathrm{C}^{\neq 0}(S,\mathbb{R}_+)$, we define a time change mapping, which is $\mathcal{F}$-measurable,

$$\begin{array}{rccc} g\cdot X : & \mathbb{D}_{\exp}(S) & \to & \mathbb{D}_{\exp}(S) \\ & x & \mapsto & g\cdot x, \end{array}$$

as follows: for $t \in \mathbb{R}_+$

$$(g\cdot X)_t := \begin{cases} X_{\tau^g_\infty -} & \text{if } t \geq A^g_{\tau^g_\infty},\ X_{\tau^g_\infty -} \text{ exists and belongs to } \{g = 0\}, \\ X_{\tau^g_t} & \text{otherwise.} \end{cases} \tag{3.2}$$

For $g \in \mathrm{C}^{\neq 0}(S,\mathbb{R}_+)$ and $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\exp}(S))$, we also define $g\cdot\mathbf{P}$ the pushforward of $\mathbf{P}$ by $x \mapsto g\cdot x$.

The fact that this mapping is measurable will be proved in the next section.

Remark 3.2. Let us stress that, by using Corollary 2.3, τ^g_t is a stopping time. In particular, the following stopping time will play a crucial role:

$$\tau^g_\infty = \tau^{\{g\neq 0\}} := \xi \wedge \inf\{t \geq 0 \mid g(X_{t-}) \wedge g(X_t) = 0\}. \tag{3.3}$$

The time of explosion of $g\cdot X$ is given by

$$\xi(g\cdot X) = \begin{cases} \infty & \text{if } \tau^g_\infty < \xi \text{ or } X_{\xi-} \text{ exists and belongs to } \{g = 0\}, \\ A^g_\xi & \text{otherwise.} \end{cases}$$

◊

Roughly speaking, $g \cdot X$ is given by $(g \cdot X)_t := X_{\tau_t^g}$ where $t \mapsto \tau_t^g$ is the solution of $\dot{\tau}_t^g = g(X_{\tau_t^g})$, on the time interval $[0, \tau_\infty^g)$.

Proposition 3.3.

1. *For $U \subset S$ an open subset, by identifying*

$$\mathrm{C}(U, \mathbb{R}_+) = \big\{ g \in \mathrm{C}^{\neq 0}(S, \mathbb{R}_+) \,|\, \{g \neq 0\} \subset U \text{ and } g \text{ is continuous on } U \big\},$$

the time change mapping

$$\begin{array}{ccc} \mathrm{C}(U, \mathbb{R}_+) \times \mathbb{D}_{exp}(S) & \to & \mathbb{D}_{exp}(S) \\ (g, x) & \mapsto & g \cdot x, \end{array}$$

is measurable between $\mathcal{B}(\mathrm{C}(U, \mathbb{R}_+)) \otimes \mathcal{F}$ and $\mathcal{F}$.

2. *If $g_1, g_2 \in \mathrm{C}^{\neq 0}(S, \mathbb{R}_+)$ and $x \in \mathbb{D}_{exp}(S)$, then $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.*

3. *If g is bounded and belongs to $\mathrm{C}^{\neq 0}(S, \mathbb{R}_+)$, and $x \in \mathbb{D}(S)$, then $g \cdot x \in \mathbb{D}(S)$.*

4. *Define*

$$\widetilde{\mathrm{C}}^{\neq 0}(S, \mathbb{R}_+) := \big\{ g \in \mathrm{C}^{\neq 0}(S, \mathbb{R}_+) \,|\, \forall K \subset S \text{ compact, } g(K) \text{ is bounded} \big\}.$$

If $g \in \widetilde{\mathrm{C}}^{\neq 0}(S, \mathbb{R}_+)$ and $x \in \mathbb{D}_{loc}(S)$, then $g \cdot x \in \mathbb{D}_{loc}(S)$.

Proof. The first point is straightforward by using Proposition 2.2, while the second point is a direct consequence of the time change definition and, in particular, of the first part of (3.2). The third point can be easily deduced because,

$$\xi(g \cdot x) \geq \int_0^\infty \frac{\mathrm{d}s}{g(x_s)} \geq \int_0^\infty \frac{\mathrm{d}s}{\|g\|} = \infty.$$

To prove the fourth point we suppose that $\xi(g \cdot x) < \infty$ and $\{g \cdot x_s\}_{s < \xi(g \cdot x)} \Subset S$. Then, $\{x_s\}_{s < \xi(x)} = \{g \cdot x_s\}_{s < \xi(g \cdot x)}$ so,

$$\infty > \xi(g \cdot x) = \int_0^{\xi(x)} \frac{\mathrm{d}s}{g(x_s)} \geq \frac{\xi(x)}{\|g\|_{\{x_s\}_{s < \xi(x)}}}.$$

Hence $\xi(x) < \infty$ and so, $g \cdot x_{\xi(g \cdot x)-} = x_{\xi(x)-}$ exists. □

Remark 3.4. It can be proved that if $g \in \mathrm{C}^{\neq 0}(S, \mathbb{R}_+)$ and $(\mathbf{P}_a)_{a \in S}$ is a strong Markov family, then $(g \cdot \mathbf{P}_a)_{a \in S}$ is a Markov family. Furthermore, if $(\mathbf{P}_a)_{a \in S}$ is a $\mathcal{F}_{t+}$-strong Markov family, then $(g \cdot \mathbf{P}_a)_{a \in S}$ is a $\mathcal{F}_{t+}$-strong Markov family. We will not use these properties here hence we skip the proofs of these statements. ◊

Another interesting fact is the following:

Theorem 3.5 (Continuity of the time change)**.** *For couples* $(g,x) \in \widetilde{\mathrm{C}}^{\neq 0}(S,\mathbb{R}_+) \times \mathbb{D}_{loc}(S)$ *consider the following two conditions:*

$$\tau^g_\infty(x) < \xi(x) \textit{ implies } \int_0^{\tau^g_\infty(x)+} \frac{\mathrm{d}s}{g(x_s)} = \infty, \tag{3.4}$$

and

$$A^g_{\tau^g_\infty(x)}(x) < \infty,\ x_{\tau^g_\infty(x)-} \textit{ exists in } S \textit{ and } g(x_{\tau^g_\infty(x)-}) = 0 \ \textit{ imply } \ x_{\tau^g_\infty(x)-} = x_{\tau^g_\infty(x)}. \tag{3.5}$$

Introduce the set

$$B_{\mathrm{tc}} := \left\{(g,x) \in \widetilde{\mathrm{C}}^{\neq 0}(S,\mathbb{R}_+) \times \mathbb{D}_{loc}(S) \,\middle|\, \textit{ conditions } (3.4) \textit{ and } (3.5) \textit{ hold } \right\}. \tag{3.6}$$

Then the time change

$$\begin{array}{ccc} \widetilde{\mathrm{C}}^{\neq 0}(S,\mathbb{R}_+) \times \mathbb{D}_{loc}(S) & \to & \mathbb{D}_{loc}(S) \\ (g,x) & \mapsto & g \cdot x \end{array}$$

is continuous on B_{tc} *when we endow respectively* $\widetilde{\mathrm{C}}^{\neq 0}(S,\mathbb{R}_+)$ *with the topology of uniform convergence on compact sets and* $\mathbb{D}_{loc}(S)$ *with the local Skorokhod topology. In particular*

$$\begin{array}{ccc} \mathrm{C}(S,\mathbb{R}_+^*) \times \mathbb{D}_{loc}(S) & \to & \mathbb{D}_{loc}(S) \\ (g,x) & \mapsto & g \cdot x \end{array}$$

is continuous for the topologies of uniform convergence on compact sets and local Skorokhod topology. Here and elsewhere we denote by $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$ *the set of positive real numbers.*

Remark 3.6. 1) It is not difficult to prove that B_{tc} is the continuity set.
2) If $(g,x) \in B_{\mathrm{tc}}$ and $h \in \widetilde{\mathrm{C}}^{\neq 0}(S,\mathbb{R}_+)$ is such that $\{h=0\} = \{g=0\}$ and $h \leq Cg$ for a constant $C \in \mathbb{R}_+$, then $(h,x) \in B_{\mathrm{tc}}$.
3) More generally, let B_0 be the set of $(g,x) \in \widetilde{\mathrm{C}}^{\neq 0}(S,\mathbb{R}_+) \times \mathbb{D}_{\mathrm{loc}}(S)$ such that

$$\begin{aligned} &\tau^g_\infty(x) < \infty \Rightarrow \forall t \geq 0,\ x_{\tau^g_\infty(x)+t} = x_{\tau^g_\infty(x)}, \\ &x_{\tau^g_\infty(x)-} \text{ exists in } S \text{ and } g(x_{\tau^g_\infty(x)-}) = 0 \Rightarrow x_{\tau^g_\infty(x)-} = x_{\tau^g_\infty(x)}. \end{aligned}$$

Then

$$\left\{(g, g\cdot x) \,\middle|\, (g,x) \in \widetilde{\mathrm{C}}^{\neq 0}(S,\mathbb{R}_+) \times \mathbb{D}_{\mathrm{loc}}(S)\right\} \subset B_0 \subset B_{\mathrm{tc}}. \tag{3.7}$$

$\Diamond$

To simplify the proof of the theorem we use a technical result containing a construction of a sequence of bi-Lipschitz bijections $(\lambda^k)_k$ useful when proving the convergence. Before stating this result let us note that, for any $(g,x) \in \widetilde{\mathrm{C}}^{\neq 0}(S,\mathbb{R}_+) \times \mathbb{D}_{\mathrm{loc}}(S)$ and any $t \leq \tau^g_\infty(x)$ such that $\{x_s\}_{s<t} \Subset \{g \neq 0\}$, by using (3.1), $A^g_t(x) < \infty$.

Lemma 3.7. *Take a metric* d *of* S*. Let* $x, x^k \in \mathbb{D}_{loc}(S)$ *and* $g, g_k \in \widetilde{\mathrm{C}}^{\neq 0}(S,\mathbb{R}_+)$ *be such that* (g_k, x^k) *converges to* (g,x)*, as* $k \to \infty$*. Let* $t \leq \tau^g_\infty(x)$ *be such that* $\{x_s\}_{s<t} \Subset \{g \neq 0\}$*. Then,*

i) there exists a sequence $(\lambda^k)_k \in \widetilde{\Lambda}^{\mathbb{N}}$ such that, for k large enough $\lambda^k_{A^g_t(x)} \leq \xi(g_k \cdot x^k)$ and

$$\|\log \dot{\lambda}^k\| \to 0, \qquad \sup_{v<A^g_t(x)} d(g\cdot x_v, g_k \cdot x^k_{\lambda^k_v}) \to 0, \quad g_k \cdot x^k_{\lambda^k_{A^g_t(x)}} \to g\cdot x_{A^g_t(x)}, \quad \text{as } k\to\infty.$$

ii) Moreover, if $\tau^g_\infty(x) < \xi(x)$ and $\int_0^{\tau^g_\infty(x)+} \frac{\mathrm{d}s}{g(x_s)} = \infty$, $(\lambda^k)_k$ may be chosen such that for any $v \geq 0$ and k large enough $\lambda^k_{A^g_t(x)+v} < \xi(g_k \cdot x^k)$ and

$$\limsup_{k\to\infty} \sup_{A^g_t(x)\leq w\leq A^g_t(x)+v} d\left(g_k \cdot x^k_{\lambda^k_w}, \{x_s\}_{t\leq s\leq \tau^g_\infty(x)}\right) = 0.$$

We postpone the proof of the lemma, and we give the proof of the continuity of time change:

Proof of Theorem 3.5. We remark first that

$$B_{\text{tc}} = B_1 \cup B_2 \cup B_3 \cup B_4,$$

with

$$B_1 := \left\{A^g_{\tau^g_\infty(x)}(x) = \infty \text{ or } \{x_s\}_{s<\tau^g_\infty(x)} \not\Subset S\right\},$$
$$B_2 := \left\{\tau^g_\infty(x) = \xi(x) < \infty \text{ and } \{x_s\}_{s<\tau^g_\infty(x)} \Subset \{g \neq 0\}\right\},$$
$$B_3 := \left\{\tau^g_\infty(x) < \xi(x),\ x_{\tau^g_\infty(x)-} = x_{\tau^g_\infty(x)},\ A^g_{\tau^g_\infty(x)}(x) < \infty \text{ and } \int_0^{\tau^g_\infty(x)^+} \frac{\mathrm{d}s}{g(x_s)} = \infty\right\},$$
$$B_4 := \left\{\tau^g_\infty(x) < \xi(x),\ g(x_{\tau^g_\infty(x)-}) \neq 0 \text{ and } \int_0^{\tau^g_\infty(x)^+} \frac{\mathrm{d}s}{g(x_s)} = \infty\right\}.$$

Let $x, x^k \in \mathbb{D}_{\text{loc}}(S)$ and $g, g_k \in \widetilde{\mathrm{C}}^{\neq 0}(S, \mathbb{R}_+)$ be such that (g_k, x^k) converge to (g, x) and $(g, x) \in B$. We need to prove that

$$g_k \cdot x^k \xrightarrow[k\to\infty]{\mathbb{D}_{\text{loc}}(S)} g\cdot x, \tag{3.8}$$

and we will decompose the proof with respect to values of i such that $(g, x) \in B_i$.

- If $(g, x) \in B_1$, we use the first part of Lemma 3.7 for all $t < \tau^g_\infty(x)$. We obtain that $A^g_t(x) < \xi(g\cdot x)$. Since $A^g_t(x)$ tends to $\xi(g\cdot x)$, when t tends to $\tau^g_\infty(x)$, by a diagonal extraction procedure we deduce (3.8).
- If $(g, x) \in B_2$, it suffices to apply the first part of Lemma 3.7 to $t := \xi(x)$ and $A^g_t(x) = \xi(g\cdot x)$.
- If $(g, x) \in B_3$, let $t < \tau^g_\infty(x)$ be. Then, by the third part of Lemma 3.7 there exists $\lambda^k \in \widetilde{\Lambda}$ such that, for any $v \geq 0$, for k large enough, $\lambda^k_{A^g_t(x)+v} < \xi(g_k \cdot x^k)$ and

  $$\|\log \dot{\lambda}^k\| \underset{k\to\infty}{\longrightarrow} 0, \quad \limsup_{k\to\infty} \sup_{w\leq A^g_t(x)+v} d(g\cdot x_w, g_k\cdot x^k_{\lambda^k_w}) \leq 2d\left(x_{\tau^g_\infty(x)}, \{x_s\}_{t\leq s\leq\tau^g_\infty(x)}\right).$$

  Since x is continuous at $\tau^g_\infty(x)$, we conclude by a diagonal extraction procedure, by letting t tends to $\tau^g_\infty(x)$ and $v \to \infty$.

- If $(g,x) \in B_4$, let $t = \tau^g_\infty(x)$ be. By the second part of Lemma 3.7 there exists $\lambda^k \in \widetilde{\Lambda}$ such that, for any $v \geq 0$, for k large enough $\lambda^k_{A^g_t(x)+v} < \xi(g_k \cdot x^k)$, and

$$\| \log \dot{\lambda}^k \| \underset{k\to\infty}{\longrightarrow} 0, \qquad \sup_{w \leq A^g_t(x)+v} d(g \cdot x_w, g_k \cdot x^k_{\lambda^k_w}) \underset{k\to\infty}{\longrightarrow} 0.$$

We conclude by a diagonal extraction procedure and letting $v \to \infty$.

□

Proof of Lemma 3.7. Let $\widetilde{\lambda}^k \in \widetilde{\Lambda}$ be as in Theorem 2.4 and to simplify notations define, for $s \geq 0$

$$\tau_s := \tau^g_s(x), \qquad A_s := A^g_s(x),$$
$$\tau^k_s := \tau^{g_k}_s(x^k), \qquad A^k_s := A^{g_k}_s(x^k),$$

and $u := A_t$. Since $\tau_u = t \leq \xi(x)$ and $\{x_s\}_{s<t} \Subset S$ we have, for k large enough $\widetilde{\lambda}^k_t \leq \xi(x^k)$, and $\| \log \dot{\widetilde{\lambda}}^k \|_t \longrightarrow 0$, $\sup_{s<t} d(x_s, x^k_{\widetilde{\lambda}^k_s}) \longrightarrow 0$ and $x^k_{\widetilde{\lambda}^k_t} \longrightarrow x_t$, as $k \to \infty$. Since $\{x_s\}_{s<t} \Subset \{g \neq 0\}$, we deduce that for k large enough $\{x^k_s\}_{s<\widetilde{\lambda}^k_t} \Subset \{g_k \neq 0\}$. Define then $\lambda^k \in \widetilde{\Lambda}$ by

$$\begin{cases} \lambda^k_v := A^k_{\widetilde{\lambda}^k_{\tau_v}} = \displaystyle\int_0^v \frac{g(x_{\tau_w})}{g_k(x^k_{\widetilde{\lambda}^k_{\tau_w}})} \dot{\widetilde{\lambda}}^k_{\tau_w} \mathrm{d}w & \text{if } v \leq u, \\ \dot{\lambda}^k_v = 1 & \text{if } v > u. \end{cases}$$

Since $\widetilde{\lambda}^k_t \leq \tau^k_\infty$ we have

$$\lambda^k_u \leq A^k_{\tau^k_\infty} \leq \xi(g_k \cdot x^k),$$

now we obtain

$$\sup_{v<u} d(g \cdot x_v, g_k \cdot x^k_{\lambda^k_v}) = \sup_{v<u} d(x_{\tau_v}, x^k_{\widetilde{\lambda}^k_{\tau_v}}) = \sup_{s<t} d(x_s, x^k_{\widetilde{\lambda}^k_s}) \underset{k\to\infty}{\longrightarrow} 0,$$
$$g_k \cdot x^k_{\lambda^k_u} = x^k_{\widetilde{\lambda}^k_t} \underset{k\to\infty}{\longrightarrow} x_t = g \cdot x_u$$
$$\| \log \dot{\lambda}^k \| = \operatorname{esssup}_{v \leq u} \left| \log \frac{\dot{\widetilde{\lambda}}^k_{\tau_v} g(x_{\tau_v})}{g_k(x^k_{\widetilde{\lambda}^k_{\tau_v}})} \right| = \operatorname{esssup}_{s \leq \tau_u} \left| \log \frac{\dot{\widetilde{\lambda}}^k_s g(x_s)}{g_k(x^k_{\widetilde{\lambda}^k_s})} \right| \underset{k\to\infty}{\longrightarrow} 0.$$

For the second part of the proposition we keep the same construction as previously. For any $v \geq 0$ we have that

$$\tau^k_{\lambda^k_{u+v}} = \inf \left\{ t \geq \lambda^k_u \,\middle|\, \int_{\lambda^k_u}^t \frac{\mathrm{d}s}{g_k(x^k_s)} \geq v \right\} \wedge \tau^k_\infty.$$

Using Fatou's lemma

$$\liminf_{k\to\infty} \int_{\lambda^k_u}^{(\widetilde{\lambda}^k_{\tau_\infty})^+} \frac{\mathrm{d}s}{g_k(x^k_s)} = \liminf_{k\to\infty} \int_t^{\tau_\infty +} \frac{\dot{\widetilde{\lambda}}^k_s \mathrm{d}s}{g_k(x^k_{\lambda^k_s})} \geq \int_t^{\tau_\infty +} \frac{\mathrm{d}s}{g(x_s)} = \infty$$

so, $\limsup_{k\to\infty} \tau^k_{\lambda^k_{u+v}} - \widetilde{\lambda}^k_{\tau_\infty} \leq 0$. Moreover, for k large enough, $\tau^k_{\lambda^k_{u+v}} \geq \tau^k_{\lambda^k_u} = \widetilde{\lambda}^k_t$, so, $\lambda^k_{u+v} < \xi(g_k \cdot x^k)$ and

$$\limsup_{k\to\infty} \sup_{u\leq w\leq u+v} d\left(g_k \cdot x^k_{\lambda^k_w}, \{x_s\}_{t\leq s\leq \tau_\infty}\right) = 0.$$

□

3.2 Connection between local and global Skorokhod topologies

Generally to take into account the explosion, one considers processes in $\mathbb{D}(S^\Delta)$, the set of cadlag processes described in Definition 2.1, associated to the space S^Δ, and endowed with the global Skorokhod topology (see Definition 2.5). More precisely, the set of cadlag paths with values in S^Δ is given by

$$\mathbb{D}(S^\Delta) = \left\{ x \in (S^\Delta)^{\mathbb{R}_+} \;\middle|\; \begin{array}{l} \forall t \geq 0,\ x_t = \lim_{s\downarrow t} x_s, \ \text{ and} \\ \forall t > 0,\ x_{t-} := \lim_{s\uparrow t} x_s \ \text{ exists in } S^\Delta \end{array} \right\}.$$

A sequence $(x^k)_k$ in $\mathbb{D}(S^\Delta)$ converges to x for the global Skorokhod topology if and only if there exists a sequence $(\lambda^k)_k$ of increasing homeomorphisms on $\mathbb{R}_+$ such that

$$\forall t \geq 0, \quad \lim_{k\to\infty} \sup_{s\leq t} d(x_s, x^k_{\lambda^k_s}) = 0 \quad \text{and} \quad \lim_{k\to\infty} \|\lambda^k - \mathrm{id}\|_t = 0.$$

In this section we give the connection between $\mathbb{D}(S^\Delta)$ with the global Skorokhod topology and $\mathbb{D}_{\mathrm{loc}}(S)$ with the local Skorokhod topology.

We first identify these two measurable subspaces

$$\begin{aligned} \mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta) &= \left\{ x \in \mathbb{D}_{\mathrm{loc}}(S) \;\middle|\; 0 < \xi(x) < \infty \Rightarrow x_{\xi(x)-} \text{ exist in } S^\Delta \right\} \\ &= \left\{ x \in \mathbb{D}(S^\Delta) \;\middle|\; \forall t \geq \tau^S,\ x_t = \Delta \right\}. \end{aligned}$$

We can summarise our trajectories spaces by

$$\begin{array}{ccccc} \mathbb{D}(S) \subset & \mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta) & \subset \mathbb{D}_{\mathrm{loc}}(S) \subset \mathbb{D}_{\exp}(S). \\ & \cap & \\ & \mathbb{D}(S^\Delta) & \end{array}$$

Hence $\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)$ will be endowed with two topologies, the local topology from $\mathbb{D}_{\mathrm{loc}}(S)$ and the global topology from $\mathbb{D}(S^\Delta)$.

Remark 3.8. 1) On $\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)$ the trace topology from $\mathbb{D}_{\mathrm{loc}}(S)$ is weaker than the trace topology from $\mathbb{D}(S^\Delta)$. Eventually, these two topologies coincide on $\mathbb{D}(S)$. Indeed, this is clear using a metric d on S^Δ and the characterisations of topologies given in Remark 2.6. The result in Corollary 3.10 below is a converse sentence of the present remark.
2) If $x \in \mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)$ then $g \cdot x$ is well-defined in $\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)$ for

$$g \in \mathrm{C}_b(S, \mathbb{R}_+^*) \subset \left\{ g \in \mathrm{C}^{\neq 0}(S^\Delta, \mathbb{R}_+) \,|\, g(\Delta) = 0 \right\}.$$

We deduce from Theorem 3.5 and the third point of Remark 3.6 that the mapping

$$\begin{array}{ccc} \mathrm{C}_b(S, \mathbb{R}_+^*) \times \mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta) & \to & \mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta) \\ (g, x) & \mapsto & g \cdot x \end{array}$$

is continuous between the topology of the uniform convergence and the global Skorokhod topology. ♢

The following result is stated in a very general form because it will be useful when studying, for instance, the martingale problems.

Proposition 3.9 (Connection between $\mathbb{D}_{\text{loc}}(S)$ and $\mathbb{D}(S^\Delta)$)**.** *Let E be an arbitrary locally compact Hausdorff space with countable base and consider*

$$\begin{array}{rccc} \mathbf{P}: & E & \to & \mathcal{P}(\mathbb{D}_{loc}(S)) \\ & a & \mapsto & \mathbf{P}_a \end{array}$$

a weakly continuous mapping for the local Skorokhod topology. Then for any open subset U of S, there exists $g \in \mathrm{C}(S, \mathbb{R}_+)$ such that $\{g \neq 0\} = U$, for all $a \in E$

$$g \cdot \mathbf{P}_a \left(0 < \xi < \infty \Rightarrow X_{\xi-} \textit{ exists in } U\right) = 1,$$

and the application

$$\begin{array}{rccc} g \cdot \mathbf{P}: & E & \to & \mathcal{P}(\{0 < \xi < \infty \Rightarrow X_{\xi-} \textit{ exists in } U\}) \\ & a & \mapsto & g \cdot \mathbf{P}_a \end{array}$$

is weakly continuous for the global Skorokhod topology from $\mathbb{D}(S^\Delta)$.

Before giving the proof of Proposition 3.9 we point out a direct application: we take $E := \mathbb{N} \cup \{\infty\}$, $U = S$ and a sequence of Dirac probability measures $\mathbf{P}_k = \delta_{x^k}$, $\mathbf{P}_\infty = \delta_x$. Then we deduce from Proposition 3.9 the following:

Corollary 3.10 (Another description of $\mathbb{D}_{\text{loc}}(S)$)**.** *Let $x, x^1, x^2, \ldots \in \mathbb{D}_{loc}(S)$. Then the sequence x^k converges to x in $\mathbb{D}_{loc}(S)$, as $k \to \infty$, if and only if there exists $g \in \mathrm{C}(S, \mathbb{R}_+^*)$ such that $g \cdot x, g \cdot x^1, g \cdot x^2, \ldots \in \mathbb{D}(S^\Delta)$, and $g \cdot x^k$ converges to $g \cdot x$ in $\mathbb{D}(S^\Delta)$, as $k \to \infty$.*

We proceed with the proof of Proposition 3.9 and, firstly we state a important result which will be our main tool:

Lemma 3.11. *Let D be a compact subset of $\mathbb{D}_{loc}(S)$ and U be an open subset of S. There exists $g \in \mathrm{C}(S, \mathbb{R}_+)$ such that:*

i) $\{g \neq 0\} = U$,

ii) for all $x \in D$, (g, x) is in the set B_{tc} given by (3.6) *in Theorem 3.5 and*

$$g \cdot x \in \left\{0 < \xi < \infty \Rightarrow X_{\xi-} \textit{ exists in } U\right\}.$$

iii) the trace topologies of $\mathbb{D}_{loc}(S)$ and $\mathbb{D}(S^\Delta)$ coincide on $\{g \cdot x \,|\, x \in D\}$.

Furthermore, if $g \in \mathrm{C}(S, \mathbb{R}_+)$ satisfies i)-iii) and if $h \in \mathrm{C}(S, \mathbb{R}_+)$ is such that $\{h \neq 0\} = U$ and $h \leq Cg$ with a non-negative constant C, then h also satisfies i)-iii).

Proof of Proposition 3.9. Let $(\widetilde{K}_n)_{n \in \mathbb{N}^*}$ be an increasing sequence of compact subset of E such that $E = \bigcup_n \widetilde{K}_n$, then $\{\mathbf{P}_a\}_{a \in \widetilde{K}_n}$ is tight, for all $n \in \mathbb{N}^*$. So, there exist subsets $D_n \subset \mathbb{D}_{\text{loc}}(S)$ which are compacts for the topology of $\mathbb{D}_{\text{loc}}(S)$, and such that

$$\sup_{a \in \widetilde{K}_n} \mathbf{P}_a(D_n^c) \leq \frac{1}{n}.$$

For any $n \in \mathbb{N}^*$, consider g_n satisfying i)-iii) of Lemma 3.11 associated to the compact set D_n. It is no difficult to see that there exists $g \in \mathrm{C}(S, \mathbb{R}_+)$ such that $\{g \neq 0\} = U$ and for all $n \in \mathbb{N}^*$, $g \leq C_n g_n$ for non-negative constants C_n. Hence, g satisfies i)-iii) for all D_n, $n \in \mathbb{N}^*$. Hence, for all $a \in E$

$$g \cdot \mathbf{P}_a \left(0 < \xi < \infty \Rightarrow X_{\xi-} \text{ exists in } U\right) \geq \mathbf{P}_a\Big(\bigcup_{n \in \mathbb{N}^*} D_n\Big) = 1.$$

Let $a_k, a \in E$ such that $a_k \underset{k\to\infty}{\longrightarrow} a$. For n large enough $\{a_k\}_k \subset \widetilde{K}_n$. Then, if F is a subset of $\{0 < \xi < \infty \Rightarrow X_{\xi-} \text{ exists in } U\}$ which is closed for the topology of $\mathbb{D}(S^\Delta)$,

$$\begin{aligned}\limsup_{k\to\infty} & \, g \cdot \mathbf{P}_{a_k}(F) - g \cdot \mathbf{P}_a(F) \\ & \leq \limsup_{k\to\infty} \mathbf{P}_{a_k}(X \in D_n,\ g \cdot X \in F) - \mathbf{P}_a(X \in D_n,\ g \cdot X \in F) + \frac{1}{n}.\end{aligned}$$

But thanks to iii) in Lemma 3.11, $\{X \in D_n,\ g \cdot X \in F\}$ is a subset of $\mathbb{D}_{\mathrm{loc}}(S)$ which is closed for the topology of $\mathbb{D}_{\mathrm{loc}}(S)$. Hence, by using the Portmanteau theorem (see for instance Theorem 2.1 from [Bil99], p. 16)

$$\limsup_{k\to\infty} \mathbf{P}_{a_k}(X \in D_n,\ g \cdot X \in F) \leq \mathbf{P}_a(X \in D_n,\ g \cdot X \in F)$$

and so, letting $n \to \infty$,

$$\limsup_{k\to\infty} g \cdot \mathbf{P}_{a_k}(F) \leq g \cdot \mathbf{P}_a(F).$$

By using the Portmanteau theorem, the proof of the proposition is complete, except for the proof of Lemma 3.11. $\square$

Proof of Lemma 3.11. Let d be a metric on S^Δ and denote

$$K_n := \left\{a \in S \;\middle|\; d(a, S^\Delta \backslash U) \geq 2^{-n}\right\}.$$

By using Theorem 2.8,there exists a sequence $(\eta_n)_n \in (0,1)^{\mathbb{N}}$ decreasing to 0 such that

$$\sup_{x \in D} \omega'_{2^n, B(\Delta, 2^{-n-2})^c, x}(\eta_n) < 2^{-n-2}. \tag{3.9}$$

Moreover, there exists $g \in \mathrm{C}(S^\Delta, [0,1])$ such that $\{g \neq 0\} = U$ and $g_{|K_n^c} \leq 2^{-n}\eta_n$. For instance, we can define

$$g(a) := \sup_{n \geq 0} \Big((2^{-n}\eta_n) \wedge d(a, S^\Delta \backslash K_n)\Big),\ a \in S^\Delta.$$

Let $x \in D$ be. We consider the following two situations:

- If $\tau^g_\infty(x) < \infty$ and $\{x_s\}_{s < \tau^g_\infty(x)}$ is not a compact of U, take $m \in \mathbb{N}$ such that $2^m \geq \tau^g_\infty(x)$, denote

$$t := \min\left\{s \geq 0 \;\middle|\; x_s \notin \mathring{K}_{m+1}\right\} < \tau^g_\infty(x)$$

and let $n \geq m$ be such that $x_t \in K_{n+2} \backslash \overset{\circ}{K}_{n+1}$. Using (3.9) there exist $t_1, t_2 \in \mathbb{R}_+$ such that $t_1 \leq t < t_2 < \tau^g_\infty(x)$, $t_2 - t_1 > \eta_n$ and $x_s \notin K_n$ for all $s \in [t_1, t_2)$. So,

$$A^g_{\tau^g_\infty(x)}(x) \geq \int_{t_1}^{t_2} \frac{\mathrm{d}s}{g(x_s)} \geq 2^m,$$

hence, letting m goes to infinity,

$$A^g_{\tau^g_\infty(x)}(x) = \infty.$$

- If $\tau^g_\infty(x) < \xi(x)$ and $g(x_{\tau^g_\infty(x)-}) \neq 0$, then $g(x_{\tau^g_\infty(x)}) = 0$. Let $m \in \mathbb{N}$ be such that $2^m \geq \tau^g_\infty(x)$ and $\{x_s\}_{s \leq \tau^g_\infty(x)} \subset B(\Delta, 2^{-m-2})^c$. Using (3.9), there exist $t_1, t_2 \in \mathbb{R}_+$ such that $t_1 \leq \tau^g_\infty(x) < t_2 < \xi(x)$, $t_2 - t_1 > \eta_m$ and $x_s \notin K_m$ for all $s \in [t_1, t_2)$. So

$$\int_0^{\tau^g_\infty(x)+\eta_m} \frac{\mathrm{d}s}{g(x_s)} \geq \int_{t_1}^{t_1+\eta_m} \frac{\mathrm{d}s}{g(x_s)} \geq 2^m$$

hence letting m tend to infinity

$$\int_0^{\tau^g_\infty(x)+} \frac{\mathrm{d}s}{g(x_s)} = \infty.$$

Hence, we obtain that $(g, x) \in B_{\text{tc}}$ and $g \cdot x \in \{0 < \xi < \infty \Rightarrow X_{\xi-} \text{ exists in } U\}$ and ii) is verified.

We proceed by proving iii). Thanks to Remark 3.8, to get the equivalence of the topologies it is enough to prove that if $x^k, x \in D$ are such that $g \cdot x^k \to g \cdot x$ for the topology from $\mathbb{D}_{\text{loc}}(S)$ and $\xi(g \cdot x) < \infty$, then the convergence also holds for the topology from $\mathbb{D}(S^\Delta)$. Let $\lambda^k \in \widetilde{\Lambda}$ be such that

$$\sup_{s \leq \xi(g \cdot x)} d(g \cdot x_s, g \cdot x^k_{\lambda^k_s}) \longrightarrow 0, \qquad \|\log \dot{\lambda}^k\|_{\xi(g \cdot x)} \longrightarrow 0, \quad \text{as } k \to \infty.$$

We may suppose that $\dot{\lambda}^k_s = 0$, for $s \geq \xi(g \cdot x)$. Denote $t_k := \lambda^k_{\xi(g \cdot x)}$ and choose $m \in \mathbb{N}$ be such that $\{g \cdot x_s\}_{s < \xi(g \cdot x)} \Subset \overset{\circ}{K}_m$ and $\xi(g \cdot x) < 2^m$. Then, for k large enough $\{g \cdot x^k_s\}_{s < t_k} \Subset \overset{\circ}{K}_m$, $g \cdot x^k_{t_k} \notin K_{m+1}$ and $t_k < 2^m$.

- Either $g \cdot x^k_{t_k} \notin U$ and so $g \cdot x^k_{\lambda^k_s} = g \cdot x^k_{t^k}$ for all $s \geq \xi(g \cdot x)$.

- Or $g \cdot x^k_{t_k} \in U$ and let $n \geq m$ be such that $g \cdot x^k_{t_k} = x_{\tau^g_{t_k}(x^k)} \in K_{n+2} \backslash \overset{\circ}{K}_{n+1}$. Using (3.9), $d(x_s, x^k_{t_k}) < 2^{-n-2}$ and so $x_s \in U \backslash K_n$ for all $s \in [\tau^g_{t_k}(x^k), \tau^g_{t_k}(x^k) + \eta_n]$. Hence, $A^g_{\tau^g_{t_k}(x^k)+\eta_n} \geq t_k + 2^n$, so $d(g \cdot x_s, g \cdot x^k_{t_k}) < 2^{-n-2}$ for all $s \in [t_k, t_k + 2^n]$.

Hence we obtain that, for k large enough,

$$\sup_{s \leq \xi(g \cdot x) + 2^m} d(g \cdot x_s, g \cdot x^k_{\lambda^k_s}) \leq \sup_{s \leq \xi(g \cdot x)} d(g \cdot x_s, g \cdot x^k_{\lambda^k_s}) + 2^{-m-2},$$

so letting m goes to the infinity we obtain that $g \cdot x^k$ converge to $g \cdot x$ for the global Skorokhod topology from $\mathbb{D}(S^\Delta)$. Hence, the proof of iii) is done.

Finally, to prove the last part of the lemma let $g \in \mathrm{C}(S, \mathbb{R}_+)$ be such that i)-iii) are satisfied and let $h \in \mathrm{C}(S, \mathbb{R}_+)$ be such that $\{h \neq 0\} = U$ and $h \leq Cg$ with a non-negative constant C. Thanks to Remark 3.6, (h, x) belongs to the set B_{tc} given by (3.6), it is also clear that $h \cdot x \in \{0 < \xi < \infty \Rightarrow X_{\xi-} \text{ exists in } U\}$. We have that $\frac{h}{g} \in \mathrm{C}_b(U, \mathbb{R}_+^*)$, so using (3.7) for S and S^Δ, the bijection

$$\begin{array}{ccc} \{g \cdot x \,|\, x \in D\} & \to & \{h \cdot x \,|\, x \in D\} \\ x & \mapsto & \frac{h}{g} \cdot x \end{array}$$

is continuous for the topology of $\mathbb{D}_{\mathrm{loc}}(S)$, but also of $\mathbb{D}(S^\Delta)$. But since $\{g \cdot x \,|\, x \in D\}$ is compact, this application is bi-continuous, and we obtain the result. Now the proof of lemma is complete. ☐

4 Proofs of main results on local Skorokhod metrics

In this section we will prove Theorem 2.4 and Theorem 2.8, by following the strategy developed in §12, pp. 121-137 from [Bil99]. To construct metrics on $\mathbb{D}_{\mathrm{loc}}(S)$, we will consider a metric d on S. To begin with, we define two families of pseudo-metrics:

Lemma 4.1 (Skorokhod metrics)**.** *For $0 \leq t < \infty$ and $K \subset S$ a compact subset, the following two expressions on $\mathbb{D}_{exp}(S)$:*

$$\rho_{t,K}(x^1, x^2) := \inf_{\substack{t_i \leq \xi(x^i) \\ \lambda \in \Lambda, \lambda_{t_1} = t_2}} \sup_{s < t_1} d(x^1_s, x^2_{\lambda_s}) \vee \|\lambda - \mathrm{id}\|_{t_1} \vee \max_{i \in \{1,2\}} \Big(d(x^i_{t_i}, K^c) \wedge (t - t_i)_+ \mathbb{1}_{t_i < \xi(x^i)} \Big),$$

$$\widetilde{\rho}_{t,K}(x^1, x^2) := \inf_{\substack{t_i \leq \xi(x^i) \\ \lambda \in \widetilde{\Lambda}, \lambda_{t_1} = t_2}} \sup_{s < t_1} d(x^1_s, x^2_{\lambda_s}) \vee \|\log \dot{\lambda}\|_{t_1} \vee \|\lambda - \mathrm{id}\|_{t_1} \vee \max_{i \in \{1,2\}} \Big(d(x^i_{t_i}, K^c) \wedge (t - t_i)_+ \mathbb{1}_{t_i < \xi(x^i)} \Big).$$

define two pseudo-metrics.

