Locally Feller processes and martingale local problems

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Abstract: This paper is devoted to the study of a certain type of martingale problems associated to general operators corresponding to processes which have finite lifetime. We analyse several properties and in particular the weak convergence of sequences of solutions for an appropriate Skorokhod topology setting. We point out the Feller-type features of the associated solutions to this type of martingale problem. Then localisation theorems for well-posed martingale problems or for corresponding generators are proved.
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1 Introduction

The theory of Lévy-type processes stays an active domain of research during the last two decades. Heuristically, a Lévy-type process X with symbol $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is a Markov process which behaves locally like a Lévy process with characteristic exponent $q(a, \cdot)$, in a neighbourhood of each point $a \in \mathbb{R}^d$. One associates to a Lévy-type process the pseudo-differential operator L given by, for $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$,

$$Lf(a) := -\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}a\cdot\alpha} q(a,\alpha)\widehat{f}(\alpha)\mathrm{d}\alpha, \quad \text{where} \quad \widehat{f}(\alpha) := (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}a\cdot\alpha} f(a)\mathrm{d}a.$$

Does a sequence $X^{(n)}$ of Lévy-type processes, having symbols q_n, converges towards some process, when the sequence of symbols q_n converges to a symbol q? What can we say about the sequence $X^{(n)}$ when the corresponding sequence of pseudo-differential operators L_n converges to an operator L? What could be the appropriate setting when one wants to approximate a Lévy-type process by a family of discrete Markov chains? This is the kind of question which naturally appears when we study Lévy-type processes.

It was a very useful observation that a unified manner to tackle a lot of questions about large classes of processes is the martingale problem approach (see, for instance, Stroock [22] for Lévy-type processes, Stroock and Varadhan [23] for diffusion processes, Kurtz [19] for Lévy-driven stochastic differential equations...). Often, convergence results are obtained under technical restrictions: for instance, when the closure of L is the generator of a Feller process (see Kallenberg [15] Thm. 19.25, p. 385, Thm. 19.28, p. 387 or Böttcher, Schilling and Wang [3], Theorem 7.6 p. 172). In a number of situations the cited condition is not satisfied. In the present paper we try to describe a general method which should be the main tool to tackle these difficulties and, even, should relax some of technical restrictions. We analyse sequences of martingale problems associated to large class of operators acting on continuous functions and we look to Feller-type features of the associated of solutions.

There exist many fundamental references where this kind of objects are studied. In a pioneer work, Courrège [6] described the form of a linear operator satisfying the positive maximum principle, as the sum of a second order differential operator and a singular integral operator, and he made the connection with the infinitesimal generator of a Feller semigroup. At the same period, Courrège and Priouret [7] studied a method of construction and of decomposition of Markov processes with continuous paths, along increasing sequences of terminal hitting times, by using the method of pasting together processes on overlapping open sets. Hoh and Jacob [11] discussed the martingale problem for a large class of pseudo-differential operators, especially the martingale problem for generators of Lévy type (see also the monograph of Jacob [14] on Feller semigroups generated by pseudo-differential operators). The concept of the symbol of a Markov process as a probabilistic counterpart of the symbol of a pseudo-differential operator is often used. Hoh [12] studied a class of pseudo-differential operators with negative-definite symbols which generate Markov processes and solved the martingale problem for this class of pseudo-differential operators, assuming the smoothness of its symbol with respect to the space variable (see also [13]). Kühn [17] considered a pseudo-differential operator with continuous negative definite symbol such that the martingale problem is well-posed on the space of smooth functions. She proved that the solution of this martingale problem is a conservative rich Feller process under a growth condition on the symbol. Böttcher and Schilling [2] gave a scheme to approximate a Feller process by Markov chains in terms of the symbol of the generator of the process (see also [2] for an application). Symbols are also used when studying SDE's driven by Lévy noises. Schilling and Schburr [21] computed the symbol of the strong solution of a SDE driven by a Lévy process and having a locally Lipschitz multiplicative coefficient and proved that this strong solution is a Feller process, provided the coefficient is bounded (see also [3]). Kühn [18] proved that if the coefficient of the SDE is continuous and satisfies a linear growth condition then a weak solution, provided that it exists, is also a Feller process.

Let us briefly describe some of the ideas developed in the present paper. To begin with, let us point out that the local Skorokhod topology on a locally compact Hausdorff space S constitutes a good setting when one needs to consider explosions in finite time (see [10]). Heuristically, we modify the global Skorokhod topology, on the space of càdlàg paths, by localising with respect to the space variable, in order to include the eventual explosions. The definition of a martingale local problem follows in a natural way: we need to stop the martingale when it exits from compact sets. Similarly, a stochastic process is locally Feller if, for any compact set of S, it coincides with a Feller process before it exits from the compact set. Let us note that a useful tool allowing to make the connection between local and global objects (Skorokhod topology, martingale, infinitesimal generator or Feller processes) is the time change transformation. Likewise, one has stability of all these local notions under the time change.

We study the existence and the uniqueness of solutions for martingale local problems and we illustrate their locally Feller-type features (see Theorem 4.5). Then we deduce a description of the generator of a locally Feller family of probabilities by using a martingale (see Theorem 4.14). Furthermore we characterise the convergence of a sequence of locally Feller processes in terms of convergence of operators, provided that the sequence of martingale local problems are well-posed (see Theorem 4.17) and without supposing that the closure of the limit operator is an infinitesimal generator. We also consider the localisation question (as described in Ethier and Kurtz [8], §4.6, pp. 216-221) and we give answers in terms of martingale local problem or in terms of generator (Theorems 4.21 and 4.23). We stress that a Feller process is locally Feller, hence our results, in particular the convergence theorems apply to Feller processes. In Theorem 4.10 we give a characterisation of the Feller property in terms of the locally Feller property plus an additional condition. As a first example, let us consider the simple one-dimensional SDE, $dX_t = dB_t + b(X_t)dt$ driven by a standard Brownian motion with $b \in \mathrm{C}(\mathbb{R}, \mathbb{R}_+^*)$. The associated martingale local problem associated to the operator defined for compact supported smooth functions f,

$$Lf(x) := \frac{1}{2}f''(x) + b(x)f'(x),$$

is well-posed by invoking the classical theory of Stroock and Varadhan [23] and the localisation theorem. Moreover it can be proved by using the scale function (see [8]) that the solution is Feller if and only if:

$$\int_{-\infty}^{0} b(y)^{-1}\mathrm{d}y = \infty = \int_{0}^{\infty} b(y)^{-1}\mathrm{d}y.$$

Our results should be useful in several situations, for instance, to analyse the convergence of a Markov chain towards a Lévy-type process under general conditions (improving the results, for instance, Thm 11.2.3 from Stroock and Varadhan [23] p. 272, Thm. 19.28 from Kallenberg [15], p. 387 or from Böttcher and Schnurr [2]). Two examples of applications are briefly presented in Remarks 4.19 and 4.24. Complete development of some of these applications and of some concrete examples (as the Euler scheme of approximation for Lévy-type process or the connection between the Sinai's random walk and the Brox diffusion describing the evolution of a Brownian particle into a Brownian potential [4]) are the object of a separate work [9]. Let us also note that in [9] we slightly modify the Brox's diffusion by considering the evolution of a Brownian particle in a very irregular potential getting in this case another interesting example of a locally Feller process (see also Remarks 4.11 and 4.12 for other examples of locally Feller processes which are not Feller processes). The method developed in the present paper should apply for other situations. In a work in progress, we apply a similar method for some singular stochastic differential equations driven by α-stable processes, other than Brownian motion.

The paper is organised as follows: in the next section we recall some notations and results obtained in our previous paper [10] on the local Skorokhod topology on spaces of càdlàg functions, tightness and time change transformation. Section 3 is devoted to the study of the martingale local problem : properties, tightness and convergence, but also the existence of solutions. The most important results are presented in Section 4. In §4.1 and §4.2 we give the definitions and point out characterisations of a locally Feller family and its connection with a Feller family, essentially in terms of martingale local problems. We also provide two corrections of a result by van Casteren [5] (see also [18], p. 2 and [17], p. 3603). In §4.3 we give a generator description of a locally Feller family and we characterise the convergence of a sequence of locally Feller families. §4.4 contains the localisation procedure for martingale problems and generators. We collect in the Appendix the most of technical proofs.

2 Preliminary notations and results

We recall here some notations and results concerning the local Skorokhod topology, the tightness criterion and a time change transformation which will be useful to state and prove our main results. Complete statements and proofs are described in an entirely dedicated paper [10].

Let S be a locally compact Hausdorff space with countable base. The space S could be endowed with a metric and so it is a Polish space. Take $\Delta \notin S$, and we will denote by $S^\Delta \supset S$ the one-point compactification of S, if S is not compact, or the topological sum $S \sqcup \{\Delta\}$, if S is compact (so Δ is an isolated point). Denote $\mathrm{C}(S) := \mathrm{C}(S, \mathbb{R})$, resp. $\mathrm{C}(S^\Delta) := \mathrm{C}(S^\Delta, \mathbb{R})$, the set of real continuous functions on S, resp. on S^Δ. If $\mathrm{C}_0(S)$ denotes the set of functions $f \in \mathrm{C}(S)$ vanishing in Δ, we will identify

$$\mathrm{C}_0(S) = \left\{ f \in \mathrm{C}(S^\Delta) \;\middle|\; f(\Delta) = 0 \right\}.$$

We endow the set $\mathrm{C}(S)$ with the topology of uniform convergence on compact sets and $\mathrm{C}_0(S)$ with the topology of uniform convergence.

The fact that a subset A is compactly embedded in an open subset $U \subset S$ will be denoted $A \Subset U$. If $x \in (S^\Delta)^{\mathbb{R}_+}$, we denote

$$\xi(x) := \inf\{t \geq 0 \,|\, \{x_s\}_{s \leq t} \not\Subset S\}.$$

Here and elsewhere we denote $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}$ and $\mathbb{R}_+^* := \{t \in \mathbb{R} : t > 0\}$.

Firstly, we introduce the set of càdlàg paths with values in S^Δ,

$$\mathbb{D}(S^\Delta) := \left\{ x \in (S^\Delta)^{\mathbb{R}_+} \;\middle|\; \begin{array}{l} \forall t \geq 0,\ x_t = \lim_{s \downarrow t} x_s, \ \text{ and} \\ \forall t > 0,\ x_{t-} := \lim_{s \uparrow t} x_s \ \text{ exists in } S^\Delta \end{array} \right\}$$

endowed with the global Skorokhod topology (see, for instance, Chap. 3 in [8], pp. 116-147) which is Polish.

Secondly, we proceed with the definition of a set of exploding càdlàg paths

$$\mathbb{D}_{\mathrm{loc}}(S) := \left\{ x \in (S^{\Delta})^{\mathbb{R}_+} \;\middle|\; \begin{array}{l} \forall t \geq \xi(x),\ x_t = \Delta, \\ \forall t \geq 0,\ x_t = \lim_{s \downarrow t} x_s, \\ \forall t > 0 \text{ s.t. } \{x_s\}_{s<t} \Subset S,\ x_{t-} := \lim_{s \uparrow t} x_s \text{ exists} \end{array} \right\}.$$

Consider d an arbitrary metric on S^{Δ}. A sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{D}_{\mathrm{loc}}(S)$ converges to x if and only if there exists a sequence $(\lambda^k)_k$ of increasing homeomorphisms on $\mathbb{R}_+$ satisfying

$$\forall t \geq 0 \text{ s.t. } \{x_s\}_{s<t} \Subset S, \quad \lim_{k\to\infty} \sup_{s\leq t} d(x_s, x^k_{\lambda^k_s}) = 0 \quad \text{and} \quad \lim_{k\to\infty} \sup_{s\leq t} |\lambda^k_s - s| = 0.$$

It can be showed that $\mathbb{D}_{\mathrm{loc}}(S)$ endowed with this convergence is a Polish space (see Theorem 2.4, p. 1187, in [10]). The topology associated to this convergence is called the local Skorokhod topology.

In fact the global Skorokhod topology is the trace (of the local) topology from $\mathbb{D}_{\mathrm{loc}}(S)$ to $\mathbb{D}(S^{\Delta})$ and a sequence $(x^k)_k$ from $\mathbb{D}(S^{\Delta})$ converges to $x \in \mathbb{D}(S^{\Delta})$ for the global Skorokhod topology if and only if there exists a sequence $(\lambda^k)_k$ in Λ such that

$$\forall t \geq 0, \quad \lim_{k\to\infty} \sup_{s\leq t} d(x_s, x^k_{\lambda^k_s}) \longrightarrow 0, \quad \text{and} \quad \lim_{k\to\infty} \sup_{s\leq t} |\lambda^k_s - s| = 0.$$

We recover the usual Skorokhod topology on $\mathbb{D}(S^{\Delta})$, as it is described, for instance, in §16 pp. 166-179 from [1]. Note that in Theorem 2.4, p. 1187, from [10] it is also proved, as for the usual Skorokhod topology, that the local Skorokhod topology does not depend on d but only on the topology on S.

We will always denote by X the canonical process on $\mathbb{D}(S^{\Delta})$ or on $\mathbb{D}_{\mathrm{loc}}(S)$, without danger of confusion. We endow each of $\mathbb{D}(S^{\Delta})$ and $\mathbb{D}_{\mathrm{loc}}(S)$ with the Borel σ-algebra $\mathcal{F} := \sigma(X_s,\ 0 \leq s < \infty)$ and a filtration $\mathcal{F}_t := \sigma(X_s,\ 0 \leq s \leq t)$. As usual, we will always denote by $\mathcal{P}(\mathbb{D}(S^{\Delta}))$ or $\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ the set of probability measures on $\mathbb{D}(S^{\Delta})$ or on $\mathbb{D}_{\mathrm{loc}}(S)$. We will always omit the argument X for the explosion time $\xi(X)$ of the canonical process. It is clear that ξ is a stopping time. Furthermore, if $U \subset S$ is an open subset,

$$\tau^U := \inf\{t \geq 0 \mid X_{t-} \notin U \text{ or } X_t \notin U\} \wedge \xi \tag{2.1}$$

is a stopping time.

There are several ways to localise processes, for instance one can stop when they leave a large compact set. Nevertheless this method does not preserve the convergence and we need to adapt this procedure in order to recover continuity. Let us describe our time change transformation.

Consider a positive continuous function $\mathfrak{g} \in \mathrm{C}(S, \mathbb{R}_+)$ and following (2.1), we can write

$$\tau^{\{\mathfrak{g}\neq 0\}}(x) := \inf\{t \geq 0 \mid \mathfrak{g}(x_{t-}) \wedge \mathfrak{g}(x_t) = 0\} \wedge \xi(x).$$

For any $x \in \mathbb{D}_{\text{loc}}(S)$ and $t \in \mathbb{R}_+$ we denote

$$\tau_t^{\mathfrak{g}}(x) := \inf \left\{ s \geq 0 \;\middle|\; s \geq \tau^{\{g \neq 0\}} \text{ or } \int_0^s \frac{\mathrm{d}u}{\mathfrak{g}(x_u)} \geq t \right\}. \tag{2.2}$$

We define a time change transformation, which is $\mathcal{F}$-measurable,

$$\begin{array}{rccc} \mathfrak{g} \cdot X : & \mathbb{D}_{\text{loc}}(S) & \to & \mathbb{D}_{\text{loc}}(S) \\ & x & \mapsto & \mathfrak{g} \cdot x, \end{array}$$

as follows: for $t \in \mathbb{R}_+$

$$(\mathfrak{g} \cdot X)_t := \begin{cases} X_{\tau^{\{\mathfrak{g} \neq 0\}}-} & \text{if } \tau_t^{\mathfrak{g}} = \tau^{\{\mathfrak{g} \neq 0\}},\ X_{\tau^{\{\mathfrak{g} \neq 0\}}-} \text{ exists and belongs to } \{\mathfrak{g} = 0\}, \\ X_{\tau_t^{\mathfrak{g}}} & \text{otherwise.} \end{cases} \tag{2.3}$$

The time change transformation will be a useful tool used to compare the local notions, as local Skorokhod topology, martingale local problems or locally Feller processes, with the usual (global) notions.

For any $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\text{loc}}(S))$, we also define $\mathfrak{g} \cdot \mathbf{P}$ the pushforward of $\mathbf{P}$ by $x \mapsto \mathfrak{g} \cdot x$. Let us stress that, $\tau_t^{\mathfrak{g}}$ is a stopping time (see Corollary 2.3 in [10]). The time of explosion of $\mathfrak{g} \cdot X$ is given by

$$\xi(\mathfrak{g} \cdot X) = \begin{cases} \infty & \text{if } \tau^{\{\mathfrak{g} \neq 0\}} < \xi \text{ or } X_{\xi-} \text{ exists and belongs to } \{\mathfrak{g} = 0\}, \\ \int_0^{\xi} \frac{\mathrm{d}u}{\mathfrak{g}(x_u)} & \text{otherwise.} \end{cases}$$

It is not difficult to see, using the definition of the time change (2.3), that

$$\forall \mathfrak{g}_1, \mathfrak{g}_2 \in \mathrm{C}(S, \mathbb{R}_+),\ \forall x \in \mathbb{D}_{\text{loc}}(S), \quad \mathfrak{g}_1 \cdot (\mathfrak{g}_2 \cdot x) = (\mathfrak{g}_1 \mathfrak{g}_2) \cdot x. \tag{2.4}$$

In [10] Proposition 3.9, p. 1199, a connection between $\mathbb{D}_{\text{loc}}(S)$ and $\mathbb{D}(S^\Delta)$ was given. We recall here this result because it will be employed several times.

Proposition 2.1 (Connection between $\mathbb{D}_{\text{loc}}(S)$ and $\mathbb{D}(S^\Delta)$)**.** *Let $\widetilde{S}$ be an arbitrary locally compact Hausdorff space with countable base and consider*

$$\begin{array}{rccc} \mathbf{P} : & \widetilde{S} & \to & \mathcal{P}(\mathbb{D}_{loc}(S)) \\ & a & \mapsto & \mathbf{P}_a \end{array}$$

a weakly continuous mapping for the local Skorokhod topology. Then for any open subset U of S, there exists $\mathfrak{g} \in \mathrm{C}(S, \mathbb{R}_+)$ such that $\{\mathfrak{g} \neq 0\} = U$, for all $a \in \widetilde{S}$

$$\mathfrak{g} \cdot \mathbf{P}_a \left(0 < \xi < \infty \Rightarrow X_{\xi-} \textit{ exists in } U\right) = 1,$$

and the application

$$\begin{array}{rccc} \mathfrak{g} \cdot \mathbf{P} : & \widetilde{S} & \to & \mathcal{P}(\{0 < \xi < \infty \Rightarrow X_{\xi-} \textit{ exists in } U\}) \\ & a & \mapsto & \mathfrak{g} \cdot \mathbf{P}_a \end{array}$$

is weakly continuous for the global Skorokhod topology of $\mathbb{D}(S^\Delta)$.

Another useful result which we would like to recall from [10] is the following version of the Aldous criterion of tightness: let $(\mathbf{P}_n)_n$ be a sequence of probability measures on $\mathcal{P}(\mathbb{D}_{\text{loc}}(S))$. If for all $t \geq 0$, $\varepsilon > 0$, and open subset $U \Subset S$, we have:

$$\limsup_{n\to\infty} \sup_{\substack{\tau_1 \leq \tau_2 \\ \tau_2 \leq (\tau_1+\delta)\wedge t \wedge \tau^U}} \mathbf{P}_n\big(d(X_{\tau_1}, X_{\tau_2}) \geq \varepsilon\big) \underset{\delta\to 0}{\longrightarrow} 0, \tag{2.5}$$

then $\{\mathbf{P}_n\}_n$ is tight for the local Skorokhod topology (see Proposition 2.9, p. 1190, from [10]). In (2.5) the supremum is taken over all $\mathcal{F}_t$-stopping times τ_1, τ_2.

Let $(\mathcal{G}_t)_{t\geq 0}$ be a filtration of $\mathcal{F}$ containing $(\mathcal{F}_t)_{t\geq 0}$. Recall that a family of probability measures $(\mathbf{P}_a)_{a\in S} \in \mathcal{P}(\mathbb{D}_{\text{loc}}(S))^S$ is called $(\mathcal{G}_t)_t$-Markov if, for any $B \in \mathcal{F}$, $a \mapsto \mathbf{P}_a(B)$ is measurable, for any $a \in S$, $\mathbf{P}_a(X_0 = a) = 1$, and for any $B \in \mathcal{F}$, $a \in S$ and $t_0 \in \mathbb{R}_+$

$$\mathbf{P}_a\left((X_{t_0+t})_t \in B \mid \mathcal{G}_{t_0}\right) = \mathbf{P}_{X_{t_0}}(B),\ \mathbf{P}_a - \text{almost surely},$$

where $\mathbf{P}_\Delta$ is the unique element of $\mathcal{P}(\mathbb{D}_{\text{loc}}(S))$ such that $\mathbf{P}_\Delta(\xi = 0) = 1$ and, as usual, $(X_{t_0+t})_t$ is the shifted process. If the latter property is also satisfied by replacing t_0 with any $(\mathcal{G}_t)_t$-stopping time, the family of probability measures is $(\mathcal{G}_t)_t$-strong Markov. If $\mathcal{G}_t = \mathcal{F}_t$ we just say that the family is (strong) Markov. If ν is a measure on S^Δ we set $\mathbf{P}_\nu := \int \mathbf{P}_a \nu(\mathrm{d}a)$. Then the distribution of X_0 under $\mathbf{P}_\nu$ is ν, and $\mathbf{P}_\nu$ satisfies the (strong) Markov property.

To finish this section let us recall the following property of the time change stated in Remark 3.4, p. 1196, from [10], used several times in the present paper, but not in that one.

Proposition 2.2 (Strong Markov property and time change)**.** *Consider $\mathfrak{g} \in \mathrm{C}(S, \mathbb{R}_+)$ and $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$. If $(\mathbf{P}_a)_a$ is a $(\mathcal{F}_{t+})_t$-strong Markov family, then $(\mathfrak{g} \cdot \mathbf{P}_a)_a$ is also $(\mathcal{F}_{t+})_t$-strong Markov family.*

For the sake of completeness we will provide the proof of Proposition 2.2 in the Appendix A.2.

3 Martingale local problem

3.1 Definition and first properties

To begin with we recall the optional sampling theorem. Its proof can be found in Theorem 2.13 and Remark 2.14. p. 61 from [8].

Theorem 3.1 (Optional sampling theorem)**.** *Let $(\Omega, (\mathcal{G}_t)_t, \mathbb{P})$ be a filtered probability space and let M be a càdlàg $(\mathcal{G}_t)_t$-martingale, then for all $(\mathcal{G}_{t+})_t$-stopping times τ and σ, with τ bounded,*

$$\mathbb{E}\left[M_\tau \mid \mathcal{G}_{\sigma+}\right] = M_{\tau\wedge\sigma}, \quad \mathbb{P}\text{-almost surely.}$$

In particular M is a $(\mathcal{G}_{t+})_t$-martingale. We denoted here $\mathcal{G}_{t+} := \cap_{\varepsilon>0} \mathcal{G}_{t+\varepsilon}$.

All along the paper the operators from $\mathrm{C}_0(S)$ to $\mathrm{C}(S)$, will be denoted as a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$, in other words its graph. This will be not a major notation constraint, since in the following most of the operators are univariate.

Definition 3.2 (Martingale local problem)**.** Let L be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$.

a) The set $\mathcal{M}(L)$ of solutions of the martingale local problem associated to L is the set of $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ such that for all $(f,g) \in L$ and open subset $U \Subset S$:

$$f(X_{t\wedge\tau^U}) - \int_0^{t\wedge\tau^U} g(X_s)\mathrm{d}s \text{ is a } \mathbf{P}\text{-martingale} \tag{3.1}$$

with respect to the filtration $(\mathcal{F}_t)_t$ or, equivalent, to the filtration $(\mathcal{F}_{t+})_t$. Recall that τ^U is given by (2.1). The martingale *local problem* should not be confused with the *local martingale* problem (see Remark 3.3 for a connection).

b) We say that there is existence of a solution for the martingale local problem if for any $a \in S$ there exists an element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathbf{P}(X_0 = a) = 1$.

c) We say that there is uniqueness of the solution for the martingale local problem if for any $a \in S$ there is at most one element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathbf{P}(X_0 = a) = 1$.

d) The martingale local problem is said well-posed if there is existence and uniqueness of the solution.

Remark 3.3. 1) The hypothesis of continuity of g ensures the fact that (3.1) is adapted to the (non-augmented) canonical filtration $(\mathcal{F}_t)_t$.
2) By using the dominated convergence when U is growing towards S, and by the previous definition (3.1), for all $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$, $(f,g) \in L \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_b(S)\big)$ and $\mathbf{P} \in \mathcal{M}(L)$, we have that

$$f(X_t) - \int_0^{t\wedge\xi} g(X_s)\mathrm{d}s \text{ is a } \mathbf{P}\text{-martingale.}$$

Indeed, if $(f,g) \in \mathrm{C}_0(S) \times \mathrm{C}_b(S)$ the quantity in (3.1) is uniformly bounded. Hence, if $L \subset \mathrm{C}_0(S) \times \mathrm{C}_b(S)$, the martingale local problem and the classical martingale problem are equivalent.
3) It can be proved that, for all $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$, $(f,g) \in L$ and $\mathbf{P} \in \mathcal{M}(L)$ such that

$$\mathbf{P}\big(\xi < \infty \text{ implies } \{X_s\}_{s<\xi} \Subset S\big) = 1,$$

we have

$$f(X_t) - \int_0^{t\wedge\xi} g(X_s)\mathrm{d}s \text{ is a } \mathbf{P}\text{-local martingale.}$$

Indeed let us denote $\Omega = \big\{\xi < \infty \text{ implies } \{X_s\}_{s<\xi} \Subset S\big\}$ and introduce the family of stopping times

$$\sigma^{U,T} := \tau^U \vee \big(T\mathbb{1}_{\{\tau^U \le T, \tau^U = \xi\}}\big), \qquad \text{with } U \Subset S,\, T \ge 0.$$

To obtain the assertion, we remark that, almost surely on Ω, $X^{\sigma^{U,T}} = X^{\tau^U}$, and, when $T \to \infty$ and U growing towards S, $\sigma^{U,T}$ grows to infinity.
4) We shall see that the uniqueness or, respectively, the existence of a solution for the martingale local problem when one starts from a fixed point implies the uniqueness or the existence of a solution for the martingale local problem when one starts with an arbitrary measure (see Proposition 3.14).
5) Consider $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ and $\mathbf{P} \in \mathcal{M}(L)$. If $(f, g) \in L$ and $U \Subset S$ is an open subset, then, by dominated convergence

$$\frac{\mathbf{E}\left[f(X_{t\wedge\tau^U}) \mid \mathcal{F}_0\right] - f(X_0)}{t} = \mathbf{E}\left[\frac{1}{t}\int_0^{t\wedge\tau^U} g(X_s)\mathrm{d}s \;\middle|\; \mathcal{F}_0\right] \underset{t\to 0}{\overset{\mathbf{P}\text{-a.s.}}{\longrightarrow}} g(X_0). \qquad \Diamond$$

Some useful properties concerning the martingale local problem are stated below:

Proposition 3.4 (Martingale local problem properties)**.** *Let $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ be.*

1. (Time change) *Take $\mathfrak{h} \in \mathrm{C}(S, \mathbb{R}_+)$ and denote*

$$\mathfrak{h}L := \{(f, \mathfrak{h}g) \mid (f, g) \in L\}. \tag{3.2}$$

Then, for all $\mathbf{P} \in \mathcal{M}(L)$,

$$\mathfrak{h} \cdot \mathbf{P} \in \mathcal{M}(\mathfrak{h}L). \tag{3.3}$$

2. (Closure property) *The closure with respect to $\mathrm{C}_0(S) \times \mathrm{C}(S)$ satisfies*

$$\mathcal{M}\left(\overline{span(L)}\right) = \mathcal{M}(L). \tag{3.4}$$

3. (Compactness and convexity property) *Suppose that $\mathrm{D}(L)$ is a dense subset of $\mathrm{C}_0(S)$, where the domain of L is defined by*

$$\mathrm{D}(L) := \{f \in \mathrm{C}_0(S) \mid \exists g \in \mathrm{C}(S),\ (f, g) \in L\}.$$

Then $\mathcal{M}(L)$ is a convex compact set for the local Skorokhod topology.

The following result provides a continuity property of the mapping $L \mapsto \mathcal{M}(L)$.

Proposition 3.5. *Let $L_n, L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ be such that*

$$\forall (f, g) \in L, \quad \exists (f_n, g_n) \in L_n, \text{ such that } \quad f_n \underset{n\to\infty}{\overset{\mathrm{C}_0}{\longrightarrow}} f,\ g_n \underset{n\to\infty}{\overset{\mathrm{C}}{\longrightarrow}} g. \tag{3.5}$$

Then:

1. (Continuity) *Let $\mathbf{P}^n, \mathbf{P} \in \mathcal{P}(\mathbb{D}_{loc}(S))$ be such that $\mathbf{P}^n \in \mathcal{M}(L_n)$ and suppose that $\{\mathbf{P}^n\}_n$ converges weakly to $\mathbf{P}$ for the local Skorokhod topology. Then $\mathbf{P} \in \mathcal{M}(L)$.*

2. (Tightness) *Suppose that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$, then for any sequence $\mathbf{P}^n \in \mathcal{M}(L_n)$, $\{\mathbf{P}^n\}_n$ is tight for the local Skorokhod topology.*

The proofs of Propositions 3.4 and 3.5 are interlaced.

Proof of part 1 of Proposition 3.4. Take $(f,g) \in L$ and an open subset $U \Subset S$. If $s_1 \leq \cdots \leq s_k \leq s \leq t$ are positive numbers and $\varphi_1, \dots, \varphi_k \in \mathrm{C}(S^\Delta)$, we need to prove that

$$\mathfrak{h} \cdot \mathbf{E}\left[\left(f(X_{t\wedge\tau^U}) - f(X_{s\wedge\tau^U}) - \int_{s\wedge\tau^U}^{t\wedge\tau^U} (\mathfrak{h}g)(X_u)\mathrm{d}u\right)\varphi_1(X_{s_1})\cdots\varphi_k(X_{s_k})\right] = 0. \quad (3.6)$$

We will proceed in two steps: firstly we suppose that $U \Subset \{\mathfrak{h} \neq 0\}$. Recalling the definition (2.2), if we denote $\tau_t := \tau_t^{\mathfrak{h}} \wedge \tau^U$, we have, for all $t \in \mathbb{R}_+$,

$$\mathfrak{h} \cdot X_{t\wedge\tau^U(\mathfrak{h}\cdot X)} = X_{\tau_t} \quad (3.7)$$

and

$$\int_0^{t\wedge\tau^U(\mathfrak{h}\cdot X)} (\mathfrak{h}g)(\mathfrak{h}\cdot X_u)\mathrm{d}u = \int_0^{t\wedge\tau^U(\mathfrak{h}\cdot X)} (\mathfrak{h}g)(X_{\tau_u})\mathrm{d}u = \int_0^{\tau_t} g(X_u)\mathrm{d}u. \quad (3.8)$$

Hence by (3.7)-(3.8) and using the optional sampling Theorem 3.1

$$\begin{aligned}
&\mathfrak{h} \cdot \mathbf{E}\left[\left(f(X_{t\wedge\tau^U}) - f(X_{s\wedge\tau^U}) - \int_{s\wedge\tau^U}^{t\wedge\tau^U} (\mathfrak{h}g)(X_u)\mathrm{d}u\right)\varphi_1(X_{s_1})\cdots\varphi_k(X_{s_k})\right] \\
&= \mathfrak{h} \cdot \mathbf{E}\left[\left(f(X_{t\wedge\tau^U}) - f(X_{s\wedge\tau^U}) - \int_{s\wedge\tau^U}^{t\wedge\tau^U} (\mathfrak{h}g)(X_u)\mathrm{d}u\right)\varphi_1(X_{s_1\wedge\tau^U})\cdots\varphi_k(X_{s_k\wedge\tau^U})\right] \\
&\qquad = \mathbf{E}\left[\left(f(X_{\tau_t}) - f(X_{\tau_s}) - \int_{\tau_s}^{\tau_t} g(X_u)\mathrm{d}u\right)\varphi_1(X_{\tau_{s_1}})\cdots\varphi_k(X_{\tau_{s_k}})\right] = 0.
\end{aligned}$$

Secondly, we suppose that $U \Subset S$. Recall that d is the metric on S^Δ and we introduce, for $n \geq 1$, $U_n := \{a \in U \,|\, d(a, \{\mathfrak{h} = 0\}) > n^{-1}\}$. It is straightforward to obtain the following pointwise convergences,

$$\begin{gathered}
\mathfrak{h} \cdot X_{t\wedge\tau^{U_n}(\mathfrak{h}\cdot X)} \underset{n\to\infty}{\longrightarrow} \mathfrak{h} \cdot X_{t\wedge\tau^U(\mathfrak{h}\cdot X)}, \\
\textstyle\int_0^{t\wedge\tau^{U_n}(\mathfrak{h}\cdot X)} (\mathfrak{h}g)(\mathfrak{h}\cdot X_u)\mathrm{d}u \underset{n\to\infty}{\longrightarrow} \int_0^{t\wedge\tau^U(\mathfrak{h}\cdot X)} (\mathfrak{h}g)(\mathfrak{h}\cdot X_u)\mathrm{d}u.
\end{gathered}$$

Therefore,

$$\begin{aligned}
&f(X_{t\wedge\tau^{U_n}}) - f(X_{s\wedge\tau^{U_n}}) - \int_{s\wedge\tau^{U_n}}^{t\wedge\tau^{U_n}} (\mathfrak{h}g)(X_u)\mathrm{d}u \\
&\qquad\qquad \underset{n\to\infty}{\overset{\mathfrak{h}\cdot\mathbf{P}\text{-a.s.}}{\longrightarrow}} f(X_{t\wedge\tau^U}) - f(X_{s\wedge\tau^U}) - \int_{s\wedge\tau^U}^{t\wedge\tau^U} (\mathfrak{h}g)(X_u)\mathrm{d}u.
\end{aligned}$$

Applying the first step to $U_n \Subset \{\mathfrak{h} \neq 0\}$ and letting $n \to \infty$, by dominated convergence we obtain (3.6). $\square$

Proof of part 1 of Proposition 3.5. By using Proposition 2.1 we know that there exists $\mathfrak{h} \in \mathrm{C}(S, \mathbb{R}_+^*)$ such that $\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)$ has probability 1 under $\mathfrak{h} \cdot \mathbf{P}^n$ and under $\mathfrak{h} \cdot \mathbf{P}$ and such that $\mathfrak{h} \cdot \mathbf{P}^n$ converges weakly to $\mathfrak{h} \cdot \mathbf{P}$ for the global Skorokhod topology of $\mathbb{D}(S^\Delta)$. Let us fix (f, g) and (f_n, g_n) arbitrary as in (3.5) and then we can modify $\mathfrak{h}$ such that it satisfies furthermore $\mathfrak{h} g_n, \mathfrak{h} g \in \mathrm{C}_0(S)$ and $\mathfrak{h} g_n \xrightarrow[n\to\infty]{\mathrm{C}_0} \mathfrak{h} g$. Indeed, for instance, we can multiply $\mathfrak{h}$ with a function from $\mathrm{C}(S; \mathbb{R}_+^*)$ which is less than $d(\cdot, \Delta)/(\sup_{n\in\mathbb{N}} \mathfrak{h}|g_n - g|)$.

Let $\mathbb{T}$ be the set of $t \in \mathbb{R}_+$ such that $\mathfrak{h} \cdot \mathbf{P}(X_{t-} = X_t) = 1$, so $\mathbb{R}_+ \backslash \mathbb{T}$ is countable. Let $s_1 \leq \cdots \leq s_k \leq s \leq t$ belonging to $\mathbb{T}$ and consider $\varphi_1, \ldots, \varphi_k \in \mathrm{C}(S^\Delta)$. By using 1 of Proposition 3.4 and the first part of Remark 3.3

$$\mathfrak{h} \cdot \mathbf{E}^n \left[\left(f_n(X_t) - f_n(X_s) - \int_s^t (\mathfrak{h} g_n)(X_u) \mathrm{d}u \right) \varphi_1(X_{s_1}) \cdots \varphi_k(X_{s_k}) \right] = 0. \tag{3.9}$$

Noting that the sequences of functions f_n and $\mathfrak{h} g_n$ converge uniformly, respectively to f and $\mathfrak{h} g$, and since $\varphi_1, \ldots \varphi_k$ are bounded, it can be deduced that the sequence of functions $\big(f_n(X_t) - f_n(X_s) - \int_s^t (\mathfrak{h} g_n)(X_u)\mathrm{d}u\big)\varphi_1(X_{s_1}) \cdots \varphi_k(X_{s_k})$ converges uniformly to the function $\big(f(X_t) - f(X_s) - \int_s^t (\mathfrak{h} g)(X_u)\mathrm{d}u\big)\varphi_1(X_{s_1}) \cdots \varphi_k(X_{s_k})$. This last function is continuous $\mathfrak{h} \cdot \mathbf{P}$-almost everywhere for the topology of $\mathbb{D}(S^\Delta)$. Hence we can take the limit, as $n \to \infty$, in (3.9) and we obtain that

$$\mathfrak{h} \cdot \mathbf{E} \left[\left(f(X_t) - f(X_s) - \int_s^t (\mathfrak{h} g)(X_u) \mathrm{d}u \right) \varphi_1(X_{s_1}) \cdots \varphi_k(X_{s_k}) \right] = 0. \tag{3.10}$$

Since $\mathbb{T}$ is dense in $\mathbb{R}_+$, since $f, \mathfrak{h} g, \varphi_1, \ldots, \varphi_k$ are bounded, by right continuity of paths of the canonical process, and by dominated convergence, (3.10) extends to $s_i, s, t \in \mathbb{R}_+$. Hence $\mathfrak{h} \cdot \mathbf{P} \in \mathcal{M}(\{(f, \mathfrak{h} g)\})$, so using (2.4) and part 1 of Proposition 3.4, $\mathbf{P} = (1/\mathfrak{h}) \cdot \mathfrak{h} \cdot \mathbf{P} \in \mathcal{M}(\{(f, g)\})$. Since $(f, g) \in L$ was chosen arbitrary, we have proved that $\mathbf{P} \in \mathcal{M}(L)$. □

Proof of part 2 of Proposition 3.4. It is straightforward that $\mathcal{M}(\mathrm{span}(L)) = \mathcal{M}(L)$. Let $\mathbf{P} \in \mathcal{M}(L)$. We apply part 1 of Proposition 3.5 to the stationary sequences $\mathbf{P}^n = \mathbf{P}$ and $L_n = \mathrm{span}(L)$ and to $\overline{\mathrm{span}(L)}$. Hence $\mathbf{P} \in \mathcal{M}(\overline{\mathrm{span}(L)})$ and the proof is done. □

Proof of part 2 of Proposition 3.5. Take $t \in \mathbb{R}_+$ and $U \Subset S$ an open subset. By using Lemma 3.8 and considering $\mathcal{K} := \overline{U}$ and $\mathcal{U} := \{(a, b) \in S \times S \mid d(a, b) < \varepsilon\}$, we have

$$\sup_{\substack{\tau_1 \leq \tau_2 \\ \tau_2 \leq (\tau_1 + \delta) \wedge \tau^U \wedge t}} \mathbf{P}_n\big(d(X_{\tau_1}, X_{\tau_2}) \geq \varepsilon\big) \underset{\substack{n\to\infty \\ \delta\to 0}}{\longrightarrow} 0,$$

hence (2.5) is satisfied and the Aldous criterion applies (Proposition 2.9 in [10]). □

Proof of part 3 of Proposition 3.4. It is straightforward that $\mathcal{M}(L)$ is convex. To prove the compactness, let $(\mathbf{P}^n)_n$ be a sequence from $\mathcal{M}(L)$. We apply part 2 of Proposition 3.5 to this sequence and to the stationary sequence $L_n = L$. Hence $(\mathbf{P}^n)$ is tight, so there exists a subsequence $(\mathbf{P}^{n_k})_k$ which converges towards some $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$. Thanks to part 1 of Proposition 3.5 we can deduce that $\mathbf{P} \in \mathcal{M}(L)$. The statement of the proposition is then obtained since $\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ is a Polish space. □

We end this section with another property concerning martingale local problems:

Proposition 3.6 (Quasi-continuity property of the martingale local problem). *Let L be a subset of $\mathrm{C}_0(S)\times\mathrm{C}(S)$ and suppose that $\mathrm{D}(L)$ is a dense subset of $\mathrm{C}_0(S)$. Then for any $\mathbf{P}\in\mathcal{M}(L)$, $\mathbf{P}$ is $(\mathcal{F}_{t+})_t$-quasi-continuous. More precisely this means that for any $(\mathcal{F}_{t+})_t$-stopping times $\tau,\tau_1,\tau_2\ldots$*

$$X_{\tau_n}\underset{n\to\infty}{\longrightarrow}X_\tau\quad\mathbf{P}\text{-almost surely on }\left\{\tau_n\underset{n\to\infty}{\longrightarrow}\tau<\infty\right\},\tag{3.11}$$

with the convention $X_\infty:=\Delta$. In particular, for any $t\geq 0$, $\mathbf{P}(X_{t-}=X_t)=1$,

$$\mathbf{P}\big(\mathbb{D}_{loc}(S)\cap\mathbb{D}(S^\Delta)\big)=\mathbf{P}\big(\xi\in(0,\infty)\Rightarrow X_{\xi_-}\text{ exists in }S^\Delta\big)=1.$$

Moreover, for any open subset $U\subset S$, we have $\mathbf{P}(\tau^U<\infty\Rightarrow X_{\tau^U}\notin U)=1$, where τ^U is given by (2.1).

Remark 3.7. Let us note that the quasi-continuity is needed to have $X_{\tau^U}\notin U$ a.s. even if the process is right-continuous. For instance, the real Markov process $X_t:=X_0+(B_t-\lfloor X_0+B_t\rfloor)\mathbb{1}_{X_0<1}$, with B a standard Brownian motion, is right-continuous and we have $X_{\tau^{(-\infty,1)}}=X_0\mathbb{1}_{X_0\geq 1}$ which belongs to $(-\infty,1)$, provided $X_0<1$. ◊

The proof of the previous proposition is technical and is postponed to the Appendix §A.1. During this proof we use the result of the next lemma concerning the property of uniform continuity along stopping times of the martingale local problem. Its proof is likewise postponed to the Appendix §A.1

Lemma 3.8. *Let $L_n,L\subset\mathrm{C}_0(S)\times\mathrm{C}(S)$ be such that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$ and assume the convergence of the operators in the sense given by* (3.5). *Consider $\mathcal{K}$ a compact subset of S and $\mathcal{U}$ an open subset of $S\times S$ containing $\{(a,a)\}_{a\in S}$. For an arbitrary $(\mathcal{F}_{t+})_t$-stopping time τ_1 we denote the $(\mathcal{F}_{t+})_t$-stopping time*

$$\tau(\tau_1):=\inf\left\{t\geq\tau_1\mid\{(X_{\tau_1},X_s)\}_{\tau_1\leq s\leq t}\not\subset\mathcal{U}\right\}.$$

Then for each $\varepsilon>0$ there exist $n_0\in\mathbb{N}$ and $\delta>0$ such that: for any $n\geq n_0$, $(\mathcal{F}_{t+})_t$-stopping times $\tau_1\leq\tau_2$ and $\mathbf{P}\in\mathcal{M}(L_n)$ satisfying $\mathbf{E}[(\tau_2-\tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}]\leq\delta$, we have

$$\mathbf{P}(X_{\tau_1}\in\mathcal{K},\ \tau(\tau_1)\leq\tau_2)\leq\varepsilon,$$

with the convention $X_\infty:=\Delta$.