Proof. Let us perform the proof for $\widetilde{\rho}_{t,K}$, the proof being similar for $\rho_{t,K}$. The non-trivial part is the triangle inequality. Let $x^1, x^2, x^3 \in \mathbb{D}_{\mathrm{exp}}(S)$ and $\varepsilon > 0$ be then there are $t_1 \leq \xi(x^1)$, $t_2, \widehat{t_2} \leq \xi(x^2)$, $\widehat{t_3} \leq \xi(x^3)$ and $\lambda^1 \in \widetilde{\Lambda}$, $\lambda^2 \in \widetilde{\Lambda}$ such that

$$\widetilde{\rho}_{t,K}(x^1, x^2) + \varepsilon \geq \sup_{s < t_1} d(x^1_s, x^2_{\lambda^1_s}) \vee \|\log \dot{\lambda}^1\|_{t_1} \vee \|\lambda^1 - \mathrm{id}\|_{t_1} \vee \max_{i \in \{1,2\}} \Big(d(x^i_{t_i}, K^c) \wedge (t - t_i)_+ \mathbb{1}_{t_i < \xi(x^i)} \Big),$$

$$\widetilde{\rho}_{t,K}(x^2, x^3) + \varepsilon \geq \sup_{s < \widehat{t_2}} d(x^2_s, x^3_{\lambda^2_s}) \vee \|\log \dot{\lambda}^2\|_{\widehat{t_2}} \vee \|\lambda^2 - \mathrm{id}\|_{\widehat{t_2}}$$

$$\vee \max_{i\in\{2,3\}} \Big(d(x^i_{\widehat{t}_i}, K^c) \wedge (t-\widehat{t}_i)_+ \mathbb{1}_{\widehat{t}_i<\xi(x^i)} \Big).$$

Define $\breve{t}_2 := t_2 \wedge \widehat{t}_2$, $\breve{t}_1 := (\lambda^1)^{-1}_{\breve{t}_2}$, $\breve{t}_3 := \lambda^2_{\breve{t}_2}$ and $\lambda := \lambda^2 \circ \lambda^1$. Then

$$\begin{aligned}
&\sup_{s<\breve{t}_1} d(x^1_s, x^3_{\lambda_s}) \le \sup_{s<t_1} d(x^1_s, x^2_{\lambda^1_s}) + \sup_{s<\widehat{t}_2} d(x^2_s, x^3_{\lambda^2_s}),\\
&\|\log \dot\lambda\|_{\breve{t}_1} \le \|\log \dot\lambda^1\|_{t_1} + \|\log \dot\lambda^2\|_{\widehat{t}_2}, \quad \|\lambda - \mathrm{id}\|_{\breve{t}_1} \le \|\lambda^1 - \mathrm{id}\|_{t_1} + \|\lambda^2 - \mathrm{id}\|_{\widehat{t}_2}.
\end{aligned}$$

Moreover, for instance, if $\breve{t}_1 \neq t_1$, then $\breve{t}_1 < t_1 \le \xi(x_1)$, $\widehat{t}_2 = \breve{t}_2 < t_2 \le \xi(x_2)$ and

$$\begin{aligned}
d(x^1_{\breve{t}_1}, K^c) \wedge (t-\breve{t}_1)_+ &\le d(x^1_{\breve{t}_1}, x^2_{\breve{t}_2}) \vee |\breve{t}_2 - \breve{t}_1| + d(x^2_{\breve{t}_2}, K^c) \wedge (t-\breve{t}_2)_+\\
&\le \sup_{s<t_1} d(x^1_s, x^2_{\lambda^1_s}) \vee \|\lambda^1 - \mathrm{id}\|_{t_1} + d(x^2_{\widehat{t}_2}, K^c) \wedge (t-\widehat{t}_2)_+.
\end{aligned}$$

Hence

$$\widetilde{\rho}_{t,K}(x^1, x^3) \le \widetilde{\rho}_{t,K}(x^1, x^2) + \widetilde{\rho}_{t,K}(x^2, x^3) + 2\varepsilon,$$

so letting $\varepsilon \to 0$, we obtain the triangular inequality. □

We prove that these pseudo-metrics are in somehow equivalent:

Lemma 4.2. *Take $x, y \in \mathbb{D}_{loc}(S)$, $t \ge 0$ and a compact subset $K \subset S$, if $\rho_{t,K}(x,y) \le \frac{1}{9}$ then*

$$\widetilde{\rho}_{t,K}(x,y) \le 6 \cdot \sqrt{\rho_{t,K}(x,y)} \vee \omega'_{t,K,x}\left(\sqrt{\rho_{t,K}(x,y)}\right).$$

Proof. Let $\varepsilon > 0$ be arbitrary. There exist $\mu \in \Lambda$ and $T \ge 0$ such that $T \le \xi(x)$, $\mu_T \le \xi(y)$ and

$$\begin{aligned}
\sup_{s<T} d(x_s, y_{\mu_s}) \vee \|\mu - \mathrm{id}\|_T &\le \rho_{t,K}(x,y) + \varepsilon,\\
d(x_T, K^c) \wedge (t-T)_+ \mathbb{1}_{T<\xi(x)} &\le \rho_{t,K}(x,y) + \varepsilon,\\
d(y_{\mu_T}, K^c) \wedge (t-\mu_T)_+ \mathbb{1}_{\mu_T<\xi(y)} &\le \rho_{t,K}(x,y) + \varepsilon.
\end{aligned}$$

Let $\delta > 2\rho_{t,K}(x,y) + 2\varepsilon$ be arbitrary, there exist $0 = t_0 < \cdots < t_N \le \xi(x)$ such that

$$\sup_{\substack{0\le i<N\\ t_i \le s_1, s_2 < t_{i+1}}} d(x_{s_1}, x_{s_2}) \le \omega'_{t,K,x}(\delta) + \varepsilon,$$

$\delta < t_{i+1} - t_i \le 2\delta$ and $(t_N, x_{t_N}) \notin [0,t] \times K$. Set $n_0 := \max\{0 \le i \le N \mid t_i \le T\}$ and $\widetilde{T} := t_{n_0}$. Define $\lambda \in \widetilde{\Lambda}$ by

$$\begin{cases} \forall i \le n_0, & \lambda_{t_i} = \mu_{t_i},\\ \forall i < n_0, & \lambda \text{ is affine on } [t_i, t_{i+1}],\\ \forall s \ge \widetilde{T}, & \dot\lambda_s = 1. \end{cases}$$

Then

$$\|\lambda - \mathrm{id}\| = \sup_{0<i\le n_0} \|\mu_{t_i} - t_i\| \le \|\mu - \mathrm{id}\|_T \le \rho_{t,K}(x,y) + \varepsilon.$$

For $0 \le i < n_0$ we have

$$\left|\frac{\mu_{t_{i+1}} - \mu_{t_i} - t_{i+1} + t_i}{t_{i+1} - t_i}\right| \le \frac{2\|\mu - \mathrm{id}\|_T}{\delta} \le \frac{2\rho_{t,K}(x,y) + 2\varepsilon}{\delta} < 1,$$

so, by the classical estimate:

$$|\log(1+r)| \le \frac{|r|}{1-|r|} \quad \text{for } |r| < 1.$$

we deduce

$$\|\log \dot{\lambda}\| = \sup_{0 \le i < n_0} \left|\log \frac{\mu_{t_{i+1}} - \mu_{t_i}}{t_{i+1} - t_i}\right| \le \frac{2\rho_{t,K}(x,y) + 2\varepsilon}{\delta - 2\rho_{t,K}(x,y) - 2\varepsilon}.$$

Since for $s < \lambda_{\widetilde{T}}$, λ_s^{-1} and μ_s^{-1} lies in the same interval $[t_i, t_{i+1})$. Therefore,

$$\sup_{s < \lambda_{\widetilde{T}}} d(x_{\lambda_s^{-1}}, y_s) \le \sup_{s < \lambda_{\widetilde{T}}} \Big(d(x_{\mu_s^{-1}}, y_s) + d(x_{\mu_s^{-1}}, x_{\lambda_s^{-1}}) \Big) \le \rho_{t,K}(x,y) + \omega'_{t,K,x}(\delta) + 2\varepsilon.$$

For the two last terms in $\widetilde{\rho}_{t,K}$ we may consider only the case were $\widetilde{T} \ne T$. If $n_0 = N$: $d(x_{\widetilde{T}}, K^c) \wedge (t - \widetilde{T})_+ = 0$, otherwise:

$$\begin{aligned} d(x_{\widetilde{T}}, K^c) \wedge (t - \widetilde{T})_+ &\le d(x_T, K^c) \wedge (t - T)_+ + d(x_{\widetilde{T}}, x_T) \vee (T - \widetilde{T}) \\ &\le \rho_{t,K}(x,y) + \omega'_{t,K,x}(\delta) \vee (2\delta) + 2\varepsilon. \end{aligned}$$

By using $\lambda_{\widetilde{T}} = \mu_{\widetilde{T}}$, we also have

$$\begin{aligned} d(y_{\lambda_{\widetilde{T}}}, K^c) \wedge (t - \lambda_{\widetilde{T}})_+ &\le d(x_{\widetilde{T}}, K^c) \wedge (t - \widetilde{T})_+ + d(x_{\widetilde{T}}, y_{\mu_{\widetilde{T}}}) \vee |\widetilde{T} - \mu_{\widetilde{T}}| \\ &\le 2\rho_{t,K}(x,y) + \omega'_{t,K,x}(\delta) \vee (2\delta) + 3\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain that for all $\delta > 2\rho_{t,K}(x,y)$,

$$\widetilde{\rho}_{t,K}(x,y) \le \big(2\rho_{t,K}(x,y) + \omega'_{t,K,x}(\delta) \vee (2\delta)\big) \vee \frac{2\rho_{t,K}(x,y)}{\delta - 2\rho_{t,K}(x,y)}.$$

Finally, by taking $\delta := \sqrt{\rho_{t,K}(x,y)}$ we have for $\rho_{t,K}(x,y) \le \frac{1}{9}$

$$\begin{aligned} \widetilde{\rho}_{t,K}(x,y) &\le \Big(2\rho_{t,K}(x,y) + \omega'_{t,K,x}(\delta) \vee (2\delta)\Big) \vee \frac{2\rho_{t,K}(x,y)}{\delta - 2\rho_{t,K}(x,y)} \\ &\le \Big(\frac{2}{3}\sqrt{\rho_{t,K}(x,y)} + \omega'_{t,K,x}\big(\sqrt{\rho_{t,K}(x,y)}\big) \vee \big(2\sqrt{\rho_{t,K}(x,y)}\big)\Big) \vee 6\sqrt{\rho_{t,K}(x,y)} \\ &\le 6 \cdot \sqrt{\rho_{t,K}(x,y)} \vee \omega'_{t,K,x}\big(\sqrt{\rho_{t,K}(x,y)}\big). \end{aligned}$$

□

At this level it can be pointed out that we obtain the definition of the local Skorokhod topology. Indeed, by using Proposition 2.7, Lemma 4.2 and the fact that $\rho_{t,K} \le \widetilde{\rho}_{t,K}$, the two families of pseudo-metrics $(\rho_{t,K})_{t,K}$ and $(\widetilde{\rho}_{t,K})_{t,K}$ define the same topology on $\mathbb{D}_{\mathrm{loc}}(S)$, the local Skorokhod topology.

If $(K_n)_{n\in\mathbb{N}}$ is an exhaustive sequence of compact sets of S, then the mapping

$$\begin{array}{ccc} \mathbb{D}_{\mathrm{loc}}(S)^2 & \to & \mathbb{R}_+ \\ (x,y) & \mapsto & \sum_{n\in\mathbb{N}} 2^{-n} \widetilde{\rho}_{n,K_n}(x,y) \wedge 1 \end{array} \tag{4.1}$$

is a metric for the local Skorokhod topology. By using a diagonal extraction procedure, it is not difficult to prove that a sequence $(x^k)_k$ converges to x for this topology if and only if there exists a sequence $(\lambda^k)_k$ in $\widetilde{\Lambda}$ such that

- either $\xi(x) < \infty$ and $\{x_s\}_{s<\xi(x)} \Subset S$: for k large enough $\lambda^k_{\xi(x)} \leq \xi(x^k)$ and

$$\sup_{s<\xi(x)} d(x_s, x^k_{\lambda^k_s}) \longrightarrow 0, \qquad x^k_{\lambda^k_{\xi(x)}} \longrightarrow \Delta, \qquad \|\log \dot{\lambda}^k\|_{\xi(x)} \longrightarrow 0, \quad \text{as } k \to \infty,$$

- or $\xi(x) = \infty$ or $\{x_s\}_{s<\xi(x)} \not\Subset S$: for all $t < \xi(x)$, for k large enough $\lambda^k_t < \xi(x^k)$ and

$$\sup_{s \leq t} d(x_s, x^k_{\lambda^k_s}) \longrightarrow 0, \qquad \|\log \dot{\lambda}^k\|_t \longrightarrow 0, \quad \text{as } k \to \infty.$$

The local Skorokhod topology can be described by a similar characterisation with $\lambda^k \in \widetilde{\Lambda}$ replaced by $\lambda^k \in \Lambda$ and respectively, $\|\log \dot{\lambda}^k\|$ replaced by $\|\lambda - \mathrm{id}\|$. The fact that the local Skorokhod topology does not depend on the distance d is a consequence of the following lemma, which states essentially that two metrics on a compact set are uniformly equivalent:

Lemma 4.3. *Let T be a set and $x, x^k \in S^T$ be such that $\{x_t\}_{t\in T} \Subset S$, then*

$$\sup_{t\in T} d(x_t, x^k_t) \longrightarrow 0, \quad \textit{as } k \to \infty,$$

if and only if

$$\forall U \subset S^2 \textit{ open subset containing } \{(y,y)\}_{y\in S}, \quad \exists k_0\ \forall k \geq k_0,\ \forall t \in T, \quad (x_t, x^k_t) \in U.$$

So the topology of the uniform convergence on $\big\{x \in S^T \mid \{x_t\}_{t\in T} \Subset S\big\}$ depends only of the topology of S.

Proof. Suppose that $\sup_{t\in T} d(x_t, x^k_t) \longrightarrow 0$ as $k \to \infty$ and take an open subset $U \subset S^2$ containing $\{(y,y)\}_{y\in S}$. By compactness there exists $\varepsilon > 0$ such that

$$\Big\{(y_1, y_2) \in S^2 \mid y_1 \in \{x_t\}_t,\ d(y_1,y_2) < \varepsilon\Big\} \subset U,$$

so, for k large enough and for all t, $(x_t, x^k_t) \in U$. To get the converse property it suffices to consider, for each $\varepsilon > 0$, the open set $U_\varepsilon := \big\{(y_1,y_2) \in S^2 \mid d(y_1,y_2) < \varepsilon\big\}$ which clearly contains $\{(y,y)\}_{y\in S}$. $\square$

In the next lemma we discuss the completeness :

Lemma 4.4. *Suppose that (S,d) is complete. Then any sequence $(x^k)_k \in (\mathbb{D}_{loc}(S))^{\mathbb{N}}$ satisfying*

$$\forall t \geq 0,\ \forall K \subset S \textit{ compact}, \quad \widetilde{\rho}_{t,K}(x^{k_1}, x^{k_2}) \underset{k_1,k_2\to\infty}{\longrightarrow} 0,$$

admits a limit for the local Skorokhod topology.

The proof of this lemma follows from the same reasoning as the proof of the triangular inequality, the proof of Theorem 12.2 pp. 128-129 from [Bil99] and the proof of Theorem 5.6 pp. 121-122 from [EK86].

Proof. It suffices to prove that $(x^k)_k$ have a converging subsequence. By taking, possibly, a subsequence we can suppose that

$$\forall t \geq 0,\ \forall K \subset S \text{ compact}, \qquad \sum_{k\geq 0} \widetilde{\rho}_{t,K}(x^k, x^{k+1}) < \infty. \tag{4.2}$$

We split our proof in five steps.
Step 1: we construct a sequence $(\lambda^k)_k \subset \widetilde{\Lambda}$. Let $\mu^k \in \widetilde{\Lambda}$ and $\widetilde{t}_k \geq 0$ be such that for all $t \geq 0$ and $K \subset S$ compact, we have for k large enough

$$\begin{aligned}
\widetilde{t}_k \leq \xi(x^k), \quad \mu^k_{\widetilde{t}_k} \leq \xi(x^{k+1}),& \\
\sup_{s<\widetilde{t}_k} d(x^k_s, x^{k+1}_{\mu^k_s}) \vee \|\log \dot{\mu}^k\|_{\widetilde{t}_k} \vee \|\mu^k - \mathrm{id}\|_{\widetilde{t}_k} &\leq 2\widetilde{\rho}_{t,K}(x^k, x^{k+1}), \\
\Big(d(x^k_{\widetilde{t}_k}, K^c) \wedge (t - \widetilde{t}_k)_+ \mathbb{1}_{\widetilde{t}_k < \xi(x^k)}\Big) &\leq 2\widetilde{\rho}_{t,K}(x^k, x^{k+1}), \\
\Big(d(x^{k+1}_{\mu^k_{\widetilde{t}_k}}, K^c) \wedge (t - \mu^k_{\widetilde{t}_k})_+ \mathbb{1}_{\mu^k_{\widetilde{t}_k} < \xi(x^{k+1})}\Big) &\leq 2\widetilde{\rho}_{t,K}(x^k, x^{k+1}).
\end{aligned} \tag{4.3}$$

For all $k \geq 0$ define

$$t_k := \bigwedge_{i\geq 0} (\mu^k)^{-1} \circ \cdots \circ (\mu^{k+i-1})^{-1}(\widetilde{t}_{k+i}), \tag{4.4}$$

so, $t_k \leq \widetilde{t}_k$ and $\mu^k_{t_k} \leq t_{k+1}$. For $k, i \geq 0$

$$\begin{aligned}
\|\log \frac{\mathrm{d}}{\mathrm{d}s}(\mu^{k+i-1} \circ \cdots \circ \mu^k(s))\|_{t_k} &\leq \sum_{\ell=k}^{k+i-1} \|\log \dot{\mu}^\ell\|_{\widetilde{t}_\ell}, \\
\|\mu^{k+i-1} \circ \cdots \circ \mu^k - \mathrm{id}\|_{t_k} &\leq \sum_{\ell=k}^{k+i-1} \|\mu^\ell - \mathrm{id}\|_{\widetilde{t}_\ell}
\end{aligned}$$

and for $j \geq i$

$$\begin{aligned}
\|\mu^{k+j-1} \circ \cdots \circ \mu^k - \mu^{k+i-1} \circ \cdots \circ \mu^k\|_{t_k} &\leq \|\mu^{k+j-1} \circ \cdots \circ \mu^{k+i} - \mathrm{id}\|_{t_{k+i}} \\
&\leq \sum_{\ell=k+i}^{k+j-1} \|\mu^\ell - \mathrm{id}\|_{\widetilde{t}_\ell}.
\end{aligned}$$

Using (4.2) and (4.3) we obtain

$$\sum_{\ell\geq 0} \|\mu^\ell - \mathrm{id}\|_{\widetilde{t}_\ell} < \infty, \qquad \sum_{\ell\geq 0} \|\log \dot{\mu}^\ell\|_{\widetilde{t}_\ell} < \infty,$$

so, the restriction to $[0, t_k]$ of continuous functions $\mu^{k+i-1} \circ \cdots \circ \mu^k$ converges uniformly to a continuous function. Set

$$\begin{cases} \lambda^k_s := \lim_{i\to\infty} \mu^{k+i-1} \circ \cdots \circ \mu^k(s), & \text{if } s \leq t_k, \\ \dot{\lambda}^k_s := 1, & \text{if } s \geq t_k. \end{cases}$$

Clearly for $s \le t_k$, $\lambda^k_s = \lambda^{k+1} \circ \mu^k(s)$. We have

$$\|\lambda^k - \mathrm{id}\| \le \sup_{i\ge 0} \|\mu^{k+i-1} \circ \cdots \circ \mu^k - \mathrm{id}\|_{t_k} \le \sum_{\ell \ge k} \|\mu^\ell - \mathrm{id}\|_{\widetilde{t}_\ell} < \infty, \tag{4.5}$$

$$\begin{aligned} \|\log \dot{\lambda}^k\| &= \sup_{0\le s_1<s_2\le t_k} \left|\log \frac{\lambda^k_{s_2} - \lambda^k_{s_1}}{s_2 - s_1}\right| \le \sup_{i\ge 0} \|\log \frac{\mathrm{d}}{\mathrm{d}s}(\mu^{k+i-1} \circ \cdots \circ \mu^k(s))\|_{t_k} \\ &\le \sum_{\ell\ge k} \|\log \dot{\mu}^\ell\|_{\widetilde{t}_\ell} < \infty, \end{aligned} \tag{4.6}$$

so, $\lambda^k \in \widetilde{\Lambda}$.

Step 2: we construct a path $x \in \mathbb{D}_{exp}(S)$. For all $k \ge 0$: $\lambda^k_{t_k} = \lambda^{k+1}_{\mu^k_{t_k}} \le \lambda^{k+1}_{t_{k+1}}$ and moreover for all $0 \le k_1 \le k_2$

$$\sup_{s<\lambda^{k_1}_{t_{k_1}}} d\left(x^{k_1}_{(\lambda^{k_1})^{-1}_s}, x^{k_2}_{(\lambda^{k_2})^{-1}_s}\right) = \sup_{s<t_{k_1}} d\left(x^{k_1}_s, x^{k_2}_{\mu^{k_1-1}\circ\cdots\circ\mu^{k_1}(s)}\right) \le \sum_{\ell=k_1}^{k_2-1} \sup_{s<\widetilde{t}_\ell} d(x^\ell_s, x^{\ell+1}_{\mu^\ell_s}).$$

By using (4.2) and (4.3) we get $\sum_{\ell\ge 0} \sup_{s<\widetilde{t}_\ell} d(x^\ell_s, x^{\ell+1}_{\mu^\ell_s}) < \infty$. By using the completeness of (S, d), we deduce that, for each $m \in \mathbb{N}$, the sequence $x^k_{(\lambda^k)^{-1}}$ converges uniformly on $[0, \lambda^m_{t_m})$. We can define $x \in \mathbb{D}_{\exp}(S)$ by setting:

$$\xi(x) := \lim_{k\to\infty} \lambda^k_{t_k} \quad \text{and} \quad \forall s < \xi(x),\ x_s := \lim_{k\to\infty} x^k_{(\lambda^k)^{-1}_s}.$$

We see that, for all $k \ge 0$

$$\sup_{s<\lambda^k_{t_k}} d(x^k_{(\lambda^k)^{-1}_s}, x_s) = \sup_{s<t_k} d\big(x^k_s, x_{\lambda^k_s}\big) \le \sum_{\ell\ge k} \sup_{s<\widetilde{t}_\ell} d(x^\ell_s, x^{\ell+1}_{\mu^\ell_s}). \tag{4.7}$$

Step 3: we prove that the infimum in (4.4) *is a minimum.* Suppose that there exists $k_0 \ge 0$ such that

$$\forall i \ge 0, \quad t_{k_0} < (\mu^{k_0})^{-1} \circ \cdots \circ (\mu^{k_0+i-1})^{-1}(\widetilde{t}_{k_0+i})$$

and we will show that one get a contradiction. Firstly, note that for all $k \ge k_0$ we necessarily have $\mu^k_{t_k} = t_{k+1}$ and $t_k < \widetilde{t}_k$ so, $\lambda^k_{t_k}$ is constant equal to $\xi(x)$ and furthermore

$$d(x^k_{t_k}, x^{k+1}_{t_{k+1}}) \le \sup_{s<\widetilde{t}_k} d(x^k_s, x^{k+1}_{\mu^k_s}).$$

Since $\sum_{k\ge 0} \sup_{s<\widetilde{t}_k} d(x^k_s, x^{k+1}_{\mu^k_s}) < \infty$, $x^k_{t_k}$ converges to an element $a \in S$. Let $\varepsilon > 0$ be arbitrary such that $K := \overline{B(a, 3\varepsilon)} \subset S$ is compact. Let $k_1 \ge k_0$ be such that

$$d(x^{k_1}_{t_{k_1}}, a) < \varepsilon, \quad \sum_{k\ge k_1} \widetilde{\rho}_{\xi(x)+4\varepsilon,K}(x^k, x^{k+1}) < \frac{\varepsilon}{2}$$

and such that (4.3) holds for all $k \ge k_1$ with $t = \xi(x) + 4\varepsilon$. Set

$$s_\ell := \bigwedge_{0\le i\le \ell} (\mu^{k_1})^{-1} \circ \cdots \circ (\mu^{k_1+i-1})^{-1}(\widetilde{t}_{k_1+i}).$$

It is clear that $s_\ell > t_{k_1}$ and s_ℓ is a decreasing sequence converging to t_{k_1} so, the set $\{\ell > 0 \mid s_\ell < s_{\ell-1}\}$ is infinite. Let $\ell > 0$ be such that $s_\ell < s_{\ell-1}$ and $s_\ell - t_{k_1} < \varepsilon$, then

$$s_\ell = (\mu^{k_1})^{-1} \circ \cdots \circ (\mu^{k_1+\ell-1})^{-1}(\widetilde{t}_{k_1+\ell}).$$

Therefore,

$$\begin{aligned}\widetilde{t}_{k_1+\ell} &= \mu^{k_1+\ell-1} \circ \cdots \circ \mu^{k_1}(s_\ell) < \mu^{k_1+\ell-1} \circ \cdots \circ \mu^{k_1}(t_{k_1} + \varepsilon) \\ &\le \sum_{i=k_1}^{k_1+\ell-1} \|\mu^i - \mathrm{id}\|_{\widetilde{t}_i} + t_{k_1} + \varepsilon \le \sum_{i=k_1}^{k_1+\ell-1} \|\mu^i - \mathrm{id}\|_{\widetilde{t}_i} + \|\lambda^{k_1} - \mathrm{id}\|_{t_{k_1}} + \xi(x) + \varepsilon \\ &\le \xi(x) + \varepsilon + 2\sum_{i\ge k_1} \|\mu^i - \mathrm{id}\|_{\widetilde{t}_i} \le \xi(x) + \varepsilon + 4\sum_{i\ge k_1} \widetilde{\rho}_{\xi(x)+4\varepsilon,K}(x^i, x^{i+1}) < \xi(x) + 3\varepsilon.\end{aligned} \tag{4.8}$$

Furthermore $\widetilde{t}_{k_1+\ell} < \mu^{k_1+\ell-1}_{\widetilde{t}_{k_1+\ell-1}} \le \xi(x^{k_1+\ell})$ and

$$d(x^{k_1+\ell}_{\widetilde{t}_{k_1+\ell}}, K^c) \wedge (\xi(x) + 4\varepsilon - \widetilde{t}_{k_1+\ell})_+ \mathbb{1}_{\widetilde{t}_{k_1+\ell} < \xi(x^{k_1+\ell})} \le 2\widetilde{\rho}_{\xi(x)+4\varepsilon,K}(x^{k_1+\ell}, x^{k_1+\ell+1}) < \varepsilon,$$

so, by (4.8),

$$d(x^{k_1+\ell}_{\widetilde{t}_{k_1+\ell}}, K^c) < \varepsilon,$$

and

$$d(x^{k_1}_{s_\ell}, K^c) \le d(x^{k_1}_{s_\ell}, x^{k_1+\ell}_{\widetilde{t}_{k_1+\ell}}) + d(x^{k_1+\ell}_{\widetilde{t}_{k_1+\ell}}, K^c) < \sum_{i=k_1}^{k_1+\ell-1} \sup_{s<\widetilde{t}_i} d(x^i_s, x^{i+1}_{\mu^i_s}) + \varepsilon < 2\varepsilon.$$

Hence we have $d(a, x^{k_1}_{s_\ell}) > \varepsilon$ and $d(a, x^{k_1}_{t_{k_1}}) < \varepsilon$. Letting $\ell \to \infty$ we get a contradiction.
Step 4: fix $t \ge 0$ and $K \subset S$ a compact set: we prove that $\lim_{k\to\infty} \widetilde{\rho}_{t,K}(x^k, x) = 0$. Taking k large enough (4.3) holds and by using (4.5), (4.6) and (4.7),

$$\begin{aligned}\sup_{s<t_k} d\big(x^k_s, x_{\lambda^k_s}\big) &\le \sum_{\ell\ge k} \sup_{s<\widetilde{t}_\ell} d(x^\ell_s, x^{\ell+1}_{\mu^\ell_s}) \le 2\sum_{\ell\ge k} \widetilde{\rho}_{t,K}(x^\ell, x^{\ell+1}), \\ \|\lambda^k - \mathrm{id}\| &\le \sum_{\ell\ge k} \|\mu^\ell - \mathrm{id}\|_{\widetilde{t}_\ell} \le 2\sum_{\ell\ge k} \widetilde{\rho}_{t,K}(x^\ell, x^{\ell+1}), \\ \|\log \dot\lambda^k\| &\le \sum_{\ell\ge k} \|\log \dot\mu^\ell\|_{\widetilde{t}_\ell} \le 2\sum_{\ell\ge k} \widetilde{\rho}_{t,K}(x^\ell, x^{\ell+1}).\end{aligned}$$

Moreover by the previous step we know that the infimum in (4.4) is a minimum and so, in the present step, we set

$$\begin{aligned} m &:= \min\Big\{\ell \ge k \;\Big|\; (\mu^k)^{-1} \circ \cdots \circ (\mu^{\ell-1})^{-1}(\widetilde{t}_\ell) = t_k\Big\} \in \mathbb{N}, \\ M &:= \sup\Big\{\ell \ge k \;\Big|\; (\mu^k)^{-1} \circ \cdots \circ (\mu^{\ell-1})^{-1}(\widetilde{t}_\ell) = t_k\Big\} \in \mathbb{N} \cup \{\infty\}.\end{aligned}$$

Then

$$\begin{aligned} d(x^k_{t_k}, K^c) \wedge (t - t_k)_+ \mathbb{1}_{t_k < \xi(x^k)} &\le \sum_{\ell=k}^{m-1} \sup_{s<\widetilde{t}_\ell} d(x^\ell_s, x^{\ell+1}_{\mu^\ell_s}) \vee \|\mu^\ell - \mathrm{id}\|_{\widetilde{t}_\ell} \\ &\quad + d(x^m_{\widetilde{t}_m}, K^c) \wedge (t - \widetilde{t}_m)_+ \mathbb{1}_{\widetilde{t}_m < \xi(x^m)} \\ &\le 2\sum_{\ell\ge k} \widetilde{\rho}_{t,K}(x^\ell, x^{\ell+1}).\end{aligned}$$

It is clear that $\lambda^k_{t_k} = \xi(x)$ if and only if $M = \infty$. If $M < \infty$

$$\begin{aligned}
d(x_{\lambda^k_{t_k}}, K^c) \wedge (t - \lambda^k_{t_k})_+ &= d(x_{\lambda^M_{t_M}}, K^c) \wedge (t - \lambda^M_{t_M})_+ \\
&\leq d(x^{M+1}_{\mu^M_{t_M}}, K^c) \wedge (t - \mu^M_{t_M})_+ + \sum_{\ell > M} \sup_{s < \widetilde{t}_\ell} d(x^\ell_s, x^{\ell+1}_{\mu^\ell_s}) \vee \|\mu^\ell - \mathrm{id}\|_{\widetilde{t}_\ell} \\
&\leq 2 \sum_{\ell \geq k} \widetilde{\rho}_{t,K}(x^\ell, x^{\ell+1}). \qquad (4.9)
\end{aligned}$$

We have proved that

$$\widetilde{\rho}_{t,K}(x^k, x) \leq 2 \sum_{\ell \geq k} \widetilde{\rho}_{t,K}(x^\ell, x^{\ell+1}) \underset{k\to\infty}{\longrightarrow} 0.$$

Step 5: we prove that $x \in \mathbb{D}_{loc}(x)$. Suppose that $\xi(x) < \infty$ and that $\{x_s\}_{s<\xi(x)} \Subset S$. Let $\varepsilon > 0$ be such that $K := \left\{ y \in S \,\middle|\, d(y, \{x_s\}_{s<\xi(x)}) \leq \varepsilon \right\}$ is compact and set $t = \xi(x) + \varepsilon$. By using (4.9) we have, for k large enough,

$$d(x_{\lambda^k_{t_k}}, K^c) \wedge (t - \lambda^k_{t_k})_+ \mathbb{1}_{\lambda^k_{t_k} < \xi(x)} \leq 2 \sum_{\ell \geq k} \widetilde{\rho}_{t,K}(x^\ell, x^{\ell+1}) < \varepsilon.$$

Then $\xi(x) = \lambda^k_{t_k}$ and we deduce that

$$\sup_{s < \xi(x)} d\big(x^k_{(\lambda^k)^{-1}s}, x_s\big) \leq 2 \sum_{\ell \geq k} \widetilde{\rho}_{t,K}(x^\ell, x^{\ell+1}) \underset{k\to\infty}{\longrightarrow} 0$$

and that the limit $x_{\xi(x)-}$ exists in S. Therefore, $x \in \mathbb{D}_{\mathrm{loc}}(S)$ and x^k converges to x for the local Skorokhod topology. $\square$

To prove the separability and the criterion of compactness we will use the following technical result:

Lemma 4.5. *Let $R \subset S$, $\delta > 0$ and $N \in \mathbb{N}$ be. Define*

$$\mathcal{E}_{R,\delta,N} := \big\{ x \in \mathbb{D}_{loc}(S) \,\big|\, \xi(x) \leq N\delta,\ \forall k \in \mathbb{N},\ x \text{ is a constant in } R \cup \{\Delta\} \text{ on } [k\delta, (k+1)\delta) \big\}.$$

Then for any $x \in \mathbb{D}_{loc}(S)$

$$\rho_{N\delta,K}(x, \mathcal{E}_{R,\delta,N}) \leq \Big(\sup_{a \in K} d(a, R) + \omega'_{N\delta,K,x}(\delta) \Big) \vee \delta.$$

Proof. For arbitrary $\varepsilon > 0$, there exist $0 = t_0 < \cdots < t_M \leq \xi(x)$ such that

$$\sup_{\substack{0 \leq i < M \\ t_i \leq s_1, s_2 < t_{i+1}}} d(x_{s_1}, x_{s_2}) \leq \omega'_{N\delta,K,x}(\delta) + \varepsilon,$$

such that for all $0 \leq i < M$, $t_{i+1} > t_i + \delta$ and $(t_M, x_{t_M-}) \notin [0, N\delta] \times K$. Denote $t^* := \min\left\{ s \geq 0 \,\middle|\, s \geq N\delta \text{ or } d(x_s, K^c) = 0 \right\} \leq t_M$ and define $\widetilde{M} := \min\{0 \leq i \leq M \mid t_i \geq t^*\}$. Define $\widetilde{t}_{\widetilde{M}} := \left\lceil \frac{t^*}{\delta} \right\rceil \delta$ where $\lceil r \rceil$ denotes the smallest integer larger or equal than the real number r. Moreover, for $0 \leq i < \widetilde{M}$ define $\widetilde{t}_i := \left\lfloor \frac{t_i}{\delta} \right\rfloor \delta$ where we recall that $\lfloor r \rfloor$ denotes the integer part of the real number r, so $0 = \widetilde{t}_0 < \ldots < \widetilde{t}_{\widetilde{M}}$. Finally, we define $\widetilde{x} \in \mathcal{E}_{R,\delta,N}$ by

$$\begin{cases} \xi(\widetilde{x}) := \widetilde{t}_{\widetilde{M}}, \\ \forall 0 \leq i < \widetilde{M} : \quad \text{we choose } \widetilde{x}_{\widetilde{t}_i} \text{ in } R \text{ such that } d(x_{t_i}, \widetilde{x}_{\widetilde{t}_i}) < d(x_{t_i}, R) + \varepsilon, \\ \forall 0 \leq i < \widetilde{M},\ \forall \widetilde{t}_i \leq s < \widetilde{t}_{i+1} : \quad \widetilde{x}_s := \widetilde{x}_{\widetilde{t}_i}, \end{cases}$$

and $\lambda \in \Lambda$ given by

$$\begin{cases} \forall 0 \leq i \leq \widetilde{M} : \quad \lambda_{t_i \wedge t^*} = \widetilde{t}_i, \\ \forall 0 \leq i < \widetilde{M} : \quad \lambda \text{ is affine on } [t_i, t_{i+1} \wedge t^*], \\ \forall s \geq t^* : \quad \dot{\lambda}_s = 1. \end{cases}$$

We can write

$$\begin{aligned} \rho_{N\delta,K}(x, \mathcal{E}_{R,\delta,N}) \leq \rho_{N\delta,K}(x, \widetilde{x}) &\leq \sup_{s < t^*} d(x_s, \widetilde{x}_{\lambda_s}) \vee \|\lambda - \mathrm{id}\| \\ &\leq \left(\sup_{a \in K} d(a, R) + \omega'_{t,K,x}(\delta) + 2\varepsilon \right) \vee \delta, \end{aligned}$$

so, letting $\varepsilon \to 0$ we obtain the result. □

The separability is an easy consequence:

Lemma 4.6. *The local Skorokhod topology on $\mathbb{D}_{loc}(S)$ is separable.*

Proof. Let R be a countable dense part of S and introduce the countable set

$$E := \bigcup_{n, N \in \mathbb{N}^*} \mathcal{E}_{R, \frac{1}{n}, N}.$$

Consider $x \in \mathbb{D}_{\mathrm{loc}}(S)$, $t \geq 0$, $K \subset S$ a compact set and let $\varepsilon > 0$ be. We choose $n \in \mathbb{N}^*$ such that $n^{-1} \leq \varepsilon$ and $\omega'_{t+1,K,x}(n^{-1}) \leq \varepsilon$ and set $N := \lceil nt \rceil$. We can write

$$\begin{aligned} \rho_{t,K}(x, \mathcal{E}_{R,\frac{1}{n},N}) \leq \rho_{\frac{N}{n},K}(x, \mathcal{E}_{R,\frac{1}{n},N}) &\leq \left(\sup_{a \in K} d(a, R) + \omega'_{\frac{N}{n},K,x}(\frac{1}{n}) \right) \vee \frac{1}{n} \\ &\leq \omega'_{t+1,K,x}(\frac{1}{n}) \vee \frac{1}{n} \leq \varepsilon. \end{aligned}$$

We deduce that E is dense, hence $\mathbb{D}_{\mathrm{loc}}(S)$ is separable. □

We have now all the ingredients to prove the characterisation of the compactness:

Proof of Theorem 2.8. First, notice that, similarly as in the proof of Lemma 4.3, the condition (2.5) is equivalent to: for all $t \geq 0$, all compact subset $K \subset S$ and all open subset $U \subset S^2$ containing the diagonal $\{(y, y) \,|\, y \in S\}$, there exists $\delta > 0$ such that for all $x \in D$ there exist $0 = t_0 < \cdots < t_N \leq \xi(x)$ such that

$$\forall 0 \leq i < N,\ s_1, s_2 \in [t_i, t_{i+1}), \quad (x_{s_1}, x_{s_2}) \in U,$$

for all $0 \leq i < N$, $t_{i+1} - t_i > \delta$, and $(t_N, x_{t_N}) \notin [0, t] \times K$. Hence, the condition (2.5) is independent to d, and we can suppose that (S, d) is complete. Suppose that D satisfy condition (2.5), then, by using Lemma 4.4, we need to prove that for all $t \geq 0$, $K \subset S$ a compact set and $\varepsilon > 0$ arbitrary, D can be recovered by a finite number of $\widetilde{\rho}_{t,K}$-balls of radius ε. Let $0 < \eta \leq \frac{1}{9}$ be such that

$$6 \cdot \sqrt{\eta} \vee \sup_{x \in D} \omega'_{t,K,x} (\sqrt{\eta}) \leq \varepsilon,$$

and let $\delta \leq \eta$ be such that

$$\sup_{x \in D} \omega'_{t+1,K,x} (\delta) \leq \frac{\eta}{2}.$$

Since K is compact we can choose a finite set $R \subset S$ such that

$$\sup_{a \in K} d(a, R) \leq \frac{\eta}{2},$$

take $N := \lceil t\delta^{-1} \rceil$. Then by using Lemma 4.5,

$$\sup_{x \in D} \rho_{t,K}(x, \mathcal{E}_{R,\delta,N}) \leq \sup_{x \in D} \rho_{N\delta,K}(x, \mathcal{E}_{R,\delta,N}) \leq \left(\sup_{a \in K} d(a, R) + \sup_{x \in D} \omega'_{N\delta,K,x}(\delta) \right) \vee \delta \leq \eta$$

and by using Lemma 4.2,

$$\sup_{x \in D} \widetilde{\rho}_{t,K}(x, \mathcal{E}_{R,\delta,N}) \leq 6 \sup_{x \in D} \left(\sqrt{\rho_{t,K}(x, \mathcal{E}_{R,\delta,N})} \vee \omega'_{t,K,x} \left(\sqrt{\rho_{t,K}(x, \mathcal{E}_{R,\delta,N})} \right) \right) \leq \varepsilon.$$

Since $\mathcal{E}_{R,\delta,N}$ is finite we can conclude that D is relatively compact.
To prove the converse sentence, thanks to the first part of Proposition 2.7 it is enough to prove that if $x^k, x \in \mathbb{D}_{\mathrm{loc}}(S)$ with x^k converging to x, then for all $t \geq 0$ and all compact subset $K \subset S$,

$$\limsup_{k \to \infty} \omega'_{t,K,x^k}(\delta) \underset{\delta \to 0}{\longrightarrow} 0.$$

This is a direct consequence of Proposition 2.7. Let us stress that although we cite Theorem 2.4 in the proof of Proposition 2.7, in reality we only need the sequential characterisation of the convergence. □

We close this section by the study of the Borel σ-algebra $\mathcal{B}(\mathbb{D}_{\mathrm{loc}}(S))$.

Lemma 4.7. *Borel σ-algebra $\mathcal{B}(\mathbb{D}_{loc}(S))$ coincides with $\mathcal{F}$.*

Proof. Let $f \in \mathrm{C}(S^{\Delta})$ and $0 \leq a < b < \infty$ be. Consider $x^k \in \mathbb{D}_{\mathrm{loc}}(S)$ converging to $x \in \mathbb{D}_{\mathrm{loc}}(S)$, with $\xi(x) > b$, and take $\lambda^k \in \widetilde{\Lambda}$ as in Theorem 2.4. Then, for k large enough $b \vee \lambda^k_b < \xi(x^k)$ and by dominated convergence,

$$\int_a^b f(x^k_s)\mathrm{d}s = \int_a^{\lambda^k_a} f(x^k_s)\mathrm{d}s + \int_a^b f(x^k_{\lambda^k_s})\dot{\lambda}^k_s\mathrm{d}s + \int_{\lambda^k_b}^b f(x^k_s)\mathrm{d}s \underset{k \to \infty}{\longrightarrow} \int_a^b f(x_s)\mathrm{d}s.$$

Hence, the set $\{x \in \mathbb{D}_{\mathrm{loc}}(S) \mid b < \xi(x)\}$ is open and on this set the function

$$x \mapsto \int_a^b f(x_s)\mathrm{d}s$$

is continuous so, for $t \geq 0$ and $\varepsilon > 0$ the mapping from $\mathbb{D}_{\text{loc}}(S)$ to $\mathbb{R}$

$$x \mapsto \begin{cases} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} f(x_s)\mathrm{d}s, & \text{if } t+\varepsilon < \xi(x), \\ f(\Delta), & \text{otherwise,} \end{cases}$$

is measurable for the Borel σ-algebra $\mathcal{B}(\mathbb{D}_{\text{loc}}(S))$ and, the same is true for the mapping $x \mapsto f(x_t)$, by taking the limit. Since f is arbitrary, $x \mapsto x_t$ is also measurable and so, $\mathcal{F} \subset \mathcal{B}(\mathbb{D}_{\text{loc}}(S))$.

Conversely, since the space is separable, it is enough to prove that for each $x^0 \in \mathbb{D}_{\text{loc}}(S)$, $t \geq 0$, $K \subset S$ compact and $\varepsilon > 0$ there exists $V \subset \mathbb{D}_{\text{loc}}(S)$, $\mathcal{F}$-measurable, such that

$$\left\{x \in \mathbb{D}_{\text{loc}}(S) \;\middle|\; \rho_{t,K}(x,x^0) \leq \varepsilon\right\} \subset V \subset \left\{x \in \mathbb{D}_{\text{loc}}(S) \;\middle|\; \rho_{t,K}(x,x^0) \leq 3\varepsilon\right\}. \tag{4.10}$$

Proposition 2.7 allows to get the existence of $0 = t_0^0 < \cdots < t_N^0 \leq \xi(x^0)$ such that

$$\sup_{\substack{0 \leq i < N \\ t_i^0 \leq s_1, s_2 < t_{i+1}^0}} d(x_{s_1}^0, x_{s_2}^0) \leq \varepsilon,$$

and $(t_N^0, x_{t_N^0}^0) \notin [0,t] \times \in K$. If we define

$$V := \left\{ x \in \mathbb{D}_{\text{loc}}(S) \;\middle|\; \begin{array}{l} \exists 0 = t_0 \leq \cdots \leq t_M \leq \xi(x),\ M \leq N \text{ such that:} \\ \qquad \forall 0 \leq i \leq M, \quad |t_i - t_i^0| \leq \varepsilon \\ \qquad \forall 0 \leq i < M,\ \forall t \in [t_i, t_{i+1}), \quad d(x_t, x_{t_i^0}^0) \leq 2\varepsilon \\ \qquad d(x_{t_M^0}^0, K^c) \wedge (t - t_M^0)_+ \mathbb{1}_{t_M^0 < \xi(x^0)} \leq 2\varepsilon \\ \qquad d(x_{t_M-}, K^c) \wedge d(x_{t_M}, K^c) \wedge (t - t_M)_+ \mathbb{1}_{t_M < \xi(x)} \leq 3\varepsilon \end{array} \right\},$$

it is straightforward to obtain (4.10). Since

$$V = \left\{ x \in \mathbb{D}_{\text{loc}}(S) \;\middle|\; \begin{array}{l} \forall \delta > 0,\ \exists 0 = q_0 \leq \cdots \leq q_M < \xi(x) - \delta,\ M \leq N \text{ such that:} \\ \qquad \forall 0 \leq i \leq M, \quad |q_i - t_i^0| \leq \varepsilon + \delta \\ \qquad \forall 0 \leq i < M,\ \forall q \in [q_i + \delta, q_{i+1} - \delta], \quad d(x_q, x_{t_i^0}^0) \leq 2\varepsilon \\ \qquad d(x_{t_M^0}^0, K^c) \wedge (t - t_M^0)_+ \mathbb{1}_{t_M^0 < \xi(x^0)} \leq 2\varepsilon \\ \qquad d(x_{q_M}, K^c) \wedge (t - q_M)_+ \mathbb{1}_{q_M < \xi(x)} \leq 3\varepsilon + \delta \end{array} \right\},$$

where q, q_i and δ are chosen to be rational, V belongs $\mathcal{F}$. The proof is now complete. $\square$

Chapter 3

Locally Feller processes and martingale local problems

Abstract: This paper is devoted to the study of a certain type of martingale problems associated to general operators corresponding to processes which have finite lifetime. We analyse several properties and in particular the weak convergence of sequences of solutions for an appropriate Skorokhod topology setting. We point out the Feller-type features of the associated solutions to this type of martingale problem. Then localisation theorems for well-posed martingale problems or for corresponding generators are proved.
Key words: martingale problem, Feller processes, weak convergence of probability measures, Skorokhod topology, generators, localisation
MSC2010 Subject Classification: Primary 60J25; Secondary 60G44, 60J35, 60B10, 60J75, 47D07

1 Introduction

The theory of Lévy-type processes stays an active domain of research during the last two decades. Heuristically, a Lévy-type process X with symbol $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is a Markov process which behaves locally like a Lévy process with characteristic exponent $q(a, \cdot)$, in a neighbourhood of each point $a \in \mathbb{R}^d$. One associates to a Lévy-type process the pseudo-differential operator L given by, for $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$Lf(a) := -\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}a\cdot\alpha} q(a, \alpha) \widehat{f}(\alpha) \mathrm{d}\alpha, \quad \text{where} \quad \widehat{f}(\alpha) := (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}a\cdot\alpha} f(a) \mathrm{d}a.$$

Does a sequence $X^{(n)}$ of Lévy-type processes, having symbols q_n, converges toward some process, when the sequence of symbols q_n converges to a symbol q? What about a sequence $X^{(n)}$, corresponding to operators L_n, when the sequence of operators converges to an operator L? What could be the appropriate setting when one wants to approximate a Lévy-type processes by a family of discrete Markov chains? This is the kind of question which naturally appears when we study the Lévy-type processes.

It was a very useful observation that a unified manner to tackle a lot of questions about large classes of processes is the martingale problem approach (see, for instance, Stroock [Str75] for Lévy-type processes, Stroock and Varadhan [SV06] for diffusion processes, Kurtz [Kur11] for Lévy-driven stochastic differential equations...). Often, convergence results are obtained under technical restrictions: for instance, when the closure of L is the generator of a Feller process (see Kallenberg [Kal02] Thm. 19.25, p. 385, Thm. 19.28, p. 387 or Böttcher, Schilling and Wang [BSW13], Theorem 7.6 p. 172). In a number of situations the cited condition is not satisfied.

In the present paper we describe a general method which should be the main tool to tackle these difficulties and, even, should relax some of these technical restrictions. We analyse sequences of martingale problems associated to large class of operators acting on continuous functions and we look to Feller-type features of the associated of solutions.

In order to be more specific, let us point out that the local Skorokhod topology on a locally compact Hausdorff space S constitutes a good setting when one needs to consider explosions in finite time (see [GH17b]). Heuristically, we modify the global Skorokhod topology, on the space of cadlag paths, by localising with respect to space variable, in order to include the eventual explosions. The definition of a martingale local problem follows in a natural way: we need to stop the martingale when it exits from compact sets. Similarly, a stochastic process is locally Feller if, for any compact set of S, it coincides with a Feller process before it exits from the compact set. Let us note that a useful tool allowing to make the connection between local and global objects (Skorokhod topology, martingale, infinitesimal generator or Feller processes) is the time change transformation. Likewise, one has stability of all these local notions under the time change.

We study the existence and the uniqueness of solutions for martingale local problems and we illustrate their locally Feller-type features (see Theorem 4.5). Then we deduce a description of the generator of a locally Feller family of probabilities by using a martingale (see Theorem 4.10 below). Furthermore we characterise the convergence of a sequence of locally Feller processes in terms of convergence of operators, provided that the sequence of martingale local problems are well-posed (see Theorem 4.13 below) and without assuming that the closure of the limit operator is an infinitesimal generator. We also consider the localisation question (as described in Ethier and Kurtz [EK86], §4.6, pp. 216-221) and we give answers in terms of martingale local problem or in terms of generator (Theorems 4.16 and 4.18). We stress that a Feller process is locally Feller, hence our results, in particular the convergence theorems apply to Feller processes. In Theorem 4.8 we give a characterisation of Feller property in terms of locally Feller property plus an additional condition.

Our results should be useful in several situations, for instance, to analyse the convergence of a Markov chain toward a Lévy-type process under general conditions (improving the results, for instance, Thm 11.2.3 from Stroock and Varadhan [SV06] p. 272, Thm. 19.28 from Kallenberg [Kal02], p. 387 or from Bötcher and Schnurr [BS11]). We develop some of these applications (as the Euler scheme of approximation for Lévy-type process or the convergence of Sinai's random walk toward the Brox diffusion) in a separate work [GH17a]. The method which we develop should apply for other situations. In a work in progress, we try to apply a similar method for some singular stochastic differential equations driven by α-stable processes other than Brownian motion.

The present paper is organised as follows: in the next section we recall some notations and results obtained in our previous paper [GH17b] on the local Skorokhod topology on spaces of cadlag functions, tightness and time change transformation. Section 3 is devoted to the study of the martingale local problem : properties, tightness and convergence, but also the existence of solutions. The most important results are presented in Section 4. In §4.1 and §4.2 we give the definitions and point out characterisations of a locally Feller family and its connection with a Feller family, essentially in terms of martingale local problems. We also provide two corrections of a result by van Casteren [vC92]. In §4.3 we give a generator description of a locally Feller family and we characterise the convergence of a sequence of locally Feller families. §4.4 contains the localisation procedure for martingale problems and generators. We collect in the Appendix the most part of technical proofs.

2 Preliminary notations and results

We recall here some notations and results concerning the local Skorokhod topology, the tightness criterion and a time change transformation which will be useful to state and prove our main results. Complete statements and proofs are described in an entirely dedicated paper [GH17b].

Let S be a locally compact Hausdorff space with countable base. The space S could be endowed with a metric and so it is a Polish space. Take $\Delta \notin S$, and we will denote by $S^\Delta \supset S$ the one-point compactification of S, if S is not compact, or the topological sum $S \sqcup \{\Delta\}$, if S is compact (so Δ is an isolated point). Denote $\mathrm{C}(S) := \mathrm{C}(S, \mathbb{R})$, resp. $\mathrm{C}(S^\Delta) := \mathrm{C}(S^\Delta, \mathbb{R})$, the set of real continuous functions on S, resp. on S^Δ. If $\mathrm{C}_0(S)$ denotes the set of functions $f \in \mathrm{C}(S)$ vanishing in Δ, we will identify

$$\mathrm{C}_0(S) = \left\{ f \in \mathrm{C}(S^\Delta) \;\middle|\; f(\Delta) = 0 \right\}.$$

We endow the set $\mathrm{C}(S)$ with the topology of uniform convergence on compact sets and $\mathrm{C}_0(S)$ with the topology of uniform convergence.

The fact that a subset A is compactly embedded in an open subset $U \subset S$ will be denoted $A \Subset U$. If $x \in (S^\Delta)^{\mathbb{R}_+}$ we denote

$$\xi(x) := \inf\{t \geq 0 \,|\, \{x_s\}_{s \leq t} \not\Subset S\}.$$

Firstly, we introduce the set of cadlag paths with values in S^Δ,

$$\mathbb{D}(S^\Delta) := \left\{ x \in (S^\Delta)^{\mathbb{R}_+} \;\middle|\; \begin{array}{l} \forall t \geq 0,\ x_t = \lim_{s \downarrow t} x_s, \ \text{ and} \\ \forall t > 0,\ x_{t-} := \lim_{s \uparrow t} x_s \ \text{ exists in } S^\Delta \end{array} \right\},$$

endowed with the global Skorokhod topology (see, for instance, Chap. 3 in [EK86], pp. 116-147) which is Polish. A sequence $(x^k)_k$ in $\mathbb{D}(S^\Delta)$ converges to x for the latter topology if and only if there exists a sequence $(\lambda^k)_k$ of increasing homeomorphisms on $\mathbb{R}_+$ such that

$$\forall t \geq 0, \quad \lim_{k \to \infty} \sup_{s \leq t} d(x_s, x^k_{\lambda^k_s}) = 0 \quad \text{and} \quad \lim_{k \to \infty} \sup_{s \leq t} |\lambda^k_s - s| = 0.$$

The global Skorokhod topology does not depend on the arbitrary metric d on S^Δ, but only on the topology on S.

Secondly, we proceed with the definition of a set of exploding cadlag paths

$$\mathbb{D}_{\mathrm{loc}}(S) := \left\{ x \in (S^\Delta)^{\mathbb{R}_+} \;\middle|\; \begin{array}{l} \forall t \geq \xi(x),\ x_t = \Delta, \\ \forall t \geq 0,\ x_t = \lim_{s \downarrow t} x_s, \\ \forall t > 0 \text{ s.t. } \{x_s\}_{s<t} \Subset S,\ x_{t-} := \lim_{s \uparrow t} x_s \text{ exists} \end{array} \right\},$$

endowed with the local Skorokhod topology (see Theorem 2.6 in[GH17b]) which is Polish. Similarly, a sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{D}_{\mathrm{loc}}(S)$ converges to x for the local Skorokhod topology if and only if there exists a sequence $(\lambda^k)_k$ of increasing homeomorphisms on $\mathbb{R}_+$ satisfying

$$\forall t\geq 0 \text{ s.t. } \{x_s\}_{s<t}\Subset S,\quad \lim_{k\to\infty}\sup_{s\leq t} d(x_s,x^k_{\lambda^k_s})=0 \quad\text{and}\quad \lim_{k\to\infty}\sup_{s\leq t}|\lambda^k_s-s|=0.$$

Once again, the local Skorokhod topology does not depend on the arbitrary metric d on S^Δ, but only on the topology on S.

We will always denote by X the canonical process on $\mathbb{D}(S^\Delta)$ or on $\mathbb{D}_{\mathrm{loc}}(S)$, without danger of confusion. We endow each of $\mathbb{D}(S^\Delta)$ and $\mathbb{D}_{\mathrm{loc}}(S)$ with the Borel σ-algebra $\mathcal{F}:=\sigma(X_s,\ 0\leq s<\infty)$ and a filtration $\mathcal{F}_t:=\sigma(X_s,\ 0\leq s\leq t)$. We will always omit the argument X for the explosion time $\xi(X)$ of the canonical process. It is clear that ξ is a stopping time. Furthermore, if $U\subset S$ is an open subset,

$$\tau^U:=\inf\{t\geq 0\mid X_{t-}\notin U \text{ or } X_t\notin U\}\wedge\xi \tag{2.1}$$

is a stopping time.

In [GH17b] we state and prove the following version of the Aldous criterion of tightness: let $(\mathbf{P}_n)_n$ be a sequence of probability measures on $\mathbb{D}_{\mathrm{loc}}(S)$. If for all $t\geq 0$, $\varepsilon>0$, and open subset $U\Subset S$, we have:

$$\limsup_{n\to\infty}\sup_{\substack{\tau_1\leq\tau_2\\ \tau_2\leq(\tau_1+\delta)\wedge t\wedge\tau^U}}\mathbf{P}_n\big(d(X_{\tau_1},X_{\tau_2})\geq\varepsilon\big)\xrightarrow[\delta\to 0]{}0, \tag{2.2}$$

then $\{\mathcal{P}_n\}_n$ is tight for the local Skorokhod topology. Here d is an arbitrary metric on S^Δ and the supremum is taken on all stopping times τ_i.

There are several ways to localise processes, for instance one can stop them when they leave a large compact set. Nevertheless this method does not preserve the convergence and we need to adapt this procedure in order to recover continuity. Let us describe our time change transformation. Since (2.1), we can write

$$\tau^{\{g\neq 0\}}(x):=\inf\{t\geq 0\mid g(x_{t-})\wedge g(x_t)=0\}\wedge\xi(x).$$

Let $g\in\mathrm{C}(S,\mathbb{R}_+)$ be. For any $x\in\mathbb{D}_{\mathrm{loc}}(S)$ and $t\in\mathbb{R}_+$ we denote

$$\tau^g_t(x):=\inf\left\{s\geq 0\ \middle|\ s\geq\tau^{\{g\neq 0\}}\text{ or }\int_0^s\frac{\mathrm{d}u}{g(x_u)}\geq t\right\}. \tag{2.3}$$

We define a time change transformation, which is $\mathcal{F}$-measurable,

$$\begin{array}{rccc} g\cdot X: & \mathbb{D}_{\mathrm{loc}}(S) & \to & \mathbb{D}_{\mathrm{loc}}(S)\\ & x & \mapsto & g\cdot x,\end{array}$$

as follows: for $t\in\mathbb{R}_+$

$$(g\cdot X)_t:=\begin{cases} X_{\tau^{\{g\neq 0\}}-} & \text{if } \tau^g_t=\tau^{\{g\neq 0\}},\ X_{\tau^{\{g\neq 0\}}-}\text{ exists and belongs to }\{g=0\},\\ X_{\tau^g_t} & \text{otherwise.}\end{cases} \tag{2.4}$$

For any $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$, we also define $g \cdot \mathbf{P}$ the pushforward of $\mathbf{P}$ by $x \mapsto g \cdot x$. Let us stress that, τ_t^g is a stopping time (see Corollary 2.3 in [GH17b]). The time of explosion of $g \cdot X$ is given by

$$\xi(g \cdot X) = \begin{cases} \infty & \text{if } \tau^{\{g \neq 0\}} < \xi \text{ or } X_{\xi-} \text{ exists and belongs to } \{g = 0\}, \\ \int_0^\xi \frac{\mathrm{d}u}{g(x_u)} & \text{otherwise.} \end{cases}$$

It is not difficult to see, using the definition of the time change (2.4), that

$$\forall g_1, g_2 \in \mathrm{C}(S, \mathbb{R}_+),\ \forall x \in \mathbb{D}_{\mathrm{loc}}(S), \quad g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x. \tag{2.5}$$

In [GH17b]) Proposition 3.8, a connection between $\mathbb{D}_{\mathrm{loc}}(S)$ and $\mathbb{D}(S^\Delta)$ was given. We recall here this result because it will employed several times.

Proposition 2.1 (Connection between $\mathbb{D}_{\mathrm{loc}}(S)$ and $\mathbb{D}(S^\Delta)$)**.** *Let $\widetilde{S}$ be an arbitrary locally compact Hausdorff space with countable base and consider*

$$\begin{array}{rccc} \mathbf{P}: & \widetilde{S} & \to & \mathcal{P}(\mathbb{D}_{loc}(S)) \\ & a & \mapsto & \mathbf{P}_a \end{array}$$

a weakly continuous mapping for the local Skorokhod topology. Then for any open subset U of S, there exists $g \in \mathrm{C}(S, \mathbb{R}_+)$ such that $\{g \neq 0\} = U$, for all $a \in \widetilde{S}$

$$g \cdot \mathbf{P}_a \left(0 < \xi < \infty \Rightarrow X_{\xi-} \textit{ exists in } U\right) = 1,$$

and the application

$$\begin{array}{rccc} g \cdot \mathbf{P}: & \widetilde{S} & \to & \mathcal{P}(\{0 < \xi < \infty \Rightarrow X_{\xi-} \textit{ exists in } U\}) \\ & a & \mapsto & g \cdot \mathbf{P}_a \end{array}$$

is weakly continuous for the global Skorokhod topology from $\mathbb{D}(S^\Delta)$.

3 Martingale local problem

3.1 Definition and first properties

To begin with we recall the optional sampling theorem. Its proof can be found in Theorem 2.13 and Remark 2.14. p. 61 from [EK86].

Theorem 3.1 (Optional sampling theorem)**.** *Let $\left(\Omega, (\mathcal{G}_t)_{t \in \mathbb{R}_+}, \mathbb{P}\right)$ be a filtered probability space and let M be a cadlag $(\mathcal{G}_t)_t$-martingale, then for all $(\mathcal{G}_{t+})_t$-stopping times τ and σ, with τ bounded,*

$$\mathbb{E}\left[M_\tau \mid \mathcal{G}_{\sigma+}\right] = M_{\tau \wedge \sigma}, \quad \mathbb{P}\textit{-almost surely.}$$

In particular M is a $(\mathcal{G}_{t+})_t$-martingale.

Definition 3.2 (Martingale local problem)**.** Let L be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$.

a) The set $\mathcal{M}(L)$ of solutions of the martingale local problem associated to L is the set of $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\text{loc}}(S))$ such that for all $(f,g) \in L$ and open subset $U \Subset S$:

$$f(X_{t\wedge\tau^U}) - \int_0^{t\wedge\tau^U} g(X_s)\mathrm{d}s \text{ is a } \mathbf{P}\text{-martingale}$$

with respect to the filtration $(\mathcal{F}_t)_t$ or, equivalent, to the filtration $(\mathcal{F}_{t+})_t$. Recall that τ^U is given by (2.1). The martingale *local problem* should not be confused with the *local martingale* problem (see Remark 3.3 below for a connection).

b) We say that there is existence of a solution for the martingale local problem if for any $a \in S$ there exists an element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathbf{P}(X_0 = a) = 1$.

c) We say that there is uniqueness of the solution for the martingale local problem if for any $a \in S$ there is at most one element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathbf{P}(X_0 = a) = 1$.

d) The martingale local problem is said well-posed if there is existence and uniqueness of the solution.

Remark 3.3. 1) By using dominated convergence, for all $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$, $(f,g) \in L \cap \mathrm{C}_0(S) \times \mathrm{C}_b(S)$ and $\mathbf{P} \in \mathcal{M}(L)$, we have that

$$f(X_t) - \int_0^{t\wedge\xi} g(X_s)\mathrm{d}s \text{ is a } \mathbf{P}\text{-martingale.}$$

Hence, if $L \subset \mathrm{C}_0(S) \times \mathrm{C}_b(S)$, the martingale local problem and the classical martingale problem are equivalent.
2) It can be proved that, for all $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$, $(f,g) \in L$ and $\mathbf{P} \in \mathcal{M}(L)$ such that

$$\mathbf{P}\big(\xi < \infty \text{ implies } \{X_s\}_{s<\xi} \Subset S\big) = 1,$$

we have

$$f(X_t) - \int_0^{t\wedge\xi} g(X_s)\mathrm{d}s \text{ is a } \mathbf{P}\text{-local martingale.}$$

Indeed, it suffices to use the family of stopping times

$$\left\{\tau^U \vee \left(T\mathbb{1}_{\{\tau^U \le T, \tau^U = \xi\}}\right) \;\middle|\; U \Subset S,\, T \ge 0\right\},$$

to obtain the assertion.
3) We shall see that the uniqueness or, respectively, the existence of a solution for the martingale local problem when one starts from a fixed point implies the uniqueness or the existence of a solution for the martingale local problem when one starts with an arbitrary measure (see Proposition 3.12 below).
4) Let $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ and $\mathbf{P} \in \mathcal{M}(L)$ be. If $(f,g) \in L$ and $U \Subset S$ is an open subset, then, by dominated convergence

$$\frac{\mathbf{E}\left[f(X_{t\wedge\tau^U}) \mid \mathcal{F}_0\right] - f(X_0)}{t} = \mathbf{E}\left[\frac{1}{t}\int_0^{t\wedge\tau^U} g(X_s)\mathrm{d}s \;\middle|\; \mathcal{F}_0\right] \xrightarrow[t\to 0]{\mathbf{P}\text{-a.s.}} g(X_0).$$

$\Diamond$

Let us point out some useful properties concerning the martingale local problem:

Proposition 3.4 (Martingale local problem properties). *Let L be a subset of* $\mathrm{C}_0(S)\times\mathrm{C}(S)$.

1. (Time change) *Take $h \in \mathrm{C}(S,\mathbb{R}_+)$ and denote*
$$hL := \{(f, hg) \mid (f,g) \in L\}\,. \tag{3.1}$$
Then, for all $\mathbf{P} \in \mathcal{M}(L)$,
$$h \cdot \mathbf{P} \in \mathcal{M}(hL). \tag{3.2}$$

2. (Closer property) *The closure with respect to* $\mathrm{C}_0(S) \times \mathrm{C}(S)$ *satisfies*
$$\mathcal{M}\left(\overline{span(L)}\right) = \mathcal{M}(L). \tag{3.3}$$

3. (Compactness and convexity property) *Suppose that* $\mathrm{D}(L)$ *is a dense subset of* $\mathrm{C}_0(S)$, *where the domain of L is defined by*
$$\mathrm{D}(L) := \{f \in \mathrm{C}_0(S) \mid \exists g \in \mathrm{C}(S),\ (f,g) \in L\}\,.$$
Then $\mathcal{M}(L)$ is a convex compact set for the local Skorokhod topology.

4. (Quasi-continuity) *Suppose that* $\mathrm{D}(L)$ *is a dense subset of* $\mathrm{C}_0(S)$, *then for any* $\mathbf{P} \in \mathcal{M}(L)$, $\mathbf{P}$ *is* $(\mathcal{F}_{t+})_t$*-quasi-continuous. More precisely this means that for any* $(\mathcal{F}_{t+})_t$*-stopping times* $\tau, \tau_1, \tau_2 \ldots$
$$X_{\tau_n} \underset{n\to\infty}{\longrightarrow} X_\tau \quad \mathbf{P}\text{-almost surely on } \left\{\tau_n \underset{n\to\infty}{\longrightarrow} \tau < \infty\right\}, \tag{3.4}$$
with the convention $X_\infty := \Delta$.
In particular for any $t \geq 0$, $\mathbf{P}(X_{t-} = X_t) = 1$,
$$\mathbf{P}\big(\mathbb{D}_{loc}(S) \cap \mathbb{D}(S^\Delta)\big) = \mathbf{P}\big(\xi \in (0,\infty) \Rightarrow X_{\xi_-} \text{ exists in } S^\Delta\big) = 1,$$
and for any open subset $U \subset S$, $\mathbf{P}(\tau^U < \infty \Rightarrow X_{\tau^U} \notin U) = 1$.

The following result tell that the mapping $L \mapsto \mathcal{M}(L)$ is somehow upper semi-continuous.

Proposition 3.5. *Let $L_n, L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ be such that*
$$\forall (f,g) \in L, \quad \exists (f_n, g_n) \in L_n, \text{ such that } \quad f_n \xrightarrow[n\to\infty]{\mathrm{C}_0} f,\ g_n \xrightarrow[n\to\infty]{\mathrm{C}} g. \tag{3.5}$$
Then:

1. (Continuity) *Let $\mathbf{P}^n, \mathbf{P} \in \mathcal{P}(\mathbb{D}_{loc}(S))$ be such that $\mathbf{P}^n \in \mathcal{M}(L_n)$ and suppose that $\{\mathbf{P}^n\}_n$ converges weakly to $\mathbf{P}$ for the local Skorokhod topology. Then $\mathbf{P} \in \mathcal{M}(L)$.*

2. (Tightness) *Suppose that* $\mathrm{D}(L)$ *is dense in* $\mathrm{C}_0(S)$, *then for any sequence $\mathbf{P}^n \in \mathcal{M}(L_n)$, $\{\mathbf{P}^n\}_n$ is tight for the local Skorokhod topology.*

The proofs of the two latter propositions are interlaced and will be developed in the appendix (see § A.1). During these proofs we use the following result concerning the property of uniform continuity along stopping times of the martingale local problem. Its proof is likewise postponed to the Appendix.

Lemma 3.6. *Let $L_n, L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ be such that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$ and assume the convergence of the operators in the sense given by (3.5). Consider $\mathcal{K}$ a compact subset of S and $\mathcal{U}$ an open subset of S^2 containing $\{(a,a)\}_{a\in S}$. For an arbitrary $(\mathcal{F}_{t+})_t$-stopping time τ_1 we denote the $(\mathcal{F}_{t+})_t$-stopping time*

$$\tau(\tau_1) := \inf\left\{t \geq \tau_1 \mid \{(X_{\tau_1}, X_s)\}_{\tau_1 \leq s \leq t} \notin \mathcal{U}\right\}.$$

Then for each $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that: for any $n \geq n_0$, $(\mathcal{F}_{t+})_t$-stopping times $\tau_1 \leq \tau_2$ and $\mathbf{P} \in \mathcal{M}(L_n)$ satisfying $\mathbf{E}[(\tau_2 - \tau_1)\mathbb{1}_{\{X_{\tau_1} \in \mathcal{K}\}}] \leq \delta$, we have

$$\mathbf{P}(X_{\tau_1} \in \mathcal{K},\ \tau(\tau_1) \leq \tau_2) \leq \varepsilon,$$

with the convention $X_\infty := \Delta$.

3.2 Existence and conditioning

Before giving the result of existence of a solution for the martingale local problem, let us recall that $X_t^\tau = X_{\tau\wedge t}$ for τ a stopping time, and the classical positive maximal principle (see [EK86], p.165):

Definition 3.7. A subset $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ satisfies the positive maximum principle if for all $(f,g) \in L$ and $a_0 \in S$ such that $f(a_0) = \sup_{a\in S} f(a) \geq 0$ then $g(a_0) \leq 0$.

Remark 3.8. 1) A linear subspace $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ satisfying the positive maximum principle is univariate. Indeed for any $(f,g_1), (f,g_2) \in L$, applying the positive maximum principle to $(0, g_2 - g_1)$ and $(0, g_1 - g_2)$ we deduce that $g_1 = g_2$.
2) Suppose furthermore that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$, then as a consequence of the second part of Proposition 3.4 and of Theorem 3.9 below, the closure $\overline{L}$ in $\mathrm{C}_0(S) \times \mathrm{C}(S)$ satisfies the positive maximum principle, too. $\diamondsuit$

The existence of a solution for the martingale local problem result will be a consequence of Theorem 5.4 p. 199 from [EK86].

Theorem 3.9 (Existence)**.** *Let L be a linear subspace of $\mathrm{C}_0(S) \times \mathrm{C}(S)$.*

1. *If there is existence of a solution for the martingale local problem associated to L, then L satisfies the positive maximum principle.*
2. *Conversely, if L satisfies the positive maximum principle and $D(L)$ is dense in $\mathrm{C}_0(S)$, then there is existence of a solution for the martingale local problem associated to L.*

Proof. Suppose that there is existence of a solution for the martingale local problem, let $(f,g) \in L$ and $a_0 \in S$ be such that $f(a_0) = \sup_{a\in S} f(a) \geq 0$. If we take $\mathbf{P} \in \mathcal{M}(L)$ such that $\mathbf{P}(X_0 = a_0) = 1$, then, by the fourth part of Remark 3.3

$$g(a_0) = \lim_{t\to 0} \frac{1}{t}(f(X_{t\wedge\tau^U}) - f(a_0)) \leq 0,$$

so L satisfies the positive maximum principle.

Let us prove the second part of Theorem 3.9. Consider $\widetilde{L}_0$ a countable dense subset of L and $L_0 := \mathrm{span}(\widetilde{L}_0)$. There exists $h \in \mathrm{C}_0(S)$ such that for all $(f,g) \in \widetilde{L}_0$: $hg \in \mathrm{C}_0$, hence $\overline{L} = \overline{L_0}$ and $hL_0 \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$. We apply Theorem 5.4 p. 199 of [EK86] to the univariate operator hL_0: for all $a \in S$, there exists $\widetilde{\mathbf{P}} \in \mathcal{P}(\mathbb{D}(S^\Delta))$ such that $\widetilde{\mathbf{P}}(X_0 = a) = 1$ and for all $(f,g) \in hL_0$

$$f(X_t) - \int_0^t g(X_s)\mathrm{d}s \text{ is a } \widetilde{\mathbf{P}}\text{-martingale.}$$

Then $\mathbf{P} := \mathscr{L}_{\widetilde{\mathbf{P}}}(X^{\tau^S}) \in \mathcal{P}\left(\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)\right)$, moreover for any $(f,g) \in hL_0$, for any open subset $U \Subset S$, for any $s_1 \leq \cdots \leq s_k \leq s \leq t$ in $\mathbb{R}_+$ and for any $\varphi_1, \ldots, \varphi_k \in \mathrm{C}(S^\Delta)$,

$$\mathbf{E}\left[\left(f(X_{t\wedge\tau^U}) - f(X_{s\wedge\tau^U}) - \int_{s\wedge\tau^U}^{t\wedge\tau^U} g(X_u)\mathrm{d}u\right)\varphi_1(X_{s_1})\cdots\varphi_k(X_{s_k})\right]$$
$$\widetilde{\mathbf{E}}\left[\left(f(X_{t\wedge\tau^U}) - f(X_{s\wedge\tau^U}) - \int_{s\wedge\tau^U}^{t\wedge\tau^U} g(X_u)\mathrm{d}u\right)\varphi_1(X_{s_1})\cdots\varphi_k(X_{s_k})\right] = 0.$$

Hence $\mathbf{P} \in \mathcal{M}(hL_0)$. To conclude we use the two first parts of Proposition 3.4:

$$\mathcal{M}(L) = \mathcal{M}(\overline{L}) = \mathcal{M}(L_0) = \left\{ \frac{1}{h} \cdot \mathbf{Q} \;\middle|\; \mathbf{Q} \in \mathcal{M}(hL_0) \right\}.$$

So $\frac{1}{h} \cdot \mathbf{P} \in \mathcal{M}(L)$ and the existence of a solution for the martingale local problem is proved. □

Remark 3.10. Since $\mathcal{F}$ is the Borel σ-algebra on the Polish space $\mathbb{D}_{\mathrm{loc}}(S)$, we can use Theorem 6.3, in [Kal02], p. 107. So, for any $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ and $(\mathcal{F}_{t+})_t$-stopping time τ, the regular conditional distribution $\mathbf{Q}_X \overset{\mathbf{P}\text{-a.s.}}{:=} \mathscr{L}_{\mathbf{P}}\big((X_{\tau+t})_{t\geq 0} \,\big|\, \mathcal{F}_{\tau+}\big)$ exists. It means that there exists

$$\begin{array}{rccc} \mathbf{Q}: & \mathbb{D}_{\mathrm{loc}}(S) & \to & \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S)) \\ & x & \mapsto & \mathbf{Q}_x \end{array}$$

such that for any $A \in \mathcal{F}$, $\mathbf{Q}_X(A)$ is $\mathcal{F}_{\tau+}$-measurable and

$$\mathbf{P}\big((X_{\tau+t})_{t\geq 0} \in A \,\big|\, \mathcal{F}_{\tau+}\big) = \mathbf{Q}_X(A) \quad \mathbf{P}\text{-almost surely.} \qquad \lozenge$$

Proposition 3.11 (Conditioning)**.** *Take $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$, $\mathbf{P} \in \mathcal{M}(L)$, and a $(\mathcal{F}_{t+})_t$-stopping time τ. As in Remark 3.10 we denote $\mathbf{Q}_X \overset{\mathbf{P}\text{-a.s.}}{:=} \mathscr{L}_{\mathbf{P}}\big((X_{\tau+t})_{t\geq 0} \,\big|\, \mathcal{F}_{\tau+}\big)$, then*

$$\mathbf{Q}_X \in \mathcal{M}(L), \quad \mathbf{P}\text{-}\textit{almost surely.}$$

Proof. Let (f,g) be in L, $s_1 \leq \cdots \leq s_k \leq s \leq t$ be in $\mathbb{R}_+$, $\varphi_1, \ldots, \varphi_k$ be in $\mathrm{C}(S^\Delta)$ and $U \Subset S$ be an open subset. Here and elsewhere we will denote by $E^{\mathbf{Q}_x}$ the expectation with respect to $\mathbf{Q}_x$. Since

$$\mathbb{1}_{\tau<\tau^U} E^{\mathbf{Q}_X}\left[\big(f(X_{t\wedge\tau^U}) - f(X_{s\wedge\tau^U}) - \int_{s\wedge\tau^U}^{t\wedge\tau^U} g(X_u)\mathrm{d}u\big)\varphi_1(X_{s_1})\cdots\varphi_k(X_{s_k})\right]$$
$$\overset{\mathbf{P}\text{-a.s.}}{=} \mathbb{1}_{\tau<\tau^U}\mathbf{E}\Big[\big(f(X_{(t+\tau)\wedge\tau^U}) - f(X_{(s+\tau)\wedge\tau^U}) - \int_{(s+\tau)\wedge\tau^U}^{(t+\tau)\wedge\tau^U} g(X_u)\mathrm{d}u\big)$$
$$\times \varphi_1(X_{s_1+\tau})\cdots\varphi_k(X_{s_k+\tau}) \,\Big|\, \mathcal{F}_{\tau+}\Big] \overset{\mathbf{P}\text{-a.s.}}{=} 0,$$

we have

$$\mathbf{E}\Big(E^{\mathbf{Q}_X}\Big[\big(f(X_{t\wedge\tau^U})-f(X_{s\wedge\tau^U})-\int_{s\wedge\tau^U}^{t\wedge\tau^U}g(X_u)\mathrm{d}u\big)\varphi_1(X_{s_1})\cdots\varphi_k(X_{s_k})\Big]\neq 0\Big) \\ \leq \mathbf{P}\big(\tau^U\leq\tau<\xi\big). \quad (3.6)$$

Let $\widetilde{L}$ be a countable dense subset of L, C be a countable dense subset of $\mathrm{C}(S^\Delta)$ and $U_n \Subset S$ be an increasing sequence of open subsets such that $S=\bigcup_n U_n$. Then $\mathbf{Q}_X\in\mathcal{M}(L)$ if and only if for all $(f,g)\in\widetilde{L}$, $k\in\mathbb{N}$, for any $s_1\leq\cdots\leq s_k\leq s\leq t$ in $\mathbb{Q}_+$, for any $\varphi_1,\ldots,\varphi_k\in C$, and for n large enough

$$E^{\mathbf{Q}_X}\Big[\big(f(X_{t\wedge\tau^{U_n}})-f(X_{s\wedge\tau^{U_n}})-\int_{s\wedge\tau^{U_n}}^{t\wedge\tau^{U_n}}g(X_u)\mathrm{d}u\big)\varphi_1(X_{s_1})\cdots\varphi_k(X_{s_k})\Big]=0.$$

Hence $\{\mathbf{Q}_X\in\mathcal{M}(L)\}$ is in $\mathcal{F}_{\tau+}$ and by (3.6), $\mathbf{P}$-almost surely $\mathbf{Q}_X\in\mathcal{M}(L)$. □

Proposition 3.12. *Set $L\subset\mathrm{C}_0(S)\times\mathrm{C}(S)$.*

1. *If there is uniqueness of the solution for the martingale local problem then for any $\mu\in\mathcal{P}(S^\Delta)$ there is at most one element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathscr{L}_{\mathbf{P}}(X_0)=\mu$.*
2. *If there is existence of a solution for the martingale local problem and $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$, then for any $\mu\in\mathcal{P}(S^\Delta)$ there exists an element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathscr{L}_{\mathbf{P}}(X_0)=\mu$.*

Proof. Suppose that we have uniqueness of the solution for the martingale local problem. Let μ be in $\mathcal{P}(S^\Delta)$ and $\mathbf{P}^1,\mathbf{P}^2\in\mathcal{M}(L)$ be such that $\mathscr{L}_{\mathbf{P}^1}(X_0)=\mathscr{L}_{\mathbf{P}^2}(X_0)=\mu$. As in Remark 3.10 let $\mathbf{Q}_\bullet,\mathbf{R}_\bullet : S^\Delta\to\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ be such that

$$\mathbf{Q}_{X_0}\overset{\mathbf{P}^1\text{-a.s.}}{:=}\mathscr{L}_{\mathbf{P}^1}\left(X\mid\mathcal{F}_0\right), \qquad \mathbf{R}_{X_0}\overset{\mathbf{P}^2\text{-a.s.}}{:=}\mathscr{L}_{\mathbf{P}^2}\left(X\mid\mathcal{F}_0\right).$$

Then, by Proposition 3.11, $\mathbf{Q}_a,\mathbf{R}_a\in\mathcal{M}(L)$ for μ-almost all a, so, by uniqueness of the solution for the martingale local problem, $\mathbf{Q}_a=\mathbf{R}_a$ for μ-almost all a. We finally obtain $\mathbf{P}^1=\int\mathbf{Q}_a\nu(\mathrm{d}a)=\int\mathbf{R}_a\nu(\mathrm{d}a)=\mathbf{P}^2$.

Suppose that we have existence of a solution for the martingale local problem and that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$. Thanks to 3 from Proposition 3.4 $\mathcal{M}(L)$ is convex and compact. Hence the set

$$C:=\{\mu\in\mathcal{P}(S^\Delta)\,|\,\exists\mathbf{P}\in\mathcal{M}(L)\ \text{ such that }\ \mathscr{L}_{\mathbf{P}}(X_0)=\mu\}$$

is convex and compact. Since there is existence of a solution for the martingale local problem we have $\big\{\delta_a\ \big|\ a\in S^\Delta\big\}\subset C$ so $C=\mathcal{P}(S^\Delta)$. □

4 Locally Feller families of probabilities

In this section we will study a local counterpart of Feller families in connection with Feller semi-groups and martingale local problems. The basic notions and facts on Feller semi-groups can be founded in Chapter 19 pp. 367-389 from [Kal02].

4.1 Feller families of probabilities

Let $(\mathcal{G}_t)_{t\geq 0}$ be a filtration containing $(\mathcal{F}_t)_{t\geq 0}$. Recall that a family of probability measures $(\mathbf{P}_a)_{a\in S} \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is called $(\mathcal{G}_t)_t$-Markov if, for any $B \in \mathcal{F}$, $a \mapsto \mathbf{P}_a(B)$ is measurable, for any $a \in S$, $\mathbf{P}_a(X_0 = a) = 1$, and for any $B \in \mathcal{F}$, $a \in S$ and $t_0 \in \mathbb{R}_+$

$$\mathbf{P}_a\left((X_{t_0+t})_t \in B \mid \mathcal{G}_{t_0}\right) = \mathbf{P}_{X_{t_0}}(B),\ \mathbf{P}_a - \text{almost surely},$$

where $\mathbf{P}_\Delta$ is the unique element of $\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ such that $\mathbf{P}_\Delta(\xi = 0) = 1$. If the latter property is also satisfied by replacing t_0 with any $(\mathcal{G}_t)_t$-stopping time, the family of probability measures is $(\mathcal{G}_t)_t$-strong Markov. If $\mathcal{G}_t = \mathcal{F}_t$ we just say that the family is (strong) Markov. If ν is a measure on S^Δ we set $\mathbf{P}_\nu := \int \mathbf{P}_a \nu(\mathrm{d}a)$. Then the distribution of X_0 under $\mathbf{P}_\nu$ is ν, and $\mathbf{P}_\nu$ satisfies the (strong) Markov property.

Definition 4.1 (Feller family)**.** A Markov family $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is said to be Feller if for all $f \in \mathrm{C}_0(S)$ and $t \in \mathbb{R}_+$ the function

$$\begin{array}{rccc} T_t f: & S & \to & \mathbb{R} \\ & a & \mapsto & \mathbf{E}_a[f(X_t)] \end{array}$$

is in $\mathrm{C}_0(S)$. In this case it is no difficult to see that $(T_t)_t$ is a Feller semi-group on $\mathrm{C}_0(S)$ (see p. 369 in [Kal02]) called the semi-group of $(\mathbf{P}_a)_a$. Its generator L is the set of $(f,g) \in \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ such that, for all $a \in S$

$$\frac{T_t f(a) - f(a)}{t} \underset{t\to 0}{\longrightarrow} g(a).$$

and we call it the $\mathrm{C}_0 \times \mathrm{C}_0$-generator of $(\mathbf{P}_a)_a$.

In [vC92] Theorem 2.5, p. 283, one states a connection between Feller families and martingale problems. Unfortunately the proof given in the cited paper is correct only on a compact space S. The fact that a Feller family of probabilities is the unique solution of an appropriate martingale problem is stated in the proposition below. We will prove the converse of this result in Theorem 4.7.