3.2 Existence and conditioning

Before giving the result of existence of a solution for the martingale local problem, let us recall that $X^\tau_t=X_{\tau\wedge t}$ for τ a stopping time, and the classical positive maximal principle (see [8], p.165):

Definition 3.9. A subset $L\subset\mathrm{C}_0(S)\times\mathrm{C}(S)$ satisfies the positive maximum principle if for all $(f,g)\in L$ and $a_0\in S$ such that $f(a_0)=\sup_{a\in S}f(a)\geq 0$ then $g(a_0)\leq 0$.

The existence of a solution for the martingale local problem result will be a consequence of Theorem 5.4 p. 199 from [8].

Theorem 3.10 (Existence)**.** *Let L be a linear subspace of* $\mathrm{C}_0(S) \times \mathrm{C}(S)$*.*

1. *If there is existence of a solution for the martingale local problem associated to L, then L satisfies the positive maximum principle.*
2. *Conversely, if L satisfies the positive maximum principle and $D(L)$ is dense in* $\mathrm{C}_0(S)$*, then there is existence of a solution for the martingale local problem associated to L.*

Remark 3.11. 1) A linear subspace $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ satisfying the positive maximum principle is univariate. Indeed for any $(f, g_1), (f, g_2) \in L$, applying the positive maximum principle to $(0, g_2 - g_1)$ and $(0, g_1 - g_2)$ we deduce that $g_1 = g_2$.
2) Suppose furthermore that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$, then as a consequence of the second part of Proposition 3.4 and of Theorem 3.10, the closure $\overline{L}$ in $\mathrm{C}_0(S) \times \mathrm{C}(S)$ satisfies the positive maximum principle, too. $\Diamond$

Proof of Theorem 3.10. Suppose that there is existence of a solution for the martingale local problem, let $(f, g) \in L$ and $a_0 \in S$ be such that $f(a_0) = \sup_{a \in S} f(a) \geq 0$. If we take $\mathbf{P} \in \mathcal{M}(L)$ such that $\mathbf{P}(X_0 = a_0) = 1$, then, by the fifth part of Remark 3.3

$$g(a_0) = \lim_{t \to 0} \frac{1}{t} \big(\mathbf{E} \left[f(X_{t \wedge \tau^U}) \mid \mathcal{F}_0 \right] - f(a_0) \big) \leq 0,$$

so L satisfies the positive maximum principle.

Let us prove the second part of Theorem 3.10. Consider $\widetilde{L}_0$ a countable dense subset of L and $L_0 := \mathrm{span}(\widetilde{L}_0)$. There exists $\mathfrak{h} \in \mathrm{C}_0(S; \mathbb{R}_+^*)$ such that for all $(f, g) \in \widetilde{L}_0$: $\mathfrak{h} g \in \mathrm{C}_0$, hence $\overline{L} = \overline{L_0}$ and $\mathfrak{h} L_0 \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$. We apply Theorem 5.4 p. 199 in [8] to the univariate operator $\mathfrak{h} L_0$: for all $a \in S$, there exists $\widetilde{\mathbf{P}} \in \mathcal{P}(\mathbb{D}(S^\Delta))$ such that $\widetilde{\mathbf{P}}(X_0 = a) = 1$ and for all $(f, g) \in \mathfrak{h} L_0$

$$f(X_t) - \int_0^t g(X_s) \mathrm{d}s \text{ is a } \widetilde{\mathbf{P}}\text{-martingale.}$$

We set $\mathbf{P} := \mathscr{L}_{\widetilde{\mathbf{P}}}(X^\xi)$ the law of X^ξ under $\widetilde{\mathbf{P}}$. Then $\mathbf{P} \in \mathcal{P}\left(\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)\right)$. Moreover, for any $(f, g) \in \mathfrak{h} L_0$, open subset $U \Subset S$, $s_1 \leq \cdots \leq s_k \leq s \leq t$ in $\mathbb{R}_+$ and $\varphi_1, \ldots, \varphi_k \in \mathrm{C}(S^\Delta)$,

$$\mathbf{E} \left[\left(f(X_{t \wedge \tau^U}) - f(X_{s \wedge \tau^U}) - \int_{s \wedge \tau^U}^{t \wedge \tau^U} g(X_u) \mathrm{d}u \right) \varphi_1(X_{s_1}) \cdots \varphi_k(X_{s_k}) \right]$$
$$\widetilde{\mathbf{E}} \left[\left(f(X_{t \wedge \tau^U}) - f(X_{s \wedge \tau^U}) - \int_{s \wedge \tau^U}^{t \wedge \tau^U} g(X_u) \mathrm{d}u \right) \varphi_1(X_{s_1}) \cdots \varphi_k(X_{s_k}) \right] = 0.$$

Hence $\mathbf{P} \in \mathcal{M}(\mathfrak{h}L_0)$. To conclude we use the first two parts of Proposition 3.4:

$$\mathcal{M}(L) = \mathcal{M}(\overline{L}) = \mathcal{M}(L_0) = \left\{ \frac{1}{\mathfrak{h}} \cdot \mathbf{Q} \,\middle|\, \mathbf{Q} \in \mathcal{M}(\mathfrak{h}L_0) \right\}.$$

So $\frac{1}{\mathfrak{h}} \cdot \mathbf{P} \in \mathcal{M}(L)$ and the existence of a solution for the martingale local problem is proved. $\square$

Remark 3.12. Since $\mathcal{F}$ is the Borel σ-algebra on the Polish space $\mathbb{D}_{\text{loc}}(S)$, we can use Theorem 6.3, in [15], p. 107. So, for any $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\text{loc}}(S))$ and $(\mathcal{F}_{t+})_t$-stopping time τ, the regular conditional distribution $\mathbf{Q}_X \overset{\mathbf{P}\text{-a.s.}}{:=} \mathscr{L}_{\mathbf{P}}\big((X_{\tau+t})_{t\geq 0} \,\big|\, \mathcal{F}_{\tau+}\big)$ exists. It means that there exists

$$\begin{array}{rccc} \mathbf{Q}: & \mathbb{D}_{\text{loc}}(S) & \to & \mathcal{P}(\mathbb{D}_{\text{loc}}(S)) \\ & x & \mapsto & \mathbf{Q}_x \end{array}$$

such that for any $A \in \mathcal{F}$, $\mathbf{Q}_X(A)$ is $\mathcal{F}_{\tau+}$-measurable and

$$\mathbf{P}\big((X_{\tau+t})_{t\geq 0} \in A \,\big|\, \mathcal{F}_{\tau+}\big) = \mathbf{Q}_X(A) \quad \mathbf{P}\text{-almost surely.} \qquad \Diamond$$

The following proposition contains a near result as Theorem 4.2, p. 184 in [8].

Proposition 3.13 (Conditioning)**.** *Take $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$, $\mathbf{P} \in \mathcal{M}(L)$, and a $(\mathcal{F}_{t+})_t$-stopping time τ. As in Remark 3.12 we denote $\mathbf{Q}_X \overset{\mathbf{P}\text{-a.s.}}{:=} \mathscr{L}_{\mathbf{P}}\big((X_{\tau+t})_{t\geq 0} \,\big|\, \mathcal{F}_{\tau+}\big)$, then*

$$\mathbf{Q}_X \in \mathcal{M}(L), \quad \mathbf{P}\text{-almost surely.}$$

Proof. Let (f, g) be in L, $s_1 \leq \cdots \leq s_k \leq s \leq t$ be in $\mathbb{R}_+$, $\varphi_1, \ldots, \varphi_k$ be in $\mathrm{C}(S^\Delta)$ and $U \Subset S$ be a open subset. Here and elsewhere we will denote by $E^{\mathbf{Q}_x}$ the expectation with respect to $\mathbf{Q}_x$. Since

$$\begin{aligned} &\mathbb{1}_{\tau<\tau^U} E^{\mathbf{Q}_X}\Big[\big(f(X_{t\wedge\tau^U}) - f(X_{s\wedge\tau^U}) - \int_{s\wedge\tau^U}^{t\wedge\tau^U} g(X_u)\mathrm{d}u\big)\varphi_1(X_{s_1})\cdots\varphi_k(X_{s_k})\Big] \\ &\quad \overset{\mathbf{P}\text{-a.s.}}{=} \mathbb{1}_{\tau<\tau^U}\mathbf{E}\Big[\big(f(X_{(t+\tau)\wedge\tau^U}) - f(X_{(s+\tau)\wedge\tau^U}) - \int_{(s+\tau)\wedge\tau^U}^{(t+\tau)\wedge\tau^U} g(X_u)\mathrm{d}u\big) \\ &\qquad\qquad \times \varphi_1(X_{s_1+\tau})\cdots\varphi_k(X_{s_k+\tau}) \,\Big|\, \mathcal{F}_{\tau+}\Big] \overset{\mathbf{P}\text{-a.s.}}{=} 0, \end{aligned}$$

we have

$$\begin{aligned} &\mathbf{P}\Big(E^{\mathbf{Q}_X}\Big[\big(f(X_{t\wedge\tau^U}) - f(X_{s\wedge\tau^U}) - \int_{s\wedge\tau^U}^{t\wedge\tau^U} g(X_u)\mathrm{d}u\big)\varphi_1(X_{s_1})\cdots\varphi_k(X_{s_k})\Big] \neq 0\Big) \\ &\qquad\qquad \leq \mathbf{P}\big(\tau^U \leq \tau < \xi\big). \end{aligned} \tag{3.12}$$

Let $\widetilde{L}$ be a countable dense subset of L, C be a countable dense subset of $\mathrm{C}(S^\Delta)$ and $U_n \Subset S$ be an increasing sequence of open subsets such that $S = \bigcup_n U_n$. Then $\mathbf{Q}_X \in$

$\mathcal{M}(L)$ if and only if for all $(f,g) \in \widetilde{L}$, $k \in \mathbb{N}$, for any $s_1 \leq \cdots \leq s_k \leq s \leq t$ in $\mathbb{Q}_+$, for any $\varphi_1, \ldots, \varphi_k \in C$, and for n large enough

$$E^{\mathbf{Q}_X}\left[\left(f(X_{t\wedge\tau^{U_n}}) - f(X_{s\wedge\tau^{U_n}}) - \int_{s\wedge\tau^{U_n}}^{t\wedge\tau^{U_n}} g(X_u)\mathrm{d}u\right)\varphi_1(X_{s_1})\cdots\varphi_k(X_{s_k})\right] = 0.$$

Hence $\{\mathbf{Q}_X \in \mathcal{M}(L)\}$ is in $\mathcal{F}_{\tau+}$ and by (3.12), $\mathbf{P}$-almost surely $\mathbf{Q}_X \in \mathcal{M}(L)$. □

Proposition 3.14. *Set $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$.*

1. *If there is uniqueness of the solution for the martingale local problem then for any $\mu \in \mathcal{P}(S^\Delta)$ there is at most one element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathscr{L}_{\mathbf{P}}(X_0) = \mu$.*
2. *If there is existence of a solution for the martingale local problem and $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$, then for any $\mu \in \mathcal{P}(S^\Delta)$ there exists an element $\mathbf{P}$ in $\mathcal{M}(L)$ such that $\mathscr{L}_{\mathbf{P}}(X_0) = \mu$.*

Proof. Suppose that we have uniqueness of the solution for the martingale local problem. Let μ be in $\mathcal{P}(S^\Delta)$ and $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{M}(L)$ be such that $\mathscr{L}_{\mathbf{P}^1}(X_0) = \mathscr{L}_{\mathbf{P}^2}(X_0) = \mu$. As in Remark 3.12 let $\mathbf{Q}^1_\bullet, \mathbf{Q}^2_\bullet : S^\Delta \to \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ be such that

$$\mathbf{Q}^1_{X_0} \overset{\mathbf{P}^1\text{-a.s.}}{:=} \mathscr{L}_{\mathbf{P}^1}\left(X \mid \mathcal{F}_0\right), \qquad \mathbf{Q}^2_{X_0} \overset{\mathbf{P}^2\text{-a.s.}}{:=} \mathscr{L}_{\mathbf{P}^2}\left(X \mid \mathcal{F}_0\right). \tag{3.13}$$

Then, by Proposition 3.13, $\mathbf{Q}^1_a, \mathbf{Q}^2_a \in \mathcal{M}(L)$ for μ-almost all a, so, by uniqueness of the solution for the martingale local problem, $\mathbf{Q}^1_a = \mathbf{Q}^2_a$ for μ-almost all a. We finally obtain, by (3.13), $\mathbf{P}^1 = \int \mathbf{Q}^1_a \mu(\mathrm{d}a) = \int \mathbf{Q}^2_a \mu(\mathrm{d}a) = \mathbf{P}^2$.

Suppose that we have existence of a solution for the martingale local problem and that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$. Thanks to the property 3 in Proposition 3.4, $\mathcal{M}(L)$ is convex and compact. Hence the set

$$C := \{\mu \in \mathcal{P}(S^\Delta) \mid \exists \mathbf{P} \in \mathcal{M}(L) \text{ such that } \mathscr{L}_{\mathbf{P}}(X_0) = \mu\}$$

is convex and compact. Since there is existence of a solution for the martingale local problem we have $\left\{\delta_a \;\middle|\; a \in S^\Delta\right\} \subset C$ so $C = \mathcal{P}(S^\Delta)$. □

4 Locally Feller families of probabilities

In this section we will study a local counterpart of Feller families in connection with Feller semi-groups and martingale local problems. The basic notions and facts on Feller semi-groups can be found in Chapter 19 pp. 367-389 from [15].

4.1 Feller families of probabilities

Definition 4.1 (Feller family)**.** A Markov family $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is said to be Feller if for all $f \in \mathrm{C}_0(S)$ and $t \in \mathbb{R}_+$ the function

$$\begin{array}{rccc} T_t f : & S & \to & \mathbb{R} \\ & a & \mapsto & \mathbf{E}_a[f(X_t)] \end{array}$$

is in $\mathrm{C}_0(S)$. In this case it is no difficult to see that $(T_t)_t$ is a Feller semi-group on $\mathrm{C}_0(S)$ (see p. 369 in [15]) called the semi-group of $(\mathbf{P}_a)_a$. Its generator L is the set of $(f,g) \in \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ such that, for all $a \in S$

$$\frac{T_t f(a) - f(a)}{t} \underset{t\to 0}{\longrightarrow} g(a).$$

and we call it the $(\mathrm{C}_0 \times \mathrm{C}_0)$-generator of $(\mathbf{P}_a)_a$.

In [5] Theorem 2.5, p. 283, one states a connection between Feller families and martingale problems. Unfortunately the proof given in the cited paper is correct only on a compact space S. The fact that a Feller family of probabilities is the unique solution of an appropriate martingale problem is stated in the proposition below. We will prove the converse of this result in Theorem 4.9.

To give this statement we need to introduce some notations. For $L \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ we define

$$L^{\Delta} := \operatorname{span}\left(L \cup \{(\mathbb{1}_{S^{\Delta}}, 0)\}\right) \subset \mathrm{C}(S^{\Delta}) \times \mathrm{C}(S^{\Delta}). \tag{4.1}$$

We recall that we identified $\mathrm{C}_0(S)$ by the set of functions $f \in \mathrm{C}(S^{\Delta})$ such that $f(\Delta) = 0$. The set of solutions $\mathcal{M}(L^{\Delta}) \subset \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S^{\Delta}))$ of the martingale problem associated to L^{Δ} satisfies

$$\forall \mathbf{P} \in \mathcal{M}(L^{\Delta}), \quad \mathbf{P}(X_0 \in S^{\Delta} \Rightarrow X \in \mathbb{D}(S^{\Delta})) = 0.$$

Without loss of the generality, to study the martingale problem associated to L^{Δ} it suffices to study the set of solution with S^{Δ}-conservative paths:

$$\mathcal{M}_{\mathrm{c}}(L^{\Delta}) := \mathcal{M}(L^{\Delta}) \cap \mathcal{P}(\mathbb{D}(S^{\Delta})) = \left\{\mathbf{P} \in \mathcal{M}(L^{\Delta}) \;\middle|\; \mathbf{P}(X_0 \in S^{\Delta}) = 1\right\}.$$

Indeed, the unique non-conservative solution of $\mathcal{M}(L^{\Delta})$ is the process which leaves S^{Δ} at time 0. In fact $\mathcal{M}_{\mathrm{c}}(L^{\Delta})$ is the set consisting of $\mathbf{P} \in \mathcal{P}(\mathbb{D}(S^{\Delta}))$ such that for all $(f,g) \in L$

$$f(X_t) - \int_0^t g(X_s)\mathrm{d}s \quad \text{ is a } \mathbf{P}\text{-martingale.} \tag{4.2}$$

The following result is well-known and, for the sake of completeness, we provide its proof below:

Proposition 4.2. *If $(T_t)_t$ is a Feller semi-group on $\mathrm{C}_0(S)$ with L its generator, then there is a unique Feller family $(\mathbf{P}_a)_a$ with semi-group $(T_t)_t$. Moreover the martingale problem associate to L^{Δ} is well-posed and*

$$\mathcal{M}_{\mathrm{c}}(L^{\Delta}) = \{\mathbf{P}_\mu\}_{\mu \in \mathcal{P}(S^{\Delta})}.$$

Remark 4.3. 1. For any $\mathbf{P} \in \mathcal{M}_{\mathrm{c}}(L^{\Delta})$ the distribution of X^{τ^S} under $\mathbf{P}$ satisfies

$$\mathscr{L}_{\mathbf{P}}(X^{\tau^S}) \in \mathcal{M}_{\mathrm{c}}(L^{\Delta}) \cap \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S)) \subset \mathcal{M}(L).$$

Moreover if $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$, thanks to Proposition 3.6

$$\mathcal{M}(L) = \mathcal{M}_{\mathrm{c}}(L^{\Delta}) \cap \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S)).$$

So if $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$ there is existence of a solution for the martingale problem associated to L if and only if there is existence of a solution to the martingale problem associated to L^{Δ}. Moreover the uniqueness of the solution for the martingale problem associated to L^{Δ} imply uniqueness of the solution for the martingale problem associated to L.
2. If S is compact and $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S) = \mathrm{C}(S)$, then it is straightforward to obtain $\mathcal{M}(L) = \mathcal{M}_{\mathrm{c}}(L^{\Delta})$. $\Diamond$

Proof of Proposition 4.2. The existence of a solution for the martingale problem is a consequence of Theorem 3.10, see for instance the Hille–Yoshida theorem (Theorem 19.11, p. 375 in [15]). Thanks to Proposition 3.13 and using chain rule for conditioning, to identify the finite dimensional distributions of solutions solving the martingale problem, we need to prove that

$$\forall \mathbf{P} \in \mathcal{M}_{\mathrm{c}}(L^{\Delta}),\, \forall t \geq 0,\, \forall f \in \mathrm{D}(L),\ \mathbf{E}\left[f(X_t)\right] = \mathbf{E}\left[T_t f(X_0)\right].$$

Let $0 = t_0 \leq \cdots \leq t_{N+1} = t$ be a subdivision of $[0,t]$, then

$$\begin{aligned}\mathbf{E}\left[f(X_t) \mid \mathcal{F}_0\right] - T_t f(X_0) &= \sum_{i=0}^{N} \mathbf{E}\left[T_{t-t_{i+1}} f(X_{t_{i+1}}) \;\middle|\; \mathcal{F}_0\right] - \mathbf{E}\left[T_{t-t_i} f(X_{t_i}) \mid \mathcal{F}_0\right] \\ &= \sum_{i=0}^{N} \mathbf{E}\left[\mathbf{E}\left[T_{t-t_{i+1}} f(X_{t_{i+1}}) \;\middle|\; \mathcal{F}_{t_i}\right] - T_{t-t_i} f(X_{t_i}) \;\middle|\; \mathcal{F}_0\right].\end{aligned}$$

Moreover for each $i \in \{0, \dots N\}$, using martingales properties for the first part and semi-groups properties, in particular that $T_t f \in D(L)$ (see for instance Theorem 19.6, p. 372 in [15]) for the second,

$$\mathbf{E}\Big[T_{t-t_{i+1}} f(X_{t_{i+1}}) \,\Big|\, \mathcal{F}_{t_i}\Big] - T_{t-t_i} f(X_{t_i}) = \mathbf{E}\Big[\int_{t_i}^{t_{i+1}} L T_{t-t_{i+1}} f(X_s) - L T_{t-s} f(X_{t_i}) \mathrm{d}s \,\Big|\, \mathcal{F}_{t_i}\Big],$$

so

$$\left|\mathbf{E}\left[f(X_t) - T_t f(X_0)\right]\right| \leq \mathbf{E} \sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} \left|L T_{t-t_{i+1}} f(X_s) - L T_{t-s} f(X_{t_i})\right| \mathrm{d}s.$$

By dominated convergence we can conclude. $\square$

Before introducing the definition of a locally Feller family, let us state a result on an application of a time change to a Feller family (see (2.3)):

Proposition 4.4. *Let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be a Feller family with $(\mathrm{C}_0 \times \mathrm{C}_0)$-generator L. Then, for any $\mathfrak{g} \in \mathrm{C}_b(S, \mathbb{R}_+^*)$, $(\mathfrak{g} \cdot \mathbf{P}_a)_a$ is a Feller family with $(\mathrm{C}_0 \times \mathrm{C}_0)$-generator $\overline{\mathfrak{g}L}$, taking the closure in $\mathrm{C}_0(S) \times \mathrm{C}_0(S)$.*

Proof. Thanks to the property 1 in Proposition 3.4 and to the Proposition 4.2, the result is only a reformulation of Theorem 2, p. 275 in [20]. For the sake of completeness we give the statement of this result in our context: if $L \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ is the generator of a Feller semi-group, then for any $\mathfrak{g} \in \mathrm{C}_b(S, \mathbb{R}_+^*)$, $\overline{\mathfrak{g}L}$ is the generator of a Feller semi-group. $\square$

4.2 Locally Feller families and connection with martingale problems

We are ready to introduce the notion of locally Feller family of probabilities. This is given in the following theorem:

Theorem 4.5 (Definition of a locally Feller family)**.** *If $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$, the following four assertions are equivalent:*

1. *(continuity) the family $(\mathbf{P}_a)_a$ is Markov and $a \mapsto \mathbf{P}_a$ is continuous for the local Skorokhod topology;*
2. *(time change) there exists $\mathfrak{g} \in \mathrm{C}(S, \mathbb{R}_+^*)$ such that $(\mathfrak{g} \cdot \mathbf{P}_a)_a$ is a Feller family;*
3. *(martingale) there exists $L \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ such that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$ and $(\mathbf{P}_a)_a$ is the unique solution solving the martingale local problem for L:*
$$\forall a \in S, \quad \mathbf{P} \in \mathcal{M}(L) \ \text{ and } \ \mathbf{P}(X_0 = a) = 1 \Longleftrightarrow \mathbf{P} = \mathbf{P}_a;$$
4. *(localisation) for any open subset $U \Subset S$ there exists a Feller family $(\widetilde{\mathbf{P}}_a)_a$ such that for any $a \in S$*
$$\mathscr{L}_{\mathbf{P}_a}\left(X^{\tau^U}\right) = \mathscr{L}_{\widetilde{\mathbf{P}}_a}\left(X^{\tau^U}\right).$$

A family satisfying one of these equivalent conditions will be called a locally Feller family. Moreover a locally Feller family $(\mathbf{P}_a)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov and for all $\mu \in \mathcal{P}(S^\Delta)$, $\mathbf{P}_\mu$ is quasi-continuous.