To give this statement we need to introduce some notations. For $L \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ we define

$$L^\Delta := \mathrm{span}\left(L \cup \{(\mathbb{1}_{S^\Delta}, 0)\}\right) \subset \mathrm{C}(S^\Delta) \times \mathrm{C}(S^\Delta). \tag{4.1}$$

We recall that we identified $\mathrm{C}_0(S)$ by the set of functions $f \in \mathrm{C}(S)$ such that $f(\Delta) = 0$. The set of solutions $\mathcal{M}(L^\Delta) \subset \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S^\Delta))$ of the martingale problem associated to L^Δ satisfies

$$\forall \mathbf{P} \in \mathcal{M}(L^\Delta), \quad \mathbf{P}(X_0 \in S^\Delta \Rightarrow X \in \mathbb{D}(S^\Delta)) = 0.$$

Without loss of generality, to study the martingale problem associated to L^Δ it suffices to study the set of solution with S^Δ-conservative paths:

$$\mathcal{M}_{\mathrm{c}}(L^\Delta) := \mathcal{M}(L^\Delta) \cap \mathcal{P}(\mathbb{D}(S^\Delta)) = \left\{\mathbf{P} \in \mathcal{M}(L^\Delta) \;\middle|\; \mathbf{P}(X_0 \in S^\Delta) = 1\right\}.$$

In fact $\mathcal{M}_{\mathrm{c}}(L^\Delta)$ is the set consisting of $\mathbf{P} \in \mathcal{P}(\mathbb{D}(S^\Delta))$ such that for all $(f,g) \in L$

$$f(X_t) - \int_0^t g(X_s)\mathrm{d}s \quad \text{is a } \mathbf{P}\text{-martingale.} \tag{4.2}$$

Proposition 4.2. *If $(T_t)_t$ is a Feller semi-group on $\mathrm{C}_0(S)$ with L its generator, then there is a unique Feller family $(\mathbf{P}_a)_a$ with semi-group $(T_t)_t$. Moreover the martingale problem associate to L^Δ is well-posed and*

$$\mathcal{M}_{\mathrm{c}}(L^\Delta) = \{\mathbf{P}_\mu\}_{\mu \in \mathcal{P}(S^\Delta)}.$$

Remark 4.3. 1. For any $\mathbf{P} \in \mathcal{M}_{\mathrm{c}}(L^\Delta)$ the distribution of X^{τ^S} under $\mathbf{P}$ satisfies

$$\mathscr{L}_{\mathbf{P}}(X^{\tau^S}) \in \mathcal{M}_{\mathrm{c}}(L^\Delta) \cap \mathbb{D}_{\mathrm{loc}}(S) \subset \mathcal{M}(L).$$

Moreover if $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$, thanks to 4 from Proposition 3.4

$$\mathcal{M}(L) = \mathcal{M}_{\mathrm{c}}(L^\Delta) \cap \mathbb{D}_{\mathrm{loc}}(S).$$

So if $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$ there is existence of a solution for the martingale problem associated to L if and only if there is existence of a solution to the martingale problem associated to L^Δ. Moreover the uniqueness of the solution for the martingale problem associated to L^Δ imply uniqueness of the solution for the martingale problem associated to L.
2. If S is compact and $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S) = \mathrm{C}(S)$, then it is straightforward to obtain $\mathcal{M}(L) = \mathcal{M}_{\mathrm{c}}(L^\Delta)$. $\Diamond$

For the sake of completeness we give:

Proof of Proposition 4.2. The existence of a solution for the martingale problem is a consequence of Theorem 3.9. Thanks to Proposition 3.11, to prove our result we need to prove that

$$\forall \mathbf{P} \in \mathcal{M}_{\mathrm{c}}(L^\Delta),\ \forall t \geq 0,\ \forall f \in \mathrm{D}(L),\ \mathbf{E}\left[f(X_t)\right] = \mathbf{E}\left[T_t f(X_0)\right]$$

Let $0 = t_0 \leq \dots \leq t_{N+1} = t$ be a subdivision of $[0,t]$, then

$$\begin{aligned}
\mathbf{E}\left[f(X_t) \mid \mathcal{F}_0\right] - T_t f(X_0) &= \sum_{i=0}^{N} \mathbf{E}\left[T_{t-t_{i+1}} f(X_{t_{i+1}}) \;\middle|\; \mathcal{F}_0\right] - \mathbf{E}\left[T_{t-t_i} f(X_{t_i}) \mid \mathcal{F}_0\right] \\
&= \sum_{i=0}^{N} \mathbf{E}\left[\mathbf{E}\left[T_{t-t_{i+1}} f(X_{t_{i+1}}) \;\middle|\; \mathcal{F}_{t_i}\right] - T_{t-t_i} f(X_{t_i}) \;\middle|\; \mathcal{F}_0\right].
\end{aligned}$$

Moreover for each $i \in \{0, \dots N\}$, using martingales properties for the first part and semi-groups properties (see for instance Theorem 19.6, p. 372 in [Kal02]) for the second

$$\mathbf{E}\Big[T_{t-t_{i+1}} f(X_{t_{i+1}}) \,\Big|\, \mathcal{F}_{t_i}\Big] - T_{t-t_i} f(X_{t_i}) = \mathbf{E}\Big[\int_{t_i}^{t_{i+1}} LT_{t-t_{i+1}} f(X_s) - LT_{t-s} f(X_{t_i}) \mathrm{d}s \,\Big|\, \mathcal{F}_{t_i}\Big],$$

so

$$\left|\mathbf{E}\left[f(X_t) - T_t f(X_0)\right]\right| \leq \mathbf{E} \sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} \left|LT_{t-t_{i+1}} f(X_s) - LT_{t-s} f(X_{t_i})\right| \mathrm{d}s.$$

By dominated convergence we can conclude. $\square$

Before introducing the definition of a locally Feller family, let us state a result on an application of a time change to a Feller family:

Proposition 4.4. *Let* $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ *be a Feller family with* $\mathrm{C}_0 \times \mathrm{C}_0$*-generator* L*. Then, for any* $g \in \mathrm{C}_b(S, \mathbb{R}_+^*)$*,* $(g \cdot \mathbf{P}_a)_a$ *is a Feller family with* $\mathrm{C}_0 \times \mathrm{C}_0$*-generator* $\overline{gL}$*, taking the closure in* $\mathrm{C}_0(S) \times \mathrm{C}_0(S)$*.*

Proof. Thanks to the first part of Proposition 3.4 and to the Proposition 4.2, the result is only a reformulation of Theorem 2, p. 275 in [Lum73]. For the sake of completeness we give the statement of this result in our context: if $L \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ is the generator of a Feller semi-group, then for any $g \in \mathrm{C}_b(S, \mathbb{R}_+^*)$, $\overline{gL}$ is the generator of a Feller semi-group. □

4.2 Local Feller families and connection with martingale problems

We are ready to introduce the notion of locally Feller family of probabilities. This is given in the following theorem whose proof is technical and it is postponed to the Appendix §A.2

Theorem 4.5 (Definition of a locally Feller family)**.** *If* $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$*, the following four assertions are equivalent:*

1. *(continuity) the family* $(\mathbf{P}_a)_a$ *is Markov and* $a \mapsto \mathbf{P}_a$ *is continuous for the local Skorokhod topology;*

2. *(time change) there exists* $g \in \mathrm{C}(S, \mathbb{R}_+^*)$ *such that* $(g \cdot \mathbf{P}_a)_a$ *is a Feller family;*

3. *(martingale) there exists* $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ *such that* $\mathrm{D}(L)$ *is dense in* $\mathrm{C}_0(S)$ *and*
$$\forall a \in S, \quad \mathbf{P} \in \mathcal{M}(L) \text{ and } \mathbf{P}(X_0 = a) = 1 \Longleftrightarrow \mathbf{P} = \mathbf{P}_a;$$

4. *(localisation) for any open subset* $U \Subset S$ *there exists a Feller family* $(\widetilde{\mathbf{P}}_a)_a$ *such that for any* $a \in S$
$$\mathscr{L}_{\mathbf{P}_a}\left(X^{\tau^U}\right) = \mathscr{L}_{\widetilde{\mathbf{P}}_a}\left(X^{\tau^U}\right).$$

We will call such a family a locally Feller family.
Moreover a locally Feller family $(\mathbf{P}_a)_a$ *is* $(\mathcal{F}_{t+})_t$*-strong Markov and for all* $\mu \in \mathcal{P}(S^\Delta)$*,* $\mathbf{P}_\mu$ *is quasi-continuous.*

Remark 4.6. A natural question is how can we construct locally Feller families? We give here answers to this question.

i) A Feller family is locally Feller.

ii) If $g \in \mathrm{C}(S, \mathbb{R}_+^*)$ and $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is locally Feller, then $(g \cdot \mathbf{P}_a)_a$ is locally Feller. This result is to be compared with the result of Proposition 4.4.

iii) If S is a compact space, a family is locally Feller if and only if it is Feller. This sentence is an easy consequence of the third part of the latter theorem and of Proposition 4.4.

iv) As consequence of the first assertion in Theorem 4.5, if $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\text{loc}}(S))^S$ is locally Feller then the family

$$\begin{array}{ccc} U & \to & \mathcal{P}(\mathbb{D}_{\text{loc}}(U)) \\ a & \mapsto & \mathscr{L}_{\mathbf{P}_a}(\widetilde{X}) \end{array}$$

is locally Feller in the space U. Indeed, it is straightforward to verify that, for any open subset $U \subset S$, the following mapping is continuous,

$$\begin{array}{ccc} \mathbb{D}_{\text{loc}}(S) & \to & \mathbb{D}_{\text{loc}}(U) \\ x & \mapsto & \widetilde{x} \end{array} \quad \text{with} \quad \widetilde{x}_s := \begin{cases} x_s & \text{if } s < \tau^U(x), \\ \Delta & \text{otherwise.} \end{cases}$$

$\Diamond$

Since a locally Feller family on S^Δ is also Feller we can deduce from Theorem 4.5 a characterisation of Feller families in terms of martingale problem. The following theorem is the converse of Proposition 4.2 and provides a first correction of the result Theorem 2.5, p. 283 in [vC92].

Theorem 4.7 (Feller families - first characterisation)**.** *Consider* $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$. *The following assertions are equivalent:*

1. $(\mathbf{P}_a)_a$ *is Feller;*

2. *the family* $(\mathbf{P}_a)_a$ *is Markov,* $\mathbf{P}_a \in \mathcal{P}(\mathbb{D}(S^\Delta))$ *for any* $a \in S$*, and* $S^\Delta \ni a \mapsto \mathbf{P}_a$ *is continuous for the global Skorokhod topology;*

3. *there exists* $L \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ *such that* $\mathrm{D}(L)$ *is dense in* $\mathrm{C}_0(S)$ *and*

$$\forall a \in S^\Delta, \quad \mathbf{P} \in \mathcal{M}_{\mathrm{c}}(L^\Delta) \text{ and } \mathbf{P}(X_0 = a) = 1 \Longleftrightarrow \mathbf{P} = \mathbf{P}_a.$$

We recall that $\mathbf{P}_\Delta$ *is defined by* $\mathbf{P}_\Delta(\forall t \geq 0,\ X_t = \Delta) = 1$.

Proof. Thanks to the fourth point of Proposition 3.4 a Feller family in $\mathcal{P}(\mathbb{D}_{\text{loc}}(S))$ continues to be Feller also in $\mathcal{P}(\mathbb{D}(S^\Delta))$, so a family $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\text{loc}}(S))^S$ is Feller if and only if the family $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}(S^\Delta))^{S^\Delta}$ is Feller. Since S^Δ is compact, using the third point of Remark 4.6, this is also equivalent to say that $(\mathbf{P}_a)_{a \in S^\Delta}$ is locally Feller in S^Δ. Hence the theorem is a consequence of Theorem 4.5 applied on the space S^Δ and to Proposition 4.2. $\square$

The following theorem provides a new relationship between the local Feller property and the Feller property. With the help of Theorem 4.5 we obtain another correction of the Theorem 2.5 p. 283 from [vC92] by adding the missing condition (4.3).

Theorem 4.8 (Feller families - second characterisation)**.** *Consider* $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$. *The following assertions are equivalent:*

1. $(\mathbf{P}_a)_a$ *is Feller;*

2. $(\mathbf{P}_a)_a$ *is locally Feller and*

$$\forall t \geq 0,\ \forall K \subset S \text{ compact set,} \quad \mathbf{P}_a(X_t \in K) \underset{a \to \Delta}{\longrightarrow} 0; \tag{4.3}$$

3. $(\mathbf{P}_a)_a$ *is locally Feller and*

$$\forall t \geq 0,\ \forall K \subset S \text{ compact set,} \quad \mathbf{P}_a\big(\tau^{S\backslash K} < t \wedge \xi\big) \underset{a \to \Delta}{\longrightarrow} 0.$$

Proof. 1⇒2. Take a compact $K \subset S$ and $t \geq 0$. There exists $f \in \mathrm{C}_0(S)$ such that $f \geq \mathbb{1}_K$. Since the family is Feller,

$$\mathbf{P}_a(X_t \in K) \leq \mathbf{E}_a[f(X_t)] \underset{a \to \Delta}{\longrightarrow} 0.$$

2⇒3. Take an open subset $U \Subset S$ such that $K \subset U$ and define

$$\tau := \inf\Big\{ s \geq 0 \,\Big|\, \{(X_0, X_u)\}_{0 \leq u \leq s} \notin U^2 \cup (S\backslash K)^2 \Big\}.$$

By the third sentence of Theorem 4.5, we can apply Lemma 3.6 to $\mathcal{K} := K$, $\mathcal{U} := U^2 \cup (S\backslash K)^2$, $\tau_1 := 0$ and $\tau_2 := \frac{t}{N}$, we get the existence of $N \in \mathbb{N}$ such that

$$\sup_{b \in K} \mathbf{P}_b\Big(\tau \leq \frac{t}{N}\Big) < 1.$$

By Theorem 4.5, $\mathbf{P}_a$ is quasi-continuous for any $a \in S$, so $\mathbf{P}_a(X_{\tau^{S\backslash K}} \in K \cup \{\Delta\}) = 1$. Denoting $\lceil r \rceil$ the smallest integer larger or equal than the real number r, we have

$$\begin{aligned}
\mathbf{P}_a\Big(\exists k \in \mathbb{N},\ k \leq N,\ X_{ktN^{-1}} \in U\Big) &\geq \mathbf{P}_a\Big(\tau^{S\backslash K} < t \wedge \xi,\ X_{tN^{-1}\lceil t^{-1}N\tau^{S\backslash K}\rceil} \in U\Big) \\
&= \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^{S\backslash K} < t \wedge \xi\}} \mathbf{E}_{X_{\tau^{S\backslash K}}}\left[X_s \in U\right]_{|s = tN^{-1}\lceil t^{-1}N\tau^{S\backslash K}\rceil - \tau^{S\backslash K}}\Big] \\
&\geq \mathbf{P}_a\Big(\tau^{S\backslash K} < t \wedge \xi\Big)\Big[1 - \sup_{b \in K} \mathbf{P}\big(\tau \leq tN^{-1}\big)\Big],
\end{aligned}$$

so

$$\mathbf{P}_a\big(\tau^{S\backslash K} < t \wedge \xi\big) \leq \frac{\sum_{k=0}^{N} \mathbf{P}_a\big(X_{ktN^{-1}} \in U\big)}{1 - \sup_{b \in K} \mathbf{P}_b\big(\tau \leq tN^{-1}\big)} \longrightarrow 0, \quad \text{as } a \to \Delta.$$

3⇒1. Consider $f \in \mathrm{C}_0(S)$ and let $t \geq 0$ and $\varepsilon > 0$ be. There exists a compact subset $K \subset S$ such that $\|f\|_{K^c} \leq \varepsilon$, and an open subset $U \Subset S$ such that $K \subset U$ and

$$\sup_{a \notin U} \mathbf{P}_a(\tau^{S\backslash K} < t \wedge \xi) \leq \varepsilon.$$

With the aim of the second assertion of Theorem 4.5 and Proposition 4.4, there exists $g \in \mathrm{C}(S, (0,1])$ such that $g(a) = 1$, for $a \in U$, and $(g \cdot \mathbf{P}_a)_a$ is Feller. Then for any $a \in S$

$$\begin{aligned}
\big|\mathbf{E}_a[f(X_t)] - \mathbf{E}_a[f((g \cdot X)_t)]\big| &\leq \mathbf{E}_a\big[\,|f(X_t) - f((g \cdot X)_t)|\,\mathbb{1}_{\{\tau^U < t\}}\big] \\
&\leq \mathbf{E}_a\big[\,|f(X_t)|\,\mathbb{1}_{\{\tau^U < t\}}\big] + \mathbf{E}_a\big[\,|f((g \cdot X)_t)|\,\mathbb{1}_{\{\tau^U < t\}}\big].
\end{aligned}$$

By Theorem 4.5, $\mathbf{P}_a$ is quasi-continuous, so $\mathbf{P}_a(X_{\tau^U} \notin U) = 1$, we have

$$\begin{aligned}
\mathbf{E}_a\Big[\big|f(X_t)\big|\mathbb{1}_{\{\tau^U < t\}}\Big] &= \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^U < t\}} \mathbf{E}_{X_{\tau^U}}\left[|f(X_s)|\right]_{|s = t - \tau^U}\Big] \\
&= \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^U < t\}} \mathbf{E}_{X_{\tau^U}}\left[|f(X_s)|\mathbb{1}_{\{\tau^{S\backslash K} < t \wedge \xi\}}\right]_{|s = t - \tau^U}\Big] \\
&\quad + \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^U < t\}} \mathbf{E}_{X_{\tau^U}}\left[|f(X_s)|\mathbb{1}_{\{\tau^{S\backslash K} \geq t \wedge \xi\}}\right]_{|s = t - \tau^U}\Big] \\
&\leq \|f\| \sup_{a \notin U} \mathbf{P}_a(\tau^{S\backslash K} < t \wedge \xi) + \|f\|_{K^c} \leq (\|f\| + 1)\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}_a\Big[\,|f(g\cdot X_t)|\,\mathbb{1}_{\{\tau^U<t\}}\Big] &= \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^U<t\}}\mathbf{E}_{X_{\tau^U}}\left[|f(g\cdot X_s)|\right]_{|s=t-\tau^U}\Big]\\ &= \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^U<t\}}\mathbf{E}_{X_{\tau^U}}\left[|f(g\cdot X_s)|\mathbb{1}_{\{\tau^{S\setminus K}<t\wedge\xi\}}\right]_{|s=t-\tau^U}\Big]\\ &\quad+\mathbf{E}_a\Big[\mathbb{1}_{\{\tau^U<t\}}\mathbf{E}_{X_{\tau^U}}\left[|f(g\cdot X_s)|\mathbb{1}_{\{\tau^{S\setminus K}\geq t\wedge\xi\}}\right]_{|s=t-\tau^U}\Big]\\ &\leq \|f\|\sup_{a\notin U}\mathbf{P}_a(\tau^{S\setminus K}<t\wedge\xi)+\|f\|_{K^c}\leq(\|f\|+1)\varepsilon.\end{aligned}$$

Hence

$$\big|\mathbf{E}_a[f(X_t)]-\mathbf{E}_a[f((g\cdot X)_t)]\big|\leq 2(\|f\|+1)\varepsilon,$$

so, since $a\mapsto\mathbf{E}_a[f((g\cdot X)_t)]$ is in $\mathrm{C}_0(S)$, letting $\varepsilon\to0$ we deduce that $a\mapsto\mathbf{E}_a[f(X_t)]$ is in $\mathrm{C}_0(S)$, hence $(\mathbf{P}_a)_a$ is Feller. $\square$

4.3 Generator description and convergence

In this subsection we analyse the generator of a locally Feller family:

Definition 4.9. Let $(\mathbf{P}_a)_a\in\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ be a locally Feller family. The $\mathrm{C}_0\times\mathrm{C}$-generator L of $(\mathbf{P}_a)_a\in\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is the set of functions $(f,g)\in\mathrm{C}_0(S)\times\mathrm{C}(S)$ such that for any $a\in S$ and any open subset $U\Subset S$

$$f(X_{t\wedge\tau^U})-\int_0^{t\wedge\tau^U}g(X_s)\mathrm{d}s\text{ is a }\mathbf{P}_a\text{-martingale.}$$

Theorem 4.10 (Description of the generator)**.** *Let $(\mathbf{P}_a)_a\in\mathcal{P}(\mathbb{D}_{loc}(S))^S$ be a locally Feller family and L its $\mathrm{C}_0\times\mathrm{C}$-generator. Then $\mathrm{D}(L)$ is dense, L is an univariate closed sub-vector space,*

$$\mathcal{M}(L)=\{\mathbf{P}_\mu\}_{\mu\in\mathcal{P}(S^\Delta)},$$

L satisfies the positive maximum principle and does not have a strict linear extension satisfying the positive maximum principle. Moreover for any $(f,g)\in\mathrm{C}_0(S)\times\mathrm{C}(S)$ we have equivalence between:

1. *$(f,g)\in L$;*
2. *for all $a\in S$, there exists an open set $U\subset S$ containing a such that*

$$\lim_{t\to0}\frac{1}{t}\Big(\mathbf{E}_a\left[f(X_{t\wedge\tau^U})\right]-f(a)\Big)=g(a);$$

3. *for all open subset $U\Subset S$ and $a\in U$*

$$\lim_{t\to0}\frac{1}{t}\Big(\mathbf{E}_a\left[f(X_{t\wedge\tau^U})\right]-f(a)\Big)=g(a).$$

Proof. Thanks to the third assertion of Theorem 4.5 and Proposition 3.12, we have $\mathcal{M}(L)=\{\mathbf{P}_\nu\}_{\nu\in\mathcal{P}(S^\Delta)}$ and $\mathrm{D}(L)$ is dense. By the point 2 of Proposition 3.4, L is a closed sub-vector space. The fourth part of Remark 3.3 allows us to conclude that: L is univariate, L satisfies the positive maximum principle, and that 1⇒3. It is straightforward that 3⇒2. Thanks to Theorem 3.9, L does not have strict linear extension satisfying the positive maximum principle. Finally the set of (f,g) satisfying the statement 2 is a linear extension of L satisfying the positive maximum principle, so by the previous assertion 2⇒1. □

Remark 4.11. One can ask, as in Remark 4.6, how can we obtain the generator of a locally Feller family? A similar statement of first one in the cited remark is Proposition 4.12 below. The second one is straightforward: if $g \in \mathrm{C}(S, \mathbb{R}_+^*)$ and if L is the $\mathrm{C}_0 \times \mathrm{C}$-generator of $(\mathbf{P}_a)_a$, then gL is the $\mathrm{C}_0 \times \mathrm{C}$-generator of $(g \cdot \mathbf{P}_a)_a$, as we can see by using 1 from Proposition 3.4. ♢

Proposition 4.12. *Let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be a Feller family, L_0 its $\mathrm{C}_0 \times \mathrm{C}_0$-generator and L its $\mathrm{C}_0 \times \mathrm{C}$-generator. Then taking the closure in $\mathrm{C}_0(S) \times \mathrm{C}(S)$*

$$L_0 = L \cap \mathrm{C}_0(S) \times \mathrm{C}_0(S), \quad \textit{and} \quad L = \overline{L_0}.$$

Proof. First, we have $L_0 \subset L \cap \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ by Proposition 4.2. Hence $L \cap \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ is an extension of L_0 satisfying the positive maximum principle, so by a maximality result (a consequence of Hille-Yoshida's, see for instance Lemma 19.12, p. 377 in [Kal02]), $L_0 = L \cap \mathrm{C}_0(S) \times \mathrm{C}_0(S)$.

Secondly, take $(f,g) \in L$. Let $h \in \mathrm{C}(S, \mathbb{R}_+^*)$ be a bounded function such that $hg \in \mathrm{C}_0(S)$. Thanks to Proposition 4.4 the $\mathrm{C}_0 \times \mathrm{C}_0$-generator of $(h \cdot \mathbf{P}_a)_a$ is $\overline{hL_0}^{\mathrm{C}_0(S) \times \mathrm{C}_0(S)}$. Moreover the $\mathrm{C}_0 \times \mathrm{C}$-generator of $(h \cdot \mathbf{P}_a)_a$ is hL. Hence applying the first step to the family $(h \cdot \mathbf{P}_a)_a$ we deduce that

$$\overline{hL_0}^{\mathrm{C}_0(S) \times \mathrm{C}_0(S)} = (hL) \cap \mathrm{C}_0(S) \times \mathrm{C}_0(S),$$

so $(f, hg) \in \overline{hL_0}^{\mathrm{C}_0(S) \times \mathrm{C}_0(S)}$ and $(f,g) \in \overline{L_0}^{\mathrm{C}_0(S) \times \mathrm{C}(S)}$. □

Theorem 4.13 (Convergence of locally Feller family)**.** *For $n \in \mathbb{N} \cup \{\infty\}$, let $(\mathbf{P}_a^n)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be a locally Feller family and let L_n be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$. Suppose that for any $n \in \mathbb{N}$, $\overline{L_n}$ is the generator of $(\mathbf{P}_a^n)_a$, suppose also that $\mathrm{D}(L_\infty)$ is dense in $\mathrm{C}_0(S)$ and*

$$\mathcal{M}(L_\infty) = \{\mathbf{P}_\mu^\infty\}_{\mu \in \mathcal{P}(S^\Delta)}.$$

Then we have equivalence between:

1. *the mapping*
$$\begin{array}{ccc} \mathbb{N} \cup \{\infty\} \times \mathcal{P}(S^\Delta) & \to & \mathcal{P}\left(\mathbb{D}_{loc}(S)\right) \\ (n, \mu) & \mapsto & \mathbf{P}_\mu^n \end{array}$$
is weakly continuous for the local Skorokhod topology;

2. *for any $a_n, a \in S$ such that $a_n \to a$, $\mathbf{P}_{a_n}^n$ converges weakly for the local Skorokhod topology to $\mathbf{P}_a^\infty$, as $n \to \infty$;*

3. *for any $(f,g) \in L_\infty$, for each n, there exists $(f_n, g_n) \in L_n$ such that $f_n \xrightarrow[n\to\infty]{\mathrm{C}_0} f$, $g_n \xrightarrow[n\to\infty]{\mathrm{C}} g$.*

Remark 4.14. 1) We may deduce a similar theorem for Feller process.
2) An improvement with respect to the classical result of convergence Theorem 19.25, p. 385, in [Kal02], is that one does not need to know that $\overline{L_\infty}$ is the generator of the family, but only the fact that the martingale local problem is well-posed. Let us point out that there are situations were the generator is not known. $\Diamond$

Proof of Theorem 4.13. It is straightforward that 1⇒2. The implication 3⇒1 is a consequence of Proposition 3.5.
We prove that 2⇒3. We can suppose that L_∞ is the generator of $(\mathbf{P}_a^\infty)_a$. It is straightforward to obtain that

$$\begin{array}{ccc} \mathbb{N} \cup \{\infty\} \times S^\Delta & \to & \mathcal{P}\left(\mathbb{D}_{\mathrm{loc}}(S)\right) \\ (n, a) & \mapsto & \mathbf{P}_a^n \end{array}$$

is weakly continuous for the local Skorokhod topology. Thanks to Proposition 2.1, on the connection between $\mathbb{D}_{\mathrm{loc}}(S)$ and $\mathbb{D}(S^\Delta)$, there exists $h \in \mathrm{C}(S, \mathbb{R}_+^*)$ such that, for any $n \in \mathbb{N} \cup \{\infty\}$ and $a \in S$,

$$h \cdot \mathbf{P}_a^n\big(\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)\big) = 1,$$

and the mapping

$$\begin{array}{ccc} \mathbb{N} \cup \{\infty\} \times S^\Delta & \to & \mathcal{P}\left(\mathbb{D}(S^\Delta)\right) \\ (n, a) & \mapsto & h \cdot \mathbf{P}_a^n \end{array}$$

is weakly continuous for the global Skorokhod topology. Thanks to Theorem 4.7, $(\mathbf{P}_a^n)_a$ is a Feller family, for all $n \in \mathbb{N} \cup \{\infty\}$. From Remark 4.11 and Proposition 4.12 we deduce that: $h\overline{L_n} \cap \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ is the $\mathrm{C}_0 \times \mathrm{C}_0$-generator of $(\mathbf{P}_a^n)_a$ for $n \in \mathbb{N}$, $hL_\infty \cap \mathrm{C}_0(S)^2$ is the $\mathrm{C}_0 \times \mathrm{C}_0$-generator of $(\mathbf{P}_a^\infty)_a$ and

$$\overline{hL_\infty \cap \mathrm{C}_0(S) \times \mathrm{C}_0(S)}^{\mathrm{C}_0(S) \times \mathrm{C}(S)} = hL_\infty.$$

Take arbitrary elements $a, a_1, a_2 \ldots \in S^\Delta$ and $t, t_1, t_2 \ldots \in \mathbb{R}_+$ such that $a_n \to a$ and $t_n \to t$, then $h \cdot \mathbf{P}_{a_n}^n$ converges weakly for the global Skorokhod topology to $h \cdot \mathbf{P}_a^\infty$. By Theorem 4.5, $h \cdot \mathbf{P}_a^\infty$ is quasi-continuous, so $h \cdot \mathbf{P}_a^\infty(X_{t-} = X_t) = 1$. Hence, for any $f \in \mathrm{C}_0(S)$

$$h \cdot \mathbf{E}_{a_n}^n[f(X_{t_n})] \underset{n \to \infty}{\longrightarrow} h \cdot \mathbf{E}_a^\infty[f(X_t)].$$

From here we can deduce that, for any $t \geq 0$

$$\lim_{n \to \infty} \sup_{s \leq t} \sup_{a \in S} \left| h \cdot \mathbf{E}_a^n[f(X_s)] - h \cdot \mathbf{E}_a^\infty[f(X_s)] \right| = 0.$$

Here and elsewhere we denote by $\mathbf{E}_a^n$ the expectation with respect to the probability measure $\mathbf{P}_a^n$. Hence by Trotter-Kato's theorem (cf. Theorem 19.25, p. 385, [Kal02]), for any $(f, g) \in hL_\infty \cap \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ there exist $(f_n, g_n) \in h\overline{L_n} \cap \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ such that $(f_n, g_n) \underset{n \to \infty}{\longrightarrow} (f, g)$, so it is straightforward to deduce statement 3. $\square$

4.4 Localisation for martingale problems and generators

We are interested to the localisation procedure. More precisely, assume that $\mathcal{U}$ is a recovering of S by open sets and, for each $U \in \mathcal{U}$, let $(\mathbf{P}_a^U)_a$ be a locally Feller family, such that for all $U_1, U_2 \in \mathcal{U}$ and $a \in S$

$$\mathscr{L}_{\mathbf{P}_a^{U_1}}\left(X^{\tau^{U_1 \cap U_2}}\right) = \mathscr{L}_{\mathbf{P}_a^{U_2}}\left(X^{\tau^{U_1 \cap U_2}}\right).$$

We wonder if there exists a locally Feller family $(\mathbf{P}_a)_a$ such that for all $U \in \mathcal{U}$ and $a \in S$

$$\mathscr{L}_{\mathbf{P}_a}\left(X^{\tau^U}\right) = \mathscr{L}_{\mathbf{P}_a^U}\left(X^{\tau^U}\right) \ ?$$

An attempt to give an answer to this question needs to reformulate it in terms of generators of locally Feller families. This reformulation is suggested by the following:

Proposition 4.15. *Let $L_1, L_2 \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ be such that $\mathrm{D}(L_1) = \mathrm{D}(L_2)$ is dense in $\mathrm{C}_0(S)$ and take an open subset $U \subset S$. Suppose that*

- *the martingale local problem associated to L_1 is well-posed, and,*
- *for all $a \in U$ there exists $\mathbf{P}^2 \in \mathcal{M}(L_2)$ with $\mathbf{P}^2(X_0 = a) = 1$.*

Then

$$\forall \mathbf{P}^2 \in \mathcal{M}(L_2),\ \exists \mathbf{P}^1 \in \mathcal{M}(L_1), \qquad \mathscr{L}_{\mathbf{P}^2}\left(X^{\tau^U}\right) = \mathscr{L}_{\mathbf{P}^1}\left(X^{\tau^U}\right) \tag{4.4}$$

if and only if

$$\forall (f,g) \in L_2, \qquad g_{|U} = (L_1 f)_{|U}.$$

We postpone the proof of this proposition and we state two main results of localisation.

Theorem 4.16 (Localisation for the martingale problem)**.** *Let L be a linear subspace of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ with $\mathrm{D}(L)$ dense in $\mathrm{C}_0(S)$. Suppose that for all $a \in S$ there exist a neighbourhood V of a and a subset $\widetilde{L}$ of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ such that the martingale local problem associated to $\widetilde{L}$ is well-posed and such that*

$$\left\{(f, g_{|V}) \ \middle|\ (f,g) \in L\right\} = \left\{(f, g_{|V}) \ \middle|\ (f,g) \in \widetilde{L}\right\}. \tag{4.5}$$

Then the martingale local problem associated to L is well-posed.

Proof. Thanks to Theorem 3.9, to prove the existence of a solution for the martingale local problem it suffices to prove that L satisfies the positive maximum principle. Let $(f,g) \in L$ and $a \in S$ be such that $f(a) = \max f \geq 0$. Then there exist a neighbourhood V of a and a subset $\widetilde{L}$ of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ such that the martingale local problem associated to $\widetilde{L}$ is well-posed and (4.5) holds true. In particular, by Theorem 3.9, $\widetilde{L}$ satisfies the positive maximum principle and so

$$g(a) = \widetilde{L} f(a) \leq 0.$$

To prove the uniqueness of the solution for the martingale local problem, we take $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{M}(L)$ and an arbitrary open subset $V \Subset S$. By hypothesis and using the relative compactness of V, there exist $N \in \mathbb{N}$, open subsets $U_1, \dots, U_N \subset S$ and subsets $L_1, \dots, L_N \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ such that $V \Subset \bigcup_n U_n$, such that for all $1 \leq n \leq N$ the martingale local problem associated to L_n is well-posed and such that

$$\left\{(f, g_{|U_n}) \ \middle|\ (f,g) \in L\right\} = \left\{(f, g_{|U_n}) \ \middle|\ (f,g) \in \widetilde{L_n}\right\}.$$

At this level of the proof we need a technical but important result:

Lemma 4.17. *Let U be an open subset of S and L be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ such that $\mathrm{D}(L)$ is dense in $\mathrm{C}(S)$ and the martingale local problem associated to L is well-posed. Then there exist a subset L_0 of L and a function h_0 of $\mathrm{C}(S, \mathbb{R}_+)$ with $\{h_0 \neq 0\} = U$ such that $\overline{L} = \overline{L_0}$, such that $h_0 L_0 \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ and such that: for any $h \in \mathrm{C}(S, \mathbb{R}_+)$ with $\{h \neq 0\} = U$ and $\sup_{a \in U}(h/h_0)(a) < \infty$, the martingale problem associated to $(hL_0)^\Delta$ is well-posed in $\mathbb{D}(S^\Delta)$. Recall that $(hL_0)^\Delta$ is defined by (4.1) and that the associated martingale problem is defined by (4.2).*

We postpone the proof of this lemma to the Appendix (see § A.3) and we proceed with the proof of our theorem.

Applying Lemma 4.17, there exist a subset D of $\mathrm{C}_0(S)$ and a function h of $\mathrm{C}(S, \mathbb{R}_+)$ with $\{h \neq 0\} = V$ such that for all $1 \leq n \leq N$: $\overline{L_n} = \overline{L_{n\,|D}}$, $hL_{n\,|D} \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ and the martingale problem associated to $\left(hL_{n\,|D}\right)^\Delta$ is well-posed. Denote $L_{N+1} := D \times \{0\}$ and $U^{N+1} := S^\Delta \backslash \overline{V}$. We may now apply Theorem 6.2 and also Theorem 6.1 pp. 216-217, in [EK86] to $hL_{|D}$ and $(U_n)_{1 \leq n \leq N+1}$ and we deduce that the martingale problem associated to $(hL_{|D})^\Delta$ is well-posed. Hence $h \cdot \mathbf{P}^1 = h \cdot \mathbf{P}^2$ so

$$\mathscr{L}_{\mathbf{P}^1}(X^{\tau^V}) = \mathscr{L}_{\mathbf{P}^2}(X^{\tau^V}).$$

We obtain the result by letting V to grow toward S. This ends the proof of the theorem except to the proof of Lemma 4.17 postponed to § A.3. □

Theorem 4.18 (Localisation of generator)**.** *Let L be a linear subspace of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ with $\mathrm{D}(L)$ dense in $\mathrm{C}_0(S)$. Suppose that for all subsets $V \Subset S$ there exists a linear subspace $\widetilde{L}$ of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ such that $\overline{\widetilde{L}}$ is the generator of a locally Feller family and*

$$\left\{(f, g_{|V}) \;\middle|\; (f,g) \in L\right\} = \left\{(f, g_{|V}) \;\middle|\; (f,g) \in \widetilde{L}\right\}.$$

Then $\overline{L}$ is the generator of a locally Feller family.

Proof. Thanks to Theorem 4.16 the martingale local problem associated to L is well-posed, let $(\mathbf{P}_a^\infty)_a$ the locally Feller family associate to L. Let L_∞ be the generator of $(\mathbf{P}_a^\infty)_a$. Let $U_n \Subset S$ be an increasing sequence of open subsets such that $S = \bigcup_n U_n$ and let $L_n \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ be such that for all $n \in \mathbb{N}$, $\overline{L_n}$ is the generator of a locally Feller family $(\mathbf{P}_a^n)_a$ and

$$\left\{(f, g_{|U_n}) \;\middle|\; (f,g) \in L\right\} = \left\{(f, g_{|U_n}) \;\middle|\; (f,g) \in L_n\right\}. \tag{4.6}$$

Then by using Proposition 4.15, for all $n \in \mathbb{N}$ and $a \in S$

$$\mathscr{L}_{\mathbf{P}_a^\infty}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}_a^n}\left(X^{\tau^{U_n}}\right). \tag{4.7}$$

At this level we use a result of localisation of the continuity stated and proved in § A.2, Lemma A.2. Therefore, by (4.7) the mapping

$$\begin{array}{ccc} \mathbb{N} \cup \{\infty\} \times S^\Delta & \to & \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S)) \\ (n,a) & \mapsto & \mathbf{P}_a^n \end{array}$$

is weakly continuous for the local Skorokhod topology. Hence by Theorem 4.13, for any $f \in \mathrm{D}(L_\infty)$ there exists $(f_n)_n \in \mathrm{D}(L)^{\mathbb{N}}$ such that $(f_n, L_n f_n) \underset{n\to\infty}{\longrightarrow} (f, L_\infty f)$, so by (4.6) $(f_n, L f_n) \underset{n\to\infty}{\longrightarrow} (f, L_\infty f)$. Hence $\overline{L} = L_\infty$ is the generator of a locally Feller family. The proof of the theorem is complete except for the proof of Proposition 4.15. $\square$

Proof of Proposition 4.15. Suppose (4.4). For each $a \in U$, take an open subset $V \subset U$, $\mathbf{P}^1 \in \mathcal{M}(L_1)$ and $\mathbf{P}^2 \in \mathcal{M}(L_2)$ such that $a \in V \Subset S$ and $\mathbf{P}^1(X_0 = a) = \mathbf{P}^2(X_0 = a) = 1$. By using the fourth part of Remark 3.3 we have for each $(f, g) \in L_2$

$$g(a) = \lim_{t\to 0} \frac{1}{t}\Big(\mathbf{E}^2\left[f(X_{t\wedge\tau^V})\right] - f(a)\Big) = \lim_{t\to 0} \frac{1}{t}\Big(\mathbf{E}^1\left[f(X_{t\wedge\tau^V})\right] - f(a)\Big) = L_1 f(a).$$

For the converse, by Lemma 4.17 there exists $h \in \mathrm{C}(S, \mathbb{R}_+)$ with $\{h \neq 0\} = U$ such that the martingale local problem associated to $hL_1 = hL_2$ is well-posed. Take $\mathbf{P}^2 \in \mathcal{M}(L_2)$ and let $\mathbf{P}^1 \in \mathcal{M}(L_1)$ be such that $\mathscr{L}_{\mathbf{P}^1}(X_0) = \mathscr{L}_{\mathbf{P}^2}(X_0)$, then $h\cdot\mathbf{P}^1, h\cdot\mathbf{P}^2 \in \mathcal{M}(hL_1)$ so $h\cdot\mathbf{P}^1 = h\cdot\mathbf{P}^2$ and hence (4.4) is verified. $\square$

A Appendix: proof of technical results

A.1 Proofs of Propositions 3.4 and 3.5

Remind that the proofs of Propositions 3.4 and 3.5 are interlaced and will be performed in several ordered steps.

Proof of Lemma 3.6. Take a metric d on S and $a_0 \in \mathcal{K}$, then there exists $\varepsilon_0 > 0$ such that $B(a_0, 4\varepsilon_0) \Subset S$ and $\left\{(a,b) \in S^2 \;\middle|\; a \in \mathcal{K},\ d(a,b) < 3\varepsilon_0\right\} \subset \mathcal{U}$. Define

$$\widetilde{f}(a) := \begin{cases} 1, & \text{if } d(a,a_0) \le \varepsilon_0, \\ 0, & \text{if } d(a,a_0) \ge 2\varepsilon_0, \\ 2 - \frac{d(a,a_0)}{\varepsilon_0}, & \text{if } \varepsilon_0 \le d(a,a_0) \le 2\varepsilon_0. \end{cases}$$

Then

$$\widetilde{f} \in \mathrm{C}_0(S), \quad 0 \le \widetilde{f} \le 1, \quad \forall\, a \in B(a_0, \varepsilon_0),\ \widetilde{f}(a) = 1 \ \text{ and } \ \{\widetilde{f} \neq 0\} \subset B(a, 3\varepsilon_0).$$

Let $\eta > 0$ be arbitrary. There exist $(f,g) \in L$ and a sequence $(f_n, g_n) \in L_n$ such that $\|f - \widetilde{f}\| \le \eta$ and the sequence $(f_n, g_n)_n$ converges to (f,g) for the topology of $\mathrm{C}_0(S)\times\mathrm{C}(S)$. Let $\tau_1 \le \tau_2$ be $(\mathcal{F}_{t+})_t$-stopping times and let n be in $\mathbb{N}$, assume that $\mathbf{P} \in \mathcal{M}(L_n)$. For $\varepsilon < 3\varepsilon_0$ we denote

$$\sigma_\varepsilon := \inf\Big\{t \ge \tau_1 \,\Big|\, t \ge \xi \text{ or } \sup_{\tau_1 \le s \le t} d(X_{\tau_1}, X_s) \ge \varepsilon\Big\}.$$

Let an open subset $V \Subset S$ be such that $V \supset B(a_0, 4\varepsilon_0)$. If $t \geq 0$ and $\varepsilon < 3\varepsilon_0$ we can write

$$\begin{aligned}
&\mathbf{E}\left[f_n(X_{t\wedge\tau^V\wedge\sigma_\varepsilon\wedge\tau_2})\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K}\}}\right] \\
&\qquad = \mathbf{E}\left[\left(f_n(X_{t\wedge\tau^V\wedge\tau_1}) + \int_{t\wedge\tau^V\wedge\tau_1}^{t\wedge\tau^V\wedge\sigma_\varepsilon\wedge\tau_2} g_n(X_s)\mathrm{d}s\right)\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K}\}}\right] \\
&\qquad \geq \mathbf{E}\left[\widetilde{f}(X_{t\wedge\tau^V\wedge\tau_1})\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K}\}}\right] - \|\widetilde{f} - f_n\| \\
&\qquad\quad + \mathbf{E}\left[\int_{t\wedge\tau^V\wedge\tau_1}^{t\wedge\tau^V\wedge\sigma_\varepsilon\wedge\tau_2} g_n(X_s)\mathrm{d}s\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K}\}}\right] \\
&\qquad \geq \mathbf{P}\big(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K}\big) - \mathbf{P}\big(t\wedge\tau^V < \tau_1 < \xi\big) - \eta - \|f - f_n\| \\
&\qquad\quad - \mathbf{E}\big[(\tau_2-\tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}\big]\cdot\|g_n\|_{B(a_0,4\varepsilon_0)}.
\end{aligned} \tag{A.1}$$

Splitting on the events $\{\sigma_\varepsilon > \tau_2\}$, $\{\sigma_\varepsilon \leq t\wedge\tau^V\wedge\tau_2\}$ and $\{t\wedge\tau^V < \sigma_\varepsilon \leq \tau_2\}$

$$\begin{aligned}
&\mathbf{E}\left[f_n(X_{t\wedge\tau^V\wedge\sigma_\varepsilon\wedge\tau_2})\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K}\}}\right] \\
&\qquad \leq \mathbf{P}\big(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K},\ \sigma_\varepsilon > \tau_2\big) + \eta + \|f - f_n\| \\
&\qquad\quad + \mathbf{E}\big[f_n(X_{\sigma_\varepsilon})\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\}}\big] + \mathbf{P}\big(X_{\tau_1}\in\mathcal{K},\ t<\tau_2\big) + \eta + \|f-f_n\|.
\end{aligned} \tag{A.2}$$

Hence by (A.1) and (A.2),

$$\begin{aligned}
&\mathbf{P}\big(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K},\ \tau(\tau_1)\leq\tau_2\big) \leq \mathbf{P}\big(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K},\ \sigma_\varepsilon\leq\tau_2\big) \\
&\quad \leq 3\eta + 3\|f-f_n\| + \mathbf{P}\big(t\wedge\tau^V<\tau_1<\xi\big) + \mathbf{E}\big[(\tau_2-\tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}\big]\cdot\|g_n\|_{B(a_0,4\varepsilon_0)} \\
&\qquad\qquad + \mathbf{E}\big[f_n(X_{\sigma_\varepsilon})\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\}}\big] + \mathbf{P}\big(X_{\tau_1}\in\mathcal{K},\ t<\tau_2\big).
\end{aligned}$$

Since the limit $\lim_{\varepsilon\uparrow 3\varepsilon_0} X_{\sigma_\varepsilon}$ exists and is in $S^\Delta\backslash B(X_{\tau_1}, 3\varepsilon_0)$ we have

$$\begin{aligned}
\limsup_{\varepsilon\uparrow3\varepsilon_0}\mathbf{E}\big[f_n(X_{\sigma_\varepsilon})\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\}}\big] &\leq \|f_n\|_{B(a_0,2\varepsilon_0)^c} \leq \|f-f_n\| + \|f-\widetilde{f}\| + \|\widetilde{f}\|_{B(a_0,2\varepsilon_0)^c} \\
&\leq \|f-f_n\| + \delta,
\end{aligned}$$

so

$$\begin{aligned}
&\mathbf{P}\big(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K},\ \tau(\tau_1)\leq\tau_2\big) \\
&\qquad\qquad \leq 4\eta + 4\|f-f_n\| + \mathbf{P}\big(t\wedge\tau^V<\tau_1<\xi\big) \\
&\qquad\qquad\quad + \mathbf{E}\big[(\tau_2-\tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}\big]\cdot\|g_n\|_{B(a_0,4\varepsilon_0)} + \mathbf{P}\big(X_{\tau_1}\in\mathcal{K},\ t<\tau_2\big).
\end{aligned}$$

Letting $t\to\infty$ and V growing to S, $\mathbf{P}\big(t\wedge\tau^V<\tau_1<\xi\big)$ tends to 0, hence

$$\begin{aligned}
&\mathbf{P}\big(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K},\ \tau(\tau_1)\leq\tau_2\big) \\
&\qquad \leq 4\eta + 4\|f-f_n\| + \mathbf{E}[(\tau_2-\tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}]\cdot\|g_n\|_{B(a_0,4\varepsilon_0)} \\
&\qquad\qquad + \mathbf{P}\big(X_{\tau_1}\in\mathcal{K},\ \tau_2=\infty\big).
\end{aligned}$$

So letting $n\to\infty$, $\mathbf{E}[(\tau_2-\tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}]\to 0$ and $\eta\to0$ we deduce that for each $\varepsilon>0$ there exist $n_0\in\mathbb{N}$ and $\delta>0$ such that: for any $n\geq n_0$, $(\mathcal{F}_{t+})_t$-stopping times $\tau_1\leq\tau_2$ and $\mathbf{P}\in\mathcal{M}(L_n)$ satisfying $\mathbf{E}[(\tau_2-\tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}]\leq\delta$ we have

$$\mathbf{P}(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K},\ \tau(\tau_1)\leq\tau_2)\leq\varepsilon.$$

We conclude since a_0 was arbitrary chosen in $\mathcal{K}$ and by using a finite recovering of the compact $\mathcal{K}$. $\square$

Proof of part 4 of Proposition 3.4.

Step 1: we prove the $(\mathcal{F}_{t+})_t$-quasi-continuity before the explosion time ξ. Let τ_n, τ be $(\mathcal{F}_{t+})_t$-stopping times and denote $\widetilde{\tau}_n := \inf_{m\geq n}\tau_m$, $\widetilde{\tau} := \sup_{n\in\mathbb{N}}\widetilde{\tau}_n$ and

$$A := \begin{cases} \lim_{n\to\infty} X_{\widetilde{\tau}_n}, & \text{if the limit exists,}\\ \Delta, & \text{otherwise.}\end{cases}$$

Let d be a metric on S^Δ and take $\varepsilon > 0$, $t \geq 0$ and an open subset $U \Subset S$. Since

$$\lim_{n\to\infty} \mathbf{E}\big[\widetilde{\tau}\wedge t\wedge\tau^U - \widetilde{\tau}_n\wedge t\wedge\tau^U\big] = 0,$$

by Lemma 3.6 applied to $\mathcal{K} := \overline{U}$ and $\mathcal{U} = \big\{(a,b)\in S^2 \;\big|\; d(a,b) < \varepsilon\big\}$ we get

$$\mathbf{P}\big(X_{\widetilde{\tau}_n\wedge t\wedge\tau^U} \in U,\ d(X_{\widetilde{\tau}_n\wedge t\wedge\tau^U}, X_{\widetilde{\tau}\wedge t\wedge\tau^U}) \geq \varepsilon\big) \underset{n\to\infty}{\longrightarrow} 0.$$

Hence

$$\begin{aligned}\mathbf{P}\big(\widetilde{\tau}\leq t\wedge\tau^U,\ d(X_{\widetilde{\tau}_n}, X_{\widetilde{\tau}}) \geq \varepsilon\big) &= \mathbf{P}\big(\widetilde{\tau}_n < \widetilde{\tau}\leq t\wedge\tau^U,\ d(X_{\widetilde{\tau}_n}, X_{\widetilde{\tau}}) \geq \varepsilon\big)\\ &\leq \mathbf{P}\big(X_{\widetilde{\tau}_n\wedge t\wedge\tau^U} \in U,\ d(X_{\widetilde{\tau}_n\wedge t\wedge\tau^U}, X_{\widetilde{\tau}\wedge t\wedge\tau^U}) \geq \varepsilon\big).\end{aligned}$$

Letting $n\to\infty$ on the both sides of the latter inequality we obtain that

$$\mathbf{P}\big(\widetilde{\tau}\leq t\wedge\tau^U,\ d(A, X_{\widetilde{\tau}}) \geq \varepsilon\big) = 0.$$

Then, successively if $t\to\infty$, U growing to S and $\varepsilon\to 0$ it follows that

$$\mathbf{P}\big(\widetilde{\tau} < \infty,\ \{X_s\}_{s<\widetilde{\tau}} \Subset S,\ A \neq X_{\widetilde{\tau}}\big) = 0.$$

We deduce

$$\begin{aligned}&\mathbf{P}\big(X_{\tau_n} \underset{n\to\infty}{\not\longrightarrow} X_\tau,\ \tau_n \underset{n\to\infty}{\longrightarrow} \tau < \infty,\ \{X_s\}_{s<\tau} \Subset S\big)\\ &\qquad = \mathbf{P}\big(A \neq X_{\widetilde{\tau}},\ \tau_n \underset{n\to\infty}{\longrightarrow} \tau = \widetilde{\tau} < \infty,\ \{X_s\}_{s<\widetilde{\tau}} \Subset S\big) = 0. \qquad \text{(A.3)}\end{aligned}$$

Step 2: we prove that $\mathbf{P}\big(\mathbb{D}_{loc}(S)\cap\mathbb{D}(S^\Delta)\big) = 1$. Let K be a compact subset of S and take an open subset $U \Subset S$ containing K. For $n\in\mathbb{N}$ define the stopping times

$$\begin{aligned}&\sigma_0 := 0,\\ &\tau_n := \inf\{t\geq\sigma_n \mid \{X_s\}_{\sigma_n\leq s\leq t} \not\Subset S\backslash K\},\\ &\sigma_{n+1} := \inf\{t\geq\tau_n \mid \{X_s\}_{\tau_n\leq s\leq t} \not\Subset U\}.\end{aligned}$$

Let $V_k \Subset S\backslash K$ be an increasing sequence of open subset such that $S\backslash K = \bigcup_k V_k$, and denote $\tau_n^k := \inf\{t\geq\sigma_n \mid \{X_s\}_{\sigma_n\leq s\leq t} \not\Subset V_k\}$. Then, by (A.3)

$$\mathbf{P}\Big(X_{\tau_n^k} \underset{k\to\infty}{\not\longrightarrow} X_{\tau_n},\ \tau_n < \infty,\ \{X_s\}_{s<\tau_n} \Subset S\Big) = 0,$$

so $\{\tau_n < \xi\} = \{X_{\tau_n} \in K\}$ **P**-almost surely. Thanks to Lemma 3.6 applied to $\mathcal{K} := K$ and $\mathcal{U} := U^2 \cup (S\backslash K)^2$

$$\sup_{n\in\mathbb{N}} \mathbf{P}\big(X_{\tau_n} \in K,\ \sigma_{n+1} < \tau_n + \varepsilon\big) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

For $\varepsilon > 0$,

$$\begin{aligned}
\mathbf{P}\big(\xi < \infty,\ \{X_s\}_{s<\xi} \not\Subset S \text{ and } \forall t < \xi, \exists s \in [t,\xi),\ X_s \in K\big) \\
\leq \mathbf{P}\big(\exists n, \forall m \geq n,\ \tau_m < \xi < \tau_m + \varepsilon\big) \leq \sup_{n\in\mathbb{N}} \mathbf{P}\big(\tau_n < \xi < \tau_n + \varepsilon\big) \\
\leq \sup_{n\in\mathbb{N}} \mathbf{P}\big([X_{\tau_n} \in K,\ \sigma_{n+1} < \tau_n + \varepsilon\big),
\end{aligned}$$

so letting $\varepsilon \to 0$ we obtain

$$\mathbf{P}\big(\xi < \infty,\ \{X_s\}_{s<\xi} \not\Subset S \text{ and } \forall t < \xi, \exists s \in [t,\xi),\ X_s \in K\big) = 0. \tag{A.4}$$

Letting K growing toward S, we deduce from (A.4) that $\mathbf{P}\big(\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)\big) = 1$.
Step 3. Let τ_n, τ be $(\mathcal{F}_{t+})$-stopping times. By the first step $X_{\tau_n} \underset{n\to\infty}{\longrightarrow} X_\tau$ **P**-almost surely on

$$\big\{\tau_n \underset{n\to\infty}{\longrightarrow} \tau < \infty,\ \{X_s\}_{s<\tau} \Subset S\big\},$$

by the second step this is also the case on

$$\big\{\tau_n \underset{n\to\infty}{\longrightarrow} \tau = \xi < \infty,\ \{X_s\}_{s<\tau} \not\Subset S\big\},$$

and this is clearly true on $\big\{\tau_n \underset{n\to\infty}{\longrightarrow} \tau > \xi\big\}$, so the proof is done. □

Proof of part 1 of Proposition 3.4. Take $(f,g) \in L$ and an open subset $U \Subset S$. If $s_1 \leq \cdots \leq s_k \leq s \leq t$ are positive numbers and $f_1, \ldots, f_k \in \mathrm{C}(S^\Delta)$, we need to prove that

$$h \cdot \mathbf{E}\left[\left(f(X_{t\wedge\tau^U}) - f(X_{s\wedge\tau^U}) - \int_{s\wedge\tau^U}^{t\wedge\tau^U} (hg)(X_u)\mathrm{d}u\right) f_1(X_{s_1}) \cdots f_k(X_{s_k})\right] = 0. \tag{A.5}$$

We will proceed in two steps: firstly we suppose that $U \Subset \{h \neq 0\}$. Recalling the definition (2.3), if we denote $\tau_t := \tau_t^h \wedge \tau^U$, then we have, for all $t \in \mathbb{R}_+$,

$$h \cdot X_{t\wedge\tau^U(h\cdot X)} = X_{\tau_t}, \tag{A.6}$$

$$\int_0^{t\wedge\tau^U(h\cdot X)} (hg)(h \cdot X_u)\mathrm{d}u = \int_0^{t\wedge\tau^U(h\cdot X)} (hg)(X_{\tau_u})\mathrm{d}u = \int_0^{\tau_t} g(X_u)\mathrm{d}u. \tag{A.7}$$

Hence by (A.6)-(A.7) and optional sampling Theorem 3.1

$$\begin{aligned}
&h \cdot \mathbf{E}\left[\left(f(X_{t\wedge\tau^U}) - f(X_{s\wedge\tau^U}) - \int_{s\wedge\tau^U}^{t\wedge\tau^U} (hg)(X_u)\mathrm{d}u\right) f_1(X_{s_1}) \cdots f_k(X_{s_k})\right] \\
&= h \cdot \mathbf{E}\left[\left(f(X_{t\wedge\tau^U}) - f(X_{s\wedge\tau^U}) - \int_{s\wedge\tau^U}^{t\wedge\tau^U} (hg)(X_u)\mathrm{d}u\right) f_1(X_{s_1\wedge\tau^U}) \cdots f_k(X_{s_k\wedge\tau^U})\right] \\
&= \mathbf{E}\left[\left(f(X_{\tau_t}) - f(X_{\tau_s}) - \int_{\tau_s}^{\tau_t} g(X_u)\mathrm{d}u\right) f_1(X_{\tau_{s_1}}) \cdots f_k(X_{\tau_{s_k}})\right] = 0.
\end{aligned}$$

Secondly, we suppose only that $U \Subset S$. Let d be a metric on S and we introduce, for $n \geq 1$ integer, $U_n := \{a \in U \mid d(a, \{h = 0\}) > n^{-1}\}$. Then it is straightforward to obtain the pointwise convergences

$$h \cdot X_{t\wedge\tau^{U_n}(h\cdot X)} \underset{n\to\infty}{\longrightarrow} h \cdot X_{t\wedge\tau^{U}(h\cdot X)},$$
$$\int_0^{t\wedge\tau^{U_n}(h\cdot X)} (hg)(h\cdot X_u)\mathrm{d}u \underset{n\to\infty}{\longrightarrow} \int_0^{t\wedge\tau^{U}(h\cdot X)} (hg)(h\cdot X_u)\mathrm{d}u,$$

so

$$\begin{aligned} f(X_{t\wedge\tau^{U_n}}) - f(X_{s\wedge\tau^{U_n}}) - \int_{s\wedge\tau^{U_n}}^{t\wedge\tau^{U_n}} (hg)(X_u)\mathrm{d}u \\ \underset{n\to\infty}{\overset{h\cdot\mathbf{P}\text{-a.s.}}{\longrightarrow}} f(X_{t\wedge\tau^{U}}) - f(X_{s\wedge\tau^{U}}) - \int_{s\wedge\tau^{U}}^{t\wedge\tau^{U}} (hg)(X_u)\mathrm{d}u. \end{aligned}$$

Applying the first step to U_n and letting $n \to \infty$, by dominated convergence we obtain (A.5). □

Proof of part 1 of Proposition 3.5. By using Proposition 2.1 we know that there exists $h \in \mathrm{C}(S, \mathbb{R}_+^*)$ such that $\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)$ has probability 1 under $h \cdot \mathbf{P}^n$ and under $h \cdot \mathbf{P}$ and such that $h \cdot \mathbf{P}^n$ converges weakly to $h \cdot \mathbf{P}$ for the global Skorokhod topology from $\mathbb{D}(S^\Delta)$. Let us fix (f, g) and (f_n, g_n) arbitrary as in (3.5) and then we can modify h such that it satisfies furthermore $hg_n, hg \in \mathrm{C}_0(S)$ and $hg_n \underset{n\to\infty}{\overset{\mathrm{C}_0}{\longrightarrow}} hg$.

Let $\mathbb{T}$ be the set of $t \in \mathbb{R}_+$ such that $h \cdot \mathbf{P}(X_{t-} = X_t) = 1$, so $\mathbb{R}_+\backslash\mathbb{T}$ is countable. Let $s_1 \leq \cdots \leq s_k \leq s \leq t$ belonging to $\mathbb{T}$ and let $\varphi_1, \ldots, \varphi_k \in \mathrm{C}(S^\Delta)$ be. By using 1 of Proposition 3.4 and the first part of Remark 3.3

$$h \cdot \mathbf{E}^n \left[\left(f_n(X_t) - f_n(X_s) - \int_s^t (hg_n)(X_u)\mathrm{d}u \right) \varphi_1(X_{s_1}) \cdots \varphi_k(X_{s_k}) \right] = 0. \tag{A.8}$$

The sequence of functions $\left(f_n(X_t) - f_n(X_s) - \int_s^t (hg_n)(X_u)\mathrm{d}u\right)\varphi_1(X_{s_1}) \cdots \varphi_k(X_{s_k})$ converges uniformly to the function $\left(f(X_t) - f(X_s) - \int_s^t (hg)(X_u)\mathrm{d}u\right)\varphi_1(X_{s_1}) \cdots \varphi_k(X_{s_k})$ which is continuous $h \cdot \mathbf{P}$-almost everywhere for the topology of $\mathbb{D}(S^\Delta)$. Hence we can take the limit, as $n \to \infty$, in (A.8) and we obtain that

$$h \cdot \mathbf{E} \left[\left(f(X_t) - f(X_s) - \int_s^t (hg)(X_u)\mathrm{d}u \right) \varphi_1(X_{s_1}) \cdots \varphi_k(X_{s_k}) \right] = 0. \tag{A.9}$$

Since $\mathbb{T}$ is dense in $\mathbb{R}_+$, by right continuity of paths of the canonical process, and by dominated convergence (A.9) extends to $s_i, s, t \in \mathbb{R}_+$. Hence $h \cdot \mathbf{P} \in \mathcal{M}(\{(f, hg)\})$, so using (2.5) and part 1 of Proposition 3.4, $\mathbf{P} = (1/h) \cdot h \cdot \mathbf{P} \in \mathcal{M}(\{(f, g)\})$. Since $(f, g) \in L$ was chosen arbitrary, we have proved that $\mathbf{P} \in \mathcal{M}(L)$. □

Proof of part 2 of Proposition 3.4. It is straightforward that $\mathcal{M}(\mathrm{span}(L)) = \mathcal{M}(L)$. Let $\mathbf{P} \in \mathcal{M}(L)$. We apply the part 1 of Proposition 3.5 to the stationary sequences $\mathbf{P}^n = \mathbf{P}$ and $L_n = \mathrm{span}(L)$ and to $\overline{\mathrm{span}(L)}$. Hence $\mathbf{P} \in \mathcal{M}(\overline{\mathrm{span}(L)})$ and the proof is finished. □

Proof of part 2 of Proposition 3.5. Take $t \in \mathbb{R}_+$ and an open subset $U \Subset S$, and let d be a metric on S^Δ. By Lemma 3.6, considering $\mathcal{K} := \overline{U}$ and $\mathcal{U} := \{(a,b) \in S^2 \mid d(a,b) < \varepsilon\}$, we have

$$\sup_{\substack{\tau_1 \leq \tau_2 \\ \tau_2 \leq (\tau_1+\delta)\wedge\tau^U\wedge t}} \mathbf{P}_n\big(d(X_{\tau_1}, X_{\tau_2}) \geq \varepsilon\big) \underset{\substack{n\to\infty \\ \delta\to 0}}{\longrightarrow} 0,$$

hence (2.2) is satisfied and we can apply the Aldous criterion (see also Proposition 2.14 in [GH17b]). □

Proof of part 3 of Proposition 3.4. It is straightforward that $\mathcal{M}(L)$ is convex. To prove the compacteness, let $(\mathbf{P}^n)_n$ be a sequence from $\mathcal{M}(L)$. We apply the part 2 of Proposition 3.5 to this sequence and to the stationary sequence $L_n = L$. Hence $(\mathbf{P}^n)$ is tight, so there exists a subsequence $(\mathbf{P}^{n_k})_k$ which converges toward some $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$. Thanks to the part 1 of Proposition 3.5 we can deduce that $\mathbf{P} \in \mathcal{M}(L)$. The statement of the proposition is then obtained since $\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ is a Polish space. □

A.2 Proof of Theorem 4.5

To prove the theorem we will use three preliminary results.

Lemma A.1. *Let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be such that $a \mapsto \mathbf{P}_a$ is continuous for the local Skorokhod topology. Suppose that for all $a \in S^\Delta$: $\mathbf{P}_a(X_0 = a) = 1$ and there exists a dense subset $\mathbb{T}_a \subset \mathbb{R}_+$ such that for any $B \in \mathcal{F}$ and $t_0 \in \mathbb{T}_a$*

$$\mathbf{P}_a\left((X_{t_0+t})_t \in B \mid \mathcal{F}_{t_0}\right) = \mathbf{P}_{X_{t_0}}(B) \quad \mathbf{P}_a\text{-almost surely.}$$

Then $(\mathbf{P}_a)_a$ is a $(\mathcal{F}_{t+})_t$-strong Markov family.

Proof. Let τ be a $(\mathcal{F}_{t+})_t$-stopping time, let $a \in S$ be and let F be a bounded continuous function from $\mathbb{D}_{\mathrm{loc}}(S)$ to $\mathbb{R}$. For each $n \in \mathbb{N}^*$ chose a discrete subspace $\mathbb{T}_a^n \subset \mathbb{T}_a$ such that $(t, t+n^{-1}] \cap \mathbb{T}_a^n$ is not empty for any $t \in \mathbb{R}_+^*$, and define

$$\tau_n := \min\left\{t \in \mathbb{T}_a^n \mid \tau < t\right\}.$$

Hence τ_n is a $(\mathcal{F}_t)_t$-stopping time with value in $\mathbb{T}_a^n$, so

$$\mathbf{E}_a\left[F\left((X_{\tau_n+t})_t\right) \mid \mathcal{F}_{\tau_n}\right] = \mathbf{E}_{X_{\tau_n}} F \quad \mathbf{P}_a\text{-almost surely.}$$

Since $\tau < \tau_n \leq \tau + n^{-1}$ on $\{\tau < \infty\}$ and $a \mapsto \mathbf{P}_a$ is continuous, $\lim_{n\to\infty} \mathbf{E}_{X_{\tau_n}} F = \mathbf{E}_{X_\tau} F$. We have

$$\begin{aligned}
\mathbf{E}_a &\left|\mathbf{E}_a\left[F\left((X_{\tau+t})_t\right) \mid \mathcal{F}_{\tau+}\right] - \mathbf{E}_a\left[F\left((X_{\tau_n+t})_t\right) \mid \mathcal{F}_{\tau_n}\right]\right| \\
&\leq \mathbf{E}_a\left|\mathbf{E}_a\left[F\left((X_{\tau+t})_t\right) \mid \mathcal{F}_{\tau+}\right] - \mathbf{E}_a\left[F\left((X_{\tau+t})_t\right) \mid \mathcal{F}_{\tau_n}\right]\right| \\
&\quad + \mathbf{E}_a\left|F\left((X_{\tau+t})_t\right) - F\left((X_{\tau_n+t})_t\right)\right|.
\end{aligned} \tag{A.10}$$

On the right hand side, the first term converges to 0 (see, for instance, Theorem 7.23, p. 132 in [Kal02]) and the second term converges to 0 by dominated convergence. Hence

$$\mathbf{E}_a\left[F\left((X_{\tau+t})_t\right) \mid \mathcal{F}_{\tau+}\right] = \mathbf{E}_{X_\tau} F \quad \mathbf{P}_a\text{-almost surely,}$$

so $(\mathbf{P}_a)_a$ is a $(\mathcal{F}_{t+})_t$-strong Markov family. □

Lemma A.2 (Localisation of continuity). *Set $\widetilde{S}$ an arbitrary metrisable topological space, consider $U_n \subset S$, an increasing sequence of open subsets such that $S = \bigcup_n U_n$. Let $(\mathbf{P}_a^n)_{a,n} \in \mathcal{P}(\mathbb{D}_{loc}(S))^{\widetilde{S}\times\mathbb{N}}$ be such that*

1. *for each $n \in \mathbb{N}$, $a \mapsto \mathbf{P}_a^n$ is weakly continuous for the local Skorokhod topology,*
2. *for each $n \le m$ and $a \in \widetilde{S}$*

$$\mathscr{L}_{\mathbf{P}_a^m}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}_a^n}\left(X^{\tau^{U_n}}\right). \tag{A.11}$$

Then there exists a unique family $(\mathbf{P}_a^\infty)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^{\widetilde{S}}$ such that for any $n \in \mathbb{N}$ and $a \in \widetilde{S}$

$$\mathscr{L}_{\mathbf{P}_a^\infty}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}_a^n}\left(X^{\tau^{U_n}}\right). \tag{A.12}$$

Furthermore the mapping

$$\begin{array}{ccc} \mathbb{N}\cup\{\infty\}\times\widetilde{S} & \to & \mathcal{P}(\mathbb{D}_{loc}(S)) \\ (n,a) & \mapsto & \mathbf{P}_a^n \end{array} \tag{A.13}$$

is weakly continuous for the local Skorokhod topology.

Before giving the proof of this lemma let us recall that in Theorem 2.15 of [GH17b] is obtained an improvement of the Aldous criterion of tightness. More precisely a subset $\mathcal{P} \subset \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ is tight if and only if

$$\forall t \ge 0,\ \forall \varepsilon > 0,\ \forall \text{ open } U \Subset S, \quad \sup_{\mathbf{P}\in\mathcal{P}} \sup_{\substack{\tau_1\le\tau_2\le\tau_3 \\ \tau_3\le(\tau_1+\delta)\wedge t\wedge\tau^U}} \mathbf{P}(R \ge \varepsilon) \xrightarrow[\delta\to 0]{} 0, \tag{A.14}$$

where the supremum is taken along τ_i stopping times and with

$$R := \begin{cases} d(X_{\tau_1}, X_{\tau_2}) \wedge d(X_{\tau_2}, X_{\tau_3}) & \text{if } 0 < \tau_1 < \tau_2, \\ d(X_{\tau_2 -}, X_{\tau_2}) \wedge d(X_{\tau_2}, X_{\tau_3}) & \text{if } 0 < \tau_1 = \tau_2, \\ d(X_{\tau_1}, X_{\tau_2}) & \text{if } 0 = \tau_1, \end{cases}$$

d being an arbitrary metric on S^Δ.

Proof of Lemma A.2. The uniqueness is straightforward using that $X^{\tau^{U_n}}$ converge to X pointwise for the local Skorokhod topology as $n \to \infty$.

Let us prove that for any compact subset $K \subset \widetilde{S}$, the set $\{\mathbf{P}_a^n \mid a \in K,\ n \in \mathbb{N}\}$ is tight. If $U \Subset S$ is an arbitrary open subset, there exists $N \in \mathbb{N}$ such that $U \subset U_N$. Let $t.\varepsilon > 0$ be. By the continuity of $a \mapsto \mathbf{P}_a^n$, the set $\{\mathbf{P}_a^n \mid a \in K,\ 0 \le n \le N\}$ is tight, so using the characterisation (A.14) we have

$$\sup_{\substack{0\le n\le N \\ a\in K}} \sup_{\substack{\tau_1\le\tau_2\le\tau_3 \\ \tau_3\le(\tau_1+\delta)\wedge t\wedge\tau^U}} \mathbf{P}_a^n(R \ge \varepsilon) \xrightarrow[\delta\to 0]{} 0.$$

Since $U \subset U_N$, for all $n \ge N$ and $a \in K$,

$$\mathscr{L}_{\mathbf{P}_a^N}\left(X^{\tau^U}\right) = \mathscr{L}_{\mathbf{P}_a^n}\left(X^{\tau^U}\right),$$

hence

$$\sup_{n\in\mathbb{N},\ a\in K}\ \sup_{\substack{\tau_1\le\tau_2\le\tau_3\\ \tau_3\le(\tau_1+\delta)\wedge t\wedge\tau^U}} \mathbf{P}^n_a(R\ge\varepsilon) = \sup_{\substack{0\le n\le N\\ a\in K}}\ \sup_{\substack{\tau_1\le\tau_2\le\tau_3\\ \tau_3\le(\tau_1+\delta)\wedge t\wedge\tau^U}} \mathbf{P}^n_a(R\ge\varepsilon) \underset{\delta\to 0}{\longrightarrow} 0.$$

So, again by (A.14), $\{\mathbf{P}^n_a \mid a\in K,\ n\in\mathbb{N}\}$ is tight.

Hence, if $a\in\widetilde{S}$, then the set $\{\mathbf{P}^n_a\}_n$ is tight. Fix such a, there exist an increasing sequence $\varphi(k)$ and a probability measure $\mathbf{P}^\infty_a\in\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ such that $\mathbf{P}^{\varphi(k)}_a$ converges to $\mathbf{P}^\infty_a$ as $k\to\infty$. Fix an arbitrary $n\in\mathbb{N}$, there exists $k_0\in\mathbb{N}$ such that $\varphi(k_0)\ge n$ and $U_n\Subset U_{\varphi(k_0)}$. Thanks to Proposition 2.1, there exists $g\in\mathrm{C}(S,\mathbb{R}_+)$ such that $U_{\varphi(k_0)}=\{g\neq 0\}$ and such that $g\cdot\mathbf{P}^{n_k}_a$ converges to $g\cdot\mathbf{P}^\infty_a$ weakly for the local Skorokhod topology, as $k\to\infty$. By using (A.11) we have, for each $k\ge k_0$, $g\cdot\mathbf{P}^{\varphi(k)}_a = g\cdot\mathbf{P}^{\varphi(k_0)}_a$, so $g\cdot\mathbf{P}^\infty_a = g\cdot\mathbf{P}^{\varphi(k_0)}_a$. Hence we deduce

$$\mathscr{L}_{\mathbf{P}^\infty_a}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}^{\varphi(k_0)}_a}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}^n_a}\left(X^{\tau^{U_n}}\right).$$

Let us prove that the mapping in (A.13) is weakly continuous for the local Skorokhod topology. Since we already verified the tightness it suffices to prove that: for any sequences $n_k\in\mathbb{N}\cup\{\infty\}$, $a_k\in\widetilde{S}$ such that $n_k\to\infty$ and $a_k\to a\in\widetilde{S}$ as $k\to\infty$ and such that the sequence $\mathbf{P}^{n_k}_{a_k}$ converges to $\mathbf{P}\in\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$, then $\mathbf{P}=\mathbf{P}^\infty_a$. Fix an arbitrary $N\in\mathbb{N}$, there exists $k_0\in\mathbb{N}$ such that $n_{k_0}\ge N$ and $U_N\Subset U_{n_{k_0}}$. As previously, by using Proposition 2.1 again, there exists $g\in\mathrm{C}(S,\mathbb{R}_+)$ such that $U_{n_{k_0}}=\{g\neq 0\}$, $g\cdot\mathbf{P}^{n_k}_{a_k}$ converges to $g\cdot\mathbf{P}$ and $g\cdot\mathbf{P}^{n_{k_0}}_{a_k}$ converges to $g\cdot\mathbf{P}^{n_{k_0}}_a$, as $k\to\infty$. Thanks to (A.12) $g\cdot\mathbf{P}^{n_k}_{a_k}=g\cdot\mathbf{P}^{n_{k_0}}_{a_k}$ for $k\ge k_0$, so $g\cdot\mathbf{P}=g\cdot\mathbf{P}^{n_{k_0}}_a=g\cdot\mathbf{P}^\infty_a$. Hence we deduce

$$\mathscr{L}_{\mathbf{P}}\left(X^{\tau^{U_N}}\right) = \mathscr{L}_{\mathbf{P}^\infty_a}\left(X^{\tau^{U_N}}\right),$$

and letting $N\to\infty$ we deduce that $\mathbf{P}=\mathbf{P}^\infty_a$. □

Lemma A.3 (Continuity and Markov property)**.** *Let*

$$\begin{array}{ccc} \mathbb{N}\cup\{\infty\}\times S^\Delta & \to & \mathcal{P}(\mathbb{D}_{loc}(S)) \\ (n,a) & \mapsto & \mathbf{P}^n_a \end{array}$$

be a weakly continuous mapping for the local Skorokhod topology such that $(\mathbf{P}^n_a)_a$ *is a Markov family for each* $n\in\mathbb{N}$*. Then* $(\mathbf{P}^\infty_a)_a$ *is a Markov family.*

Before giving the proof of the result recall the following property of the time change stated in the fifth part of Proposition 3.3 of [GH17b]: for any $(\mathbf{P}_a)_a\in\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ and $g\in\mathrm{C}(S,\mathbb{R}_+)$,

$$(\mathbf{P}_a)_a \text{ is } (\mathcal{F}_{t+})_t\text{-strong Markov } \Rightarrow (g\cdot\mathbf{P}_a)_a \text{ is } (\mathcal{F}_{t+})_t\text{-strong Markov}\ . \tag{A.15}$$

Proof. Using Proposition 2.1, there exists $g\in\mathrm{C}(S,\mathbb{R}_+^*)$ such that for all $(n,a)\in\mathbb{N}\cup\{\infty\}\times S^\Delta$, $\mathbf{P}^n_a(\mathbb{D}_{\mathrm{loc}}(S)\cap\mathbb{D}(S^\Delta))=1$ and such that $(n,a)\mapsto\mathbf{P}^n_a$ is weakly continuous for the global Skorokhod topology from $\mathbb{D}(S^\Delta)$. For all $n\in\mathbb{N}$, by Lemma A.1, $(\mathbf{P}^n_a)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov, so, by (A.15), $(g\cdot\mathbf{P}^n_a)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov.

Take $a \in S$ and denote $\mathbb{T}_a := \left\{t \in \mathbb{R}_+ \,\middle|\, g \cdot \mathbf{P}^\infty_a(X_{t-} = X_t) = 1\right\}$, so $\mathbb{T}_a$ is dense in $\mathbb{R}_+$. Let $t \in \mathbb{T}_a$ be and consider F, G two bounded function from $\mathbb{D}(S^\Delta)$ to $\mathbb{R}$ continuous for the global Skorokhod topology, we want to prove that

$$g \cdot \mathbf{E}^\infty_a \left[F\left((X_{t+s})_s\right) G\left((X_{t\wedge s})_s\right)\right] = g \cdot \mathbf{E}^\infty_a \Big[g \cdot \mathbf{E}^\infty_{X_t}[F] G\left((X_{t\wedge s})_s\right)\Big]. \tag{A.16}$$

For any $n \in \mathbb{N}$, by the Markov property we have

$$g \cdot \mathbf{E}^n_a \Big[F\left((X_{t+s})_s\right) G\left((X_{t\wedge s})_s\right)\Big] = g \cdot \mathbf{E}^n_a \Big[g \cdot \mathbf{E}^n_{X_t}[F] G\left((X_{t\wedge s})_s\right)\Big]. \tag{A.17}$$

The mappings

$$\begin{array}{rcl} \mathbb{D}(S^\Delta) & \to & \mathbb{R} \\ x & \mapsto & F\left((x_{t+s})_s\right) G\left((x_{t\wedge s})_s\right) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{D}(S^\Delta) & \to & \mathbb{R} \\ x & \mapsto & g \cdot \mathbf{E}^\infty_{x_t}[F] G\left((x_{t\wedge s})_s\right) \end{array}$$

are continuous on the set $\{X_{t-} = X_t\}$ for the global topology. Hence, since $g \cdot \mathbf{E}^n_a$ converges to $g \cdot \mathbf{E}^\infty_a$ weakly for the global topology and $g \cdot \mathbf{P}^\infty_a(X_{t-} = X_t) = 1$, we have

$$g \cdot \mathbf{E}^n_a \Big[F\left((X_{t+s})_s\right) G\left((X_{t\wedge s})_s\right)\Big] \underset{n\to\infty}{\longrightarrow} g \cdot \mathbf{E}^\infty_a \Big[F\left((X_{t+s})_s\right) G\left((X_{t\wedge s})_s\right)\Big], \tag{A.18}$$

$$g \cdot \mathbf{E}^n_a \Big[g \cdot \mathbf{E}^\infty_{X_t}[F] G\left((X_{t\wedge s})_s\right)\Big] \underset{n\to\infty}{\longrightarrow} g \cdot \mathbf{E}^\infty_a \Big[g \cdot \mathbf{E}^\infty_{X_t}[F] G\left((X_{t\wedge s})_s\right)\Big]. \tag{A.19}$$

Since $(n, b) \mapsto g \cdot \mathbf{P}^n_b$ is continuous for the global topology, using the compactness of S^Δ we have

$$\sup_{a \in S^\Delta} \left|g \cdot \mathbf{E}^n_a F - g \cdot \mathbf{E}^\infty_a F\right| \underset{n\to\infty}{\longrightarrow} 0. \tag{A.20}$$

We deduce (A.16) from (A.17)-(A.20) and so

$$g \cdot \mathbf{E}^\infty_a \big[F\left((X_{t+s})_s\right) \,\big|\, \mathcal{F}_t\big] = g \cdot \mathbf{E}^\infty_{X_t}[F], \quad g \cdot \mathbf{P}^\infty_a\text{-almost surely,}$$

so, by Lemma A.1, $(g \cdot \mathbf{P}^\infty_a)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov. Applying (A.15) to $(g \cdot \mathbf{P}^\infty_a)_a$ and $1/g$, and using (2.5), we deduce that $(\mathbf{P}^\infty_a)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov. □

Proof of Theorem 4.5. 1⇒2 Thanks to Proposition 2.1 there exists $g \in \mathrm{C}(S, \mathbb{R}^*_+)$ such that for all $a \in S^\Delta$, $\mathbf{P}_a(\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)) = 1$ and such that the mapping $a \mapsto \mathbf{P}_a$ is weakly continuous for the global Skorokhod topology from $\mathbb{D}(S^\Delta)$. Lemma A.1 insure that $(\mathbf{P}_a)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov. By (A.15) we can deduce that $(g \cdot \mathbf{P}_a)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov. Take $a \in S$ and $t \in \mathbb{R}^*_+$, we will prove that $g \cdot \mathbf{P}_a(X_{t-} = X_t) = 1$. For any $f \in \mathrm{C}(S^\Delta)$, $s < t$ and $\varepsilon > 0$, by the Markov property

$$g \cdot \mathbf{E}_a \Big[\frac{1}{\varepsilon} \int_s^{s+\varepsilon} f(X_u) \mathrm{d}u \,\Big|\, \mathcal{F}_s\Big] \overset{g\cdot\mathbf{P}_a\text{-a.s.}}{=} g \cdot \mathbf{E}_{X_s} \Big[\frac{1}{\varepsilon} \int_0^{\varepsilon} f(X_u) \mathrm{d}u\Big].$$

Since $a \mapsto g \cdot \mathbf{P}_a$ is weakly continuous for the global topology and since $x \mapsto \frac{1}{\varepsilon} \int_0^\varepsilon f(X_u) \mathrm{d}u$ is continuous for the global topology,

$$g \cdot \mathbf{E}_{X_s} \Big[\frac{1}{\varepsilon} \int_0^{\varepsilon} f(X_u) \mathrm{d}u\Big] \underset{\substack{s\to t\\ s<t}}{\longrightarrow} g \cdot \mathbf{E}_{X_{t-}} \Big[\frac{1}{\varepsilon} \int_0^{\varepsilon} f(X_u) \mathrm{d}u\Big].$$

By a similar reasoning as in (A.10) we have

$$g \cdot \mathbf{E}_a \left| g \cdot \mathbf{E}_a \Big[\frac{1}{\varepsilon} \int_t^{t+\varepsilon} f(X_u) \mathrm{d}u \,\Big|\, \mathcal{F}_{t-} \Big] - g \cdot \mathbf{E}_a \Big[\frac{1}{\varepsilon} \int_s^{s+\varepsilon} f(X_u) \mathrm{d}u \,\Big|\, \mathcal{F}_s \Big] \right| \underset{\substack{s \to t \\ s < t}}{\longrightarrow} 0$$

so

$$g \cdot \mathbf{E}_a \Big[\frac{1}{\varepsilon} \int_t^{t+\varepsilon} f(X_u) \mathrm{d}u \,\Big|\, \mathcal{F}_{t-} \Big] \overset{g \cdot \mathbf{P}_a\text{-a.s.}}{=} g \cdot \mathbf{E}_{X_{t-}} \Big[\frac{1}{\varepsilon} \int_0^{\varepsilon} f(X_u) \mathrm{d}u \Big].$$

Hence letting $\varepsilon \to 0$ we deduce $g \cdot \mathbf{E}_a \left[f(X_t) \mid \mathcal{F}_{t-} \right] \overset{g \cdot \mathbf{P}_a\text{-a.s.}}{=} f(X_{t-})$. Since f is arbitrary, this is also true for f^2 so we deduce

$$\begin{aligned} g \cdot \mathbf{E}_a \left(f(X_t) - f(X_{t-}) \right)^2 &= g \cdot \mathbf{E}_a \left[g \cdot \mathbf{E}_a \left[f^2(X_t) \mid \mathcal{F}_{t-} \right] - f^2(X_{t-}) \right] \\ &\quad - 2g \cdot \mathbf{E}_a \left[f(X_{t-}) \left(g \cdot \mathbf{E}_a \left[f(X_t) \mid \mathcal{F}_{t-} \right] - f(X_{t-}) \right) \right] \\ &= 0. \end{aligned}$$

Since f is arbitrary, taking a dense sequence of $\mathrm{C}(S^\Delta)$, we get $g \cdot \mathbf{P}_a(X_{t-} = X_t) = 1$. Finally, for any $t \in \mathbb{R}_+$ and $f \in \mathrm{C}(S^\Delta)$, since $x \mapsto f(x_t)$ is continuous for the global Skorokhod topology on $\{X_{t-} = X_t\}$, the function

$$\begin{array}{ccc} S^\Delta & \to & \mathbb{R} \\ a & \mapsto & g \cdot \mathbf{E}_a f(X_t) \end{array}$$

is continuous, so $(g \cdot \mathbf{P}_a)_a$ is a Feller family.
2⇒3. Let L be the $\mathrm{C}_0 \times \mathrm{C}_0$-generator of $(g \cdot \mathbf{P}_a)_a$, then, by Proposition 4.2, $\mathcal{M}(L) = \{g \cdot \mathbf{P}_\mu\}_{\mu \in \mathcal{P}(S^\Delta)}$ so by the first part of Proposition 3.4 and by (2.5),

$$\mathcal{M}\left(\frac{1}{g} L \right) = \{\mathbf{P}_\mu\}_{\mu \in \mathcal{P}(S^\Delta)}.$$

3⇒1. Thanks to 3 from Proposition 3.4, for the local Skorokhod topology,

$$\begin{array}{ccc} \{\mathbf{P}_a\}_{a \in S} & \to & S \\ \mathbf{P}_a & \mapsto & a \end{array}$$

is a continuous injective function defined on a compact set, so $a \mapsto \mathbf{P}_a$ is also continuous. Let τ be a $(\mathcal{F}_{t+})_t$-stopping time and a be in S. As in Remark 3.10 we denote

$$\mathbf{Q}_X \overset{\mathbf{P}_a\text{-a.s.}}{:=} \mathscr{L}_{\mathbf{P}_a} \left((X_{\tau+t})_{t \geq 0} \mid \mathcal{F}_{\tau+} \right).$$

By using Proposition 3.11, $\mathbf{Q}_X \in \mathcal{M}(L)$, $\mathbf{P}_a$-almost surely, so $\mathbf{Q}_X = \mathbf{P}_{X_\tau}$, $\mathbf{P}_a$-almost surely, hence $(\mathbf{P}_a)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov. The quasi-continuity is a consequence of 4 from Proposition 3.4.
2⇒4. Take an open subset $U \Subset S$ and define for all $a \in S$

$$\widetilde{\mathbf{P}}_a := h \cdot \mathbf{P}_a \quad \text{where} \quad h := \frac{g \wedge \min_U g}{\min_U g}.$$

By Proposition 4.4, $(\widetilde{\mathbf{P}}_a)_a$ is Feller, and moreover, since $X^{\tau^U} = (h \cdot X)^{\tau^U}$,

$$\forall a \in S, \quad \mathscr{L}_{\mathbf{P}_a}\left(X^{\tau^U} \right) = \mathscr{L}_{\widetilde{\mathbf{P}}_a}\left(X^{\tau^U} \right).$$

4⇒1. Let $U_n \Subset S$ be an increasing sequence of open subsets such that $S = \bigcup_n U_n$. For each $n \in \mathbb{N}$ there exists a Feller family $(\mathbf{P}^n_a)_a$ such that

$$\forall a \in S, \quad \mathscr{L}_{\mathbf{P}_a}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}^n_a}\left(X^{\tau^{U_n}}\right).$$

Denote $\mathbf{P}^\infty_a := \mathbf{P}_a$, then thanks to Lemma A.2 the mapping

$$\begin{array}{ccc} \mathbb{N} \cup \{\infty\} \times S^\Delta & \to & \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S)) \\ (n,a) & \mapsto & \mathbf{P}^n_a \end{array}$$

is continuous and thanks to Lemma A.3 $(\mathbf{P}^\infty_a)_a$ is a Markov family. □

A.3 Proof of Lemma 4.17

Before proving the Lemma 4.17 let us note that thanks to Proposition 2.1 and (A.15), if $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is locally Feller then for any open subset $U \subset S$ there exists $h_0 \in \mathrm{C}(S, \mathbb{R}_+)$ such that $U = \{h_0 \neq 0\}$ and $(h_0 \cdot \mathbf{P}_a)_a$ is locally Feller. This fact does not ensure that the martingale local problem associate to $h_0 L$ is well-posed as is stated in lemma. During the proof we will use two preliminary results.