We give below the proof of Theorem 4.5 but first let us make some remarks.

Remark 4.6. A natural question is how can we construct locally Feller families? We give here answers to this question.

i) A Feller family is locally Feller.

ii) If $\mathfrak{g} \in \mathrm{C}(S, \mathbb{R}_+^*)$ and $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is locally Feller, then $(\mathfrak{g} \cdot \mathbf{P}_a)_a$ is locally Feller. This result is to be compared with the result of Proposition 4.4.

iii) If S is a compact space, a family is locally Feller if and only if it is Feller. This statement is an easy consequence of the third part of the latter theorem and of Proposition 4.4.

iv) As consequence of the first assertion in Theorem 4.5, if $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is locally Feller then the family

$$\begin{array}{rcl} U & \to & \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(U)) \\ a & \mapsto & \mathscr{L}_{\mathbf{P}_a}(\widetilde{X}) \end{array}$$

is locally Feller in the space U. Indeed, it is straightforward to verify that, for any open subset $U \subset S$, the following mapping is continuous,

$$\begin{array}{rcl} \mathbb{D}_{\mathrm{loc}}(S) & \to & \mathbb{D}_{\mathrm{loc}}(U) \\ x & \mapsto & \widetilde{x} \end{array} \quad \text{with} \quad \widetilde{x}_s := \begin{cases} x_s & \text{if } s < \tau^U(x), \\ \Delta & \text{otherwise.} \end{cases} \qquad \Diamond$$

Proof of Theorem 4.5.
1⇒2 Thanks to Proposition 2.1 there exists $\mathfrak{g} \in \mathrm{C}(S, \mathbb{R}_+^*)$ such that for all $a \in S^\Delta$, $\mathfrak{g} \cdot \mathbf{P}_a(\mathbb{D}_{\mathrm{loc}}(S) \cap (S^\Delta)) = 1$ and such that the mapping

$$\begin{array}{rcl} S^\Delta & \to & \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)) \\ a & \mapsto & \mathfrak{g} \cdot \mathbf{P}_a \end{array}$$

is weakly continuous for the global Skorokhod topology of $\mathbb{D}(S^\Delta)$. Moreover we can deduce that $(\mathbf{P}_a)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov by using the following result

Lemma 4.7. *Let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be such that $a \mapsto \mathbf{P}_a$ is continuous for the local Skorokhod topology. Suppose that for all $a \in S^\Delta$: $\mathbf{P}_a(X_0 = a) = 1$ and there exists a dense subset $\mathbb{T}_a \subset \mathbb{R}_+$ such that for any $B \in \mathcal{F}$ and $t_0 \in \mathbb{T}_a$*

$$\mathbf{P}_a\left((X_{t_0+t})_t \in B \mid \mathcal{F}_{t_0}\right) = \mathbf{P}_{X_{t_0}}(B) \quad \mathbf{P}_a\text{-almost surely.}$$

Then $(\mathbf{P}_a)_a$ is a $(\mathcal{F}_{t+})_t$-strong Markov family.

The proof of Lemma 4.7 is postponed in Appendix A.2 and we proceed with the proof of Theorem 4.5. By Proposition 2.2 we can deduce that $(\mathfrak{g} \cdot \mathbf{P}_a)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov. Take $a \in S$ and $t \in \mathbb{R}_+^*$, we will prove that $\mathfrak{g} \cdot \mathbf{P}_a(X_{t-} = X_t) = 1$. For any $f \in \mathrm{C}(S^\Delta)$, $s < t$ and $\varepsilon > 0$, by the Markov property

$$\mathfrak{g} \cdot \mathbf{E}_a\Big[\frac{1}{\varepsilon}\int_s^{s+\varepsilon} f(X_u)\mathrm{d}u \,\Big|\, \mathcal{F}_s\Big] \overset{\mathfrak{g}\cdot\mathbf{P}_a\text{-a.s.}}{=} \mathfrak{g} \cdot \mathbf{E}_{X_s}\Big[\frac{1}{\varepsilon}\int_0^{\varepsilon} f(X_u)\mathrm{d}u\Big].$$

Since $a \mapsto \mathfrak{g} \cdot \mathbf{P}_a$ is weakly continuous for the global topology and since $x \mapsto \frac{1}{\varepsilon}\int_0^\varepsilon f(X_u)\mathrm{d}u$ is continuous for the global topology,

$$\mathfrak{g} \cdot \mathbf{E}_{X_s}\Big[\frac{1}{\varepsilon}\int_0^{\varepsilon} f(X_u)\mathrm{d}u\Big] \underset{\substack{s\to t\\ s<t}}{\longrightarrow} \mathfrak{g} \cdot \mathbf{E}_{X_{t-}}\Big[\frac{1}{\varepsilon}\int_0^{\varepsilon} f(X_u)\mathrm{d}u\Big].$$

By the triangle inequality and the dominated convergence theorem (see a similar reasoning following (A.5) in Appendix A.2) we have

$$\mathfrak{g} \cdot \mathbf{E}_a\left|\mathfrak{g} \cdot \mathbf{E}_a\Big[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} f(X_u)\mathrm{d}u \,\Big|\, \mathcal{F}_{t-}\Big] - \mathfrak{g} \cdot \mathbf{E}_a\Big[\frac{1}{\varepsilon}\int_s^{s+\varepsilon} f(X_u)\mathrm{d}u \,\Big|\, \mathcal{F}_s\Big]\right| \underset{\substack{s\to t\\ s<t}}{\longrightarrow} 0,$$

so

$$\mathfrak{g}\cdot\mathbf{E}_a\Big[\frac{1}{\varepsilon}\int_t^{t+\varepsilon} f(X_u)\mathrm{d}u\,\Big|\,\mathcal{F}_{t-}\Big] \overset{\mathfrak{g}\cdot\mathbf{P}_a\text{-a.s.}}{=} \mathfrak{g}\cdot\mathbf{E}_{X_{t-}}\Big[\frac{1}{\varepsilon}\int_0^{\varepsilon} f(X_u)\mathrm{d}u\Big].$$

Hence letting $\varepsilon\to 0$ we deduce $\mathfrak{g}\cdot\mathbf{E}_a\left[f(X_t)\mid\mathcal{F}_{t-}\right] \overset{\mathfrak{g}\cdot\mathbf{P}_a\text{-a.s.}}{=} f(X_{t-})$. Since f is arbitrary, this is also true for f^2 so we deduce

$$\begin{aligned}\mathfrak{g}\cdot\mathbf{E}_a\left(f(X_t)-f(X_{t-})\right)^2 &= \mathfrak{g}\cdot\mathbf{E}_a\left[\mathfrak{g}\cdot\mathbf{E}_a\left[f^2(X_t)\mid\mathcal{F}_{t-}\right]-f^2(X_{t-})\right]\\ &\quad -2\mathfrak{g}\cdot\mathbf{E}_a\left[f(X_{t-})\left(\mathfrak{g}\cdot\mathbf{E}_a\left[f(X_t)\mid\mathcal{F}_{t-}\right]-f(X_{t-})\right)\right]=0.\end{aligned}$$

Since f is arbitrary, taking a dense sequence of $\mathrm{C}(S^\Delta)$, we get $\mathfrak{g}\cdot\mathbf{P}_a(X_{t-}=X_t)=1$. Finally, for any $t\in\mathbb{R}_+$ and $f\in\mathrm{C}(S^\Delta)$, since $x\mapsto f(x_t)$ is continuous for the global Skorokhod topology on $\{X_{t-}=X_t\}$, the function

$$\begin{array}{ccc} S^\Delta & \to & \mathbb{R}\\ a & \mapsto & \mathfrak{g}\cdot\mathbf{E}_a f(X_t)\end{array}$$

is continuous, so $(\mathfrak{g}\cdot\mathbf{P}_a)_a$ is a Feller family.
2⇒3. Let L be the $(\mathrm{C}_0\times\mathrm{C}_0)$-generator of $(\mathfrak{g}\cdot\mathbf{P}_a)_a$, then, by Proposition 4.2, $\mathcal{M}(L)=\{\mathfrak{g}\cdot\mathbf{P}_\mu\}_{\mu\in\mathcal{P}(S^\Delta)}$ so by the first part of Proposition 3.4 and by (2.4),

$$\mathcal{M}\left(\frac{1}{\mathfrak{g}}L\right)=\{\mathbf{P}_\mu\}_{\mu\in\mathcal{P}(S^\Delta)}.$$

3⇒1. Thanks to 3 from Proposition 3.4, for the local Skorokhod topology,

$$\begin{array}{ccc} \{\mathbf{P}_a\}_{a\in S} & \to & S\\ \mathbf{P}_a & \mapsto & a\end{array}$$

is a continuous injective function defined on a compact set, so $a\mapsto\mathbf{P}_a$ is also continuous. Let τ be a $(\mathcal{F}_{t+})_t$-stopping time and a be in S. As in Remark 3.12 we denote

$$\mathbf{Q}_X \overset{\mathbf{P}_a\text{-a.s.}}{:=} \mathscr{L}_{\mathbf{P}_a}\left((X_{\tau+t})_{t\geq 0}\mid\mathcal{F}_{\tau+}\right).$$

By using Proposition 3.13, $\mathbf{Q}_X\in\mathcal{M}(L)$, $\mathbf{P}_a$-almost surely, so $\mathbf{Q}_X=\mathbf{P}_{X_\tau}$, $\mathbf{P}_a$-almost surely, hence $(\mathbf{P}_a)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov. The quasi-continuity is a consequence of Proposition 3.6
2⇒4. Take an open subset $U\Subset S$ and define for all $a\in S$

$$\widetilde{\mathbf{P}}_a:=\mathfrak{h}\cdot\mathbf{P}_a\quad\text{where}\quad\mathfrak{h}:=\frac{\mathfrak{g}\wedge\min_U\mathfrak{g}}{\min_U\mathfrak{g}}.$$

By Proposition 4.4, $(\widetilde{\mathbf{P}}_a)_a$ is Feller, and moreover, since $X^{\tau^U}=(\mathfrak{h}\cdot X)^{\tau^U}$,

$$\forall a\in S,\quad \mathscr{L}_{\mathbf{P}_a}\left(X^{\tau^U}\right)=\mathscr{L}_{\widetilde{\mathbf{P}}_a}\left(X^{\tau^U}\right).$$

4⇒1. Let $U_n \Subset S$ be an increasing sequence of open subsets such that $S = \bigcup_n U_n$. For each $n \in \mathbb{N}$ there exists a Feller family $(\mathbf{P}^n_a)_a$ such that

$$\forall a \in S, \quad \mathscr{L}_{\mathbf{P}_a}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}^n_a}\left(X^{\tau^{U_n}}\right).$$

Denote $\mathbf{P}^\infty_a := \mathbf{P}_a$, then thanks to LemmaA.1 stated in Appendix A.2 the mapping

$$\begin{array}{ccc} \big(\mathbb{N} \cup \{\infty\}\big) \times S^\Delta & \to & \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S)) \\ (n,a) & \mapsto & \mathbf{P}^n_a \end{array}$$

is continuous. We can conclude that $(\mathbf{P}^\infty_a)_a$ is a Markov family by using:

Lemma 4.8 (Continuity and Markov property)**.** *Let*

$$\begin{array}{ccc} \big(\mathbb{N} \cup \{\infty\}\big) \times S^\Delta & \to & \mathcal{P}(\mathbb{D}_{loc}(S)) \\ (n,a) & \mapsto & \mathbf{P}^n_a \end{array}$$

be a weakly continuous mapping for the local Skorokhod topology such that $(\mathbf{P}^n_a)_a$ is a Markov family for each $n \in \mathbb{N}$. Then $(\mathbf{P}^\infty_a)_a$ is a Markov family.

The proof of this lemma is postponed one more time to Appendix A.2. The proof of Theorem 4.5 is now complete. □

Since a locally Feller family on S^Δ is also Feller we can deduce from Theorem 4.5 a characterisation of Feller families in terms of martingale problem. The following theorem is the converse of Proposition 4.2 and provide a first correction of Theorem 2.5, p. 283 in [5] (see also [18], p. 2 and [17], p. 3603).

Theorem 4.9 (Feller families – First characterisation)**.** *Let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be, the following assertions are equivalent:*

1. *$(\mathbf{P}_a)_a$ is Feller;*
2. *the family $(\mathbf{P}_a)_a$ is Markov, $\mathbf{P}_a \in \mathcal{P}(\mathbb{D}(S^\Delta))$ for any $a \in S$, and $S^\Delta \ni a \mapsto \mathbf{P}_a$ is continuous for the global Skorokhod topology;*
3. *there exists $L \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ such that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$ and*

$$\forall a \in S^\Delta, \quad \mathbf{P} \in \mathcal{M}_{\mathrm{c}}(L^\Delta) \text{ and } \mathbf{P}(X_0 = a) = 1 \Longleftrightarrow \mathbf{P} = \mathbf{P}_a.$$

We recall that $\mathbf{P}_\Delta$ is defined by $\mathbf{P}_\Delta(\forall t \geq 0,\ X_t = \Delta) = 1$.

Proof. Thanks to Proposition 3.6 a Feller family in $\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ continues to be Feller also in $\mathcal{P}(\mathbb{D}(S^\Delta))$, so a family $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is Feller if and only if the family $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}(S^\Delta))^{S^\Delta}$ is Feller. Since S^Δ is compact, using the third point of Remark 4.6, this is also equivalent to say that $(\mathbf{P}_a)_{a \in S^\Delta}$ is locally Feller in S^Δ. Hence the theorem is a consequence of Theorem 4.5 applied on the space S^Δ and to Proposition 4.2. □

The following theorem provides a new relationship between the local Feller property and the Feller property. With the help of Theorem 4.5 we obtain another correction of Theorem 2.5 p. 283 from [5] by adding the missing condition (4.3) (see again [18], p. 2 and [17], p. 3603).

Theorem 4.10 (Feller families – Second characterisation)**.** *Let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be, the following assertions are equivalent:*

1. *$(\mathbf{P}_a)_a$ is Feller;*

2. *$(\mathbf{P}_a)_a$ is locally Feller and*
$$\forall t \geq 0,\ \forall K \subset S \text{ compact set},\quad \mathbf{P}_a(X_t \in K) \underset{a\to\Delta}{\longrightarrow} 0; \tag{4.3}$$

3. *$(\mathbf{P}_a)_a$ is locally Feller and*
$$\forall t \geq 0,\ \forall K \subset S \text{ compact set},\quad \mathbf{P}_a\big(\tau^{S\backslash K} < t \wedge \xi\big) \underset{a\to\Delta}{\longrightarrow} 0.$$

Proof. 1⇒2. Take a compact $K \subset S$ and $t \geq 0$. There exists $f \in \mathrm{C}_0(S)$ such that $f \geq \mathbb{1}_K$. Since the family is Feller,
$$\mathbf{P}_a(X_t \in K) \leq \mathbf{E}_a[f(X_t)] \underset{a\to\Delta}{\longrightarrow} 0.$$

2⇒3. Take an open subset $U \Subset S$ such that $K \subset U$ and define
$$\tau := \inf\Big\{s \geq 0 \,\Big|\, \{(X_0, X_u)\}_{0\leq u\leq s} \not\Subset U^2 \cup \big((S\backslash K)\times(S\backslash K)\big)\Big\}.$$

By the third assertion of Theorem 4.5, and applying Lemma 3.8 to $\mathcal{K} := K$, $\mathcal{U} := U^2 \cup \big((S\backslash K)\times(S\backslash K)\big)$, $\tau_1 := 0$ and $\tau_2 := \frac{t}{N}$, we get the existence of $N \in \mathbb{N}$ such that
$$\sup_{b\in K} \mathbf{P}_b\Big(\tau \leq \frac{t}{N}\Big) < 1.$$

By Theorem 4.5, $\mathbf{P}_a$ is quasi-continuous for any $a \in S$, so $\mathbf{P}_a(X_{\tau^{S\backslash K}} \in K \cup \{\Delta\}) = 1$. Denoting $\lceil r \rceil$ the smallest integer larger or equal than the real number r, we have
$$\begin{aligned}\mathbf{P}_a\Big(\exists k \in \mathbb{N},\ k \leq N,\ X_{ktN^{-1}} \in U\Big) &\geq \mathbf{P}_a\Big(\tau^{S\backslash K} < t\wedge\xi,\ X_{tN^{-1}\lceil t^{-1}N\tau^{S\backslash K}\rceil} \in U\Big)\\ &= \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^{S\backslash K}<t\wedge\xi\}}\mathbf{E}_{X_{\tau^{S\backslash K}}}[X_s \in U]_{|s=tN^{-1}\lceil t^{-1}N\tau^{S\backslash K}\rceil-\tau^{S\backslash K}}\Big]\\ &\geq \mathbf{P}_a\Big(\tau^{S\backslash K} < t\wedge\xi\Big)\Big[1 - \sup_{b\in K}\mathbf{P}\big(\tau \leq tN^{-1}\big)\Big],\end{aligned}$$

so
$$\mathbf{P}_a\big(\tau^{S\backslash K} < t\wedge\xi\big) \leq \frac{\sum_{k=0}^N \mathbf{P}_a\big(X_{ktN^{-1}} \in U\big)}{1 - \sup_{b\in K}\mathbf{P}_b\big(\tau \leq tN^{-1}\big)} \longrightarrow 0, \quad \text{as } a \to \Delta.$$

3⇒1. Consider $f \in \mathrm{C}_0(S)$, $t \geq 0$ and $\varepsilon > 0$. There exists a compact subset $K \subset S$ such that $\|f\|_{K^c} \leq \varepsilon$, and an open subset $U \Subset S$ such that $K \subset U$ and

$$\sup_{a \notin U} \mathbf{P}_a(\tau^{S \setminus K} < t \wedge \xi) \leq \varepsilon.$$

Employing the second assertion of Theorem 4.5 we see that there exists $\mathfrak{g}_1 \in \mathrm{C}(S; \mathbb{R}_+^*)$ such that $(\mathfrak{g}_1 \cdot \mathbf{P}_a)_a$ is Feller. Since $U \Subset S$, there exists $\mathfrak{g}_2 \in \mathrm{C}_b(S, \mathbb{R}_+^*)$ such that $\mathfrak{g} := \mathfrak{g}_1 \mathfrak{g}_2$ satisfies $\mathfrak{g} \in \mathrm{C}(S, (0,1])$ and $\mathfrak{g}(a) = 1$, for $a \in U$. Applying Proposition 4.4 to $\mathfrak{g}_2$ we obtain that $(\mathfrak{g} \cdot \mathbf{P}_a)_a$ is Feller. Then for any $a \in S$

$$\begin{aligned}
\big|\mathbf{E}_a[f(X_t)] - \mathbf{E}_a[f((\mathfrak{g} \cdot X)_t)]\big| &\leq \mathbf{E}_a\big[\,|f(X_t) - f((\mathfrak{g} \cdot X)_t)|\,\mathbb{1}_{\{\tau^U < t\}}\big] \\
&\leq \mathbf{E}_a\big[\,|f(X_t)|\,\mathbb{1}_{\{\tau^U < t\}}\big] + \mathbf{E}_a\big[\,|f((\mathfrak{g} \cdot X)_t)|\,\mathbb{1}_{\{\tau^U < t\}}\big].
\end{aligned}$$