Lemma A.4. *Let L be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ such that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$ and U be an open subset of S, then there exist a subset L_0 of L and a function h_0 of $\mathrm{C}(S, \mathbb{R}_+)$ with $\{h_0 \neq 0\} = U$ such that $\overline{L} = \overline{L_0}$, such that $h_0 L_0 \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ and such that: for any $h \in \mathrm{C}(S, \mathbb{R}_+)$ with $\{h \neq 0\} = U$ and $\sup_{a \in U}(h/h_0)(a) < \infty$ and any $\mathbf{P} \in \mathcal{M}_{\mathrm{c}}\left((hL_0)^\Delta\right)$, $\mathbf{P}(X = X^{\tau^U}) = 1$.*

Proof. Take L_0 a countable dense subset of L and let d be a metric on S^Δ. For any $n \in \mathbb{N}^*$ there exist $M_n \in \mathbb{N}$ and $(a_{n,m})_{1 \le m \le M_n} \in (S^\Delta \backslash U)^{M_n}$ such that

$$S^\Delta \backslash U \subset \bigcup_{m=1}^{M_n} B(a_{n,m}, n^{-1}).$$

For each $1 \le m \le M_n$ there exists $(f_{n,m}, g_{n,m}) \in L_0$ such that

$$f_{n,m}(a) \in \begin{cases} [1 - n^{-1}, 1 + n^{-1}] & \text{if } d(a, a_{n,m}) \ge 2n^{-1}, \\ [-n^{-1}, 1 + n^{-1}] & \text{if } n^{-1} \le d(a, a_{n,m}) \ge 2n^{-1}, \\ [-n^{-1}, n^{-1}] & \text{if } n^{-1} \le d(a, a_{n,m}). \end{cases}$$

Take $h_0 \in \mathrm{C}_0(S, \mathbb{R}_+)$ with $\{h_0 \neq 0\} = U$, such that $h_0 g \in \mathrm{C}_0(S)$ for any $(f, g) \in L_0$ and such that for any $n \in \mathbb{N}^*$ and $1 \le m \le M_n$

$$\|h_0\|_{B(a_{n,m}, 4n^{-1})} \|g_{n,m}\| \le \frac{1}{n}.$$

Hence $\overline{L} = \overline{L_0}$ and $hL_0 \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$. Let $h \in \mathrm{C}(S, \mathbb{R}_+)$ be such that $\{h \neq 0\} = U$ and $C := \sup_{a \in U}(h/h_0)(a) < \infty$. Let $\mathbf{P} \in \mathcal{M}_{\mathrm{c}}\left((hL)^\Delta\right)$ be such that there exists $a \in S^\Delta \backslash U$ with $\mathbf{P}(X_0 = a) = 1$. We will prove that

$$\mathbf{P}(\forall s \ge 0,\ X_s = a) = 1. \tag{A.21}$$

Take $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$. There exists $m \leq M_n$ such that $d(a, a_{n,m}) < \frac{1}{n}$. If we denote

$$\tau := \tau^{B(a,3n^{-1})},$$

then

$$\begin{aligned}\mathbf{E}[f_{n,m}(X_{t\wedge\tau})] &= f_{n,m}(a) + \mathbf{E}\left[\int_0^{t\wedge\tau} h(X_s)g_{n,m}(X_s)\mathrm{d}s\right] \\ &\leq f_{n,m}(a) + t\|h\|_{B(a_{n,m},4n^{-1})}\|g_{n,m}\| \leq \frac{1+tC}{n}\end{aligned}$$

Since by 4 from Proposition 3.4 we have $\mathbf{P}\big(\tau < \infty \Rightarrow d(X_\tau, a) \geq \frac{3}{n}\big) = 1$,

$$\begin{aligned}\mathbf{E}[f_{n,m}(X_{t\wedge\tau})] &= \mathbf{E}[f_{n,m}(X_\tau)\mathbb{1}_{\{\tau\leq t\}}] + \mathbf{E}[f_{n,m}(X_t)\mathbb{1}_{\{t<\tau\}}] \\ &\geq (1-\frac{1}{n})\mathbf{P}(\tau \leq t) - \frac{1}{n}\mathbf{P}(t < \tau) = \mathbf{P}(\tau \leq t) - \frac{1}{n},\end{aligned}$$

so

$$\mathbf{P}(\tau \leq t) \leq \frac{2+tC}{n}.$$

Hence we obtain

$$\mathbf{P}\big(\forall s \in [0,t],\ d(X_s, a) \leq \frac{3}{n}\big) \geq \mathbf{P}(t < \tau) \geq 1 - \frac{2+tC}{n}.$$

By taking the limit with respect to n and t we obtain (A.21).

To complete the proof let us consider an arbitrary $\mathbf{P} \in \mathcal{M}_{\mathrm{c}}\left((hL_0)^\Delta\right)$. As in Remark 3.10 we denote

$$\mathbf{Q}_X \overset{\mathbf{P}\text{-a.s.}}{:=} \mathscr{L}_{\mathbf{P}}\left((X_{\tau^U+t})_{t\geq 0} \;\middle|\; \mathcal{F}_{\tau^U}\right).$$

Thanks to Proposition 3.11 $\mathbf{P}$-almost surely $\mathbf{Q}_X \in \mathcal{M}\left((hL)^\Delta\right)$, and thanks to 4 from Proposition 3.4 $\mathbf{P}$-almost surely $\mathbf{Q}_X(X_0 = a) = 1$ with $a = X_\tau \in S^\Delta\backslash U$ on $\{\tau^U < \infty\}$. By using the previous situation and by applying (A.21) we get that $\mathbf{P}$-almost surely $\mathbf{Q}_X(\forall s \geq 0,\ X_s = a) = 1$, with $a = X_\tau \in S^\Delta\backslash U$ on $\{\tau^U < \infty\}$. Hence $\mathbf{P}(X = X^{\tau^U}) = 1$. □

Lemma A.5. *Let L be a subset of $\mathrm{C}_0(S) \times \mathrm{C}_0(S)$ such that the martingale problem associated to L is well-posed. Then the martingale problem associated to L^Δ is well-posed if and only if $\mathbf{P}(X = X^{\tau^S}) = 1$ for all $\mathbf{P} \in \mathcal{M}_{\mathrm{c}}(L^\Delta)$ (in other words $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{loc}(S))$).*

Proof. Assume that the martingale problem associated to L^Δ is well-posed and take $\mathbf{P} \in \mathcal{M}_{\mathrm{c}}(L^\Delta)$. Then $\mathscr{L}_{\mathbf{P}}(X^{\tau^S}) \in \mathcal{M}_{\mathrm{c}}(L^\Delta)$, so by uniqueness of the solution $\mathbf{P} = \mathscr{L}_{\mathbf{P}}(X^{\tau^S})$ and so $\mathbf{P}(X = X^{\tau^S}) = 1$. For the converse, let $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{M}_{\mathrm{c}}(L^\Delta)$ be such that $\mathscr{L}_{\mathbf{P}^1}(X_0) = \mathscr{L}_{\mathbf{P}^2}(X_0)$. Then $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ so $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{M}(L)$, hence $\mathbf{P}^1 = \mathbf{P}^2$. □

Proof of Lemma 4.17. Let L_0 and h_0 be as in Lemma A.4 and take $h \in \mathrm{C}(S, \mathbb{R}_+)$ with $\{h \neq 0\} = U$ and $\sup_{a\in U}(h/h_0)(a) < \infty$. The existence of a solution for the martingale problem associated to $(hL_0)^\Delta$ is given by the existence of a solution for the martingale problem associated to L. Let $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{M}_{\mathrm{c}}((hL_0)^\Delta)$ be such that $\mathscr{L}_{\mathbf{P}^1}(X_0) = \mathscr{L}_{\mathbf{P}^2}(X_0)$. Thanks to Lemma A.4 and Lemma A.5, for an open subset $V \Subset U$, there exist $k \in \mathrm{C}(S, \mathbb{R}_+^*)$ and a dense subset L_1 of L_0 such that $k(a) = h(a)$ for any $a \in V$, $kL_1 \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$

and the martingale problem associated to $(kL_1)^\Delta$ is well-posed. Hence we may apply Theorem 6.1 p. 216 from [EK86] and deduce that $\mathscr{L}_{\mathbf{P}^1}(X^{\tau^V}) = \mathscr{L}_{\mathbf{P}^2}(X^{\tau^V})$. Letting V growing toward U we deduce that $\mathscr{L}_{\mathbf{P}^1}(X^{\tau^U}) = \mathscr{L}_{\mathbf{P}^2}(X^{\tau^U})$ and so, since $\mathbf{P}^i(X = X^{\tau^U}) = 1$ for $i \in \{1, 2\}$, we conclude that $\mathbf{P}^1 = \mathbf{P}^2$. $\square$

Chapter 4

Lévy-type processes: convergence and discrete schemes

Abstract: We characterise the convergence of a certain class of discrete time Markov processes toward locally Feller processes in terms of convergence of associated operators. The theory of locally Feller processes is applied to Lévy-type processes in order to obtain convergence results on discrete and continuous time indexed processes, simulation methods and Euler schemes. We also apply the same theory to a slightly different situation, in order to get results of convergence of diffusions or random walks toward singular diffusions. As a consequence we deduce the convergence of random walks in random medium toward diffusions in random potential.
Key words: Lévy-type processes, random walks and diffusions in random and non-random environment, weak convergence of probability measures, discrete schemes, Skorokhod topology, martingale problem, Feller processes, generators
MSC2010 Subject Classification: Primary 60J25; Secondary 60J75, 60B10, 60G44, 60J35, 60J05, 60K37, 60E07, 47D07

1 Introduction

Lévy-type processes constitute a large class of processes allowing to build models for many phenomena. Heuristically, a Lévy-type process is a Markov process taking its values in the one-point compactification $\mathbb{R}^{d\Delta}$, such that in each point of $a \in \mathbb{R}^d$,

- it is drifted by the value of a vector $\delta(a)$,
- it diffuse with the value $\gamma(a)$, a symmetric positive semi-definite matrix,
- it jumps into a Borel subset B of $\mathbb{R}^{d\Delta}$ with the rate $\nu(a, B)$, a positive measure satisfying
$$\int (1 \wedge |b-a|^2)\nu(a, db) < \infty.$$

For a Lévy-type process we will call (δ, γ, ν) its Lévy triplet.

In general, the study of the convergence of sequences of general Markov processes is an important question. The present paper considers this question, among others, in the setting of the preceding two models. The approximating Markov sequences could have continuous or discrete time parameter in order to cover scaling transformations or discrete schemes.

A usual way to obtain such results is the use of the theory of Feller processes. In this context there exist two corresponding results of convergence (see, for instance Kallenberg [Kal02], Thms. 19.25, p. 385 and 19.27, p. 387). However, on one hand, when one needs to consider unbounded coefficients, for instance the Lévy triplet (δ, γ, ν) for Lévy-type processes, technical difficulties could appear in the framework of Feller processes. On the other hand the cited results of convergence impose the knowledge of a core of the generator. This could not be the case in some probabilistic constructions.

Our method to tackle these difficulties is to consider the context of the martingale local problems and of locally Feller processes, introduced in [GH17c]. In this general framework we have already analysed the question of convergence of sequences of locally Feller processes. In the present paper we add the study of the convergence for processes indexed by a discrete time parameter toward processes indexed by a continuous time parameter. We obtain the characterisation of the convergence in terms of convergence of associated operators, by using the uniform convergence on compact sets, and hence operators with unbounded coefficients could be considered. Likewise, we do not impose that the operator is a generator, but we assume only the well-posed feature of the associated martingale local problem. Indeed, it could be more easy to verify the well-posed feature (see for instance, Stroock [Str75] for Lévy-type processes, Stroock and Varadhan [SV06] for diffusion processes, Kurtz [Kur11] for Lévy-driven stochastic differential equations and forward equations...).

We apply our abstract results and we obtain sharp results of convergence for discrete and continuous time sequences of processes toward Lévy-type process, in terms of Lévy parameters (δ, γ, ν). We prefer the use of the Lévy triplet than the symbol associated to the operator, since the results are more precise in the situation of possibly instantaneous explosions. This is due essentially to the fact that the vague convergence of bounded measures cannot be characterised in terms of characteristic function. Our results can also be used to simulate Lévy-type processes and we improve Theorem 7.6 from Böttcher, Schilling and Wang [BSW13], p. 172, which is an approximation result of type Euler scheme. We state the results in terms of convergence of operators, but essentially one can deduce the convergence of the associated processes.

Another well known model is the dynamic of a Brownian particle in a potential. It is often given by the solution of the one-dimensional stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{2}V'(X_t)\mathrm{d}t,$$

where $V : \mathbb{R} \to \mathbb{R}$. The process X is also a Lévy-type process, and, thanks to the regularising property of the Brownian motion one can consider very general potentials, for example cadlag functions (see Mandle [Man68]). In particular, it can be supposed that the potential is a Brownian path (see Brox [Bro86]), a Lévy path (see Carmona[Car97]) or other random paths (Gaussian and/or fractional process...).

When studying a Brownian particle in a potential, we prove the continuous dependence of the diffusion with respect to the potential, using again our abstract results, even this is a different situation. We point out that it can be possible to consider potentials with very few constraints. In particular we consider diffusions in random potentials as limits of random walks in random mediums, as an application of an approximation of the diffusion by random walks on $\mathbb{Z}$. An important example is the convergence of Sinai's random walk [Sin82] toward the diffusion in a Poisson potential (recovering Thm. 2 from Seignourel [Sei00], p. 296), toward the diffusion in a Brownian potential, also called Brox's diffusion (improving Thm. 1 from Seignourel [Sei00], p. 295) and, more generally, toward the diffusion in a Lévy potential.

Let us describe the organisation of the paper. The next section contains notations and statements from our previous paper [GH17c], which are useful for an easy reading of the present paper. In particular, we give the statements concerning the existence of solutions for martingale local problems and concerning the convergence of continuous time locally Feller processes. Section 3 is devoted to the limits of sequences of discrete time processes, while Section 4 contains two results of convergence toward general Lévy-type processes. The diffusions evolving in a potential are studied in Section 5. The appendix contains the statements of several technical results already proved in [GH17c].

2 Martingale local problem setting and related results

Let S be a locally compact Polish space. Take $\Delta \notin S$, and we will denote by $S^\Delta \supset S$ the one-point compactification of S, if S is not compact, or the topological sum $S \sqcup \{\Delta\}$, if S is compact (so Δ is an isolated point). The fact that a subset A is compactly embedded in an open subset $U \subset S^\Delta$ will be denoted by $A \Subset U$. If $x \in (S^\Delta)^{\mathbb{R}_+}$ we denote the explosion time by

$$\xi(x) := \inf\{t \geq 0 \,|\, \{x_s\}_{s \leq t} \not\Subset S\}.$$

The set of exploding cadlag paths is defined by

$$\mathbb{D}_{\text{loc}}(S) := \left\{ x \in (S^\Delta)^{\mathbb{R}_+} \;\middle|\; \begin{array}{l} \forall t \geq \xi(x),\ x_t = \Delta, \\ \forall t \geq 0,\ x_t = \lim_{s \downarrow t} x_s, \\ \forall t > 0 \text{ s.t. } \{x_s\}_{s<t} \Subset S,\ x_{t-} := \lim_{s \uparrow t} x_s \text{ exists} \end{array} \right\},$$

and is endowed with the local Skorokhod topology (see Theorem 2.4 in[GH17b]) which is also Polish. A sequence $(x^k)_{k \in \mathbb{N}}$ in $\mathbb{D}_{\text{loc}}(S)$ converges to x for the local Skorokhod topology if and only if there exists a sequence $(\lambda^k)_k$ of increasing homeomorphisms on $\mathbb{R}_+$ satisfying

$$\forall t \geq 0 \text{ s.t. } \{x_s\}_{s<t} \Subset S, \quad \lim_{k \to \infty} \sup_{s \leq t} d(x_s, x^k_{\lambda^k_s}) = 0 \quad \text{and} \quad \lim_{k \to \infty} \sup_{s \leq t} |\lambda^k_s - s| = 0.$$

The local Skorokhod topology does not depend on the arbitrary metric d on S^Δ, but only on the topology on S. We will always denote by X the canonical process on $\mathbb{D}_{\text{loc}}(S)$. We endow $\mathbb{D}_{\text{loc}}(S)$ with the Borel σ-algebra $\mathcal{F} := \sigma(X_s,\ 0 \leq s < \infty)$ and a filtration $\mathcal{F}_t := \sigma(X_s,\ 0 \leq s \leq t)$.

Denote by $\mathrm{C}(S) := \mathrm{C}(S, \mathbb{R})$, respectively by $\mathrm{C}(S^\Delta) := \mathrm{C}(S^\Delta, \mathbb{R})$, the set of real continuous functions on S, respectively on S^Δ, and by $\mathrm{C}_0(S)$ the set of functions $f \in \mathrm{C}(S)$ vanishing in Δ. We endow the set $\mathrm{C}(S)$ with the topology of uniform convergence on compact sets and $\mathrm{C}_0(S)$ with the topology of uniform convergence.

We proceed by recalling the notion of martingale local problem. Let L be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$. The set $\mathcal{M}(L)$ of solutions of the martingale local problem associated to L is the set of probabilities $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\text{loc}}(S))$ such that for all $(f, g) \in L$ and an open subset

$U \Subset S$:

$$f(X_{t\wedge\tau^U}) - \int_0^{t\wedge\tau^U} g(X_s)\mathrm{d}s \text{ is a } \mathbf{P}\text{-martingale}$$

with respect to the filtration $(\mathcal{F}_t)_t$ or, equivalent, to the filtration $(\mathcal{F}_{t+})_t$. Here τ^U is the stopping time given by

$$\tau^U := \inf\{t \geq 0 \mid X_t \notin U \text{ or } X_{t-} \notin U\}. \tag{2.1}$$

In [GH17c] the following result of existence of solutions for martingale local problem was proved:

Theorem 2.1. *Let L be a linear subspace of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ such that its domain $D(L) := \{f \in \mathrm{C}_0(S) \mid \exists g \in \mathrm{C}(S),\ (f,g) \in L\}$ is dense in $\mathrm{C}_0(S)$. Then, there is equivalence between*

i) existence of a solution for the martingale local problem: for any $a \in S$ there exists an element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathbf{P}(X_0 = a) = 1$;

ii) L satisfies the positive maximum principle: for all $(f,g) \in L$ and $a_0 \in S$, if $f(a_0) = \sup_{a\in S} f(a) \geq 0$ then $g(a_0) \leq 0$.

Let us note that a linear subspace $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ satisfying the positive maximum principle is univariate, so that it can be equivalently considered as a linear operator

$$L : \mathrm{D}(L) \to \mathrm{C}(S).$$

The martingale local problem is said well-posed if there is existence and uniqueness of the solution, which means that for any $a \in S$ there exists an unique element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathbf{P}(X_0 = a) = 1$.

A family of probabilities $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is called locally Feller if there exists $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ such that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$ and

$$\forall a \in S : \qquad \mathbf{P} \in \mathcal{M}(L) \text{ and } \mathbf{P}(X_0 = a) = 1 \Longleftrightarrow \mathbf{P} = \mathbf{P}_a.$$

The $\mathrm{C}_0 \times \mathrm{C}$-generator of a locally Feller family $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is the set of functions $(f,g) \in \mathrm{C}_0(S) \times \mathrm{C}(S)$ such that, for any $a \in S$ and any open subset $U \Subset S$,

$$f(X_{t\wedge\tau^U}) - \int_0^{t\wedge\tau^U} g(X_s)\mathrm{d}s \text{ is a } \mathbf{P}_a\text{-martingale.}$$

It was noticed in Remark 4.11 in [GH17c] that if $h \in \mathrm{C}(S, \mathbb{R}_+^*)$ and if L is the $\mathrm{C}_0 \times \mathrm{C}$-generator of a locally Feller family, then

$$hL := \{(f, hg) \mid (f,g) \in L\} \text{ is the } \mathrm{C}_0 \times \mathrm{C}\text{-generator of a locally Feller family.} \tag{2.2}$$

A family of probability measures associated to a Feller semi-group constitutes a natural example of a locally Feller family (see Theorem 4.8 from [GH17c]). We recall that a Feller semi-group $(T_t)_{t\in\mathbb{R}_+}$ is a strongly continuous semi-group of positive linear contractions on $\mathrm{C}_0(S)$. Its $\mathrm{C}_0 \times \mathrm{C}_0$-generator is the set L_0 of $(f,g) \in \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ such that, for all $a \in S$

$$\lim_{t\to 0} \frac{1}{t}\big(T_t f(a) - f(a)\big) = g(a).$$

It was proved in Propositions 4.2 and 4.12 from [GH17c], that the martingale problem associated to L_0 admits a unique solution and, if L denotes its $\mathrm{C}_0(S)\times\mathrm{C}(S)$-generator then, taking the closure in $\mathrm{C}_0(S)\times\mathrm{C}(S)$,

$$L_0 = L\cap \mathrm{C}_0(S)\times\mathrm{C}_0(S) \quad \text{and} \quad L=\overline{L_0}. \tag{2.3}$$

The following result of convergence is essential for our further development and it was proved in [GH17c]. As was already pointed out in the introduction, an improvement with respect to the classical result of convergence (for instance Theorem 19.25, p. 385, in [Kal02]), is that one does not need to know the generator of the limit family, but only the fact that a martingale local problem is well-posed.

Theorem 2.2 (Convergence of locally Feller family)**.** *For $n\in\mathbb{N}\cup\{\infty\}$, let $(\mathbf{P}^n_a)_a\in\mathcal{P}(\mathbb{D}_{loc}(S))^S$ be a locally Feller family and let L_n be a subset of $\mathrm{C}_0(S)\times\mathrm{C}(S)$. Suppose that for any $n\in\mathbb{N}$, $\overline{L_n}$ is the generator of $(\mathbf{P}^n_a)_a$, suppose also that $\mathrm{D}(L_\infty)$ is dense in $\mathrm{C}_0(S)$ and*

$$\forall a\in S: \qquad \mathbf{P}\in\mathcal{M}(L_\infty) \ \text{and}\ \mathbf{P}(X_0=a)=1 \Longleftrightarrow \mathbf{P}=\mathbf{P}^\infty_a.$$

Then we have equivalence between:

a) the mapping

$$\begin{array}{ccc} \mathbb{N}\cup\{\infty\}\times\mathcal{P}(S^\Delta) & \to & \mathcal{P}\left(\mathbb{D}_{loc}(S)\right) \\ (n,\mu) & \mapsto & \mathbf{P}^n_\mu \end{array}$$

is weakly continuous for the local Skorokhod topology, where $\mathbf{P}_\mu:=\int\mathbf{P}_a\mu(\mathrm{d}a)$ and $\mathbf{P}_\Delta(X_0=\Delta)=1$;

b) for any $a_n,a\in S$ such that $a_n\to a$, $\mathbf{P}^n_{a_n}$ converges weakly for the local Skorokhod topology to $\mathbf{P}^\infty_a$, as $n\to\infty$;

c) for any $f\in\mathrm{D}(L_\infty)$, there exists for each n, $f_n\in\mathrm{D}(L_n)$ such that $f_n\xrightarrow[n\to\infty]{\mathrm{C}_0}f$, $L_nf_n\xrightarrow[n\to\infty]{\mathrm{C}}L_\infty f$.

3 Convergence of families indexed by discrete time

We start our study by giving a discrete time version of the notion of locally Feller family.

Definition 3.1 (Discrete time locally Feller family)**.** We denote by Y the discrete time canonical process on $(S^\Delta)^{\mathbb{N}}$ and we endow $(S^\Delta)^{\mathbb{N}}$ with the canonical σ-algebra. A family $(\mathbf{P}_a)_a\in\mathcal{P}\left((S^\Delta)^{\mathbb{N}}\right)^S$ is said to be a discrete time locally Feller family if there exists an operator $T:\mathrm{C}_0(S)\to\mathrm{C}_b(S)$, called transition operator, such that for any $a\in S$: $\mathbf{P}_a(Y_0=a)=1$ and

$$\forall n\in\mathbb{N},\ \forall f\in\mathrm{C}_0(S),\quad \mathbf{E}_a\left(f(Y_{n+1})\mid Y_0,\dots,Y_n\right)=\mathbb{1}_{\{Y_n\neq\Delta\}}Tf(Y_n)\quad \mathbf{P}_a\text{-a.s.} \tag{3.1}$$

If we denote $\mathbf{P}_\Delta$ the probability defined by $\mathbf{P}_\Delta(\forall n\in\mathbb{N},\ Y_n=\Delta)=1$, then for $\mu\in\mathcal{P}(S^\Delta)$, $\mathbf{P}_\mu:=\int\mathbf{P}_a\mu(\mathrm{d}a)$ satisfies also (3.1).

Now we can state the main result of this section which, similarly, is an improvement with respect to Theorem 19.27, p. 387, in [Kal02], in the sense that one does not need to know the generator of the limit family, but only the fact that a martingale local problem is well-posed.

Theorem 3.2 (Convergence)**.** *Let L be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ with $\mathrm{D}(L)$ a dense subset of $\mathrm{C}_0(S)$, such that the martingale local problem associated to L is well-posed, and let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be the associated continuous time locally Feller family. For each $n \in \mathbb{N}$ we introduce $(\mathbf{P}^n_a)_a \in \mathcal{P}((S^\Delta)^\mathbb{N})^S$ a discrete time locally Feller families having for transition operator T_n. We denote by L_n the operator $(T_n - \mathrm{id})/\varepsilon_n$, where $(\varepsilon_n)_n$ is a sequence of positive constants converging to 0, as $n \to \infty$. There is equivalence between:*

a) for any $\mu_n, \mu \in \mathcal{P}(S^\Delta)$ such that $\mu_n \to \mu$ weakly, as $n \to \infty$,

$$\mathscr{L}_{\mathbf{P}^n_{\mu_n}}\big((Y_{\lfloor t/\varepsilon_n \rfloor})_t\big) \underset{n\to\infty}{\overset{\mathcal{P}(\mathbb{D}_{loc}(S))}{\longrightarrow}} \mathbf{P}_\mu \,;$$

b) for any $a_n, a \in S$ such that $a_n \to a$, as $n \to \infty$,

$$\mathscr{L}_{\mathbf{P}^n_{a_n}}\big((Y_{\lfloor t/\varepsilon_n \rfloor})_t\big) \underset{n\to\infty}{\overset{\mathcal{P}(\mathbb{D}_{loc}(S))}{\longrightarrow}} \mathbf{P}_a \,;$$

c) for any $f \in \mathrm{D}(L)$, there exists $(f_n)_n \in \mathrm{C}_0(S)^\mathbb{N}$ such that $f_n \underset{n\to\infty}{\overset{\mathrm{C}_0(S)}{\longrightarrow}} f$, $L_n f_n \underset{n\to\infty}{\overset{\mathrm{C}(S)}{\longrightarrow}} Lf$.

Here $\lfloor r \rfloor$ denotes the integer part of the real number r.

Proof. Set $\Omega := (S^\Delta)^\mathbb{N} \times \mathbb{R}_+^\mathbb{N}$ and $\mathcal{G} := \mathcal{B}(S^\Delta)^{\otimes\mathbb{N}} \otimes \mathcal{B}(\mathbb{R}_+)^{\otimes\mathbb{N}}$. For any $\mu \in \mathcal{P}(S^\Delta)$ and $n \in \mathbb{N}$, define $\mathbb{P}^n_\mu := \mathbf{P}^n_\mu \otimes \mathcal{E}(1)^{\otimes\mathbb{N}}$, where $\mathcal{E}(1)$ is the exponential distribution with expectation 1. Define

$$Y_n : \begin{array}{ccc} \Omega & \to & S \\ (((y_k)_k, (s_k)_k)) & \mapsto & y_n \end{array} \qquad \text{and} \qquad E_n : \begin{array}{ccc} \Omega & \to & \mathbb{R}_+ \\ (((y_k)_k, (s_k)_k)) & \mapsto & s_n, \end{array}$$

and introduce the standard Poisson process

$$\forall t \geq 0, \quad N_t := \inf\Big\{ n \in \mathbb{N} \,\Big|\, \sum_{k=1}^{n+1} E_k > t \Big\}.$$

Step 1) For each $n \in \mathbb{N}$ define $Z^n_t := Y_{N_{t/\varepsilon_n}}$. Consider the modified assertions:

$a')$ for any $\mu_n, \mu \in \mathcal{P}(S^\Delta)$ such that $\mu_n \to \mu$,

$$\mathscr{L}_{\mathbb{P}^n_{\mu_n}}(Z^n) \underset{n\to\infty}{\overset{\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))}{\longrightarrow}} \mathbf{P}_\mu;$$

$b')$ for any $a_n, a \in S$ such that $a_n \to a$,

$$\mathscr{L}_{\mathbb{P}^n_{a_n}}(Z^n) \underset{n\to\infty}{\overset{\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))}{\longrightarrow}} \mathbf{P}_a,$$

We will verify that $a') \Leftrightarrow b') \Leftrightarrow c)$. We need to prove that for all $\mu \in \mathcal{P}(S^\Delta)$, $\mathscr{L}_{\mathbb{P}^n_\mu}(Z^n) \in \mathcal{M}(L_n)$. Taking $\mathcal{G}^n_t := \sigma(N_{s/\varepsilon_n}, Z^n_s,\ s \le t)$, it is enough to prove that, for each $f \in \mathrm{C}_0(S)$ and $0 \le s \le t$,

$$\mathbb{E}^n_\mu\left[f(Z^n_t) - f(Z^n_s) - \int_s^t L_n f(Z^n_u)\mathrm{d}u \;\middle|\; \mathcal{G}^n_s\right] = 0.$$

Let us introduce the $(\mathcal{G}^n_t)_t$-stopping times $\tau^n_k := \inf\left\{u \ge 0 \,\middle|\, N_{u/\varepsilon_n} = k\right\}$. Then, for all $k \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}^n_\mu\left[f(Z^n_{t\wedge(\tau^n_{k+1}\vee s)}) - f(Z^n_{t\wedge(\tau^n_k\vee s)}) \;\middle|\; \mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\right]\\
&\quad= \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}\mathbb{E}^n_\mu\left[(f(Y_{k+1}) - f(Y_k))\mathbb{1}_{\{\tau^n_{k+1}\le t\}} \;\middle|\; \mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\right]\\
&\quad= \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}\mathbb{E}^n_\mu\left[(f(Y_{k+1}) - f(Y_k))\mathbb{1}_{\{\tau^n_{k+1}-\tau^n_k\vee s\le t-\tau^n_k\vee s\}} \;\middle|\; \mathcal{G}^n_{\tau^n_k\vee s}\right]\\
&\quad= \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}(T_n f(Y_k) - f(Y_k))\big(1 - \exp(-(t-\tau^n_k\vee s)/\varepsilon_n)\big)\\
&\quad= \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}L_n f(Z^n_{\tau^n_k\vee s})\varepsilon_n\big(1 - \exp(-(t-\tau^n_k\vee s)/\varepsilon_n)\big),
\end{aligned}$$

where we used the fact that $(N_{u/\varepsilon_n})_u$ is a Poisson process. Similarly,

$$\begin{aligned}
&\mathbb{E}^n_\mu\left[\int_{t\wedge(\tau^n_k\vee s)}^{t\wedge(\tau^n_{k+1}\vee s)} L_n f(Z^n_u)\mathrm{d}u \;\middle|\; \mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\right]\\
&\quad= \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}L_n f(Z^n_{\tau^n_k\vee s})\mathbb{E}^n_\mu\left[t\wedge\tau^n_{k+1} - \tau^n_k\vee s \;\middle|\; \mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\right]\\
&\quad= \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}L_n f(Z^n_{\tau^n_k\vee s})\mathbb{E}^n_\mu\left[(t-\tau^n_k\vee s)\wedge(\tau^n_{k+1}-\tau^n_k\vee s) \;\middle|\; \mathcal{G}^n_{\tau^n_k\vee s}\right]\\
&\quad= \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}L_n f(Z^n_{\tau^n_k\vee s})\int_0^\infty (1/\varepsilon_n)\exp(-u/\varepsilon_n)((t-\tau^n_k\vee s)\wedge u)\mathrm{d}u\\
&\quad= \mathbb{1}_{\{t>\tau^n_k, s<\tau^n_{k+1}\}}L_n f(Z^n_{\tau^n_k\vee s})\varepsilon_n\big(1 - \exp(-(t-\tau^n_k\vee s)/\varepsilon_n)\big).
\end{aligned}$$

Hence

$$\mathbb{E}^n_\mu\left[f(Z^n_{t\wedge(\tau^n_{k+1}\vee s)}) - f(Z^n_{t\wedge(\tau^n_k\vee s)}) - \int_{t\wedge(\tau^n_k\vee s)}^{t\wedge(\tau^n_{k+1}\vee s)} L_n f(Z^n_u)\mathrm{d}u \;\middle|\; \mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\right] = 0.$$

Hence

$$\begin{aligned}
&\mathbb{E}^n_\mu\left[f(Z^n_t) - f(Z^n_s) - \int_s^t L_n f(Z^n_u)\mathrm{d}u \;\middle|\; \mathcal{G}^n_s\right]\\
&= \mathbb{E}^n_\mu\left[\sum_{k\ge 0}\left((Z^n_{t\wedge(\tau^n_{k+1}\vee s)}) - f(Z^n_{t\wedge(\tau^n_k\vee s)}) - \int_{t\wedge(\tau^n_k\vee s)}^{t\wedge(\tau^n_{k+1}\vee s)} L_n f(Z^n_u)\mathrm{d}u\right) \;\middle|\; \mathcal{G}^n_s\right]\\
&= \sum_{k\ge 0}\mathbb{E}^n_\mu\left[\mathbb{E}^n_\mu\left[f(Z^n_{t\wedge(\tau^n_{k+1}\vee s)}) - f(Z^n_{t\wedge(\tau^n_k\vee s)}) - \int_{t\wedge(\tau^n_k\vee s)}^{t\wedge(\tau^n_{k+1}\vee s)} L_n f(Z^n_u)\mathrm{d}u \;\middle|\; \mathcal{G}^n_{t\wedge(\tau^n_k\vee s)}\right] \;\middle|\; \mathcal{G}^n_s\right]\\
&= 0,
\end{aligned}$$

so that $\mathscr{L}_{\mathbb{P}^n_\mu}(Z^n) \in \mathcal{M}(L_n)$. Hence applying the convergence Theorem 2.2 to L_n and L, we obtain the equivalences between $a')$, $b')$ and $c)$.
Step 2. To carry on with the proof we need the following technical result

Lemma 3.3. *For $n \in \mathbb{N}$, let $(\Omega^n, \mathcal{G}^n, \mathbb{P}^n)$ be a probability space, let $Z^n : \Omega^n \to \mathbb{D}_{loc}(S)$ and $\Gamma^n : \Omega^n \to \mathrm{C}(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing random bijection. Define $\widetilde{Z}^n := Z^n \circ \Gamma^n$. Suppose that for each $\varepsilon > 0$ and $t \in \mathbb{R}_+$*

$$\mathbb{P}^n\left(\sup_{s\leq t}|\Gamma^n_s - s| \geq \varepsilon\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Then for any $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{loc}(S))$,

$$\mathscr{L}_{\mathbb{P}^n}(Z^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P} \quad \Leftrightarrow \quad \mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P},$$

where the limits are for the weak topology associated to the local Skorokhod topology.

We postpone the proof of this lemma and we finish the proof of the theorem. Let us note that for any $t \geq 0$ and $n \in \mathbb{N}$, $Y_{\lfloor t/\varepsilon_n \rfloor} = Z^n_{\Gamma^n_t}$ with

$$\Gamma^n_t := \varepsilon_n \left(\sum_{k=1}^{\lfloor t/\varepsilon_n \rfloor} E_k + (t/\varepsilon_n - \lfloor t/\varepsilon_n \rfloor) E_{\lfloor t/\varepsilon_n \rfloor + 1}\right).$$

Assuming that

$$\forall t \geq 0,\ \forall \varepsilon > 0, \qquad \sup_{\mu \in \mathcal{P}(S^\Delta)} \mathbb{P}^n_\mu\left(\sup_{s\leq t}|\Gamma^n_s - s| \geq \varepsilon\right) \underset{n\to\infty}{\longrightarrow} 0, \tag{3.2}$$

then, by the latter lemma we get $a) \Leftrightarrow a')$ and $b \Leftrightarrow b')$, so that $a) \Leftrightarrow b) \Leftrightarrow c)$.

Let us prove (3.2). Fix $t \geq 0$, $\varepsilon > 0$, $n \in \mathbb{N}$ and $\mu \in \mathcal{P}(S^\Delta)$, then since Γ^n is a continuous piecewise affine function, we have

$$\sup_{s\leq t}|\Gamma^n_s - s| \leq \sup_{\substack{k\in\mathbb{N}\\ k\leq \lceil t/\varepsilon_n \rceil}} |\Gamma^n_{k\varepsilon_n} - k\varepsilon_n| = \sup_{\substack{k\in\mathbb{N}\\ k\leq \lceil t/\varepsilon_n \rceil}} \left|\varepsilon_n \sum_{i=1}^k E_i - k\varepsilon_n\right| = \varepsilon_n \sup_{\substack{k\in\mathbb{N}\\ k\leq \lceil t/\varepsilon_n \rceil}} |M_k|$$

where $M_k := \sum_{i=1}^k E_i - k$ and $\lceil r \rceil$ denotes the smallest integer larger or equal than the real number r. Since the E_i are independent random variables with exponential distribution $\mathcal{E}(1)$ we have

$$\mathbb{E}^n_\mu[M_k^2] = k\mathbb{E}^n_\mu[(E_1 - 1)^2] = k.$$

By Markov's inequality and by the maximal Doob inequality applied to the discrete time martingale $(M_k)_k$ we can write

$$\begin{aligned}\mathbb{P}^n_\mu\left(\sup_{s\leq t}|\Gamma^n_s - s| \geq \varepsilon\right) &\leq \mathbb{P}^n_\mu\left(\varepsilon_n \sup_{k\leq \lceil t/\varepsilon_n \rceil} |M_k| \geq \varepsilon\right) \leq \frac{\mathbb{E}^n_\mu\left[\sup_{k\leq \lceil t/\varepsilon_n \rceil} M_k^2\right]\varepsilon_n^2}{\varepsilon^2} \\ &\leq \frac{4\mathbb{E}^n_\mu\left[M^2_{\lceil t/\varepsilon_n \rceil}\right]\varepsilon_n^2}{\varepsilon^2} = \frac{4\lceil t/\varepsilon_n \rceil \varepsilon_n^2}{\varepsilon^2} \leq \frac{4(t+\varepsilon_n)\varepsilon_n}{\varepsilon^2}.\end{aligned}$$

We deduce (3.2) and the proof of the theorem is complete except for the proof of Lemma 3.3.

□

Before giving the proof of Lemma 3.3 we state and prove a more general result:

Lemma 3.4. *Let E be a Polish topological space, for $n \in \mathbb{N}$, let $(\Omega^n, \mathcal{G}^n, \mathbb{P}^n)$ be a probability space and consider $Z^n, \widetilde{Z}^n : \Omega^n \to E$ random variables. Suppose that for each compact subset $K \subset E$ and each open subset $U \subset E^2$ containing the diagonal $\{(z,z) \,|\, z \in E\}$,*

$$\mathbb{P}^n\left(Z^n \in K,\ (Z^n, \widetilde{Z}^n) \notin U\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{3.3}$$

Then, for any $\mathbf{P} \in \mathcal{P}(E)$,

$$\mathscr{L}_{\mathbb{P}^n}(Z^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P} \quad \textit{implies} \quad \mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P},$$

where the limits are for the weak topology on $\mathcal{P}(E)$.

Proof. Suppose that $\mathscr{L}_{\mathbb{P}^n}(Z^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$ so that for any bounded continuous function $f : E \to \mathbb{R}$, $\lim_{n\to\infty} \mathbb{E}^n[f(Z^n)] = \int f \mathrm{d}\mathbf{P}$. Since E is a Polish space the sequence $(\mathscr{L}_{\mathbb{P}^n}(Z^n))_n$ is tight. Take an arbitrary $\varepsilon > 0$ and let K be a compact subset of E such that

$$\forall n \in \mathbb{N}, \quad \mathbb{P}^n(Z^n \notin K) \leq \varepsilon. \tag{3.4}$$

By (3.3) applied to K and $U := \{(z, \widetilde{z}) \mid |f(\widetilde{z}) - f(z)| < \varepsilon\}$, we have

$$\mathbb{P}^n\left(Z^n \in K,\ |f(\widetilde{Z}^n) - f(Z^n)| \geq \varepsilon\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Hence by (3.4)

$$\begin{aligned}
\left|\mathbb{E}^n[f(\widetilde{Z}^n)] - \int f\mathrm{d}\mathbf{P}\right| &\leq \left|\mathbb{E}^n[f(Z^n)] - \int f\mathrm{d}\mathbf{P}\right| + \mathbb{E}^n\left|f(\widetilde{Z}^n) - f(Z^n)\right| \\
&\leq \left|\mathbb{E}^n[f(Z^n)] - \int f\mathrm{d}\mathbf{P}\right| + \mathbb{E}^n\left[\left|f(\widetilde{Z}^n) - f(Z^n)\right| \mathbb{1}_{\{Z^n \in K, |f(\widetilde{Z}^n) - f(Z^n)| \geq \varepsilon\}}\right] \\
&\quad + \mathbb{E}^n\left[\left|f(\widetilde{Z}^n) - f(Z^n)\right| \mathbb{1}_{\{Z^n \in K, |f(\widetilde{Z}^n) - f(Z^n)| < \varepsilon\}}\right] + \mathbb{E}^n\left[\left|f(\widetilde{Z}^n) - f(Z^n)\right| \mathbb{1}_{\{Z^n \notin K\}}\right] \\
&\leq \left|\mathbb{E}^n[f(Z^n)] - \int f\mathrm{d}\mathbf{P}\right| + 2\|f\|\,\mathbb{P}^n\left(Z^n \in K,\ |f(\widetilde{Z}^n) - f(Z^n)| \geq \varepsilon\right) + \varepsilon(1 + 2\|f\|).
\end{aligned}$$

Letting successively $n \to \infty$ and $\varepsilon \to 0$, we deduce that

$$\mathbb{E}^n[f(\widetilde{Z}^n)] \underset{n\to\infty}{\longrightarrow} \int f\mathrm{d}\mathbf{P},$$

hence, since f is an arbitrary bounded continuous function, we have $\mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$. $\square$

Proof of Lemma 3.3. We denote by $\widetilde{\Lambda}$ the space of increasing bijections λ from $\mathbb{R}_+$ to $\mathbb{R}_+$, and for $t \in \mathbb{R}_+$ we denote $\|\lambda - \mathrm{id}\|_t := \sup_{s \leq t} |\lambda_s - s|$. Since

$$\forall \lambda \in \widetilde{\Lambda},\ \forall t \in \mathbb{R}_+,\ \forall \varepsilon > 0, \quad \|\lambda - \mathrm{id}\|_{t+\varepsilon} < \varepsilon \Rightarrow \|\lambda^{-1} - \mathrm{id}\|_t < \varepsilon,$$

the hypotheses of Lemma 3.3 are symmetric with respect to Z and $\widetilde{Z}$, so it suffices to prove only one implication. Hence we suppose $\mathscr{L}_{\mathbb{P}^n}(Z^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$ and, by applying Lemma 3.4, we prove $\mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$. Let K be a compact subset of $\mathbb{D}_{\mathrm{loc}}(S)$ and U be an open subset of $\mathbb{D}_{\mathrm{loc}}(S)^2$ containing the diagonal $\{(z,z)\,|\,z \in \mathbb{D}_{\mathrm{loc}}(S)\}$. We prove the assertion

$$\exists t \geq 0,\ \exists \varepsilon > 0,\ \forall z \in K,\ \forall \lambda \in \widetilde{\Lambda}, \quad \|\lambda - \mathrm{id}\|_t < \varepsilon \Rightarrow (z, z\circ\lambda) \in U. \tag{3.5}$$

If we suppose that (3.5) is false, then we can find two sequences $(z^n)_n \in K^{\mathbb{N}}$ and $(\lambda^n)_n \in \widetilde{\Lambda}^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $(z^n, z^n \circ \lambda^n) \notin U$ and for all $t \geq 0$, $\|\lambda_n - \mathrm{id}\|_t \to 0$, as $n \to \infty$. By compactness of K, possibly by taking a subsequence, we may suppose the existence of $z \in K$ such that $z^n \to z$ as $n \to \infty$. Then, it is straightforward to obtain

$$U \not\ni (z^n, z^n \circ \lambda^n) \underset{n\to\infty}{\longrightarrow} (z,z) \in U.$$

This is in contradiction with the fact that U is open, so we have proved (3.5). Take t and ε given by (3.5), then

$$\mathbb{P}^n\left(Z^n \in K,\ (Z^n, \widetilde{Z}^n) \notin U\right) \leq \mathbb{P}^n\left(\|\Gamma^n - \mathrm{id}\|_t \geq \varepsilon\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Hence by Lemma 3.4, $\mathscr{L}_{\mathbb{P}^n}(\widetilde{Z}^n) \underset{n\to\infty}{\longrightarrow} \mathbf{P}$. □

4 Lévy-type processes: convergence and discrete scheme

In this section we take $d \in \mathbb{N}^*$, we denote by $|\cdot|$ the Euclidean norm and by $\mathbb{R}^{d\Delta}$ the one point compactification of $\mathbb{R}^d$. Let also $\mathrm{C}_c^\infty(\mathbb{R}^d)$ be the set of compactly supported infinitely differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$. We are interested in the dynamics which locally look like as Lévy's dynamics. To simplify notations all along of the present section, let us introduce a linear functional on $\mathrm{C}_c^\infty(\mathbb{R}^d)$ which describes a dynamic in a neighbourhood of a point $a \in \mathbb{R}^d$: for $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$T_{\chi,a}(\delta,\gamma,\nu)f := \frac{1}{2}\sum_{i,j=1}^{d} \gamma_{ij}\partial^2_{ij} f(a) + \delta\cdot\nabla f(a) + \int_{\mathbb{R}^{d\Delta}} (f(b) - f(a) - \chi(a,b)\cdot\nabla f(a))\nu(\mathrm{d}b),$$

where

– *the compensation function* $\chi : \mathbb{R}^d \times \mathbb{R}^{d\Delta} \to \mathbb{R}^d$ *is a bounded measurable function satisfying, for any compact subset* $K \subset \mathbb{R}^d$,

$$\sup_{b,c\in K,\, b\neq c} \frac{|\chi(b,c) - (c-b)|}{|c-b|^2} < \infty; \tag{H1}$$

– *the drift vector is* $\delta \in \mathbb{R}^d$*, the diffusion matrix* $\gamma \in \mathbb{R}^{d\times d}$ *is symmetric positive semi-definite and the jump measure* ν *is a measure on* $\mathbb{R}^{d\Delta}$ *satisfying* $\nu(\{a\}) = 0$ *and*

$$\int_{\mathbb{R}^{d\Delta}} (1 \wedge |b-a|^2)\nu(db) < \infty. \tag{H2(a)}$$

Usually we take for compensation function

$$\chi_1(a,b) := (b-a)/(1+|b-a|^2) \quad \text{or} \quad \chi_2(a,b) := (b-a)\mathbb{1}_{|b-a|<1}. \tag{4.1}$$

It is well known (see for instance Theorem 2.12 pp. 21-22 from [Hoh98]) that for any linear operator $L : \mathrm{C}_c^\infty(\mathbb{R}^d) \to \mathrm{C}(\mathbb{R}^d)$ satisfying the positive maximum principle and for any χ satisfying (H1): for each $a \in \mathbb{R}^d$ there exist $\delta(a)$, $\gamma(a)$ and $\nu(a)$ satisfying (H2(a)) such that

$$\forall f \in \mathrm{C}_c^\infty(\mathbb{R}^d),\ \forall a \in \mathbb{R}^d, \quad Lf(a) = T_{\chi,a}(\delta(a), \gamma(a), \nu(a))f.$$

In the following we will call such an expression of L a *Lévy-type operator*.

In order to obtain a converse sentence and to get the convergence of sequences of Lévy-type operators, we have to make a more restrictive hypothesis on the couple (χ, ν): for $a \in \mathbb{R}^d$

– *the compensation function* $\chi : \mathbb{R}^d \times \mathbb{R}^{d\Delta} \to \mathbb{R}^d$ *is a bounded measurable function satisfying, for any compact subset* $K \subset \mathbb{R}^d$,

$$\sup_{b,c\in K,\, 0<|c-b|\le\varepsilon} \frac{|\chi(b,c) - (c-b)|}{|c-b|^2} \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{H3(a)}$$

and $\nu\left(\left\{b \in \mathbb{R}^{d\Delta} \;\middle|\; \chi \text{ is not continuous at } (a,b)\right\}\right) = 0$.

For example, χ_1 given in (4.1) satisfies (H3(a)) for any ν and $\chi_2(a,b)$ satisfies (H3(a)) whenever $\nu\big(\{b \in \mathbb{R}^d \,|\, |b-a| = 1\}\big) = 0$.

The following theorem contains a necessary and sufficient condition for the convergence of sequences of Lévy-type operators (and processes) in terms of their Lévy triplet. Before we introduce some notations.

- Let $\chi : \mathbb{R}^d \times \mathbb{R}^{d\Delta} \to \mathbb{R}^d$ be a compensation function. For each $a \in \mathbb{R}^d$ let $(\chi, \nu(a))$ and $(\delta(a), \gamma(a), \nu(a))$ satisfying respectively (H2(a)) and (H3(a)). Set

$$Lf(a) := T_{\chi,a}(\delta(a), \gamma(a), \nu(a))f, \qquad \text{for any } f \in \mathrm{C}_c^\infty(\mathbb{R}^d). \tag{4.2}$$

- For each $n \in \mathbb{N}$ and $a \in \mathbb{R}^d$ let $(\delta_n(a), \gamma_n(a), \nu_n(a))$ satisfying (H2(a)). Set

$$L_n f(a) := T_{\chi,a}(\delta_n(a), \gamma_n(a), \nu_n(a))f, \qquad \text{for any } f \in \mathrm{C}_c^\infty(\mathbb{R}^d). \tag{4.3}$$

- For each $n \in \mathbb{N}$ and $a \in \mathbb{R}^d$ let $\mu_n(a)$ be a probability measure on $\mathbb{R}^{d\Delta}$ and let $\varepsilon_n > 0$ be a sequence converging to 0. Set

$$T_n f(a) := \int f(b)\mu_n(a, \mathrm{d}b), \qquad \text{for any } f \in \mathrm{C}(\mathbb{R}^{d\Delta}). \tag{4.4}$$

Theorem 4.1 (Characterisation of the convergence toward Lévy-type operators)**.**
1) The function Lf *is continuous for any* $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ *if and only if*

- $a \mapsto \delta(a)$ *is continuous on* $\mathbb{R}^d$,
- $a \mapsto \int f(b)\nu(a, \mathrm{d}b)$ *is continuous on the interior of* $\{f = 0\} \cap \mathbb{R}^d$, *for any* $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$,
- $a \mapsto \gamma_{ij}(a) + \int \chi_i(a,b)\chi_j(a,b)\nu(a, \mathrm{d}b)$ *is continuous on* $\mathbb{R}^d$, *for any* $1 \le i, j \le d$.

2) Assume that Lf is continuous for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. The uniform convergence on compact sets, $L_n f \to Lf$, as $n \to \infty$, holds for all $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ if and only if

- *$\delta_n(a) \to \delta(a)$, uniformly for $a \in \mathbb{R}^d$ varying in any compact subset,*
- *$\int f(b)\nu_n(a, \mathrm{d}b) \to \int f(b)\nu(a, \mathrm{d}b)$, uniformly for $a \in \mathbb{R}^d$ varying in any compact subset of the interior of $\{f = 0\} \cap \mathbb{R}^d$, for any $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$,*
- *$\gamma_{n,ij}(a)+\int(\chi_i\chi_j)(a, b)\nu_n(a, \mathrm{d}b) \to \gamma_{ij}(a)+\int(\chi_i\chi_j)(a, b)\nu(a, \mathrm{d}b)$, uniformly for $a \in \mathbb{R}^d$ varying in any compact subset, for any $1 \leq i, j \leq d$.*

3) Assume that Lf is continuous for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. The uniform convergence on compact sets, $\varepsilon_n^{-1}(T_n f - f) \to Lf$, as $n \to \infty$, holds for all $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ if and only if

- *$\varepsilon_n^{-1} \int_{\mathbb{R}^{d\Delta}\setminus\{a\}} \chi(a, b)\mu_n(a, \mathrm{d}b) \to \delta(a)$, uniformly for $a \in \mathbb{R}^d$ varying in any compact subset,*
- *$\varepsilon_n^{-1} \int f(b)\mu_n(a, \mathrm{d}b) \to \int f(b)\nu(a, \mathrm{d}b)$, uniformly for $a \in \mathbb{R}^d$ varying in any compact subset of the interior of $\{f = 0\} \cap \mathbb{R}^d$, for any $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$,*
- *$\varepsilon_n^{-1} \int_{\mathbb{R}^{d\Delta}\setminus\{a\}}(\chi_i\chi_j)(a, b)\mu_n(a, \mathrm{d}b) \to \gamma_{ij}(a) + \int(\chi_i\chi_j)(a, b)\nu(a, \mathrm{d}b)$, uniformly for $a \in \mathbb{R}^d$ varying in any compact subset, for any $1 \leq i, j \leq d$.*

Remark 4.2. 1) Thanks to Theorems 2.2 and 3.2 we can deduce from Theorem 4.1 sharp results of convergence for the processes associated to L_n, T_n and L. In particular, the third part of Theorem 4.1 is, somehow, an improvement of the classical Donsker theorem, and, for instance allows us to simulate Lévy-type processes. We illustrate this fact by the following example.
2) The previous theorem may be adapted in the context where S is a manifold. In this case

$$Lf(a) := \frac{1}{2}D^2 f(a) \cdot \gamma(a) + Df(a) \cdot \delta(a) + \int (f(b) - f(a) - Df(a) \cdot \chi(a, b))\nu(a, \mathrm{d}b),$$

where for each $a \in \mathbb{R}^d$, $Df(a) \in T_a^*S$ and $D^2 f(a) \in T_a^*S^{\otimes 2}$ are first and second order differentials at a, the compensation function $\chi(a, b) \in T_aS$ is defined for $b \in S^\Delta$, the drift vector $\delta(a)$ is in T_aS, the diffusion $\gamma(a) \in T_aS^{\otimes 2}$ is symmetric positive semi-definite and the jump measure $\nu(a)$ is a measure on S^Δ. Here T_aS and T_a^*S are the tangent and the cotangent spaces of S at a. $\diamondsuit$

Example 4.3 (Symmetric stable type operator)**.** Consider $c \in \mathrm{C}(\mathbb{R}^d, \mathbb{R}_+)$ and $\alpha \in \mathrm{C}(\mathbb{R}^d, (0, 2))$, and denote, for $f \in \mathrm{C}_0(\mathbb{R}^d)$ and $a \in \mathbb{R}^d$,

$$Lf(a) := \int_{\mathbb{R}^d} (f(b) - f(a) - (b - a) \cdot \nabla f(a)\mathbb{1}_{|b-a|\leq 1})c(a)|b - a|^{-d-\alpha(a)}\mathrm{d}b.$$

As a consequence of the first part of Theorem 4.1, L maps $\mathrm{C}_0(\mathbb{R}^d)$ to $\mathrm{C}(\mathbb{R}^d)$. For $a \in \mathbb{R}^d$ and $n \in \mathbb{N}^*$, define the probability measure

$$\mu_n(a, \mathrm{d}b) := \frac{c(a)}{n}|b - a|^{-d-\alpha(a)}\mathbb{1}_{|b-a|\geq\varepsilon_n(a)}\mathrm{d}b, \quad \text{with} \quad \varepsilon_n(a) := \left(\frac{c(a)S_{d-1}}{n\alpha(a)}\right)^{1/\alpha(a)}$$

where $S_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the measure of the unity sphere in $\mathbb{R}^d$. Thanks to the third part of Theorem 4.1, for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$\lim_{n\to\infty} n\left(\int f(b)\mu_n(a,\mathrm{d}b) - f(a)\right) = Lf(a), \text{ uniformly for } a \text{ in compact subsets of } \mathbb{R}^d.$$

To go further, it is straightforward that for any $a \in \mathbb{R}^d$ and $n \in \mathbb{N}^*$, $\mu_n(a)$ is the distribution of the random variable

$$a + Q\left(\frac{c(a)S_{d-1}}{n\alpha(a)U}\right)^{1/\alpha(a)}, \text{ with independent } Q \sim \mathcal{U}(\mathbb{S}^{d-1}),\ U \sim \mathcal{U}([0,1]),$$

where $\mathcal{U}(\mathbb{S}^{d-1})$ and $\mathcal{U}([0,1])$ are the uniform distributions, respectively on the unity sphere of $\mathbb{R}^d$ and on $[0,1]$. To simulate the discrete time locally Feller processes associated to $(\mu_n(a))_a$ we can proceed as follows. Let $(Q_k, U_k)_k$ be an i.i.d. sequence of random variables with distributions $\mathcal{U}(\mathbb{S}^{d-1}) \otimes \mathcal{U}([0,1])$ and define, for $n \in \mathbb{N}^*$ and $k \in \mathbb{N}$,

$$Z_{k+1}^n := Z_k^n + Q_k\left(\frac{c(Z_k^n)S_{d-1}}{n\alpha(Z_k^n)U_k}\right)^{1/\alpha(Z_k^n)}.$$

Hence thanks to Theorem 3.2, if the martingale local problem associated to L is well-posed, then $(Z_{\lfloor nt\rfloor}^n)_t$ converges in distribution to the solution of the martingale local problem. $\Diamond$

Remark 4.4. This example is adaptable when we want to simulate more general Lévy-type processes. The heuristics is as follows: first we approximate the Lévy measure by finite measures, we renormalise them and then we convolute with a Gaussian measure having well chosen parameters. $\Diamond$

Before proceeding to the proof Theorem 4.1, we give a second approximation result inspired from [BSW13], Theorem 7.6 p. 172. Let $L : \mathrm{C}_c^\infty(\mathbb{R}^d) \to \mathrm{C}(\mathbb{R}^d)$ be an operator satisfying the positive maximum principle. Let the translation of f by $h \in \mathbb{R}^d$ be the mapping $\tau_h f(a) = f(a+h)$. For $a_0 \in \mathbb{R}^d$, we introduce the operator

$$L(a_0) : \mathrm{C}_c^\infty(\mathbb{R}^d) \to \mathrm{C}_0(\mathbb{R}^d) \quad \text{by} \quad L(a_0)f(a) := L(\tau_{a-a_0}f)(a_0). \tag{4.5}$$

Clearly $Lf(a) = L(a)f(a)$. Since $L(a_0)$ is invariant with respect to the translation and satisfies the positive maximum principle then its closure in $\mathrm{C}_0(\mathbb{R}^d)\times\mathrm{C}_0(\mathbb{R}^d)$ is the $\mathrm{C}_0\times\mathrm{C}_0$-generator of a Lévy family (see for instance, Section 2.1 pp. 32-41 from [BSW13]). We denote by $(T_t(a_0))_{t\geq 0}$ its Feller semi-group and we state:

Theorem 4.5 (Approximation with Lévy increments)**.**
Let $(\varepsilon_n)_n$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ and define the transition operators T_n by

$$T_n f(a) := T_{\varepsilon_n}(a)f(a), \quad \textit{for } f \in \mathrm{C}_0(\mathbb{R}^d).$$

Then, for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$\frac{1}{\varepsilon_n}(T_n f - f) \underset{n\to\infty}{\longrightarrow} Lf, \quad \textit{uniformly on compact sets.}$$

Remark 4.6. If the martingale local problem associated to L is well-posed, by Theorem 3.2, one deduces the convergence of the associated probability families. $\Diamond$

Excepting the fact that the present convergence is for the local Skorokhod topology, Theorem 4.5 is an improvement of Theorem 7.6 p. 172 from [BSW13]. More precisely, we do not need that the closure of L is a generator of a Feller semi-group, but we only suppose that the martingale local problem is well-posed. We postpone the proof of the latter theorem to the end of this section.

The proof of Theorem 4.1 is obtained as a straightforward application of the following proposition whose result is somehow similar to Theorem 8.7, pp.41-42 of [Sat13].

Proposition 4.7. *For each $n \in \mathbb{N} \cup \{\infty\}$ take $a_n \in \mathbb{R}^d$ such that $a_n \to a_\infty$ and consider $(\delta_n, \gamma_n, \nu_n)$ satisfying* (H2(a_n))*. Let also χ be such that the couple (χ, ν_∞) satisfies* (H3(a_∞))*. Then*

$$\forall f \in \mathrm{C}_c^\infty(\mathbb{R}^d), \quad T_{\chi,a_n}(\delta_n, \gamma_n, \nu_n) f \underset{n\to\infty}{\longrightarrow} T_{\chi,a_\infty}(\delta_\infty, \gamma_\infty, \nu_\infty) f, \tag{4.6}$$

if and only if the following three conditions hold

$$\begin{cases} \delta_n \underset{n\to\infty}{\longrightarrow} \delta_\infty, \\ \forall f \in \mathrm{C}(\mathbb{R}^{d\Delta}) \text{ vanishing in a neighbourhood of } a_\infty, \int f(b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int f(b)\nu_\infty(\mathrm{d}b), \\ \Big(\gamma_{n,ij} + \int (\chi_i\chi_j)(a_n, b)\nu_n(\mathrm{d}b)\Big)_{i,j} \underset{n\to\infty}{\longrightarrow} \Big(\gamma_{\infty,ij} + \int (\chi_i\chi_j)(a_\infty, b)\nu_\infty(\mathrm{d}b)\Big)_{i,j}. \end{cases} \tag{4.7}$$

Proof of Theorem 4.1. Parts 1) and 2) are direct consequences of the latter proposition. To verify Part 3) we remark that for any $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$ and $a \in \mathbb{R}^d$, we have

$$(T_n f(a) - f(a))/\varepsilon_n = T_{\chi,a}(\delta_n(a), 0, \nu_n(a)) f$$

with

$$\delta_n(a) := \varepsilon_n^{-1} \int_{\mathbb{R}^{d\Delta}\setminus\{a\}} \chi(a,b)\mu_n(a,\mathrm{d}b) \quad \text{and} \quad \nu_n(a,\mathrm{d}b) := \varepsilon_n^{-1} \mathbb{1}_{\mathbb{R}^{d\Delta}\setminus\{a\}}(b)\mu_n(a,\mathrm{d}b),$$

hence we can apply again Proposition 4.7. □

In order, to prove Proposition 4.7 we need the following lemma on the convergence of measures:

Lemma 4.8. *For $n \in \mathbb{N} \cup \{\infty\}$ let $a_n \in \mathbb{R}^d$ be such that $a_n \to a_\infty$ and let ν_n be Radon measures on $\mathbb{R}^{d\Delta} \setminus \{a_n\}$. Suppose that, for any $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$ such that f vanishes in a neighbourhood of a_∞, is constant in a neighbourhood of Δ and is infinitely differentiable in $\mathbb{R}^d$, we have*

$$\int f(b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int f(b)\nu_\infty(\mathrm{d}b).$$

i) Then, for any sequence $(f_n)_{n\in\mathbb{N}\cup\{\infty\}}$ of measurable uniformly bounded functions from $\mathbb{R}^{d\Delta}$ to $\mathbb{R}$ such that the f_n vanish in the same neighbourhood of a_∞, for $n \in \mathbb{N}\cup\{\infty\}$, and such that

$$\nu_\infty\Big(\mathbb{R}^{d\Delta} \setminus \big\{b_0 \in \mathbb{R}^{d\Delta} \mid \lim_{n\to\infty, b\to b_0} f_n(b) = f_\infty(b_0)\big\}\Big) = 0, \tag{4.8}$$

we have

$$\int f_n(b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int f_\infty(b)\nu_\infty(\mathrm{d}b).$$

ii) Assume, moreover, that there exists $\eta > 0$ such that

$$\sup_{n\in\mathbb{N}\cup\{\infty\}} \int |b-a_n|^2 \mathbb{1}_{|b-a_n|\leq\eta}\,\nu_n(\mathrm{d}b) < \infty.$$

Then, for any sequence $(f_n)_{n\in\mathbb{N}\cup\{\infty\}}$ of measurable uniformly bounded functions from $\mathbb{R}^{d\Delta}$ to $\mathbb{R}$ satisfying $f_n(a_n) = 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{0<|h|\leq\delta} \frac{f_n(a_n+h)}{|h|^2} = 0, \tag{4.9}$$

and (4.8)*, we have*

$$\int f_n(b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int f_\infty(b)\nu_\infty(\mathrm{d}b).$$

Proof. Consider a sequence of functions $(f_n)_{n\in\mathbb{N}\cup\{\infty\}}$ as in the first part of lemma. Let U_1 be an open subset such that $U_1 \Subset \mathbb{R}^{d\Delta} \setminus \{a_\infty\}$ and

$$U_1 \supset \bigcup_{n\in\mathbb{N}\cup\{\infty\}} \{f_n \neq 0\} \supset \mathbb{R}^{d\Delta} \setminus \Big\{b_0 \in \mathbb{R}^{d\Delta} \,\Big|\, \lim_{n\to\infty, b\to b_0} f_n(b) = f_\infty(b_0)\Big\}.$$

Let $\varphi_1 \in \mathrm{C}(\mathbb{R}^{d\Delta})$ be such that φ_1 vanishes in a neighbourhood of a_∞ and is constant in a neighbourhood of Δ, φ_1 is infinitely differentiable in $\mathbb{R}^d$ and such that $\varphi_1 \geq \mathbb{1}_{U_1}$. Then

$$\int \varphi_1(b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int \varphi_1(b)\nu_\infty(\mathrm{d}b),$$

so that

$$\sup_{n\in\mathbb{N}\cup\{\infty\}} \nu_n(U_1) \leq \sup_{n\in\mathbb{N}\cup\{\infty\}} \int \varphi_1(b)\nu_n(\mathrm{d}b) < \infty.$$

Since $\mathbb{R}^{d\Delta} \setminus \{a_\infty\}$ is a Polish space, the measure ν_∞ is inner regular on this set. Hence, if $\varepsilon > 0$ is chosen arbitrary, there exists a compact subset $K_\varepsilon \subset U_1$ satisfying

$$K_\varepsilon \subset \Big\{b_0 \in \mathbb{R}^{d\Delta} \mid \lim_{n\to\infty, b\to b_0} f_n(b) = f_\infty(b_0)\Big\} \quad \text{and} \quad \nu_\infty(K_\varepsilon) \geq \nu_\infty(U_1) - \varepsilon. \tag{4.10}$$

Hence f_∞ is continuous on K_ε and f_n converges uniformly to f_∞ on K_ε. There exists a function $\varphi_2 \in \mathrm{C}(\mathbb{R}^{d\Delta})$ such that φ_2 is constant in a neighbourhood of Δ, is infinitely differentiable in $\mathbb{R}^d$ and such that $\{\varphi_2 \neq 0\} \subset U_1$, $\|\varphi_2\| \leq \|f_\infty\|$ and $\|\varphi_2 - f_\infty\|_{K_\varepsilon} \leq \varepsilon$. Since (4.10), by compactness there exists an open subset $U_2 \subset U_1$ such that

$$K_\varepsilon \subset U_2 \subset \Big\{b_0 \in \mathbb{R}^{d\Delta} \mid \limsup_{n\to\infty, b\to b_0} \big|f_n(b) - \varphi_2(b_0)\big| \leq 2\varepsilon\Big\}.$$

By dominated convergence there exists a function $\varphi_3 \in \mathrm{C}(\mathbb{R}^{d\Delta})$ such that φ_3 vanishes in a neighbourhood of a_∞, is constant in a neighbourhood of Δ, is infinitely differentiable in $\mathbb{R}^d$, and such that $\mathbb{1}_{U_2} \geq \varphi_3$ and $\int \varphi_3(b)\nu_\infty(\mathrm{d}b) \geq \nu_\infty(U_2) - \varepsilon$. Hence

$$\begin{aligned}\liminf_{n\to\infty} \nu_n(U_2) &\geq \liminf_{n\to\infty} \int \varphi_3(b)\nu_n(\mathrm{d}b) = \int \varphi_3(b)\nu_\infty(\mathrm{d}b) \geq \nu_\infty(U_2) - \varepsilon \\ &\geq \nu_\infty(K_\varepsilon) - \varepsilon \geq \nu_\infty(U_1) - 2\varepsilon.\end{aligned}$$

Therefore we have

$$\begin{aligned}
\limsup_{n\to\infty}\left|\int f_n(b)\nu_n(\mathrm{d}b)-\int f_\infty(b)\nu_\infty(\mathrm{d}b)\right| &\le \limsup_{n\to\infty}\left|\int \varphi_2(b)\nu_n(\mathrm{d}b)-\int \varphi_2(b)\nu_\infty(\mathrm{d}b)\right| \\
&+\limsup_{n\to\infty}\left|\int_{U_2}(f_n(b)-\varphi_2(b))\nu_n(\mathrm{d}b)\right|+\limsup_{n\to\infty}\left|\int_{U_1\setminus U_2}(f_n(b)-\varphi_2(b))\nu_n(\mathrm{d}b)\right| \\
&+\limsup_{n\to\infty}\left|\int_{K_\varepsilon}(f_\infty(b)-\varphi_2(b))\nu_\infty(\mathrm{d}b)\right|+\limsup_{n\to\infty}\left|\int_{U_1\setminus K_\varepsilon}(f_\infty(b)-\varphi_2(b))\nu_\infty(\mathrm{d}b)\right| \\
&\le 0+2\varepsilon\sup_{n\in\mathbb{N}}\nu_n(U_1)+4\varepsilon\sup_{n\in\mathbb{N}\cup\{\infty\}}\|f_n\|+\varepsilon\nu_\infty(U_1)+2\varepsilon\|f_\infty\| \\
&\le 3\varepsilon(\sup_{n\in\mathbb{N}\cup\{\infty\}}\nu_n(U_1)+2\sup_{n\in\mathbb{N}\cup\{\infty\}}\|f_n\|).
\end{aligned}$$

Letting $\varepsilon\to 0$ we obtain that

$$\int f_n(b)\nu_n(\mathrm{d}b)\underset{n\to\infty}{\longrightarrow}\int f_\infty(b)\nu_\infty(\mathrm{d}b).$$

We proceed with part *ii)* of the lemma. Fix $\eta>0$ as in the statement and choose an arbitrary $\varepsilon>0$. By (4.9), there exists $0<\delta<\eta/2$ such that

$$\limsup_{n\to\infty}\sup_{0<|h|\le 2\delta}\frac{f_n(a_n+h)}{|h|^2}\le\frac{\varepsilon}{1\vee\sup_{n\in\mathbb{N}\cup\{\infty\}}\int|b-a_n|^2\mathbb{1}_{|b-a_n|\le\eta}\nu_n(\mathrm{d}b)}.$$

Consider a function $\varphi\in\mathrm{C}(\mathbb{R}^{d\Delta},[0,1])$ which vanishes in a neighbourhood of a_∞ and such that $\varphi(a)=1$ for any a satisfying $|a-a_\infty|\ge\delta$. Then, by the first part *i)*,

$$\int\varphi(b)f_n(b)\nu_n(\mathrm{d}b)\underset{n\to\infty}{\longrightarrow}\int\varphi(b)f_\infty(b)\nu_\infty(\mathrm{d}b).$$

For $n\in\mathbb{N}$ large enough, $|a-a_n|\le\delta$, hence

$$\left|\int(1-\varphi(b))f_n(b)\nu_n(\mathrm{d}b)\right|\le\int|b-a_n|^2\mathbb{1}_{|b-a_n|\le\eta}\nu_n(\mathrm{d}b)\sup_{0<|h|\le 2\delta}\frac{f_n(a_n+h)}{|h|^2},$$

so $\limsup_{n\to\infty}\left|\int(1-\varphi(b))f_n(b)\nu_n(\mathrm{d}b)\right|\le\varepsilon$. We also have

$$\begin{aligned}
&\left|\int(1-\varphi(b))f_\infty(b)\nu_\infty(\mathrm{d}b)\right| \\
&\qquad\le\int|b-a_\infty|^2\mathbb{1}_{|b-a_\infty|\le\eta}\nu_n(\mathrm{d}b)\limsup_{n\to\infty}\sup_{0<|h|\le 2\delta}\frac{f_n(a_n+h)}{|h|^2}\le\varepsilon,
\end{aligned}$$

so that

$$\limsup_{n\to\infty}\left|\int f_n(b)\nu_n(\mathrm{d}b)-\int f_\infty(b)\nu_\infty(\mathrm{d}b)\right|\le 2\varepsilon.$$

Letting $\varepsilon\to 0$ we can conclude. □

Proof of Proposition 4.7. Suppose first (4.6). Let $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$ be such that f vanishes in a neighbourhood of a_∞, is constant in a neighbourhood of Δ and is infinitely differentiable in $\mathbb{R}^d$. Hence $f - f(\Delta) \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, and

$$T_{\chi,a_\infty}(\delta_\infty, \gamma_\infty, \nu_\infty)(f - f(\Delta)) = \int f(b)\nu_\infty(\mathrm{d}b),$$

while, for n large enough,

$$T_{\chi,a_n}(\delta_n, \gamma_n, \nu_n)(f - f(\Delta)) = \int f(b)\nu_n(\mathrm{d}b).$$

We deduce that

$$\int f(b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int f(b)\nu_\infty(\mathrm{d}b).$$

Therefore we can apply the first part of Lemma 4.8 and in particular, for any $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$ vanishing in a neighbourhood of a,

$$\int f(b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int f(b)\nu_\infty(\mathrm{d}b).$$

Consider $\theta \in \mathrm{C}(\mathbb{R}_+, [0,1])$ such that $\theta(r) = 1$ for $r \leq 1$ and $\theta(r) = 0$ for $r \geq 2$. For $(a,b) \in \mathbb{R}^d \times \mathbb{R}^{d\Delta}$ and $n \in \mathbb{N} \cup \{\infty\}$, define

$$\widetilde{\chi}(a,b) := \theta(|b-a|)(b-a)\mathbb{1}_{b\neq\Delta} \quad \text{and} \quad \widetilde{\delta}_n := \delta_n + \int (\widetilde{\chi}(a_n,b) - \chi(a_n,b))\nu_n(\mathrm{d}b). \tag{4.11}$$

Therefore, for all $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ and all $n \in \mathbb{N} \cup \{\infty\}$

$$T_{\chi,a_n}(\delta_n, \gamma_n, \nu_n)f = T_{\widetilde{\chi},a_n}(\widetilde{\delta}_n, \gamma_n, \nu_n)f.$$

Let ϕ be an arbitrary linear form on $\mathbb{R}^d$ and consider $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ such that $f(b) = \phi \cdot (b - a_\infty)$ in a neighbourhood of a_∞. Then

$$T_{\widetilde{\chi},a_\infty}(\widetilde{\delta}_\infty, \gamma_\infty, \nu_\infty)f = \phi \cdot \widetilde{\delta}_\infty + \int (f(b) - \phi \cdot \widetilde{\chi}(a_\infty, b))\nu_\infty(\mathrm{d}b)$$

and for n large enough

$$T_{\widetilde{\chi},a_n}(\widetilde{\delta}_n, \gamma_n, \nu_n)f = \phi \cdot \widetilde{\delta}_n + \int (f(b) - f(a_n) - \phi \cdot \widetilde{\chi}(a_n, b))\nu_n(\mathrm{d}b).$$

Thanks to the first part of Lemma 4.8

$$\int (f(b) - f(a_n) - \phi \cdot \widetilde{\chi}(a_n,b))\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int (f(b) - \phi \cdot \widetilde{\chi}(a_\infty,b))\nu_\infty(\mathrm{d}b),$$

so that $\phi \cdot \widetilde{\delta}_n \underset{n\to\infty}{\longrightarrow} \phi \cdot \widetilde{\delta}_\infty$ and since ϕ was chosen arbitrary, $\widetilde{\delta}_n \underset{n\to\infty}{\longrightarrow} \widetilde{\delta}_\infty$.

Let Φ be an arbitrary symmetric bilinear form on $\mathbb{R}^d$ and if $(e_1, \ldots, e_d)$ is the canonical basis of $\mathbb{R}^d$, we denote $\Phi_{ij} = \Phi(e_i, e_j)$, $i, j = 1, \ldots, d$. Consider $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ such that

$f(b) = \Phi(b - a_\infty, b - a_\infty)$ in a neighbourhood of a_∞. Then, for n large enough,

$$\begin{aligned}
&T_{\widetilde{\chi},a_n}(\widetilde{\delta}_n, \gamma_n, \nu_n)f \\
&= \sum_{i,j=1}^d \Phi_{ij}\gamma_{n,ij} + 2\Phi(a_n - a_\infty, \widetilde{\delta}_n) + \int (f(b) - f(a_n) - 2\Phi(a_n - a_\infty, \widetilde{\chi}(a_n, b))\nu_n(\mathrm{d}b) \\
&= \sum_{i,j=1}^d \Phi_{ij}\left(\gamma_{n,ij} + \int (\widetilde{\chi}_i\widetilde{\chi}_j)(a_n, b)\nu_n(\mathrm{d}b)\right) + 2\Phi(a_n - a_\infty, \widetilde{\delta}_n) \\
&\quad + \int \Big(f(b) - f(a_n) - 2\Phi(a_n - a_\infty, \widetilde{\chi}(a_n, b)) - \sum_{i,j=1}^d \Phi_{ij}\,(\widetilde{\chi}_i\widetilde{\chi}_j)(a_n, b)\Big)\nu_n(\mathrm{d}b).
\end{aligned}$$

A similar equality holds with the index n replaced by ∞:

$$T_{\widetilde{\chi},a_\infty}(\widetilde{\delta}_\infty, \gamma_\infty, \nu_\infty)f = \sum_{i,j=1}^d \Phi_{ij}\gamma_{\infty,ij} + \int f(b)\nu_\infty(\mathrm{d}b)$$

$$= \sum_{i,j=1}^d \Phi_{ij}\left(\gamma_{\infty,ij} + \int (\widetilde{\chi}_i\widetilde{\chi}_j)(a_\infty, b)\nu_\infty(\mathrm{d}b)\right) + \int \Big(f(b) - \sum_{i,j=1}^d \Phi_{ij}\,(\widetilde{\chi}_i\widetilde{\chi}_j)(a_\infty, b)\Big)\nu_\infty(\mathrm{d}b).$$

Thanks to the first part of Lemma 4.8

$$\begin{aligned}
&\int \Big(f(b) - f(a_n) - 2\Phi(a_n - a_\infty, \widetilde{\chi}(a_n, b)) - \sum_{i,j=1}^d \Phi_{ij}\,(\widetilde{\chi}_i\widetilde{\chi}_j)(a_n, b)\Big)\nu_n(\mathrm{d}b) \\
&\qquad\qquad \underset{n\to\infty}{\longrightarrow} \int \Big(f(b) - \sum_{i,j=1}^d \Phi_{ij}\,(\widetilde{\chi}_i\widetilde{\chi}_j)(a_\infty, b)\Big)\nu_\infty(\mathrm{d}b),
\end{aligned}$$

so that

$$\sum_{i,j=1}^d \Phi_{ij}\Big(\gamma_{n,ij} + \int (\widetilde{\chi}_i\widetilde{\chi}_j)(a_n, b)\nu_n(\mathrm{d}b)\Big) \underset{n\to\infty}{\longrightarrow} \sum_{i,j=1}^d \Phi_{ij}\Big(\gamma_{\infty,ij} + \int (\widetilde{\chi}_i\widetilde{\chi}_j)(a_\infty, b)\nu_\infty(\mathrm{d}b)\Big).$$

Since Φ was chosen arbitrary, for all $1 \leq i, j \leq d$

$$\gamma_{n,ij} + \int (\widetilde{\chi}_i\widetilde{\chi}_j)(a_n, b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \gamma_{\infty,ij} + \int (\widetilde{\chi}_i\widetilde{\chi}_j)(a_\infty, b)\nu_\infty(\mathrm{d}b)).$$

So, we can apply the second part of Lemma 4.8 and in particular

$$\lim_{n\to\infty} \int (\widetilde{\chi}(a_n, b) - \chi(a_n, b))\nu_n(\mathrm{d}b) = \int (\widetilde{\chi}(a_\infty, b) - \chi(a_\infty, b))\nu_\infty(\mathrm{d}b),$$

so that, by (4.11), $\delta_n \underset{n\to\infty}{\longrightarrow} \delta_\infty$. By the second part of Lemma 4.8 we also have, for all $1 \leq i, j \leq d$,

$$\int ((\widetilde{\chi}_i\widetilde{\chi}_j)(a_n, b) - (\chi_i\chi_j)(a_n, b))\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int ((\widetilde{\chi}_i\widetilde{\chi}_j)(a_\infty, b) - (\chi_i\chi_j)(a_\infty, b))\nu_\infty(\mathrm{d}b),$$

so, we deduce

$$\gamma_{n,ij} + \int (\chi_i \chi_j)(a_n, b)\nu_n(\mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \gamma_{\infty,ij} + \int (\chi_i \chi_j)(a_\infty, b)\nu_\infty(\mathrm{d}b).$$

We prove the converse, by supposing (4.7) and applying the second part of Lemma 4.8. Let $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ be, for all $n \in \mathbb{N} \cup \{\infty\}$,

$$\begin{aligned} T_{\chi,a_n}(\delta_n, \gamma_n, \nu_n)f :=& \frac{1}{2} \sum_{i,j=1}^d \gamma_{n,ij} \partial_{ij}^2 f(a_n) + \delta_n \cdot \nabla f(a_n) \\ &+ \int (f(b) - f(a_n) - \chi(a_n, b) \cdot \nabla f(a_n))\nu_n(\mathrm{d}b) \\ =& \frac{1}{2} \sum_{i,j=1}^d \left(\gamma_{n,ij} + \int (\chi_i \chi_j)(a_n, b)\nu_n(\mathrm{d}b) \right) \partial_{ij}^2 f(a_n) + \delta_n \cdot \nabla f(a_n) \\ &+ \int \left(f(b) - f(a_n) - \chi(a_n, b) \cdot \nabla f(a_n) - \sum_{i,j=1}^d (\chi_i \chi_j)(a_n, b) \partial_{ij}^2 f(a_n) \right) \nu_n(\mathrm{d}b). \end{aligned}$$

Applying the second part of Lemma 4.8 to the last term of the previous equation, we deduce

$$T_{\chi,a_n}(\delta_n, \gamma_n, \nu_n)f \underset{n\to\infty}{\longrightarrow} T_{\chi,a_\infty}(\delta_\infty, \gamma_\infty, \nu_\infty)f. \qquad \square$$

We finish the section with the proof of Theorem 4.5. Recall that $\chi_1(a, b)$ is given by (4.1). Thanks to Theorem 2.12 pp. 21-22 from [Hoh98], for each $a \in \mathbb{R}^d$ there exists a triplet $(\delta(a), \gamma(a), \nu(a))$ satisfying (H2(a)) such that, $Lf(a) := T_{\chi_1,a}(\delta(a), \gamma(a), \nu(a))f$, for all $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. It is clear that for any $a_0, a \in \mathbb{R}^d$ and $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, the Lévy operator $L(a_0)$ defined by (4.5) satisfies also

$$L(a_0)f(a) = T_{\chi_1,a}(\delta(a_0), \gamma(a_0), \nu_a(a_0)).$$

Here and elsewhere $\nu_a(a_0)$ is the pushforward measure of $\nu(a_0)$ with respect to the translation $b \mapsto b - a_0 + a$.