By Theorem 4.5, $\mathbf{P}_a$ is quasi-continuous, so $\mathbf{P}_a(X_{\tau^U} \notin U) = 1$, we have

$$\begin{aligned}
\mathbf{E}_a\Big[|f(X_t)|\mathbb{1}_{\{\tau^U < t\}}\Big] &= \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^U < t\}} \mathbf{E}_{X_{\tau^U}}\big[|f(X_s)|\big]_{|s = t - \tau^U}\Big] \\
&= \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^U < t\}} \mathbf{E}_{X_{\tau^U}}\big[|f(X_s)|\mathbb{1}_{\{\tau^{S\setminus K} < t \wedge \xi\}}\big]_{|s = t - \tau^U}\Big] \\
&\quad + \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^U < t\}} \mathbf{E}_{X_{\tau^U}}\big[|f(X_s)|\mathbb{1}_{\{\tau^{S\setminus K} \geq t \wedge \xi\}}\big]_{|s = t - \tau^U}\Big] \\
&\leq \|f\| \sup_{a \notin U} \mathbf{P}_a(\tau^{S\setminus K} < t \wedge \xi) + \|f\|_{K^c} \leq (\|f\| + 1)\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}_a\Big[\,|f(\mathfrak{g} \cdot X_t)|\,\mathbb{1}_{\{\tau^U < t\}}\Big] &= \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^U < t\}} \mathbf{E}_{X_{\tau^U}}\big[|f(\mathfrak{g} \cdot X_s)|\big]_{|s = t - \tau^U}\Big] \\
&= \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^U < t\}} \mathbf{E}_{X_{\tau^U}}\big[|f(\mathfrak{g} \cdot X_s)|\mathbb{1}_{\{\tau^{S\setminus K} < t \wedge \xi\}}\big]_{|s = t - \tau^U}\Big] \\
&\quad + \mathbf{E}_a\Big[\mathbb{1}_{\{\tau^U < t\}} \mathbf{E}_{X_{\tau^U}}\big[|f(\mathfrak{g} \cdot X_s)|\mathbb{1}_{\{\tau^{S\setminus K} \geq t \wedge \xi\}}\big]_{|s = t - \tau^U}\Big] \\
&\leq \|f\| \sup_{a \notin U} \mathbf{P}_a(\tau^{S\setminus K} < t \wedge \xi) + \|f\|_{K^c} \leq (\|f\| + 1)\varepsilon.
\end{aligned}$$

Hence

$$\big|\mathbf{E}_a[f(X_t)] - \mathbf{E}_a[f((\mathfrak{g} \cdot X)_t)]\big| \leq 2(\|f\| + 1)\varepsilon,$$

so, since $a \mapsto \mathbf{E}_a[f((\mathfrak{g} \cdot X)_t)]$ is in $\mathrm{C}_0(S)$, letting $\varepsilon \to 0$ we deduce that $a \mapsto \mathbf{E}_a[f(X_t)]$ is in $\mathrm{C}_0(S)$, hence $(\mathbf{P}_a)_a$ is Feller. □

Remark 4.11. There exist processes which are locally Feller, but not Feller. We recall here two examples, the first provided by [16], p. 157 (see also [3], p. 52 or [17], p. 3603) and the second by [21], p. 1379 (see also [18], p. 3). A third example is given in Remark 4.12. The first example is the (deterministic) process

$$x_t = \mathrm{sgn}(x_0)(2\,t + x_0^{-2})^{-1/2}, \quad t \geq 0$$

which is the unique solution of the ODE

$$\dot{x}_t = -x_t^3,\ t > 0, \qquad \text{starting from } x_0.$$

This process is locally Feller as the unique solution of the martingale problem, but the associated semi-group does not satisfy the Feller property, since $\lim_{|x_0|\to\infty} \mathbf{E}_{x_0}[f(X_t)] = f(1/\sqrt{2t}) \neq 0$, for f a suitable continuous positive function vanishing at infinity. The second example is the strong solution of the stochastic integral equation

$$X_t = x_0 - \int_0^t X_{s-} dN_s,\ t \geq 0, \text{ where } N \text{ is a standard Poisson process.}$$

Again this process is locally Feller as the unique solution of the martingale problem, but the associated semi-group does not satisfy the Feller property. Indeed, it can be shown that $\lim_{|x_0|\to\infty} \mathbf{E}_{x_0}[f(X_t)] \neq 0$, for f a suitable continuous positive function vanishing at infinity (see [21], p. 1379 for details). $\diamondsuit$

Remark 4.12. One can ask what is the connection between locally Feller family (process) and a Markov family of probabilities whose associated semi-group maps $\mathrm{C_b}(S)$, the set of bounded continuous functions on S, into $\mathrm{C_b}(\mathrm{S})$? We will call this kind of family $\mathrm{C_b}(S)$-Feller. Here is an example of family $\mathrm{C}_b(S)$-Feller which is not locally Feller. Define a Markov family on $\mathbb{R}$ as follows: let e_1 and e_2 be two independent exponential random variables with expectation 1, and define, for $t \geq 0$:

$$X_t := \begin{cases} X_0 & \text{if } X_0 \in \{-1,0,1\}, \\ X_0 \mathbb{1}_{t<\mathsf{e}_1/(|X_0|^{-1}-1)} + X_0^{-1}\mathbb{1}_{0\leq t-\mathsf{e}_1/(|X_0|^{-1}-1)<\mathsf{e}_2/(|X_0|^{-1}-1)} & \text{if } 0 < |X_0| < 1, \\ X_0 \mathbb{1}_{t<\mathsf{e}_1/(|X_0|-1)} & \text{if } 1 < |X_0|. \end{cases}$$

This process jumps to X_t^{-1} with intensity $|X_t|^{-1} - 1$, provided $0 < |X_t| < 1$, and jumps to 0 with intensity $|X_t| - 1$, provided $1 < |X_t|$. We can see that its semi-group is given by:

$$T_t f(x) = \begin{cases} f(x) & \text{if } x \in \{-1,0,1\}, \\ f(x)\mathrm{e}^{-(|x|^{-1}-1)t} + f(x^{-1})(1-\mathrm{e}^{-(|x|^{-1}-1)t})\mathrm{e}^{-(|x|^{-1}-1)t} \\ \qquad\qquad + f(0)(1-\mathrm{e}^{-(|x|^{-1}-1)t})^2 & \text{if } 0 < |x| < 1, \\ f(x)\mathrm{e}^{-(|x|-1)t} + f(0)(1-\mathrm{e}^{-(|x|-1)t}) & \text{if } 1 < |x|. \end{cases}$$

Since T_t maps $\mathrm{C_b}(\mathbb{R})$ to $\mathrm{C_b}(\mathbb{R})$, the family is $\mathrm{C_b}(\mathbb{R})$. But the Feller family is not tight in the neighbourhood of $X_0 = 0$, so the process is not locally Feller.

Finally, we recall an example already given in [10], p.1184, of a locally Feller process which is not a Feller process. Consider the ODE

$$\dot{x}_t = (1-t)x_t^2, \quad t > 0, \quad x_0 \in \mathbb{R}.$$

For any initial condition x_0, the unique maximal solution is the deterministic process

$$x_t = \left(\frac{t^2}{2} - t + \frac{1}{x_0}\right)^{-1} \quad \text{before} \quad t_{\max} = \begin{cases} \infty, & \text{if } x_0 \in [0,2), \\ 1 - \sqrt{1 - 2/x_0}, & \text{if } x_0 \geq 2, \\ 1 + \sqrt{1 - 2/x_0}, & \text{if } x_0 < 0, \end{cases}$$

and $x_t := \Delta$, after $t_{\max}$. This trajectory is not continuous with respect to the initial condition in the neighbourhood of $x_0 = 2$, hence the process is not $\mathrm{C_b}(\mathbb{R})$-Feller. Clearly, the process is not $\mathrm{C}_0(\mathbb{R})$-Feller since it explode in finite time. ♢

4.3 Generator description and convergence

In this subsection we analyse the generator of a locally Feller family:

Definition 4.13. Let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ be a locally Feller family. The $(\mathrm{C}_0 \times \mathrm{C})$-generator L of $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is the set of functions $(f, g) \in \mathrm{C}_0(S) \times \mathrm{C}(S)$ such that for any $a \in S$ and any open subset $U \Subset S$

$$f(X_{t \wedge \tau^U}) - \int_0^{t \wedge \tau^U} g(X_s) \mathrm{d}s \text{ is a } \mathbf{P}_a\text{-martingale.}$$

We provide in Proposition 4.16 that, for Feller families, the $(\mathrm{C}_0 \times \mathrm{C})$-generator is the extension of the $(\mathrm{C}_0 \times \mathrm{C}_0)$-generator. Some authors call it the "extended generator". In the following we will always recall the space of which the graph of operator is a subset.

Theorem 4.14 (Generator's description)**.** *Let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be a locally Feller family and L its $(\mathrm{C}_0 \times \mathrm{C})$-generator. Then $\mathrm{D}(L)$ is dense, L is a univariate closed sub-vector space,*

$$\mathcal{M}(L) = \{\mathbf{P}_\mu\}_{\mu \in \mathcal{P}(S^\Delta)},$$

L satisfies the positive maximum principle and does not have a strict linear extension satisfying the positive maximum principle. Moreover for any $(f, g) \in \mathrm{C}_0(S) \times \mathrm{C}(S)$ we have equivalence between:

1. *$(f, g) \in L$;*
2. *for all $a \in S$, there exists an open set $U \subset S$ containing a such that*

$$\lim_{t \to 0} \frac{1}{t}\Big(\mathbf{E}_a \left[f(X_{t \wedge \tau^U})\right] - f(a)\Big) = g(a);$$

3. *for all open subset $U \Subset S$ and $a \in U$*

$$\lim_{t \to 0} \frac{1}{t}\Big(\mathbf{E}_a \left[f(X_{t \wedge \tau^U})\right] - f(a)\Big) = g(a).$$

Proof. Let us denote by $\widehat{L}_2$ the set of $(f,g) \in \mathrm{C}_0(S) \times \mathrm{C}(S)$ satisfying the statement 2 and $\widehat{L}_3$ the set of $(f,g) \in \mathrm{C}_0(S) \times \mathrm{C}(S)$ satisfying the statement 3.
Thanks to the third assertion of Theorem 4.5 and Proposition 3.14, we have $\mathcal{M}(L) = \{\mathbf{P}_\nu\}_{\nu \in \mathcal{P}(S^\Delta)}$ and $\mathrm{D}(L)$ is dense. By the point 2 of Proposition 3.4, L is a closed linear subspace. The fourth part of Remark 3.3 allows us to conclude that L is univariate, L satisfies the positive maximum principle, and $L \subset \widehat{L}_3$.
It is straightforward that $\widehat{L}_3 \subset \widehat{L}_2$. Thanks to Theorem 3.10, L does not have strict linear extension satisfying the positive maximum principle. We already proved that $L \subset \widehat{L}_3 \subset \widehat{L}_2$, and it can be verified, by using its definition, that $\widehat{L}_2$ satisfies the positive maximum principle. Hence $\widehat{L}_2 = L = \widehat{L}_3$. □

Remark 4.15. One can ask, as in Remark 4.6, how can we obtain the generator of a locally Feller family? A similar statement of first one in the cited remark is Proposition 4.16. The second one is straightforward: if $\mathfrak{g} \in \mathrm{C}(S, \mathbb{R}_+^*)$ and if L is the $(\mathrm{C}_0 \times \mathrm{C})$-generator of $(\mathbf{P}_a)_a$, then $\mathfrak{g}L$ is the $(\mathrm{C}_0 \times \mathrm{C})$-generator of $(\mathfrak{g} \cdot \mathbf{P}_a)_a$, as we can see by using 1 from Proposition 3.4. ♢

Proposition 4.16. *Let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be a Feller family, L_0 its $(\mathrm{C}_0 \times \mathrm{C}_0)$-generator and L its $(\mathrm{C}_0 \times \mathrm{C})$-generator. Then taking the closure in $\mathrm{C}_0(S) \times \mathrm{C}(S)$*

$$L_0 = L \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_0(S)\big), \quad \textit{and} \quad L = \overline{L_0}.$$

Proof. Firstly, we have $L_0 \subset L \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_0(S)\big)$ by Proposition 4.2. Hence $L \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_0(S)\big)$ is an extension of L_0 satisfying the positive maximum principle, so by a maximality result (a consequence of Hille–Yoshida's theorem, see for instance Lemma 19.12, p. 377 in [15]), $L_0 = L \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_0(S)\big)$.

Secondly, take $(f,g) \in L$. Let $\mathfrak{h} \in \mathrm{C}(S, \mathbb{R}_+^*)$ be a bounded function such that $\mathfrak{h}g \in \mathrm{C}_0(S)$. Thanks to Proposition 4.4 the $(\mathrm{C}_0 \times \mathrm{C}_0)$-generator of $(\mathfrak{h} \cdot \mathbf{P}_a)_a$ is $\overline{\mathfrak{h}L_0}^{\mathrm{C}_0(S)\times\mathrm{C}_0(S)}$. Moreover the $(\mathrm{C}_0 \times \mathrm{C})$-generator of $(\mathfrak{h} \cdot \mathbf{P}_a)_a$ is $\mathfrak{h}L$. Hence applying the first step to the family $(\mathfrak{h} \cdot \mathbf{P}_a)_a$ we deduce that

$$\overline{\mathfrak{h}L_0}^{\mathrm{C}_0(S)\times\mathrm{C}_0(S)} = (\mathfrak{h}L) \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_0(S)\big),$$

so $(f, \mathfrak{h}g) \in \overline{hL_0}^{\mathrm{C}_0(S)\times\mathrm{C}_0(S)}$ and $(f,g) \in \overline{L_0}^{\mathrm{C}_0(S)\times\mathrm{C}(S)}$. □

Theorem 4.17 (Convergence of locally Feller family)**.** *For $n \in \mathbb{N} \cup \{\infty\}$, let $(\mathbf{P}^n_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be a locally Feller family and let L_n be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$. Suppose that for any $n \in \mathbb{N}$, $\overline{L_n}$ is the generator of $(\mathbf{P}^n_a)_a$, suppose also that $\mathrm{D}(L_\infty)$ is dense in $\mathrm{C}_0(S)$ and*

$$\mathcal{M}(L_\infty) = \{\mathbf{P}^\infty_\mu\}_{\mu \in \mathcal{P}(S^\Delta)}.$$

Then we have equivalence between:

1. *the mapping*
$$\begin{array}{ccc} \big(\mathbb{N} \cup \{\infty\}\big) \times \mathcal{P}(S^\Delta) & \to & \mathcal{P}\left(\mathbb{D}_{loc}(S)\right) \\ (n, \mu) & \mapsto & \mathbf{P}^n_\mu \end{array}$$
is weakly continuous for the local Skorokhod topology;

2. *for any $a_n, a \in S$ such that $a_n \to a$, $\mathbf{P}^n_{a_n}$ converges weakly for the local Skorokhod topology to $\mathbf{P}^\infty_a$, as $n \to \infty$;*

3. *for any $(f, g) \in L_\infty$, there exist $(f_n, g_n) \in L_n$ such that $f_n \xrightarrow[n\to\infty]{\mathrm{C}_0} f$, $g_n \xrightarrow[n\to\infty]{\mathrm{C}} g$.*

Remark 4.18. 1) For Feller processes a convergence theorem of same type could be deduced by using the previous result and some argument to get tightness for global Skorokhod topology from tightness for local Skorokhod topology (see also Remark 2.12, p. 1191 in [10]).
2) An improvement with respect to the classical result of convergence Theorem 19.25, p. 385, in [15], is that one does not need to know that $\overline{L_\infty}$ is the generator of the family, but only the fact that the martingale local problem is well-posed. Let us point out that it is, in general, not known that $\overline{L_\infty}$ is a generator. $\Diamond$

Proof of Theorem 4.17. It is straightforward that 1⇒2. The implication 3⇒1 is a consequence of Proposition 3.5.
We prove that 2⇒3. We can suppose that L_∞ is the generator of $(\mathbf{P}^\infty_a)_a$. It is straightforward to obtain that

$$\begin{array}{ccc} \big(\mathbb{N} \cup \{\infty\}\big) \times S^\Delta & \to & \mathcal{P}\left(\mathbb{D}_{\mathrm{loc}}(S)\right) \\ (n, a) & \mapsto & \mathbf{P}^n_a \end{array}$$

is weakly continuous for the local Skorokhod topology. Thanks to Proposition 2.1, on the connection between $\mathbb{D}_{\mathrm{loc}}(S)$ and $\mathbb{D}(S^\Delta)$, there exists $\mathfrak{h} \in \mathrm{C}(S, \mathbb{R}^*_+)$ such that, for any $n \in \mathbb{N} \cup \{\infty\}$ and $a \in S$,

$$\mathfrak{h} \cdot \mathbf{P}^n_a\big(\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)\big) = 1,$$

and the mapping

$$\begin{array}{ccc} \big(\mathbb{N} \cup \{\infty\}\big) \times S^\Delta & \to & \mathcal{P}\left(\mathbb{D}(S^\Delta)\right) \\ (n, a) & \mapsto & \mathfrak{h} \cdot \mathbf{P}^n_a \end{array}$$

is weakly continuous for the global Skorokhod topology. Thanks to Theorem 4.9, $(\mathbf{P}^n_a)_a$ is a Feller family, for all $n \in \mathbb{N} \cup \{\infty\}$. From Remark 4.15 and Proposition 4.16 we deduce that: $\mathfrak{h}\overline{L_n} \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_0(S)\big)$ is the $(\mathrm{C}_0 \times \mathrm{C}_0)$-generator of $(\mathbf{P}^n_a)_a$ for $n \in \mathbb{N}$, $hL_\infty \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_0(S)\big)$ is the $(\mathrm{C}_0 \times \mathrm{C}_0)$-generator of $(\mathbf{P}^\infty_a)_a$ and

$$\overline{\mathfrak{h}L_\infty \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_0(S)\big)}^{\mathrm{C}_0(S)\times\mathrm{C}(S)} = \mathfrak{h}L_\infty.$$

Take arbitrary elements $a, a_1, a_2 \ldots \in S^\Delta$ and $t, t_1, t_2 \ldots \in \mathbb{R}_+$ such that $a_n \to a$ and $t_n \to t$, then $\mathfrak{h} \cdot \mathbf{P}^n_{a_n}$ converges weakly for the global Skorokhod topology to $\mathfrak{h} \cdot \mathbf{P}^\infty_a$. By Theorem 4.5, $\mathfrak{h} \cdot \mathbf{P}^\infty_a$ is quasi-continuous, so $\mathfrak{h} \cdot \mathbf{P}^\infty_a(X_{t-} = X_t) = 1$. Hence, for any $f \in \mathrm{C}_0(S)$

$$\mathfrak{h} \cdot \mathbf{E}^n_{a_n}[f(X_{t_n})] \underset{n\to\infty}{\longrightarrow} \mathfrak{h} \cdot \mathbf{E}^\infty_a[f(X_t)].$$

From here we can deduce that, for any $t \geq 0$

$$\lim_{n\to\infty} \sup_{s\leq t} \sup_{a\in S} \left|\mathfrak{h} \cdot \mathbf{E}^n_a[f(X_s)] - \mathfrak{h} \cdot \mathbf{E}^\infty_a[f(X_s)]\right| = 0.$$

Here and elsewhere we denote by $\mathbf{E}_a^n$ the expectation with respect to the probability measure $\mathbf{P}_a^n$. Hence by Trotter–Kato's theorem (cf. Theorem 19.25, p. 385, [15]), for any $(f,g) \in \mathfrak{h}L_\infty \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_0(S)\big)$ there exist $(f_n,g_n) \in \mathfrak{h}\overline{L_n} \cap \big(\mathrm{C}_0(S) \times \mathrm{C}_0(S)\big)$ such that $(f_n,g_n) \underset{n\to\infty}{\longrightarrow} (f,g)$, so it is straightforward to deduce statement 3. $\square$

Remark 4.19. We present here an application of Theorem 4.17. Let us denote by Y the discrete time canonical process on $(S^\Delta)^\mathbb{N}$ and we endow $(S^\Delta)^\mathbb{N}$ with the canonical σ-algebra. A family $(\mathbf{P}_a)_a \in \mathcal{P}\left((S^\Delta)^\mathbb{N}\right)^S$ is said to be a discrete time locally Feller family if there exists an operator $T : \mathrm{C}_0(S) \to \mathrm{C}_b(S)$, called transition operator, such that for any $a \in S$: $\mathbf{P}_a(Y_0 = a) = 1$ and

$$\forall n \in \mathbb{N},\ \forall f \in \mathrm{C}_0(S), \quad \mathbf{E}_a\left(f(Y_{n+1}) \mid Y_0,\dots,Y_n\right) = \mathbb{1}_{\{Y_n \neq \Delta\}} Tf(Y_n) \quad \mathbf{P}_a\text{-a.s.} \tag{4.4}$$

We set, for $\mu \in \mathcal{P}(S^\Delta)$, $\mathbf{P}_\mu := \int \mathbf{P}_a \mu(\mathrm{d}a)$, where $\mathbf{P}_\Delta$ the probability defined by $\mathbf{P}_\Delta(\forall n \in \mathbb{N},\ Y_n = \Delta) = 1$. The following result can be thought as an improvement of Theorem 19.28, p. 387 in [15]:

Theorem (Discrete-time approximation) *Let L be a subset of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ with $\mathrm{D}(L)$ a dense subset of $\mathrm{C}_0(S)$, such that the martingale local problem associated to L is well-posed, and let $(\mathbf{P}_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^S$ be the associated continuous time locally Feller family. For each $n \in \mathbb{N}$ we introduce $(\mathbf{P}_a^n)_a \in \mathcal{P}((S^\Delta)^\mathbb{N})^S$ a discrete time locally Feller family having the transition operator T_n. Denote by L_n the operator $(T_n - \mathrm{id})/\varepsilon_n$, where $(\varepsilon_n)_n$ is a sequence of positive constants converging to 0. There is equivalence between,*

a) for any $\mu_n, \mu \in \mathcal{P}(S^\Delta)$ such that $\mu_n \to \mu$ weakly, $\mathscr{L}_{\mathbf{P}_{\mu_n}^n}\left((Y_{\lfloor t/\varepsilon_n \rfloor})_t\right) \underset{n\to\infty}{\overset{\mathcal{P}(\mathbb{D}_{loc}(S))}{\longrightarrow}} \mathbf{P}_\mu$;

b) for any $a_n, a \in S$ such that $a_n \to a$, $\mathscr{L}_{\mathbf{P}_{a_n}^n}\left((Y_{\lfloor t/\varepsilon_n \rfloor})_t\right) \underset{n\to\infty}{\overset{\mathcal{P}(\mathbb{D}_{loc}(S))}{\longrightarrow}} \mathbf{P}_a$;

c) for any $f \in \mathrm{D}(L)$, there exists a sequence $(f_n)_n \in \mathrm{C}_0(S)^\mathbb{N}$ such that $f_n \underset{n\to\infty}{\overset{\mathrm{C}_0(S)}{\longrightarrow}} f$, $L_n f_n \underset{n\to\infty}{\overset{\mathrm{C}(S)}{\longrightarrow}} Lf$.

The detailed proof of this result is developed in §3 from [9] and it is based on the application of Theorem 4.17. Furthermore, this theorem is useful to deduce a characterisation of the convergence towards Lévy-type operators, and also a classical Donsker's type theorem which allows to simulate Lévy-type processes (see also [9]). $\Diamond$

4.4 Localisation for martingale problems and generators

We are interested to the localisation procedure. More precisely, assume that $\mathcal{U}$ is a covering of S by open sets and let $(\mathbf{P}_a^U)_{a\in S, U\in\mathcal{U}}$ be a doubly indexed probability family, such that: for each $U \in \mathcal{U}$, $(\mathbf{P}_a^U)_a$ is a locally Feller family, and, for all $U_1, U_2 \in \mathcal{U}$ and $a \in S$

$$\mathscr{L}_{\mathbf{P}_a^{U_1}}\left(X^{\tau^{U_1\cap U_2}}\right) = \mathscr{L}_{\mathbf{P}_a^{U_2}}\left(X^{\tau^{U_1\cap U_2}}\right).$$

We wonder if there exists a locally Feller family $(\mathbf{P}_a)_a$ such that for all $U \in \mathcal{U}$ and $a \in S$

$$\mathscr{L}_{\mathbf{P}_a}\big(X^{\tau^U}\big) = \mathscr{L}_{\mathbf{P}_a^U}\big(X^{\tau^U}\big) \ \ ?$$

An attempt to give an answer to this question needs to reformulate it in terms of generators of locally Feller families. This reformulation is suggested by the following:

Proposition 4.20. *Let $L_1, L_2 \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ be such that $\mathrm{D}(L_1) = \mathrm{D}(L_2)$ is dense in $\mathrm{C}_0(S)$ and take an open subset $U \subset S$. Suppose that*

- *the martingale local problem associated to L_1 is well-posed, and,*
- *for all $a \in U$ there exists $\mathbf{P}^2 \in \mathcal{M}(L_2)$ with $\mathbf{P}^2(X_0 = a) = 1$.*

Then

$$\forall \mathbf{P}^2 \in \mathcal{M}(L_2),\ \exists \mathbf{P}^1 \in \mathcal{M}(L_1), \qquad \mathscr{L}_{\mathbf{P}^2}\left(X^{\tau^U}\right) = \mathscr{L}_{\mathbf{P}^1}\left(X^{\tau^U}\right) \tag{4.5}$$

if and only if

$$\forall (f,g) \in L_2, \qquad g_{|U} = (L_1 f)_{|U}.$$

We postpone the proof of this proposition and we state two results of localisation.

Theorem 4.21 (Localisation for the martingale problem)**.** *Let L be a linear subspace of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ with $\mathrm{D}(L)$ dense in $\mathrm{C}_0(S)$. Suppose that for all $a \in S$ there exist a neighbourhood V of a and a subset $\widetilde{L}$ of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ such that the martingale local problem associated to $\widetilde{L}$ is well-posed and such that*

$$\big\{(f, g_{|V}) \ \big|\ (f,g) \in L\big\} = \left\{(f, g_{|V}) \ \middle|\ (f,g) \in \widetilde{L}\right\}. \tag{4.6}$$

Then the martingale local problem associated to L is well-posed.

Proof. Thanks to Theorem 3.10, to prove the existence of a solution for the martingale local problem it suffices to prove that L satisfies the positive maximum principle. Let $(f,g) \in L$ and $a \in S$ be such that $f(a) = \max f \geq 0$. Then there exist a neighbourhood V of a and a subset $\widetilde{L}$ of $\mathrm{C}_0(S) \times \mathrm{C}(S)$ such that the martingale local problem associated to $\widetilde{L}$ is well-posed and (4.6). In particular, by Theorem 3.10, $\widetilde{L}$ satisfies the positive maximum principle and so

$$g(a) = \widetilde{L} f(a) \leq 0.$$

To prove the uniqueness of the solution for the martingale local problem, we take $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{M}(L)$ and an arbitrary open subset $V \Subset S$. By hypothesis and using the relative compactness of V, there exist $N \in \mathbb{N}$, open subsets $U_1, \ldots, U_N \subset S$ and subsets $L_1, \ldots, L_N \subset \mathrm{C}_0(S) \times \mathrm{C}(S)$ such that $V \Subset \bigcup_n U_n$, such that for all $1 \leq n \leq N$ the martingale local problem associated to L_n is well-posed and such that

$$\big\{(f, g_{|U_n}) \ \big|\ (f,g) \in L\big\} = \left\{(f, g_{|U_n}) \ \middle|\ (f,g) \in \widetilde{L_n}\right\}.$$

At this level of the proof we need a technical but important result:

Lemma 4.22. *Let U be an open subset of S and L be a subset of $\mathrm{C}_0(S)\times\mathrm{C}(S)$ such that $\mathrm{D}(L)$ is dense in $\mathrm{C}(S)$ and the martingale local problem associated to L is well-posed. Then there exist a subset L_0 of L and a function $\mathfrak{h}_0$ of $\mathrm{C}(S,\mathbb{R}_+)$ with $\{\mathfrak{h}_0\neq 0\}=U$ such that $\overline{L}=\overline{L_0}$, such that $\mathfrak{h}_0L_0\subset\mathrm{C}_0(S)\times\mathrm{C}_0(S)$ and such that: for any $\mathfrak{h}\in\mathrm{C}(S,\mathbb{R}_+)$ with $\{\mathfrak{h}\neq 0\}=U$ and $\sup_{a\in U}(\mathfrak{h}/\mathfrak{h}_0)(a)<\infty$, the martingale problem associated to $(\mathfrak{h}L_0)^\Delta$ is well-posed in $\mathbb{D}(S^\Delta)$. Recall that $(\mathfrak{h}L_0)^\Delta$ is defined by* (4.1) *and that the associated martingale problem is defined by* (4.2).

We postpone the proof of this lemma to the Appendix (see §A.3) and we proceed with the proof of our theorem.

Applying Lemma 4.22, there exist a subset D of $\mathrm{C}_0(S)$ and a function $\mathfrak{h}$ of $\mathrm{C}(S,\mathbb{R}_+)$ with $\{\mathfrak{h}\neq 0\}=V$ such that for all $1\leq n\leq N$: $\overline{L_n}=\overline{L_{n\,|D}}$, $\mathfrak{h}L_{n\,|D}\subset\mathrm{C}_0(S)\times\mathrm{C}_0(S)$ and the martingale problem associated to $\left(\mathfrak{h}L_{n\,|D}\right)^\Delta$ is well-posed. Denote $L_{N+1}:=D\times\{0\}$ and $U^{N+1}:=S^\Delta\backslash\overline{V}$. We may now apply Theorem 6.2 and also Theorem 6.1 pp. 216-217, in [8] to $\mathfrak{h}L_{|D}$ and $(U_n)_{1\leq n\leq N+1}$ and we deduce that the martingale problem associated to $(\mathfrak{h}L_{|D})^\Delta$ is well-posed. Hence $\mathfrak{h}\cdot\mathbf{P}^1=\mathfrak{h}\cdot\mathbf{P}^2$ so

$$\mathscr{L}_{\mathbf{P}^1}(X^{\tau^V})=\mathscr{L}_{\mathbf{P}^2}(X^{\tau^V}).$$

We obtain the result by letting V to grow towards S. This ends the proof of the theorem except to the proof of Lemma 4.22 which is postponed to §A.3. $\square$

Theorem 4.23 (Localisation of generator)**.** *Let L be a linear subspace of $\mathrm{C}_0(S)\times\mathrm{C}(S)$ with $\mathrm{D}(L)$ dense in $\mathrm{C}_0(S)$. Suppose that for all subsets $V\Subset S$ there exists a linear subspace $\widetilde{L}$ of $\mathrm{C}_0(S)\times\mathrm{C}(S)$ such that $\overline{\widetilde{L}}$ is the generator of a locally Feller family and*

$$\left\{(f,g_{|V})\;\middle|\;(f,g)\in L\right\}=\left\{(f,g_{|V})\;\middle|\;(f,g)\in\widetilde{L}\right\}.$$

Then $\overline{L}$ is the generator of a locally Feller family.

Proof. Thanks to Theorem 4.21 the martingale local problem associated to L is well-posed, let $(\mathbf{P}^\infty_a)_a$ the locally Feller family associate to L. Let L_∞ be the generator of $(\mathbf{P}^\infty_a)_a$. Let $U_n\Subset S$ be an increasing sequence of open subsets such that $S=\bigcup_n U_n$ and let $L_n\subset\mathrm{C}_0(S)\times\mathrm{C}(S)$ be such that for all $n\in\mathbb{N}$, $\overline{L_n}$ is the generator of a locally Feller family $(\mathbf{P}^n_a)_a$ and

$$\left\{(f,g_{|U_n})\;\middle|\;(f,g)\in L\right\}=\left\{(f,g_{|U_n})\;\middle|\;(f,g)\in L_n\right\}. \tag{4.7}$$

Then by using Proposition 4.20, for all $n\in\mathbb{N}$ and $a\in S$

$$\mathscr{L}_{\mathbf{P}^\infty_a}\left(X^{\tau^{U_n}}\right)=\mathscr{L}_{\mathbf{P}^n_a}\left(X^{\tau^{U_n}}\right). \tag{4.8}$$

At this level we use a result of localisation of the continuity stated and proved in §A.2, Lemma A.1. Therefore, by (4.8) the mapping

$$\begin{array}{ccc}\left(\mathbb{N}\cup\{\infty\}\right)\times S^\Delta & \to & \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))\\ (n,a) & \mapsto & \mathbf{P}^n_a\end{array}$$

is weakly continuous for the local Skorokhod topology. Hence by Theorem 4.17, for any $f \in \mathrm{D}(L_\infty)$ there exists a sequence $(f_n)_n \in \mathrm{D}(L)^{\mathbb{N}}$ such that $(f_n, L_n f_n) \underset{n\to\infty}{\longrightarrow} (f, L_\infty f)$, so by (4.7) $(f_n, L f_n) \underset{n\to\infty}{\longrightarrow} (f, L_\infty f)$. Hence $\overline{L} = L_\infty$ is the generator of a locally Feller family. The proof of the theorem is complete except for the proof of Proposition 4.20. $\square$

Proof of Proposition 4.20. Suppose (4.5). For each $a \in U$, take an open subset $V \subset U$, $\mathbf{P}^1 \in \mathcal{M}(L_1)$ and $\mathbf{P}^2 \in \mathcal{M}(L_2)$ such that $a \in V \Subset S$ and $\mathbf{P}^1(X_0 = a) = \mathbf{P}^2(X_0 = a) = 1$. By using the fifth part of Remark 3.3 we have for each $(f, g) \in L_2$

$$g(a) = \lim_{t\to 0} \frac{1}{t}\Big(\mathbf{E}^2\left[f(X_{t\wedge\tau^V})\right] - f(a)\Big) = \lim_{t\to 0} \frac{1}{t}\Big(\mathbf{E}^1\left[f(X_{t\wedge\tau^V})\right] - f(a)\Big) = L_1 f(a).$$

For the converse, by Lemma 4.22 there exists $\mathfrak{h} \in \mathrm{C}(S, \mathbb{R}_+)$ with $\{\mathfrak{h} \neq 0\} = U$ such that the martingale local problem associated to $\mathfrak{h}L_1 = \mathfrak{h}L_2$ is well-posed. Take $\mathbf{P}^2 \in \mathcal{M}(L_2)$ and let $\mathbf{P}^1 \in \mathcal{M}(L_1)$ be such that $\mathscr{L}_{\mathbf{P}^1}(X_0) = \mathscr{L}_{\mathbf{P}^2}(X_0)$, then $\mathfrak{h}\cdot\mathbf{P}^1, \mathfrak{h}\cdot\mathbf{P}^2 \in \mathcal{M}(\mathfrak{h}L_1)$ so $\mathfrak{h}\cdot\mathbf{P}^1 = \mathfrak{h}\cdot\mathbf{P}^2$ and hence (4.5) is verified. $\square$

Remark 4.24. We present here an application of Theorem 4.23 by using symbols. We say that a function $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is bi-continuous and negative definite if $(a, \alpha) \mapsto q(a, \alpha)$ is continuous and, for each $a \in \mathbb{R}^d$, $\alpha \mapsto q(a, \alpha)$ is negative definite. Then, for $f \in \mathrm{C}_c^\infty(\mathbb{R}^d)$, the formula

$$-q(a, \nabla) f(a) := -\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}a\cdot\alpha} q(a, \alpha)\widehat{f}(\alpha)\mathrm{d}\alpha, \quad \text{where} \quad \widehat{f}(\alpha) := (2\pi)^{-d}\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}a\cdot\alpha} f(a)\mathrm{d}a.$$

defines a pseudo-differential operator $-q(\cdot, \nabla)$ which maps $\mathrm{C}_c^\infty(\mathbb{R}^d)$ into $\mathrm{C}(\mathbb{R}^\mathrm{d})$ and it satisfies the positive maximum principle. The following result can be thought as an improvement of Theorem 11.2.3, p. 272, in [23]:

Theorem (Well-posedness and localisation under ellipticity) *Let $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a bi-continuously negative definite function satisfying the following ellipticity condition:*

$$\forall a \in \mathbb{R}^d,\ \exists \beta, \eta > 0,\ \forall \alpha \in \mathbb{R}^d, \quad |q(a, \alpha)| \geq \beta|\alpha|^\eta.$$

Then the martingale local problem associated to $-q(\cdot, \nabla)$ is well-posed.

Let us sketch the proof of this result. Take $a_0 \in \mathbb{R}^d$ and $\varepsilon > 0$. Set $\psi(\alpha) := q(a_0, \alpha)$ and

$$q_\varepsilon(a, \alpha) := \begin{cases} q(a, \alpha), & \text{if } |a - a_0| \leq \varepsilon/2, \\ (2 - 2|a - a_0|/\varepsilon)q(a, \alpha) + (2|a - a_0|/\varepsilon - 1)\psi(\alpha), & \text{if } \varepsilon/2 \leq |a - a_0| \leq \varepsilon, \\ \psi(\alpha), & \text{if } \varepsilon \leq |a - a_0|. \end{cases}$$

Thanks to Theorem 4.23, to get the result it suffices to prove that, for ε small enough, the martingale local problem associated to $-q_\varepsilon(\cdot, \nabla)$ is well-posed. Clearly $-\psi(\nabla)$ is the generator of a positive semi-group on $\big(\mathrm{C}_0(\mathbb{R}^d) \cap \mathrm{L}^2(\mathbb{R}^d), \|\cdot\|_\infty + \|\cdot\|_2\big)$. We prove that $-q_\varepsilon(\cdot, \nabla)$ is a small perturbation of $-\psi(\nabla)$, or, more precisely we show that

$$(\star) \quad \|\psi(\nabla)f - q_\varepsilon(\cdot, \nabla)f\|_2 \leq (2\varepsilon)^d \|\psi(\nabla)f - q_\varepsilon(\cdot, \nabla)f\|_\infty$$

and

$$(\star\star)\quad \|\psi(\nabla)f - q_\varepsilon(\cdot,\nabla)f\|_\infty \le (2\pi)^{-d/2} C\omega(\varepsilon)\|(\psi(\nabla)^n+1)f\|_2,$$

where n is an appropriate integer, $C\in(0,\infty)$ is a constant depending on n,β,η, and

$$\omega(\varepsilon) := \sup_{a,\alpha\in\mathbb{R}^d,\ |a-a_0|\le\varepsilon} \frac{|q(a,\alpha)-q(a_0,\alpha)|}{1+|\alpha|^2}$$

decreases to 0, as $\varepsilon\to 0$. We deduce that $-q_\varepsilon(\cdot,\nabla)$ is the generator of a semi-group on $\mathrm{C}_0(\mathbb{R}^d)\cap\mathrm{L}^2(\mathbb{R}^d)$ hence in particular the generator of a Feller semi-group. The inequality $(\star)$ is a simple consequence of Hölder's inequality. To get $(\star\star)$ we can write,

$$\forall a\in\mathbb{R}^d,\ n\in\mathbb{N}^*,\ \ |\psi(\nabla)f(a)-q_\varepsilon(a,\nabla)f(a)| \le \Big\|\frac{\psi-q_\varepsilon(a,\cdot)}{\psi^n+1}\Big\|_2 \Big\|(\psi^n+1)\widehat{f}\Big\|_2.$$

Thanks to Plancherel theorem, $\|(\psi^n+1)\widehat{f}\|_2 = (2\pi)^{-d/2}\|(\psi(\nabla)^n+1)f\|_2$, and by the ellipticity hypothesis, since the real part of ψ is positive,

$$\Big|\frac{\psi(\alpha)-q_\varepsilon(a,\alpha)}{\psi(\alpha)^n+1}\Big| \le \frac{\omega(\varepsilon)(1+|\alpha|^2)}{1\vee(\beta^n|\alpha|^{n\eta}-1)}.$$

To get $(\star\star)$ we choose $n := \lfloor(4+d)/2\eta\rfloor+1$ and we set $C^2 := \int_{\mathbb{R}^d}\big(\frac{1+|\alpha|^2}{1\vee(\beta^n|\alpha|^{n\eta}-1)}\big)^2\mathrm{d}\alpha.$

$\Diamond$

A Appendix: proof of technical results

A.1 Proofs of Proposition 3.6 and Lemma 3.8

Proof of Lemma 3.8. Take a metric ρ on S and $a_0\in\mathcal{K}$, then there exists $\varepsilon_0>0$ such that $B(a_0,4\varepsilon_0)\Subset S$ and $\big\{(a,b)\in S^2\ \big|\ a\in\mathcal{K},\ \rho(a,b)<3\varepsilon_0\big\}\subset\mathcal{U}$. Define

$$\widetilde{f}(a) := \begin{cases} 1, & \text{if } \rho(a,a_0)\le\varepsilon_0,\\ 0, & \text{if } \rho(a,a_0)\ge 2\varepsilon_0,\\ 2-\frac{\rho(a,a_0)}{\varepsilon_0}, & \text{if } \varepsilon_0\le\rho(a,a_0)\le 2\varepsilon_0.\end{cases}$$

Then

$$\widetilde{f}\in\mathrm{C}_0(S),\quad 0\le\widetilde{f}\le 1,\quad \forall\, a\in B(a_0,\varepsilon_0),\ \widetilde{f}(a)=1\ \text{ and }\ \{\widetilde{f}\ne 0\}\subset B(a,3\varepsilon_0).$$

Take $\eta>0$ be arbitrary. There exist $(f,g)\in L$ and a sequence $(f_n,g_n)\in L_n$ such that $\|f-\widetilde{f}\|\le\eta$ and the sequence $(f_n,g_n)_n$ converges to (f,g) for the topology of $\mathrm{C}_0(S)\times\mathrm{C}(S)$. Consider $\tau_1\le\tau_2$, $(\mathcal{F}_{t+})_t$-stopping times and take $n\in\mathbb{N}$. Assume that $\mathbf{P}\in\mathcal{M}(L_n)$. For $\varepsilon<3\varepsilon_0$ we denote

$$\sigma_\varepsilon := \inf\Big\{t\ge\tau_1\ \Big|\ t\ge\xi \text{ or } \sup_{\tau_1\le s\le t}\rho(X_{\tau_1},X_s)\ge\varepsilon\Big\}.$$

If $V \Subset S$ is an open subset such that $V \supset B(a_0, 4\varepsilon_0)$, if $t \geq 0$ and $\varepsilon < 3\varepsilon_0$, we can write

$$\begin{aligned}
&\mathbf{E}\left[f_n(X_{t\wedge\tau^V\wedge\sigma_\varepsilon\wedge\tau_2})\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K}\}}\right] \\
&\qquad\qquad = \mathbf{E}\left[\Big(f_n(X_{t\wedge\tau^V\wedge\tau_1}) + \int_{t\wedge\tau^V\wedge\tau_1}^{t\wedge\tau^V\wedge\sigma_\varepsilon\wedge\tau_2} g_n(X_s)\mathrm{d}s\Big)\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K}\}}\right] \\
&\qquad\qquad \geq \mathbf{E}\left[\widetilde{f}(X_{t\wedge\tau^V\wedge\tau_1})\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K}\}}\right] - \|\widetilde{f} - f_n\| \\
&\qquad\qquad\quad + \mathbf{E}\left[\int_{t\wedge\tau^V\wedge\tau_1}^{t\wedge\tau^V\wedge\sigma_\varepsilon\wedge\tau_2} g_n(X_s)\mathrm{d}s\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K}\}}\right] \\
&\qquad\qquad \geq \mathbf{P}\big(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K}\big) - \mathbf{P}\big(t\wedge\tau^V < \tau_1 < \xi\big) - \eta - \|f - f_n\| \\
&\qquad\qquad\quad - \mathbf{E}\big[(\tau_2 - \tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}\big]\cdot\|g_n\|_{B(a_0,4\varepsilon_0)}.
\end{aligned} \tag{A.1}$$