Proof of Theorem 4.5. It suffices to prove that for any function $f_0 \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ and any sequence $a_n \in \mathbb{R}^d$ converging to $a_\infty \in \mathbb{R}^d$, the sequence $(T_n f_0(a_n) - f_0(a_n))/\varepsilon_n$ converges to $Lf_0(a_\infty)$. Thanks to Proposition 4.7 we have, $\delta(a_n) \underset{n\to\infty}{\longrightarrow} \delta(a_\infty)$,

$$\forall f \in \mathrm{C}(\mathbb{R}^{d\Delta}) \text{ vanishing in a neighbourhood of } a_\infty, \int f(b)\nu(a_n, \mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \int f(b)\nu(a_\infty, \mathrm{d}b),$$

and for all $1 \le i, j \le d$

$$\gamma_{ij}(a_n) + \int (\chi_i \chi_j)(a_n, b)\nu(a_n, \mathrm{d}b) \underset{n\to\infty}{\longrightarrow} \gamma_{ij}(a_\infty) + \int (\chi_i \chi_j)(a_\infty, b)\nu(a_\infty, \mathrm{d}b).$$

It is not difficult to deduce that, there exists $C \in \mathbb{R}_+$ such that, for all $n \in \mathbb{N} \cup \{\infty\}$ and $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$\|L(a_n)f\| \le C\|f\| \vee \max_{1\le i\le d} \|\partial_i f\| \vee \max_{1\le i,j\le d} \|\partial_{ij}^2 f\|.$$

Hence, $\sup_{n\in\mathbb{N}\cup\{\infty\}} \|L(a_n)f_0\| < \infty$. Consider $b_\infty \in \mathbb{R}^d$, a sequence $b_n \to b_\infty$ and a function $f \in \mathrm{C}(\mathbb{R}^{d\Delta})$ vanishing in a neighbourhood of b_∞. By using the first part of Lemma 4.8,

$$\int f(b)\nu_{b_n}(a_n, \mathrm{d}b) = \int f(b - a_n + b_n)\nu(a_n, \mathrm{d}b)$$
$$\underset{n\to\infty}{\longrightarrow} \int f(b - a_\infty + b_\infty)\nu(a_\infty, \mathrm{d}b) = \int f(b)\nu_{b_\infty}(a_\infty, \mathrm{d}b).$$

Hence, by the second part of Theorem 4.1, $L(a_n)f$ converges uniformly on compact sets toward $L(a_\infty)f$, for all $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. In particular, for each $\varepsilon > 0$ there exists an open neighbourhood U of a_∞ and $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0,\ \forall a \in U, \quad |L(a_n)f_0(a) - L(a_\infty)f_0(a_\infty)| \leq \varepsilon.$$

Let $\mathbf{P}_n$ be the unique element of $\mathcal{M}(L(a_n))$ such that $\mathbf{P}_n(X_0 = a_n) = 1$. Then, for all $n \geq n_0$

$$\begin{aligned}
\left|\frac{T_n f_0(a_n) - f_0(a_n)}{\varepsilon_n} - L(a_\infty)f_0(a_\infty)\right| &= \left|\frac{\mathbf{E}_n[f_0(X_{\varepsilon_n})] - f_0(a_n)}{\varepsilon_n} - L(a_\infty)f_0(a_\infty)\right| \\
&= \left|\mathbf{E}_n \frac{1}{\varepsilon_n}\int_0^{\varepsilon_n} \big(L(a_n)f_0(X_s) - L(a_\infty)f_0(a_\infty)\big)\mathrm{d}s\right| \\
&\leq \mathbf{E}_n\left[\mathbb{1}_{\{\tau^U < \varepsilon_n\}}\frac{1}{\varepsilon_n}\int_0^{\varepsilon_n} \big|L(a_n)f_0(X_s) - L(a_\infty)f_0(a_\infty)\big|\mathrm{d}s\right] \\
&\quad + \mathbf{E}_n\left[\mathbb{1}_{\{\tau^U \geq \varepsilon_n\}}\frac{1}{\varepsilon_n}\int_0^{\varepsilon_n} \big|L(a_n)f_0(X_s) - L(a_\infty)f_0(a_\infty)\big|\mathrm{d}s\right] \\
&\leq 2\mathbf{P}_n(\tau^U < \varepsilon_n) \sup_{m\in\mathbb{N}\cup\{\infty\}} \|L(a_m)f_0\| + \varepsilon.
\end{aligned}$$

We apply Lemma A.1 of uniform continuity along stopping times with a compact neighbourhood $\mathcal{K} \subset U$ of a_∞ and with $\mathcal{U} := \mathbb{R}^d \times U \cup (\mathbb{R}^d \setminus \mathcal{K}) \times \mathbb{R}^d$. We deduce that

$$\lim_{n\to\infty} \mathbf{P}_n(\tau^U < \varepsilon_n) = 0.$$

Hence

$$\limsup_{n\to\infty} \left|\frac{T_n f_0(a_n) - f_0(a_n)}{\varepsilon_n} - L(a_\infty)f_0(a_\infty)\right| \leq \varepsilon,$$

and we conclude by letting $\varepsilon \to 0$. □

5 Diffusion in a potential

We recall that $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ denotes the space of locally Lebesgue integrable functions. A real continuous function f is called locally absolutely continuous if its distributional derivative f' belongs to $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$. We introduce the set of potential functions

$$\mathscr{V} := \left\{V : \mathbb{R} \to \mathbb{R} \text{ measurable} \;\middle|\; \mathrm{e}^{|V|} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})\right\}.$$

It is straightforward to prove that there exists a unique Polish topology on $\mathscr{V}$ such that a sequence $(V_n)_n$ in $\mathscr{V}$ converges to $V \in \mathscr{V}$ if and only if

$$\forall M \in \mathbb{R}_+, \quad \lim_{n\to\infty} \int_{-M}^{M} |\mathrm{e}^{V(a)} - \mathrm{e}^{V_n(a)}| \vee |\mathrm{e}^{-V(a)} - \mathrm{e}^{-V_n(a)}| \,\mathrm{d}a = 0.$$

For a potential $V \in \mathscr{V}$, the operator

$$L^V := \frac{1}{2}\mathrm{e}^V \frac{\mathrm{d}}{\mathrm{d}a} \mathrm{e}^{-V} \frac{\mathrm{d}}{\mathrm{d}a} \tag{5.1}$$

is the set of couples $(f, g) \in \mathrm{C}_0(\mathbb{R}) \times \mathrm{C}(\mathbb{R})$ such that f and $\mathrm{e}^{-V} f'$ are locally absolutely continuous and $g = \frac{1}{2}\mathrm{e}^V(\mathrm{e}^{-V} f')'$. Let us notice that it is a particular case of the operator $\boldsymbol{D}_m \boldsymbol{D}_p^+$ described in [Man68], pp. 21-22. Heuristically, the solutions of the martingale local problem associated to L^V are solutions of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{2} V'(X_t)\mathrm{d}t,$$

where B is a standard Brownian motion.

Proposition 5.1 (Diffusions on potential and random walks on $\mathbb{Z}$)**.**

1. *For any potential $V \in \mathscr{V}$, the operator L^V is the generator of a locally Feller family.*
2. *For any sequence of potentials $(V_n)_n$ in $\mathscr{V}$ converging to $V \in \mathscr{V}$ for the topology of $\mathscr{V}$, the sequence of operators L^{V_n} converges to L^V, in the sense of the third statement of the convergence Theorem 2.2.*
3. *For $(n, k) \in \mathbb{N} \times \mathbb{Z}$, let $q_{n,k} \in \mathbb{R}$ and $\varepsilon_n > 0$ be. For all $n \in \mathbb{N}$, accordingly with Definition 3.1, let $(\mathbf{P}^n_k)_k \in \mathcal{P}(\mathbb{Z}^\mathbb{N})^\mathbb{Z}$ be the unique discrete time locally Feller family such that*
$$\mathbf{P}^n_k(Y_1 = k+1) = 1 - \mathbf{P}^n_k(Y_1 = k-1) = \frac{1}{\mathrm{e}^{q_{n,k}} + 1}.$$
Denote the sequence of potential in $\mathscr{V}$ by
$$V_n(a) := \sum_{k=1}^{\lfloor a/\varepsilon_n \rfloor} q_{n,k} \mathbb{1}_{a \geq \varepsilon_n} - \sum_{k=0}^{-\lfloor a/\varepsilon_n \rfloor - 1} q_{n,-k} \mathbb{1}_{a<0}\,.$$
Let V be a potential in $\mathscr{V}$ and let $(\mathbf{P}_a)_a$ be the locally Feller family associated with L^V. Assume that V_n converges to V for the topology of $\mathscr{V}$, and $\varepsilon_n \to 0$ as $n \to \infty$. Then, for any sequence $\mu_n \in \mathcal{P}(\mathbb{Z})$ such that their pushforwards with respect to the mappings $k \mapsto \varepsilon_n k$ converge to a probability measure $\mu \in \mathcal{P}(\mathbb{R})$, we have
$$\mathscr{L}_{\mathbf{P}^n_{\mu_n}} \left((\varepsilon_n Y_{\lfloor t/\varepsilon_n^2 \rfloor})_t \right) \underset{n\to\infty}{\overset{\mathcal{P}(\mathbb{D}_{loc}(S))}{\longrightarrow}} \mathbf{P}_\mu.$$

Before proving this proposition, we give an important consequence concerning a random walk and a diffusion in random environment. For $n \in \mathbb{N}$, let $(\Omega^n, \mathcal{G}^n, \mathbb{P}^n)$ be a probability space and consider the random variables

$$(q_{n,k})_k : \Omega^n \to \mathbb{R}^\mathbb{Z}, \quad (Z^n_k)_k : \Omega^n \to \mathbb{Z}^\mathbb{N} \quad \text{and} \quad \varepsilon_n : \Omega^n \to \mathbb{R}^*_+\,.$$

Suppose that for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, $\mathbb{P}^n$-almost surely,

$$\mathbb{P}^n \left(Z_{k+1}^n = Z_k^n + 1 \mid \varepsilon_n, (q_{n,\ell})_{\ell \in \mathbb{Z}}, (Z_\ell^n)_{0 \le \ell \le k} \right) = \frac{1}{\mathrm{e}^{q_{n,Z_k}} + 1}$$
$$\mathbb{P}^n \left(Z_{k+1}^n = Z_k^n - 1 \mid \varepsilon_n, (q_{n,\ell})_{\ell \in \mathbb{Z}}, (Z_\ell^n)_{0 \le \ell \le k} \right) = \frac{1}{\mathrm{e}^{-q_{n,Z_k}} + 1} = 1 - \frac{1}{\mathrm{e}^{q_{n,Z_k}} + 1}.$$

For any $n \in \mathbb{N}$ and $a \in \mathbb{R}$, denote the random potential in $\mathscr{V}$ by

$$W_n(a) := \sum_{k=1}^{\lfloor a/\varepsilon_n \rfloor} q_{n,k} \mathbb{1}_{a \ge \varepsilon_n} - \sum_{k=0}^{-\lfloor a/\varepsilon_n \rfloor - 1} q_{n,-k} \mathbb{1}_{a<0} \,. \tag{5.2}$$

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space and consider random variables

$$W : \Omega \to \mathscr{V} \quad \text{and} \quad Z : \Omega \to \mathbb{D}_{\mathrm{loc}}(\mathbb{R}).$$

Suppose that the conditional distribution of Z with respect to W satisfies, $\mathbb{P}$-a.s.

$$\mathscr{L}_{\mathbb{P}}\left(Z \mid W\right) \in \mathcal{M}(L^W).$$

Proposition 5.2. *Suppose that ε_n converges in distribution to 0, that $\varepsilon_n Z_0^n$ converges in distribution to Z_0 and that W_n converges in distribution to W for the topology of $\mathscr{V}$. Then $(\varepsilon_n Z^n_{\lfloor t/\varepsilon_n^2 \rfloor})_t$ converges in distribution to Z for the local Skorokhod topology.*

Example 5.3. 1) Let $(q_k)_k$ be an i.i.d sequence of centered real random variables with finite variance σ^2 and suppose that $q_{n,k} = \sqrt{\varepsilon_n} q_k$. Suppose also that W is a Brownian motion with variance σ^2. Then, by Donsker's theorem, W_n given by (5.2) converges in distribution to W, so that we can apply Proposition 5.2 to deduce the convergence of the random walk in a random i.i.d. medium (introduced by Sinai in [Sin82]) to the diffusion in a Brownian potential (introduced in [Bro86]). Hence we recover Theorem 1 from [Sei00], p. 295, without the hypothesis that the distribution of q_0 is compactly supported.

2) Fix deterministic $q \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+^*$. Suppose that for each $n \in \mathbb{N}$, $(q_{n,k})_k$ is an i.i.d sequence of random variables such that $\mathbb{P}^n(q_{n,k} = q) = 1 - \mathbb{P}^n(q_{n,k} = 0) = \lambda \varepsilon_n$. Suppose also that $W(a) = qN_{\lambda a}$, where N is a standard Poisson process on $\mathbb{R}$. Then, it is classical (see for instance [Car97]), that W_n given by (5.2) converges in distribution to W, so that we can apply Proposition 5.2 to deduce the convergence of Sinai's random walk to the diffusion in a Poisson potential. Hence we recover Theorem 2 from [Sei00], p. 296.

3) More generally, suppose that for each $n \in \mathbb{N}$, $(q_{n,k})_k$ is an i.i.d sequence of random variables. Likewise, suppose that W_n given again by (5.2), converges in distribution to a Lévy process W. We can apply Proposition 5.2 to deduce the convergence of Sinai's random walk to the diffusion in a Lévy potential (introduced in [Car97]). $\Diamond$

Proof of Proposition 5.2. Let F be a bounded continuous function from $\mathbb{D}_{\mathrm{loc}}(\mathbb{R})$ to $\mathbb{R}$. Define the bounded mapping

$$G : \mathbb{R} \times \mathscr{V} \times \mathbb{R}_+ \to \mathbb{R}$$

as follows: for any $a \in \mathbb{R}$, $V \in \mathscr{V}$ and $\varepsilon \in \mathbb{R}_+^*$, let $\mathbf{P}^{a,V,\varepsilon} \in \mathcal{P}(\mathbb{Z}^{\mathbb{N}})$ be the unique probability such that $\mathbf{P}^{a,V,\varepsilon}(Y_0 = \lfloor a/\varepsilon \rfloor) = 1$ and, $\mathbf{P}^{a,V,\varepsilon}$-almost surely, for all $k \in \mathbb{N}$,

$$\begin{aligned}\mathbf{P}^{a,V,\varepsilon}\left(Y_{k+1} = Y_k + 1 \mid Y_0, \dots, Y_k\right) &= 1 - \mathbf{P}^{a,V,\varepsilon}\left(Y_{k+1} = Y_k - 1 \mid Y_0, \dots, Y_k\right) \\ &= \int_{\varepsilon Y_k - \varepsilon}^{\varepsilon Y_k} \mathrm{e}^{V(a)} \mathrm{d}a \Big/ \int_{\varepsilon Y_k - \varepsilon}^{\varepsilon Y_k + \varepsilon} \mathrm{e}^{V(a)} \mathrm{d}a.\end{aligned}$$

Let $\mathbf{P}^{a,V,0} \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(\mathbb{R}))$ be the unique element belonging to $\mathcal{M}(L^V)$ and starting from a. We set

$$G(a, V, \varepsilon) := \mathbf{E}^{a,V,\varepsilon}\left[F\left((\varepsilon Y_{\lfloor t/\varepsilon^2 \rfloor})_t\right)\right] \quad \text{and} \quad G(a, V, 0) := \mathbf{E}^{a,V,0}\left[F(X)\right].$$

By Proposition 5.1, the mapping G is continuous at every point of $\mathbb{R} \times \mathscr{V} \times \{0\}$. Thus,

$$\mathbb{E}^n\left[G(\varepsilon_n Z_0^n, W_n, \varepsilon_n)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[G(Z_0, W, 0)\right].$$

Hence

$$\begin{aligned}\mathbb{E}^n\left[F\left((\varepsilon_n Z^n_{\lfloor t/\varepsilon_n^2 \rfloor})_t\right)\right] &= \mathbb{E}^n\left[\mathbb{E}^n\left[F\left((\varepsilon_n Z^n_{\lfloor t/\varepsilon_n^2 \rfloor})_t\right) \,\Big|\, \varepsilon_n,\ Z_0^n,\ (q_{n,\ell})_{\ell\in\mathbb{Z}}\right]\right] \\ &= \mathbb{E}^n\left[G(\varepsilon_n Z_0^n, W_n, \varepsilon_n)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[G(Z_0, W, 0)\right] = \mathbb{E}\left[\mathbb{E}\left[F(Z) \mid Z_0,\ W\right]\right] = \mathbb{E}\left[F(Z)\right].\end{aligned}$$

Then $(\varepsilon_n Z^n_{\lfloor t/\varepsilon_n^2 \rfloor})_t$ converges in distribution to Z. $\square$

Before starting the proof of Proposition 5.1, let us give a preliminary computation. Consider $a_1, a_2 \in \mathbb{R}$ and a measurable function $V : [a_1 \wedge a_2, a_1 \vee a_2] \to \mathbb{R}$, such that $\mathrm{e}^{|V|} \in \mathrm{L}^1([a_1 \wedge a_2, a_1 \vee a_2])$. For any absolutely continuous function $f \in \mathrm{C}([a_1 \wedge a_2, a_1 \vee a_2], \mathbb{R})$ such that $\mathrm{e}^{-V} f'$ is absolutely continuous and $g := \frac{1}{2}\mathrm{e}^V(\mathrm{e}^{-V} f')'$ is continuous, we have

$$\begin{aligned}f(a_2) = f(a_1) + \int_{a_1}^{a_2} f'(b)\mathrm{d}b &= f(a_1) + \int_{a_1}^{a_2} \mathrm{e}^{V(b)}\left((\mathrm{e}^{-V} f')(a_1) + \int_{a_1}^{b} (\mathrm{e}^{-V} f')'(c)\mathrm{d}c\right)\mathrm{d}b \\ &= f(a_1) + \int_{a_1}^{a_2} \mathrm{e}^{V(b)}\left((\mathrm{e}^{-V} f')(a_1) + 2\int_{a_1}^{b} \mathrm{e}^{-V(c)} g(c)\mathrm{d}c\right)\mathrm{d}b \qquad (5.3) \\ &= f(a_1) + (\mathrm{e}^{-V} f')(a_1)\int_{a_1}^{a_2} \mathrm{e}^{V(b)}\mathrm{d}b + 2g(a_1)\int_{a_1}^{a_2}\int_{a_1}^{b} \mathrm{e}^{V(b)-V(c)}\mathrm{d}c\,\mathrm{d}b \\ &\quad + 2\int_{a_1}^{a_2}\int_{a_1}^{b} \mathrm{e}^{V(b)-V(c)}(g(c) - g(a_1))\mathrm{d}c\,\mathrm{d}b. \qquad (5.4)\end{aligned}$$

Proof of Proposition 5.1. Proof of 1. This part is essentially an application of the second chapter of [Man68]. For the sake of completeness we give here some details. Let $h \in \mathrm{C}(\mathbb{R}, \mathbb{R}_+^*)$ be such that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\inf_{n \le |a| \le n+1} h(a) \le \frac{1}{n}\Big[&\int_n^{n+1}\int_0^a \mathrm{e}^{V(b)-V(a)}\mathrm{d}b\,\mathrm{d}a \bigwedge \int_{n+1}^{n+2}\int_n^{n+1} \mathrm{e}^{V(a)-V(b)}\mathrm{d}b\,\mathrm{d}a \\ &\bigwedge \int_{-n-1}^{-n}\int_a^0 \mathrm{e}^{V(b)-V(a)}\mathrm{d}b\,\mathrm{d}a \bigwedge \int_{-n-2}^{-n-1}\int_{-n-1}^{-n} \mathrm{e}^{V(a)-V(b)}\mathrm{d}b\,\mathrm{d}a\Big]. \qquad (5.5)\end{aligned}$$

The operator hL^V coincides on $\mathrm{C}_0(\mathbb{R}) \times \mathrm{C}_0(\mathbb{R})$ with the operator $D_m D_p^+ \subset \mathrm{C}(\overline{\mathbb{R}}) \times \mathrm{C}(\overline{\mathbb{R}})$ on the extended real line $\overline{\mathbb{R}}$, described in [Man68], pp. 21-22, where

$$\mathrm{d}m(a) := \frac{2\mathrm{e}^{-V(a)}}{h(a)}\mathrm{d}a \quad \text{and} \quad \mathrm{d}p(a) := \mathrm{e}^{V(a)}\mathrm{d}a.$$

Applying (5.5) we have

$$\begin{aligned}
\int_0^\infty \int_0^a \mathrm{d}m(b)\mathrm{d}p(a) &\geq \limsup_{n\to\infty} \int_{n+1}^{n+2} \int_n^{n+1} \mathrm{d}m(b)\mathrm{d}p(a) \geq \limsup_{n\to\infty} 2n = \infty, \\
\int_0^\infty \int_0^a \mathrm{d}p(b)\mathrm{d}m(a) &\geq \limsup_{n\to\infty} \int_n^{n+1} \int_0^a \mathrm{d}p(b)\mathrm{d}m(a) \geq \limsup_{n\to\infty} 2n = \infty, \\
\int_{-\infty}^0 \int_a^0 \mathrm{d}m(b)\mathrm{d}p(a) &\geq \limsup_{n\to\infty} \int_{-n-2}^{-n-1} \int_{-n-1}^{-n} \mathrm{d}m(b)\mathrm{d}p(a) \geq \limsup_{n\to\infty} 2n = \infty, \\
\int_{-\infty}^0 \int_a^0 \mathrm{d}p(b)\mathrm{d}m(a) &\geq \limsup_{n\to\infty} \int_{-n-1}^{-n} \int_a^0 \mathrm{d}p(b)\mathrm{d}m(a) \geq \limsup_{n\to\infty} 2n = \infty.
\end{aligned}$$

Thus according to the definition given in [Man68], pp. 24-25, the boundary points $-\infty$ and $+\infty$ are natural. Thanks to Theorem 1 and Remark 2 p. 38 of [Man68], $D_m D_p^+$ is the generator of a conservative Feller semi-group on $\mathrm{C}(\overline{\mathbb{R}})$. Furthermore by steps 7 and 8 in [Man68], pp. 31-32,

$$D_m D_p^+ f(-\infty) = D_m D_p^+ f(+\infty) = 0, \quad \forall f \in \mathrm{D}(D_m D_p^+),$$

so that the operator

$$D_m D_p^+ \cap \mathrm{C}_0(\mathbb{R}) \times \mathrm{C}_0(\mathbb{R}) = (hL^V) \cap \mathrm{C}_0(\mathbb{R}) \times \mathrm{C}_0(\mathbb{R})$$

is the $\mathrm{C}_0 \times \mathrm{C}_0$-generator of a Feller semi-group. Hence, by (2.3) and (2.2) we deduce that the operator

$$\widetilde{L} := \frac{1}{h}\overline{(hL^V) \cap \mathrm{C}_0(\mathbb{R}) \times \mathrm{C}_0(\mathbb{R})}$$

is the generator of a locally Feller family. Here the closure is taken in $\mathrm{C}_0(\mathbb{R}) \times \mathrm{C}(\mathbb{R})$, and it is clear that $\widetilde{L} \subset \overline{L^V}$. Thanks to (5.3) it is straightforward to obtain $L^V = \overline{L^V}$ and thanks to (5.4) it is straightforward to obtain that L^V satisfies the positive maximum principle. Thanks to Theorem 2.1 we deduce the existence for the martingale local problem associated to L^V. Hence $L^V = \widetilde{L}$ is the generator of a locally Feller family.

Proof of 2. Denote by $(\mathbf{P}_a^n)_a$ and $(\mathbf{P}_a^\infty)_a$ the locally Feller families associated, respectively, to L^{V_n} and L^V. By Theorem 2.2 it is enough to prove that for each sequence $a_n \in \mathbb{R}$ converging to $a_\infty \in \mathbb{R}$, $\mathbf{P}_{a_n}^n$ converges weakly to $\mathbf{P}_{a_\infty}^\infty$ for the local Skorokhod topology. Thanks to Lemma A.3, for $M \in \mathbb{N}^*$, there exists $h_M \in \mathrm{C}(\mathbb{R}, [0,1])$ such that

$$\{h_M \neq 0\} = (-2M, 2M), \quad \{h_M = 1\} = [-M, M]$$

and, for all $n \in \mathbb{N}$, the martingale local problems associated to $h_M L^V$ and to $h_M L^{V_n}$ are well-posed. For $n \in \mathbb{N}$ and $M \in \mathbb{N}^*$, denote by $(\mathbf{P}_a^{n,M})_a$ and $(\mathbf{P}_a^{\infty,M})_a$ the locally Feller families associated, respectively with $h_M L^{V_n}$ and $h_M L^V$. For $n \in \mathbb{N}$, define the extension of $h_M L^{V_n}$:

$$\widetilde{L_{n,M}} := \left\{(f,g) \in \mathrm{C}_0(\mathbb{R}) \times \mathrm{C}(\mathbb{R}) \,\middle|\, g = \frac{1}{2} h_M \mathrm{e}^{V_n} (\mathrm{e}^{-V_n} f')' \mathbb{1}_{(-2M,2M)}\right\},$$

where f and $\mathrm{e}^{-V_n} f'$ are supposed locally absolutely continuous only on $(-2M, 2M)$. By (5.4) it is straightforward to obtain that $\widetilde{L_{n,M}}$ satisfies the positive maximum principle, so that by Theorem 2.1, $\widetilde{L_{n,M}}$ is a linear subspace of the generator of the family $(\mathbf{P}_a^{n,M})_a$. We will prove that the sequence of operators $\widetilde{L_{n,M}}$ converges to the operator $h_M L^V$ in the sense of the third statement of Theorem 2.2. Consider $f \in \mathrm{D}(L)$ and define $f_n \in \mathrm{C}_0(\mathbb{R})$ by

$$f_n(a) := \begin{cases} f(a), & a \notin (-2M - n^{-1}, 2M + n^{-1}) \\ f(0) + \displaystyle\int_0^a \mathrm{e}^{V_n(b)} \Big[(\mathrm{e}^{-V} f')(0) + 2 \int_0^b \mathrm{e}^{-V_n(c)} L^V f(c) \mathrm{d}c\Big] \mathrm{d}b, & a \in [-2M, 2M], \end{cases}$$

with f_n affine on $[-2M - n^{-1}, -2M]$ and on $[2M, 2M + n^{-1}]$. Hence $f_n \in \mathrm{D}(\widetilde{L_{n,M}})$ and $\widetilde{L_{n,M}} f_n = h_M L^V f$. We have

$$\|f_n - f\| \leq \sup_{a \in [-2M,2M]} |f_n(a) - f(a)| + \sup_{\substack{2M \leq |a_1|, |a_2| \leq 2M + n^{-1} \\ 0 \leq a_1 a_2}} |f(a_2) - f(a_1)|.$$

Since f is continuous, the second supremum in the latter equation tends to 0. It is straightforward to deduce from (5.3), by using the expression of f_n and the convergence $V_n \to V$, that

$$\sup_{a \in [-2M,2M]} |f_n(a) - f(a)| \underset{n \to \infty}{\longrightarrow} 0.$$

Hence $\|f_n - f\| \to 0$ as $n \to \infty$, so the by the Theorem 2.2:

$$\mathbf{P}_{a_n}^{n,M} \underset{n \to \infty}{\longrightarrow} \mathbf{P}_{a_\infty}^{\infty,M}. \tag{5.6}$$

Using Lemma A.2, for all $M \in \mathbb{N}^*$ and $n \in \mathbb{N} \cup \{\infty\}$,

$$\mathscr{L}_{\mathbf{P}_{a_n}^{n,M}} \left(X^{\tau^{(-M,M)}}\right) = \mathscr{L}_{\mathbf{P}_{a_n}^{n}} \left(X^{\tau^{(-M,M)}}\right). \tag{5.7}$$

At this level we use a result of localisation of the continuity contained in Lemma A.4. Therefore, from (5.6) and (5.7), letting $M \to \infty$, we deduce

$$\mathbf{P}_{a_n}^{n} \underset{n \to \infty}{\longrightarrow} \mathbf{P}_{a_\infty}^{\infty}.$$

Proof of 3. For $n \in \mathbb{N}$, define the continuous function $\varphi_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ given by

$$\varphi_n(a,h) := 2 \int_a^{a+h} \int_a^b \mathrm{e}^{V_n(b) - V_n(c)} \mathrm{d}c \, \mathrm{d}b.$$

For each $a \in \mathbb{R}$, it is clear that $\varphi_n(a, \cdot)$ is strictly increasing on $\mathbb{R}_+$ and $\varphi_n(a,0) = 0$. Furthermore, since V_n is constant on the interval $\big[\varepsilon_n \lceil a/\varepsilon_n \rceil, \varepsilon_n(\lceil a/\varepsilon_n \rceil + 1)\big)$,

$$\varphi_n(a, 2\varepsilon_n) \geq 2 \int_{\varepsilon_n \lceil a/\varepsilon_n \rceil}^{\varepsilon_n(\lceil a/\varepsilon_n \rceil + 1)} \int_{\varepsilon_n \lceil a/\varepsilon_n \rceil}^{b} \mathrm{e}^{V_n(b) - V_n(c)} \mathrm{d}c \, \mathrm{d}b = \varepsilon_n^2.$$

Hence, there exists a unique $\psi_{1,n}(a) \in (0, 2\varepsilon_n]$ such that

$$\varphi_n(a, \psi_{1,n}(a)) = \varepsilon_n^2. \tag{5.8}$$

Using the continuity of φ_n and the compactness of $[0, 2\varepsilon_n]$, it is straightforward to obtain that $\psi_{1,n}$ is continuous. In the same manner, we may prove that, for each $a \in \mathbb{R}$, there exists a unique $\psi_{2,n}(a) \in (0, 2\varepsilon_n]$ such that

$$\varphi_n(a, -\psi_{2,n}(a)) = \varepsilon_n^2, \tag{5.9}$$

and that $\psi_{2,n}$ is continuous. Introduce the continuous function $p_n : \mathbb{R} \to (0, 1)$ given by

$$p_n(a) := \int_{a-\psi_{2,n}(a)}^{a} \mathrm{e}^{V_n(b)} \mathrm{d}b \Bigg/ \int_{a-\psi_{2,n}(a)}^{a+\psi_{1,n}(a)} \mathrm{e}^{V_n(b)} \mathrm{d}b, \tag{5.10}$$

and define a transition operator $T_n : \mathrm{C}_0(\mathbb{R}) \to \mathrm{C}_0(\mathbb{R})$ by

$$T_n f(a) := p_n(a) f(a + \psi_{1,n}(a)) + (1 - p_n(a)) f(a - \psi_{2,n}(a)).$$

According to Definition 3.1, let $(\widetilde{\mathbf{P}}_a^n)_a \in \mathcal{P}\left(\mathbb{R}^{\mathbb{N}}\right)^{\mathbb{R}}$ be the discrete time locally Feller family with transition operator T_n. For any $k \in \mathbb{Z}$, since V_n is constant on $[\varepsilon_n k, \varepsilon_n(k+1))$ and on $[\varepsilon_n(k-1), \varepsilon_n k)$, we have

$$\varphi_n(\varepsilon_n k, \pm\varepsilon_n) = 2 \int_{\varepsilon_n k}^{\varepsilon_n(k\pm 1)} \int_{\varepsilon_n k}^{b} \mathrm{d}c \, \mathrm{d}b = \varepsilon_n^2$$

and therefore $\psi_{1,n}(\varepsilon_n k) = \psi_{2,n}(\varepsilon_n k) = \varepsilon_n$. Furthermore

$$p_n(\varepsilon_n k) := \frac{\int_{\varepsilon_n(k-1)}^{\varepsilon_n k} \mathrm{e}^{V_n(b)} \mathrm{d}b}{\int_{\varepsilon_n(k-1)}^{\varepsilon_n(k+1)} \mathrm{e}^{V_n(b)} \mathrm{d}b} = \frac{\varepsilon_n \mathrm{e}^{V_n(\varepsilon_n(k-1))}}{\varepsilon_n \mathrm{e}^{V_n(\varepsilon_n(k-1))} + \varepsilon_n \mathrm{e}^{V_n(\varepsilon_n k)}} = \frac{1}{1 + \mathrm{e}^{q_{n,k}}},$$

hence for any $f \in \mathrm{C}_0(\mathbb{R})$,

$$T_n f(\varepsilon_n k) := \frac{1}{1 + \mathrm{e}^{q_{n,k}}} f(\varepsilon_n(k+1)) + \frac{1}{1 + \mathrm{e}^{-q_{n,k}}} f(\varepsilon_n(k-1)).$$

We deduce that for any $\mu \in \mathcal{P}(\mathbb{Z})$ and $n \in \mathbb{N}$, $\mathscr{L}_{\mathbf{P}_\mu^n}(\varepsilon_n Y) = \widetilde{\mathbf{P}}_{\widetilde{\mu}}^n$, where $\widetilde{\mu}$ is the pushforward measure of μ with respect to the mapping $k \mapsto \varepsilon_n k$.

We shall use the Theorem 3.2 of convergence of discrete time Markov families. If $f \in \mathrm{D}(L^V)$, we need to prove that there exists a sequence of continuous functions $f_n \in \mathrm{C}_0(\mathbb{R})$ converging to f such that $(T_n f_n - f_n)/\varepsilon_n^2$ converges to $L^V f$. By the second part of Proposition 5.1, there exists a sequence of continuous functions $f_n \in \mathrm{D}(L^{V_n})$ such that f_n converges to f and $L^{V_n} f_n$ converges to $L^V f$. Applying (5.4) to f_n and V_n and recalling (5.8) and (5.9), we have for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\begin{aligned} f(a + \psi_{1,n}(a)) =& f(a) + (\mathrm{e}^{-V} f')(a) \int_a^{a+\psi_{1,n}(a)} \mathrm{e}^{V(b)} \mathrm{d}b + \varepsilon_n^2 L^{V_n} f_n(a) \\ &+ 2 \int_a^{a+\psi_{1,n}(a)} \int_a^{b} \mathrm{e}^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) \mathrm{d}c \, \mathrm{d}b, \end{aligned}$$

$$
\begin{aligned}
f(a-\psi_{2,n}(a)) =& f(a) - (\mathrm{e}^{-V} f')(a) \int_{a-\psi_{2,n}(a)}^{a} \mathrm{e}^{V(b)} \mathrm{d}b + \varepsilon_n^2 L^{V_n} f_n(a) \\
&+ 2 \int_a^{a-\psi_{2,n}(a)} \int_a^b \mathrm{e}^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) \mathrm{d}c \, \mathrm{d}b.
\end{aligned}
$$

Hence by (5.10), for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
&\Big| \frac{T_n f_n(a) - f_n(a)}{\varepsilon_n^2} - L^{V_n} f_n(a) \Big| \\
&\qquad \le \frac{2p_n(a)}{\varepsilon_n^2} \Big| \int_a^{a+\psi_{1,n}(a)} \int_a^b \mathrm{e}^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) \mathrm{d}c \, \mathrm{d}b \Big| \\
&\qquad\quad + \frac{2(1-p_n(a))}{\varepsilon_n^2} \Big| \int_a^{a-\psi_{2,n}(a)} \int_a^b \mathrm{e}^{V(b)-V(c)} (L^{V_n} f_n(c) - L^{V_n} f_n(a)) \mathrm{d}c \, \mathrm{d}b \Big| \\
&\qquad \le \sup_{|h| \le 2\varepsilon_n} |L^{V_n} f_n(a+h) - L^{V_n} f_n(a)|.
\end{aligned}
$$

It is not difficult to deduce that $(T_n f_n - f_n)/\varepsilon_n^2$ converges to $L^V f$. Finally, by Theorem 3.2 for the convergence of discrete time Markov families, for any sequence $\mu_n \in \mathcal{P}(\mathbb{Z})$ such that $\widetilde{\mu}_n$ converges to a probability measure $\mu \in \mathcal{P}(\mathbb{R})$, we have

$$
\mathscr{L}_{\mathbf{P}^n_{\mu_n}} \big((\varepsilon_n Y_{\lfloor t/\varepsilon_n^2 \rfloor})_t \big) = \mathscr{L}_{\widetilde{\mathbf{P}}^n_{\widetilde{\mu}_n}} \big((Y_{\lfloor t/\varepsilon_n^2 \rfloor})_t \big) \underset{n\to\infty}{\overset{\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))}{\longrightarrow}} \mathbf{P}_\mu,
$$

where $\widetilde{\mu}_n$ are the pushforwards of μ_n with respect to the mappings $k \mapsto \varepsilon_n k$. □

A Appendix

We collect in this appendix several results already proved in [GH17c] containing technical statements and used in the previous sections. We refer the interested reader to the paper [GH17c] for the introductory contexts and proofs of each lemma.

Lemma A.1 (cf. Lemma 3.6 in [GH17c]). *Let $L_n, L_\infty \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ be such that $\mathrm{D}(L_\infty)$ is dense in $\mathrm{C}_0(S)$ and assume that, for any $f \in \mathrm{D}(L_\infty)$, there exists, for each n, $f_n \in \mathrm{D}(L_n)$ such that $f_n \underset{n\to\infty}{\overset{\mathrm{C}_0}{\longrightarrow}} f$, $L_n f_n \underset{n\to\infty}{\overset{\mathrm{C}}{\longrightarrow}} L_\infty f$. Consider $\mathcal{K}$ a compact subset of S and $\mathcal{U}$ an open subset of $S \times S$ containing $\{(a,a) \,|\, a \in S\}$. For an arbitrary $(\mathcal{F}_{t+})_t$-stopping time τ_1 we denote the $(\mathcal{F}_{t+})_t$-stopping time*

$$
\tau(\tau_1) := \inf \{ t \ge \tau_1 \mid \{(X_{\tau_1}, X_s)\}_{\tau_1 \le s \le t} \not\subset \mathcal{U} \}.
$$

Then for each $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that: for any $n \ge n_0$, for any $\tau_1 \le \tau_2$, $(\mathcal{F}_{t+})_t$-stopping times, and for any $\mathbf{P} \in \mathcal{M}(L_n)$ satisfying $\mathbf{E}[(\tau_2 - \tau_1)\mathbb{1}_{\{X_{\tau_1} \in \mathcal{K}\}}] \le \delta$, we have

$$
\mathbf{P}(X_{\tau_1} \in \mathcal{K},\ \tau(\tau_1) \le \tau_2) \le \varepsilon,
$$

with the convention $X_\infty := \Delta$.

Lemma A.2 (cf. Proposition 4.15 in [GH17c]). *Let $L_1, L_2 \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ be such that $\mathrm{D}(L_1) = \mathrm{D}(L_2)$ is dense in $\mathrm{C}_0(S)$ and assume that the martingale local problems associated* *to L_1 and L_2 are well-posed. Let $\mathbf{P}^1 \in \mathcal{M}(L_1)$ and $\mathbf{P}^2 \in \mathcal{M}(L_2)$ be two solutions of these problems having the same initial distribution and let $U \subset S$ be an open subset. If*

$$\forall f \in D(L_1),\ (L_2 f)_{|U} = (L_1 f)_{|U}\,, \qquad \textit{then} \qquad \mathscr{L}_{\mathbf{P}^2}\left(X^{\tau^U}\right) = \mathscr{L}_{\mathbf{P}^1}\left(X^{\tau^U}\right).$$

Lemma A.3 (cf. Lemma 4.17 in [GH17c])**.** *Let U be an open subset of S and L be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ with $\mathrm{D}(L)$ is dense in $\mathrm{C}(S)$. Assume that the martingale local problem associated to L is well-posed. Then there exists a function $h_0 \in \mathrm{C}(S, \mathbb{R}_+)$ satisfying $\{h_0 \neq 0\} = U$, such that for all $h \in \mathrm{C}(S, \mathbb{R}_+)$ with $\{h \neq 0\} = U$ and $\sup_{a \in U}(h/h_0)(a) < \infty$, the martingale local problem associated to hL is well-posed.*

Lemma A.4 (cf. Lemma A.2 in [GH17c])**.** *Let $(U_m)_{m \in \mathbb{N}}$ be an increasing sequence of open subsets such that $S = \bigcup_m U_m$. For $n, m \in \mathbb{N} \cup \{\infty\}$, let $\mathbf{P}^{n,m} \in \mathcal{P}(\mathbb{D}_{loc}(S))$ be such that*

i) for each $m \in \mathbb{N}$, $\mathbf{P}^{n,m} \underset{n \to \infty}{\longrightarrow} \mathbf{P}^{\infty,m}$, weakly for the local Skorokhod topology,

ii) for each $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$, $\mathscr{L}_{\mathbf{P}^{n,m}}\left(X^{\tau^{U_m}}\right) = \mathscr{L}_{\mathbf{P}^{n,\infty}}\left(X^{\tau^{U_m}}\right)$.

Then $\mathbf{P}^{n,\infty} \underset{n \to \infty}{\longrightarrow} \mathbf{P}^{\infty,\infty}$, weakly for the local Skorokhod topology.

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