Splitting on the events $\{\sigma_\varepsilon > \tau_2\}$, $\{\sigma_\varepsilon \leq t\wedge\tau^V\wedge\tau_2\}$ and $\{t\wedge\tau^V < \sigma_\varepsilon \leq \tau_2\}$

$$\begin{aligned}
&\mathbf{E}\left[f_n(X_{t\wedge\tau^V\wedge\sigma_\varepsilon\wedge\tau_2})\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K}\}}\right] \\
&\qquad \leq \mathbf{P}\big(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K},\ \sigma_\varepsilon > \tau_2\big) + \eta + \|f - f_n\| \\
&\qquad\quad + \mathbf{E}\big[f_n(X_{\sigma_\varepsilon})\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\}}\big] + \mathbf{P}\big(X_{\tau_1}\in\mathcal{K},\ t<\tau_2\big) + \eta + \|f - f_n\|.
\end{aligned} \tag{A.2}$$

Hence by (A.1) and (A.2),

$$\begin{aligned}
&\mathbf{P}\big(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K},\ \tau(\tau_1)\leq\tau_2\big) \leq \mathbf{P}\big(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K},\ \sigma_\varepsilon\leq\tau_2\big) \\
&\quad \leq 3\eta + 3\|f - f_n\| + \mathbf{P}\big(t\wedge\tau^V < \tau_1 < \xi\big) + \mathbf{E}\big[(\tau_2-\tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}\big]\cdot\|g_n\|_{B(a_0,4\varepsilon_0)} \\
&\qquad\qquad\qquad\qquad + \mathbf{E}\big[f_n(X_{\sigma_\varepsilon})\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\}}\big] + \mathbf{P}\big(X_{\tau_1}\in\mathcal{K},\ t<\tau_2\big).
\end{aligned}$$

Since the limit $\lim_{\varepsilon\uparrow 3\varepsilon_0} X_{\sigma_\varepsilon}$ exists and it belongs to $S^\Delta\backslash B(X_{\tau_1}, 3\varepsilon_0)$ we have

$$\begin{aligned}
&\limsup_{\varepsilon\uparrow 3\varepsilon_0}\mathbf{E}\big[f_n(X_{\sigma_\varepsilon})\mathbb{1}_{\{X_{\tau_1}\in B(a_0,\varepsilon_0)\}}\big] \leq \|f_n\|_{B(a_0,2\varepsilon_0)^c} \\
&\quad \leq \|f - f_n\| + \|f - \widetilde{f}\| + \|\widetilde{f}\|_{B(a_0,2\varepsilon_0)^c} \leq \|f - f_n\| + \delta,
\end{aligned}$$

so

$$\begin{aligned}
\mathbf{P}\big(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K},\ \tau(\tau_1)\leq\tau_2\big) &\leq 4\eta + 4\|f - f_n\| + \mathbf{P}\big(t\wedge\tau^V < \tau_1 < \xi\big) \\
&\quad + \mathbf{E}\big[(\tau_2-\tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}\big]\cdot\|g_n\|_{B(a_0,4\varepsilon_0)} + \mathbf{P}\big(X_{\tau_1}\in\mathcal{K},\ t<\tau_2\big).
\end{aligned}$$

Letting $t\to\infty$ and V growing to S, $\mathbf{P}\big(t\wedge\tau^V < \tau_1 < \xi\big)$ tends to 0, hence

$$\begin{aligned}
&\mathbf{P}\big(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K},\ \tau(\tau_1)\leq\tau_2\big) \\
&\quad \leq 4\eta + 4\|f - f_n\| + \mathbf{E}[(\tau_2-\tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}]\cdot\|g_n\|_{B(a_0,4\varepsilon_0)} + \mathbf{P}\big(X_{\tau_1}\in\mathcal{K},\ \tau_2=\infty\big).
\end{aligned}$$

So letting $n\to\infty$, $\mathbf{E}[(\tau_2-\tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}]\to 0$ and $\eta\to 0$, we deduce that for each $\varepsilon>0$ there exist $n_0\in\mathbb{N}$ and $\delta>0$ such that: for any $n\geq n_0$, $(\mathcal{F}_{t+})_t$-stopping times $\tau_1\leq\tau_2$ and $\mathbf{P}\in\mathcal{M}(L_n)$ satisfying $\mathbf{E}[(\tau_2-\tau_1)\mathbb{1}_{\{X_{\tau_1}\in\mathcal{K}\}}]\leq\delta$ we have

$$\mathbf{P}(X_{\tau_1}\in B(a_0,\varepsilon_0)\cap\mathcal{K},\ \tau(\tau_1)\leq\tau_2)\leq\varepsilon.$$

We conclude since a_0 was arbitrary chosen in $\mathcal{K}$ and by using a finite recovering of the compact $\mathcal{K}$. $\square$

Proof of Proposition3.6.

Step 1: we prove the $(\mathcal{F}_{t+})_t$-quasi-continuity before the explosion time ξ. Let τ_n, τ be $(\mathcal{F}_{t+})_t$-stopping times and denote $\widetilde{\tau}_n := \inf_{m \geq n} \tau_m$, $\widetilde{\tau} := \sup_{n \in \mathbb{N}} \widetilde{\tau}_n$ and

$$A := \begin{cases} \lim_{n\to\infty} X_{\widetilde{\tau}_n}, & \text{if the limit exists,} \\ \Delta, & \text{otherwise.} \end{cases}$$

Let d be the metric on S^Δ and take $\varepsilon > 0$, $t \geq 0$ and an open subset $U \Subset S$. Since

$$\lim_{n\to\infty} \mathbf{E}\big[\widetilde{\tau} \wedge t \wedge \tau^U - \widetilde{\tau}_n \wedge t \wedge \tau^U\big] = 0,$$

by Lemma 3.8 applied to $\mathcal{K} := \overline{U}$ and $\mathcal{U} = \big\{(a,b) \in S^2 \;\big|\; d(a,b) < \varepsilon\big\}$ we get

$$\mathbf{P}\big(X_{\widetilde{\tau}_n \wedge t \wedge \tau^U} \in U,\ d(X_{\widetilde{\tau}_n \wedge t \wedge \tau^U}, X_{\widetilde{\tau} \wedge t \wedge \tau^U}) \geq \varepsilon\big) \underset{n\to\infty}{\longrightarrow} 0.$$

Hence

$$\begin{aligned} \mathbf{P}\big(\widetilde{\tau} \leq t \wedge \tau^U,\ d(X_{\widetilde{\tau}_n}, X_{\widetilde{\tau}}) \geq \varepsilon\big) &= \mathbf{P}\big(\widetilde{\tau}_n < \widetilde{\tau} \leq t \wedge \tau^U,\ d(X_{\widetilde{\tau}_n}, X_{\widetilde{\tau}}) \geq \varepsilon\big) \\ &\leq \mathbf{P}\big(X_{\widetilde{\tau}_n \wedge t \wedge \tau^U} \in U,\ d(X_{\widetilde{\tau}_n \wedge t \wedge \tau^U}, X_{\widetilde{\tau} \wedge t \wedge \tau^U}) \geq \varepsilon\big). \end{aligned}$$

Letting $n \to \infty$ on the both sides of the latter inequality we obtain that

$$\mathbf{P}\big(\widetilde{\tau} \leq t \wedge \tau^U,\ d(A, X_{\widetilde{\tau}}) \geq \varepsilon\big) = 0.$$

Then, successively if $t \to \infty$, U growing to S and $\varepsilon \to 0$ it follows that

$$\mathbf{P}\big(\widetilde{\tau} < \infty,\ \{X_s\}_{s<\widetilde{\tau}} \Subset S,\ A \neq X_{\widetilde{\tau}}\big) = 0.$$

We deduce

$$\begin{aligned} &\mathbf{P}\big(X_{\tau_n} \underset{n\to\infty}{\not\longrightarrow} X_\tau,\ \tau_n \underset{n\to\infty}{\longrightarrow} \tau < \infty,\ \{X_s\}_{s<\tau} \Subset S\big) \\ &\qquad = \mathbf{P}\big(A \neq X_{\widetilde{\tau}},\ \tau_n \underset{n\to\infty}{\longrightarrow} \tau = \widetilde{\tau} < \infty,\ \{X_s\}_{s<\widetilde{\tau}} \Subset S\big) = 0. \end{aligned} \tag{A.3}$$

Step 2: we prove that $\mathbf{P}\big(\mathbb{D}_{loc}(S) \cap \mathbb{D}(S^\Delta)\big) = 1$. Let K be a compact subset of S and take an open subset $U \Subset S$ containing K. For $n \in \mathbb{N}$ define the stopping times

$$\begin{aligned} &\sigma_0 := 0, \\ &\tau_n := \inf\left\{t \geq \sigma_n \mid \{X_s\}_{\sigma_n \leq s \leq t} \not\Subset S \backslash K\right\}, \\ &\sigma_{n+1} := \inf\left\{t \geq \tau_n \mid \{X_s\}_{\tau_n \leq s \leq t} \not\Subset U\right\}. \end{aligned}$$

Let $V_k \Subset S\backslash K$ be an increasing sequence of open subset such that $S\backslash K = \bigcup_k V_k$, and denote $\tau_n^k := \inf\left\{t \geq \sigma_n \mid \{X_s\}_{\sigma_n \leq s \leq t} \not\Subset V_k\right\}$. Then, by (A.3)

$$\mathbf{P}\Big(X_{\tau_n^k} \underset{k\to\infty}{\not\longrightarrow} X_{\tau_n},\ \tau_n < \infty,\ \{X_s\}_{s<\tau_n} \Subset S\Big) = 0,$$

so $\{\tau_n < \xi\} = \{X_{\tau_n} \in K\}$ **P**-almost surely. Thanks to Lemma 3.8 applied to $\mathcal{K} := K$ and $\mathcal{U} := U^2 \cup \big((S\backslash K) \times (S\backslash K)\big)$

$$\sup_{n\in\mathbb{N}} \mathbf{P}\big(X_{\tau_n} \in K,\ \sigma_{n+1} < \tau_n + \varepsilon\big) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

For $\varepsilon > 0$,

$$\begin{aligned}\mathbf{P}\big(\xi < \infty,\ \{X_s\}_{s<\xi} \not\Subset S \text{ and } \forall t < \xi, \exists s \in [t,\xi),\ X_s \in K\big)& \\ \le \mathbf{P}\big(\exists n, \forall m \ge n,\ \tau_m < \xi < \tau_m + \varepsilon\big) &\le \sup_{n\in\mathbb{N}} \mathbf{P}\big(\tau_n < \xi < \tau_n + \varepsilon\big) \\ &\le \sup_{n\in\mathbb{N}} \mathbf{P}\big(X_{\tau_n} \in K,\ \sigma_{n+1} < \tau_n + \varepsilon\big),\end{aligned}$$

so letting $\varepsilon \to 0$ we obtain

$$\mathbf{P}\big(\xi < \infty,\ \{X_s\}_{s<\xi} \not\Subset S \text{ and } \forall t < \xi, \exists s \in [t,\xi),\ X_s \in K\big) = 0. \tag{A.4}$$

Letting K growing towards S, we deduce from (A.4) that $\mathbf{P}\big(\mathbb{D}_{\mathrm{loc}}(S) \cap \mathbb{D}(S^\Delta)\big) = 1$.
Step 3. Let τ_n, τ be $(\mathcal{F}_{t+})$-stopping times. By the first step $X_{\tau_n} \underset{n\to\infty}{\longrightarrow} X_\tau$ **P**-a.s. on

$$\big\{\tau_n \underset{n\to\infty}{\longrightarrow} \tau < \infty,\ \{X_s\}_{s<\tau} \Subset S\big\},$$

by the second step this is also the case on

$$\big\{\tau_n \underset{n\to\infty}{\longrightarrow} \tau = \xi < \infty,\ \{X_s\}_{s<\tau} \not\Subset S\big\},$$

and this is clearly true on $\big\{\tau_n \underset{n\to\infty}{\longrightarrow} \tau > \xi\big\}$, so the proof is done. □

A.2 Proofs of auxiliary results used to define locally Feller families

We provide here proofs of Lemmas 4.7 and4.8, the statement and the proof of Lemma A.1, but also the proof of Proposition 2.2, all used during the proof of Theorem 4.5.

Proof of Lemma 4.7. Let τ be a $(\mathcal{F}_{t+})_t$-stopping time, let $a \in S$ be and let F be a bounded continuous function from $\mathbb{D}_{\mathrm{loc}}(S)$ to $\mathbb{R}$. For each $n \in \mathbb{N}^*$ chose a discrete subspace $\mathbb{T}_a^n \subset \mathbb{T}_a$ such that $(t, t+n^{-1}] \cap \mathbb{T}_a^n$ is not empty for any $t \in \mathbb{R}_+^*$, and define

$$\tau_n := \min\left\{t \in \mathbb{T}_a^n \mid \tau < t\right\}.$$

Hence τ_n is a $(\mathcal{F}_t)_t$-stopping time with value in $\mathbb{T}_a^n$, so

$$\mathbf{E}_a\left[F\left((X_{\tau_n+t})_t\right) \mid \mathcal{F}_{\tau_n}\right] = \mathbf{E}_{X_{\tau_n}} F \quad \mathbf{P}_a\text{-almost surely.}$$

Since $\tau < \tau_n \le \tau + n^{-1}$ on $\{\tau < \infty\}$ and $a \mapsto \mathbf{P}_a$ is continuous, $\lim_{n\to\infty} \mathbf{E}_{X_{\tau_n}} F = \mathbf{E}_{X_\tau} F$. We have

$$\begin{aligned}\mathbf{E}_a\left|\mathbf{E}_a\left[F\left((X_{\tau+t})_t\right) \mid \mathcal{F}_{\tau+}\right] - \mathbf{E}_a\left[F\left((X_{\tau_n+t})_t\right) \mid \mathcal{F}_{\tau_n}\right]\right|& \\ \le \mathbf{E}_a\left|\mathbf{E}_a\left[F\left((X_{\tau+t})_t\right) \mid \mathcal{F}_{\tau+}\right] - \mathbf{E}_a\left[F\left((X_{\tau+t})_t\right) \mid \mathcal{F}_{\tau_n}\right]\right|& \\ + \mathbf{E}_a\left|F\left((X_{\tau+t})_t\right) - F\left((X_{\tau_n+t})_t\right)\right|.&\end{aligned} \tag{A.5}$$

On the right hand side, the first term converges to 0 (see, for instance, Theorem 7.23, p. 132 in [15]) and the second term converges to 0 by dominated convergence. Hence

$$\mathbf{E}_a\left[F\left((X_{\tau+t})_t\right) \mid \mathcal{F}_{\tau+}\right] = \mathbf{E}_{X_\tau} F \quad \mathbf{P}_a\text{-almost surely,}$$

so $(\mathbf{P}_a)_a$ is a $(\mathcal{F}_{t+})_t$-strong Markov family. $\square$

Lemma A.1 (Localisation of continuity)**.** *Set $\widetilde{S}$ an arbitrary metrisable topological space, consider $U_n \subset S$, an increasing sequence of open subsets such that $S = \bigcup_n U_n$. Let $(\mathbf{P}^n_a)_{a,n} \in \mathcal{P}(\mathbb{D}_{loc}(S))^{\widetilde{S}\times\mathbb{N}}$ be such that*

1. *for each $n \in \mathbb{N}$, $a \mapsto \mathbf{P}^n_a$ is weakly continuous for the local Skorokhod topology,*
2. *for each $n \leq m$ and $a \in \widetilde{S}$*

$$\mathscr{L}_{\mathbf{P}^m_a}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}^n_a}\left(X^{\tau^{U_n}}\right). \tag{A.6}$$

Then there exists a unique family $(\mathbf{P}^\infty_a)_a \in \mathcal{P}(\mathbb{D}_{loc}(S))^{\widetilde{S}}$ such that for any $n \in \mathbb{N}$ and $a \in \widetilde{S}$

$$\mathscr{L}_{\mathbf{P}^\infty_a}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}^n_a}\left(X^{\tau^{U_n}}\right). \tag{A.7}$$

Furthermore the mapping

$$\begin{array}{ccc} \left(\mathbb{N}\cup\{\infty\}\right)\times\widetilde{S} & \to & \mathcal{P}(\mathbb{D}_{loc}(S)) \\ (n,a) & \mapsto & \mathbf{P}^n_a \end{array} \tag{A.8}$$

is weakly continuous for the local Skorokhod topology.

Before giving the proof of this lemma let us recall that in Theorem 2.11, p. 1190, in [10], one obtains an improvement of the Aldous criterion of tightness stated in (2.5). More precisely a subset $\mathcal{P} \subset \mathcal{P}\left(\mathbb{D}_{\mathrm{loc}}(S)\right)$ is tight if and only if

$$\forall t \geq 0,\ \forall \varepsilon > 0,\ \forall \text{ open } U \Subset S,\quad \sup_{\mathbf{P}\in\mathcal{P}} \sup_{\substack{\tau_1\leq\tau_2\leq\tau_3 \\ \tau_3\leq(\tau_1+\delta)\wedge t\wedge\tau^U}} \mathbf{P}(R \geq \varepsilon) \xrightarrow[\delta\to 0]{} 0, \tag{A.9}$$

where the supremum is taken along τ_i stopping times and with

$$R := \begin{cases} d(X_{\tau_1}, X_{\tau_2}) \wedge d(X_{\tau_2}, X_{\tau_3}) & \text{if } 0 < \tau_1 < \tau_2, \\ d(X_{\tau_2-}, X_{\tau_2}) \wedge d(X_{\tau_2}, X_{\tau_3}) & \text{if } 0 < \tau_1 = \tau_2, \\ d(X_{\tau_1}, X_{\tau_2}) & \text{if } \tau_1 = 0. \end{cases}$$

Proof of Lemma A.1. The uniqueness is straightforward using that $X^{\tau^{U_n}}$ converge to X pointwise for the local Skorokhod topology as $n \to \infty$.

Let us prove that for any compact subset $K \subset \widetilde{S}$, the set $\{\mathbf{P}^n_a \mid a \in K,\ n \in \mathbb{N}\}$ is tight. If $U \Subset S$ is an arbitrary open subset, there exists $N \in \mathbb{N}$ such that $U \subset U_N$.

Consider t and ε two strictly positive real numbers. By the continuity of $a \mapsto \mathbf{P}_a^n$, the set $\{\mathbf{P}_a^n \mid a \in K,\ 0 \le n \le N\}$ is tight, so using the characterisation (A.9) we have

$$\sup_{\substack{0\le n\le N\\ a\in K}} \sup_{\substack{\tau_1\le\tau_2\le\tau_3\\ \tau_3\le(\tau_1+\delta)\wedge t\wedge\tau^U}} \mathbf{P}_a^n(R\ge\varepsilon) \underset{\delta\to 0}{\longrightarrow} 0.$$

Since $U \subset U_N$, for all $n \ge N$ and $a \in K$,

$$\mathscr{L}_{\mathbf{P}_a^N}\left(X^{\tau^U}\right) = \mathscr{L}_{\mathbf{P}_a^n}\left(X^{\tau^U}\right),$$

hence

$$\sup_{n\in\mathbb{N},\ a\in K} \sup_{\substack{\tau_1\le\tau_2\le\tau_3\\ \tau_3\le(\tau_1+\delta)\wedge t\wedge\tau^U}} \mathbf{P}_a^n(R\ge\varepsilon) = \sup_{\substack{0\le n\le N\\ a\in K}} \sup_{\substack{\tau_1\le\tau_2\le\tau_3\\ \tau_3\le(\tau_1+\delta)\wedge t\wedge\tau^U}} \mathbf{P}_a^n(R\ge\varepsilon) \underset{\delta\to 0}{\longrightarrow} 0.$$

So, again by (A.9), $\{\mathbf{P}_a^n \mid a \in K,\ n \in \mathbb{N}\}$ is tight.

Hence, if $a \in \widetilde{S}$, then the set $\{\mathbf{P}_a^n\}_n$ is tight. Fix such a, there exist an increasing sequence $\varphi(k)$ and a probability measure $\mathbf{P}_a^\infty \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ such that $\mathbf{P}_a^{\varphi(k)}$ converges to $\mathbf{P}_a^\infty$ as $k \to \infty$. Fix an arbitrary $n \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that $\varphi(k_0) \ge n$ and $U_n \Subset U_{\varphi(k_0)}$. Thanks to Proposition 2.1, there exists $\mathfrak{g} \in \mathrm{C}(S, \mathbb{R}_+)$ such that $U_{\varphi(k_0)} = \{\mathfrak{g} \ne 0\}$ and such that $\mathfrak{g}\cdot\mathbf{P}_a^{n_k}$ converges to $\mathfrak{g}\cdot\mathbf{P}_a^\infty$ weakly for the local Skorokhod topology, as $k \to \infty$. By using (A.6) we have, for each $k \ge k_0$, $\mathfrak{g}\cdot\mathbf{P}_a^{\varphi(k)} = \mathfrak{g}\cdot\mathbf{P}_a^{\varphi(k_0)}$, so $\mathfrak{g}\cdot\mathbf{P}_a^\infty = \mathfrak{g}\cdot\mathbf{P}_a^{\varphi(k_0)}$. Hence we deduce

$$\mathscr{L}_{\mathbf{P}_a^\infty}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}_a^{\varphi(k_0)}}\left(X^{\tau^{U_n}}\right) = \mathscr{L}_{\mathbf{P}_a^n}\left(X^{\tau^{U_n}}\right).$$

Let us prove that the mapping in (A.8) is weakly continuous for the local Skorokhod topology. Since we already verified the tightness it suffices to prove that: for any sequences $n_k \in \mathbb{N}\cup\{\infty\}$, $a_k \in \widetilde{S}$ such that $n_k \to \infty$ and $a_k \to a \in \widetilde{S}$ as $k \to \infty$ and such that the sequence $\mathbf{P}_{a_k}^{n_k}$ converges to $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$, then $\mathbf{P} = \mathbf{P}_a^\infty$. Fix an arbitrary $N \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that $n_{k_0} \ge N$ and $U_N \Subset U_{n_{k_0}}$. As previously, by using Proposition 2.1 again, there exists $\mathfrak{g} \in \mathrm{C}(S, \mathbb{R}_+)$ such that $U_{n_{k_0}} = \{\mathfrak{g} \ne 0\}$, $\mathfrak{g}\cdot\mathbf{P}_{a_k}^{n_k}$ converges to $\mathfrak{g}\cdot\mathbf{P}$ and $\mathfrak{g}\cdot\mathbf{P}_{a_k}^{n_{k_0}}$ converges to $\mathfrak{g}\cdot\mathbf{P}_a^{n_{k_0}}$, as $k \to \infty$. Thanks to (A.7) $\mathfrak{g}\cdot\mathbf{P}_{a_k}^{n_k} = \mathfrak{g}\cdot\mathbf{P}_{a_k}^{n_{k_0}}$ for $k \ge k_0$, so $\mathfrak{g}\cdot\mathbf{P} = \mathfrak{g}\cdot\mathbf{P}_a^{n_{k_0}} = \mathfrak{g}\cdot\mathbf{P}_a^\infty$. This yields

$$\mathscr{L}_{\mathbf{P}}\left(X^{\tau^{U_N}}\right) = \mathscr{L}_{\mathbf{P}_a^\infty}\left(X^{\tau^{U_N}}\right),$$

and letting $N \to \infty$ we obtain that $\mathbf{P} = \mathbf{P}_a^\infty$. □

Proof of Lemma 4.8. Using Proposition 2.1, there exists $\mathfrak{g} \in \mathrm{C}(S, \mathbb{R}_+^*)$ such that for all $(n,a) \in \big(\mathbb{N}\cup\{\infty\}\big)\times S^\Delta$, $\mathbf{P}_a^n(\mathbb{D}_{\mathrm{loc}}(S)\cap\mathbb{D}(S^\Delta)) = 1$ and such that $(n,a) \mapsto \mathbf{P}_a^n$ is weakly continuous for the global Skorokhod topology from $\mathbb{D}(S^\Delta)$. For all $n \in \mathbb{N}$, by Lemma 4.7, $(\mathbf{P}_a^n)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov, so, by Proposition 2.2, $(\mathfrak{g}\cdot\mathbf{P}_a^n)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov.

Take $a \in S$ and denote $\mathbb{T}_a := \big\{t \in \mathbb{R}_+ \,\big|\, \mathfrak{g} \cdot \mathbf{P}_a^\infty(X_{t-} = X_t) = 1\big\}$, so $\mathbb{T}_a$ is dense in $\mathbb{R}_+$. Let $t \in \mathbb{T}_a$ be and consider F, G two bounded function from $\mathbb{D}(S^\Delta)$ to $\mathbb{R}$ continuous for the global Skorokhod topology, we want to prove that

$$\mathfrak{g} \cdot \mathbf{E}_a^\infty \left[F\left((X_{t+s})_s\right) G\left((X_{t\wedge s})_s\right)\right] = \mathfrak{g} \cdot \mathbf{E}_a^\infty \Big[\mathfrak{g} \cdot \mathbf{E}_{X_t}^\infty[F] G\left((X_{t\wedge s})_s\right)\Big]. \tag{A.10}$$

For any $n \in \mathbb{N}$, by the Markov property we have

$$\mathfrak{g} \cdot \mathbf{E}_a^n \Big[F\left((X_{t+s})_s\right) G\left((X_{t\wedge s})_s\right)\Big] = \mathfrak{g} \cdot \mathbf{E}_a^n \Big[\mathfrak{g} \cdot \mathbf{E}_{X_t}^n[F] G\left((X_{t\wedge s})_s\right)\Big]. \tag{A.11}$$

The mappings

$$\begin{array}{rcl} \mathbb{D}(S^\Delta) & \to & \mathbb{R} \\ x & \mapsto & F\left((x_{t+s})_s\right) G\left((x_{t\wedge s})_s\right) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{D}(S^\Delta) & \to & \mathbb{R} \\ x & \mapsto & \mathfrak{g} \cdot \mathbf{E}_{x_t}^\infty[F] G\left((x_{t\wedge s})_s\right) \end{array}$$

are continuous on the set $\{X_{t-} = X_t\}$ for the global topology. Hence, since $\mathfrak{g} \cdot \mathbf{E}_a^n$ converges to $\mathfrak{g} \cdot \mathbf{E}_a^\infty$ weakly for the global topology and $\mathfrak{g} \cdot \mathbf{P}_a^\infty(X_{t-} = X_t) = 1$, we have

$$\mathfrak{g} \cdot \mathbf{E}_a^n \Big[F\left((X_{t+s})_s\right) G\left((X_{t\wedge s})_s\right)\Big] \underset{n\to\infty}{\longrightarrow} \mathfrak{g} \cdot \mathbf{E}_a^\infty \Big[F\left((X_{t+s})_s\right) G\left((X_{t\wedge s})_s\right)\Big], \tag{A.12}$$

$$\mathfrak{g} \cdot \mathbf{E}_a^n \Big[\mathfrak{g} \cdot \mathbf{E}_{X_t}^\infty[F] G\left((X_{t\wedge s})_s\right)\Big] \underset{n\to\infty}{\longrightarrow} \mathfrak{g} \cdot \mathbf{E}_a^\infty \Big[\mathfrak{g} \cdot \mathbf{E}_{X_t}^\infty[F] G\left((X_{t\wedge s})_s\right)\Big]. \tag{A.13}$$

Since $(n, b) \mapsto \mathfrak{g} \cdot \mathbf{P}_b^n$ is continuous for the global topology, using the compactness of S^Δ we have

$$\sup_{a \in S^\Delta} \left|\mathfrak{g} \cdot \mathbf{E}_a^n F - \mathfrak{g} \cdot \mathbf{E}_a^\infty F\right| \underset{n\to\infty}{\longrightarrow} 0. \tag{A.14}$$

We deduce (A.10) from (A.11)-(A.14) and so

$$\mathfrak{g} \cdot \mathbf{E}_a^\infty \big[F\left((X_{t+s})_s\right) \,\big|\, \mathcal{F}_t\big] = \mathfrak{g} \cdot \mathbf{E}_{X_t}^\infty[F], \quad \mathfrak{g} \cdot \mathbf{P}_a^\infty\text{-almost surely,}$$

so, by Lemma 4.7, $(\mathfrak{g} \cdot \mathbf{P}_a^\infty)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov. Applying Proposition 2.2 to $(\mathfrak{g} \cdot \mathbf{P}_a^\infty)_a$ and $1/\mathfrak{g}$, and using (2.4), we deduce that $(\mathbf{P}_a^\infty)_a$ is $(\mathcal{F}_{t+})_t$-strong Markov. $\square$

To be complete, we finally provide the

Proof of Proposition 2.2. Let us first verify that if $(\mathbf{P}_a)_a$ is a $(\mathcal{F}_t)_t$-strong Markov family, then $(\mathfrak{g} \cdot \mathbf{P}_a)_a$ is a $(\mathcal{F}_t)_t$-Markov family, for any $\mathfrak{g} \in \mathrm{C}^{\neq 0}(S, \mathbb{R}_+)$, where

$$\mathrm{C}^{\neq 0}(S, \mathbb{R}_+) := \{\mathfrak{g} : S \to \mathbb{R}_+ \mid \{\mathfrak{g} = 0\} \text{ is closed and } \mathfrak{g} \text{ is continuous on } \{\mathfrak{g} \neq 0\}\}.$$

Recall that by (2.3) $(\mathfrak{g} \cdot X)_t = X_{\tau_t^{\mathfrak{g}}}$, where $t \mapsto \tau_t^{\mathfrak{g}}$ is the solution of $\tau_t^{\mathfrak{g}} = g(X_{\tau_t^{\mathfrak{g}}})$ (see also Remark 3.2, p. 1195, in [10]). Then clearly $(\mathfrak{g} \cdot X)_t \in \mathcal{F}_{\tau_t^{\mathfrak{g}}}$ and $\big\{(\mathfrak{g} \cdot X)_t \neq X_{\tau_t^{\mathfrak{g}}}\big\} \in \mathcal{F}_{\tau_t^{\mathfrak{g}}}$. Moreover it is straightforward to prove that

$$(\mathfrak{g} \cdot X)_t \neq X_{\tau_t^{\mathfrak{g}}} \text{ implies that } \mathfrak{g}((\mathfrak{g} \cdot X)_t) = 0, \text{ and } (\mathfrak{g} \cdot X_{t+\bullet})_s \text{ is constant,}$$

and

$$(\mathfrak{g}\cdot X)_t = X_{\tau_t^{\mathfrak{g}}} \text{ implies that } (\mathfrak{g}\cdot X_{t+\bullet})_s = (\mathfrak{g}\cdot X_{\tau_t^{\mathfrak{g}}+\bullet})_s.$$

Assume that $(\mathbf{P}_a)_{a\in S}$ is a $(\mathcal{F}_t)_t$-strong Markov family. Using the latter remarks, for any $t\in\mathbb{R}_+$, $a\in S^\Delta$ and $B\in\mathcal{F}$, we can write, $\mathbf{P}_a$-a.s.,

$$\begin{aligned}\mathbf{P}_a\left((\mathfrak{g}\cdot X_{t+\bullet})_s\in B\;\middle|\;\mathcal{F}_{\tau_t^{\mathfrak{g}}}\right) &= \mathbf{P}_a\left((\mathfrak{g}\cdot X_{t+\bullet})_s\in B,\ \mathfrak{g}\cdot X_t = X_{\tau_t^{\mathfrak{g}}}\;\middle|\;\mathcal{F}_{\tau_t^{\mathfrak{g}}}\right)\\ &\quad+\mathbf{P}_a\left((\mathfrak{g}\cdot X_{t+\bullet})_s\in B,\ \mathfrak{g}\cdot X_t \neq X_{\tau_t^{\mathfrak{g}}}\;\middle|\;\mathcal{F}_{\tau_t^{\mathfrak{g}}}\right)\\ = \mathbf{P}_a\left(\mathfrak{g}\cdot(X_{\tau_t+\bullet})_s\in B\;\middle|\;\mathcal{F}_{\tau_t^{\mathfrak{g}}}\right)\mathbb{1}_{\{\mathfrak{g}\cdot X_t = X_{\tau_t^{\mathfrak{g}}}\}} &+ \mathbf{P}_a\left((\mathfrak{g}\cdot X_t)_s\in B\;\middle|\;\mathcal{F}_{\tau_t^{\mathfrak{g}}}\right)\mathbb{1}_{\{\mathfrak{g}\cdot X_t\neq X_{\tau_t^{\mathfrak{g}}}\}}\\ &= \mathbf{P}_{X_{\tau_t^{\mathfrak{g}}}}(\mathfrak{g}\cdot X\in B) = \mathfrak{g}\cdot\mathbf{P}_{\mathfrak{g}\cdot X_t}(B).\end{aligned}$$

Hence $(\mathfrak{g}\cdot\mathbf{P}_a)_{a\in S}$ is a $(\mathcal{F}_t)_t$-Markov family.

If $(\mathbf{P}_a)_a$ is a $(\mathcal{F}_{t+})_t$-strong Markov family, then for any $(\mathcal{F}_{(\tau_t^{\mathfrak{g}})+})_t$-stopping time σ,

$$\{\tau_\sigma^{\mathfrak{g}} < t\} = \bigcup_{q\in\mathbb{Q}_+}\{\sigma<q,\ \tau_q^{\mathfrak{g}}<t\}\in\mathcal{F}_t,$$

so $\tau_\sigma^{\mathfrak{g}}$ is a $(\mathcal{F}_{t+})_t$-stopping time. Using the same argument as before we obtain that $(\mathfrak{g}\cdot\mathbf{P}_a)_a$ is a $(\mathcal{F}_{t+})_t$-strong Markov family. □

A.3 Proof of Lemma 4.22

Before proving the Lemma 4.22 let us note that thanks to Propositions 2.1 and 2.2, if $(\mathbf{P}_a)_a\in\mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))^S$ is locally Feller then for any open subset $U\subset S$ there exists $\mathfrak{h}_0\in\mathrm{C}(S,\mathbb{R}_+)$ such that $U=\{\mathfrak{h}_0\neq 0\}$ and $(\mathfrak{h}_0\cdot\mathbf{P}_a)_a$ is locally Feller. This fact does not ensure that the martingale local problem associated to $\mathfrak{h}_0 L$ is well-posed as is stated in Lemma 4.22. During the proof we will use two preliminary results.

Lemma A.2. *Let L be a subset of $\mathrm{C}_0(S)\times\mathrm{C}(S)$ such that $\mathrm{D}(L)$ is dense in $\mathrm{C}_0(S)$ and U be an open subset of S, then there exist a subset L_0 of L and a function $\mathfrak{h}_0$ of $\mathrm{C}(S,\mathbb{R}_+)$ with $\{\mathfrak{h}_0\neq 0\}=U$ such that $\overline{L}=\overline{L_0}$, such that $\mathfrak{h}_0 L_0\subset\mathrm{C}_0(S)\times\mathrm{C}_0(S)$ and such that: for any $\mathfrak{h}\in\mathrm{C}(S,\mathbb{R}_+)$ with $\{\mathfrak{h}\neq 0\}=U$ and $\sup_{a\in U}(\mathfrak{h}/\mathfrak{h}_0)(a)<\infty$ and any $\mathbf{P}\in\mathcal{M}_{\mathrm{c}}\left((\mathfrak{h}L_0)^\Delta\right)$, $\mathbf{P}(X=X^{\tau^U})=1$.*

Proof. Take L_0 a countable dense subset of L and let d be a metric on S^Δ. For any $n\in\mathbb{N}^*$ there exist $M_n\in\mathbb{N}$ and $(a_{n,m})_{1\le m\le M_n}\in(S^\Delta\backslash U)^{M_n}$ such that

$$S^\Delta\backslash U\subset\bigcup_{m=1}^{M_n}B(a_{n,m},n^{-1}).$$

For each $1\le m\le M_n$ there exist $(f_{n,m},g_{n,m})\in L_0$ such that

$$f_{n,m}(a)\in\begin{cases}[1-n^{-1},1+n^{-1}] & \text{if } d(a,a_{n,m})\ge 2n^{-1},\\ [-n^{-1},1+n^{-1}] & \text{if } n^{-1}\le d(a,a_{n,m})\ge 2n^{-1},\\ [-n^{-1},n^{-1}] & \text{if } n^{-1}\le d(a,a_{n,m}).\end{cases}$$

Take $\mathfrak{h}_0 \in \mathrm{C}_0(S, \mathbb{R}_+)$ with $\{\mathfrak{h}_0 \neq 0\} = U$, such that $\mathfrak{h}_0 g \in \mathrm{C}_0(S)$ for any $(f, g) \in L_0$ and such that for any $n \in \mathbb{N}^*$ and $1 \leq m \leq M_n$

$$\|\mathfrak{h}_0\|_{B(a_{n,m}, 4n^{-1})} \|g_{n,m}\| \leq \frac{1}{n}.$$

Hence $\overline{L} = \overline{L_0}$ and $\mathfrak{h}L_0 \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$. Let $\mathfrak{h} \in \mathrm{C}(S, \mathbb{R}_+)$ be such that $\{\mathfrak{h} \neq 0\} = U$ and $C := \sup_{a \in U} (\mathfrak{h}/\mathfrak{h}_0)(a) < \infty$. Let $\mathbf{P} \in \mathcal{M}_{\mathrm{c}}\left((\mathfrak{h}L)^\Delta\right)$ be such that there exists $a \in S^\Delta \backslash U$ with $\mathbf{P}(X_0 = a) = 1$. We will prove that

$$\mathbf{P}(\forall s \geq 0,\ X_s = a) = 1. \tag{A.15}$$

Take $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$. There exists $m \leq M_n$ such that $d(a, a_{n,m}) < \frac{1}{n}$. If we denote

$$\tau := \tau^{B(a, 3n^{-1})},$$

then

$$\begin{aligned}\mathbf{E}[f_{n,m}(X_{t\wedge\tau})] &= f_{n,m}(a) + \mathbf{E}\left[\int_0^{t\wedge\tau} \mathfrak{h}(X_s) g_{n,m}(X_s) \mathrm{d}s\right] \\ &\leq f_{n,m}(a) + t\|\mathfrak{h}\|_{B(a_{n,m}, 4n^{-1})} \|g_{n,m}\| \leq \frac{1 + tC}{n}.\end{aligned}$$

Since $\mathbf{P}\big(\tau < \infty \Rightarrow d(X_\tau, a) \geq \frac{3}{n}\big) = 1$, by (3.6) in the proof of Proposition 3.4, we have

$$\begin{aligned}\mathbf{E}[f_{n,m}(X_{t\wedge\tau})] &= \mathbf{E}[f_{n,m}(X_\tau)\mathbb{1}_{\{\tau \leq t\}}] + \mathbf{E}[f_{n,m}(X_t)\mathbb{1}_{\{t < \tau\}}] \\ &\geq (1 - \frac{1}{n})\mathbf{P}(\tau \leq t) - \frac{1}{n}\mathbf{P}(t < \tau) = \mathbf{P}(\tau \leq t) - \frac{1}{n},\end{aligned}$$

so

$$\mathbf{P}(\tau \leq t) \leq \frac{2 + tC}{n}.$$

Hence we obtain

$$\mathbf{P}\big(\forall s \in [0, t],\ d(X_s, a) \leq 3/n\big) \geq \mathbf{P}(t < \tau) \geq 1 - \frac{2 + tC}{n}.$$

By taking the limit with respect to n and t we obtain (A.15).

To complete the proof let us consider an arbitrary $\mathbf{P} \in \mathcal{M}_{\mathrm{c}}\left((\mathfrak{h}L_0)^\Delta\right)$. As in Remark 3.12 we denote

$$\mathbf{Q}_X \overset{\mathbf{P}\text{-a.s.}}{:=} \mathscr{L}_{\mathbf{P}}\left((X_{\tau^U + t})_{t \geq 0} \;\middle|\; \mathcal{F}_{\tau^U}\right).$$

Thanks to Proposition 3.13 $\mathbf{P}$-almost surely $\mathbf{Q}_X \in \mathcal{M}\left((\mathfrak{h}L)^\Delta\right)$, and thanks to 3.6 from Proposition 3.4 $\mathbf{P}$-almost surely $\mathbf{Q}_X(X_0 = a) = 1$ with $a = X_\tau \in S^\Delta \backslash U$ on $\{\tau^U < \infty\}$. By using the previous case and by applying (A.15) we get that $\mathbf{P}$-almost surely $\mathbf{Q}_X(\forall s \geq 0, X_s = a) = 1$, with $a = X_\tau \in S^\Delta \backslash U$ on $\{\tau^U < \infty\}$. Hence $\mathbf{P}(X = X^{\tau^U}) = 1$. $\square$

Lemma A.3. *Let L be a subset of $\mathrm{C}_0(S) \times \mathrm{C}_0(S)$ such that the martingale problem associated to L is well-posed. Then the martingale problem associated to L^Δ is well-posed if and only if $\mathbf{P}(X = X^{\tau^S}) = 1$ for all $\mathbf{P} \in \mathcal{M}_{\mathrm{c}}(L^\Delta)$ (in other words $\mathbf{P} \in \mathcal{P}(\mathbb{D}_{loc}(S))$) .*

Proof. Assume that the martingale problem associated to L^Δ is well-posed and take $\mathbf{P} \in \mathcal{M}_c(L^\Delta)$. Then $\mathscr{L}_{\mathbf{P}}(X^{\tau^S}) \in \mathcal{M}_c(L^\Delta)$, so by uniqueness of the solution $\mathbf{P} = \mathscr{L}_{\mathbf{P}}(X^{\tau^S})$ and so $\mathbf{P}(X = X^{\tau^S}) = 1$. For the converse, let $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{M}_c(L^\Delta)$ be such that $\mathscr{L}_{\mathbf{P}^1}(X_0) = \mathscr{L}_{\mathbf{P}^2}(X_0)$. Then $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{P}(\mathbb{D}_{\mathrm{loc}}(S))$ so $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{M}(L)$, hence $\mathbf{P}^1 = \mathbf{P}^2$. □

Proof of Lemma 4.22. Let L_0 and $\mathfrak{h}_0$ be as in Lemma A.2 and take $\mathfrak{h} \in \mathrm{C}(S, \mathbb{R}_+)$ with $\{\mathfrak{h} \neq 0\} = U$ and $\sup_{a \in U}(\mathfrak{h}/\mathfrak{h}_0)(a) < \infty$. The existence of a solution for the martingale problem associated to $(\mathfrak{h}L_0)^\Delta$ is given by the existence of a solution for the martingale problem associated to L. Let $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{M}_c((\mathfrak{h}L_0)^\Delta)$ be such that $\mathscr{L}_{\mathbf{P}^1}(X_0) = \mathscr{L}_{\mathbf{P}^2}(X_0)$. Thanks to Lemma A.2 and Lemma A.3, for an open subset $V \Subset U$, there exist $\mathfrak{g} \in \mathrm{C}(S, \mathbb{R}_+^*)$ and a dense subset L_1 of L_0 such that $\mathfrak{g}(a) = \mathfrak{h}(a)$ for any $a \in V$, $\mathfrak{k}L_1 \subset \mathrm{C}_0(S) \times \mathrm{C}_0(S)$ and the martingale problem associated to $(\mathfrak{g}L_1)^\Delta$ is well-posed. Hence we may apply Theorem 6.1 p. 216 from [8] and deduce that $\mathscr{L}_{\mathbf{P}^1}(X^{\tau^V}) = \mathscr{L}_{\mathbf{P}^2}(X^{\tau^V})$. Letting V growing towards U we deduce that $\mathscr{L}_{\mathbf{P}^1}(X^{\tau^U}) = \mathscr{L}_{\mathbf{P}^2}(X^{\tau^U})$ and so, since $\mathbf{P}^i(X = X^{\tau^U}) = 1$ for $i \in \{1, 2\}$, we conclude that $\mathbf{P}^1 = \mathbf{P}^2$. □

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