# Homework #5 : final examination

Due on March 31th : individual work

**Exercise 1** *Warm-up : behaviour of characteristic exponent*
Let $\eta(u)$ be the characteristic exponent of a real-valued Lévy process :

$$\eta(u) = ibu - \frac{1}{2}\Gamma u^2 + \int_{\mathbb{R}} \left[e^{iuy} - 1 - iuy\mathbf{1}_{\{|y|<1\}}\right]\nu(dy).$$

1. Show that $\lim_{|u|\to\infty} \frac{\eta(u)}{u^2} = -\frac{1}{2}\Gamma$, by proving that $\lim_{|u|\to\infty} \frac{1}{u^2}\int_{\mathbb{R}} \left[e^{iuy} - 1 - iuy\mathbf{1}_{\{|y|<1\}}\right]\nu(dy) = 0$.
2. Show that $X$ has bounded variations on every time interval a.s. if and only if $\Gamma = 0$ and $\int_{\mathbb{R}}(1 \wedge |y|)\nu(dy) < \infty$. Use Lévy-Itô decomposition or the exponential formula for PRM. In that case $\lim_{|u|\to\infty} \frac{\eta(u)}{u} = i\,\mathtt{d}$, where $\mathtt{d} = b - \int_{-1}^{1} y\,\nu(dy)$ is the drift. Use the dominated convergence as in the first point.
3. Show that the characteristic exponent $\eta$ is bounded if and only if $X$ is a compound Poisson process. If you assume that $\eta$ is bounded, show that

$$(\operatorname{Re}\eta)(u) = \int_{\mathbb{R}^*} \big(\cos(uy) - 1\big)\nu(dy)$$

and that

$$\int_{\mathbb{R}^*} \big(e^{-ty^2/2} - 1\big)\nu(dy) = \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-\lambda^2/2t}(\operatorname{Re}\eta)(\lambda)d\lambda \le \sup_{\lambda\in\mathbb{R}}(\operatorname{Re}\eta)(\lambda).$$

Deduce further that $X$ has bounded variations (in the sense of the preceding point).

**Exercise 2** *First round : subordinators*
A non-decreasing Lévy process with values in $\mathbb{R}_+$ is called a *subordinator*.

1. Show that a real valued Lévy process $X = (X_t : t \ge 0)$ is a subordinator if its characteristic exponent has the form

$$\eta(u) = ibu + \int_0^{\infty} \big(e^{iyu} - 1\big)\nu(dy), \tag{1}$$

where $b \ge 0$ and the Lévy measure has the support in $\mathbb{R}_+$ and satisfies

$$\int_0^1 y\,\nu(dy) < \infty \quad \text{or equivalently} \quad \int_0^{\infty} \frac{y}{1+y}\nu(dy) < \infty. \tag{2}$$

Moreover $X_t = tb + \sum_{s\le t}\Delta X_s$. Use the positivity and the monotonicity to prove that $X$ contains no Brownian part. Then use the existence of all moments of the third term in the Lévy-Itô decomposition of $X$ to obtain (2).

2. Show that for a subordinator
$$\mathbb{E}\big(e^{-\lambda X_t}\big) = e^{-t\phi(\lambda)}, \tag{3}$$
where the Laplace exponent
$$\phi(\lambda) = \eta(i\lambda) = -b\lambda + \int_0^\infty \big(1 - e^{-\lambda y}\big)\nu(dy). \tag{4}$$
Analyse the analytic continuation of $u \mapsto \mathbb{E}\big(e^{iuX_t}\big)$.
3. A subordinator $X$ is called one-sided stable process if for each $a \geq 0$ there corresponds a constant $c(a) \geq 0$ such that $aX_t$ and $X_{t\,c(a)}$ have the same law.
   (a) Show that $c(\cdot)$ in this definition is continuous ans satisfies the equation $c(a\tilde{a}) = c(a)c(\tilde{a})$. Then deduce that $c(a) = a^\alpha$, with some $\alpha > 0$ (the index).
   (b) Deduce further that $\phi(a) = c(a)\phi(1)$ and hence
   $$\mathbb{E}\big(e^{-\lambda X_t}\big) = e^{-t\,r\,\lambda^\alpha},\ r > 0 \ \text{(the rate)}. \tag{5}$$
   By using the concavity of $\phi$ deduce that $\alpha \in (0, 1)$.
   (c) Prove (or assume) that for $\alpha \in (0, 1)$,
   $$\int_0^\infty (1 - e^{-\lambda y})\frac{1}{y^{1+\alpha}}\, dy = \frac{\Gamma(1-\alpha)}{\alpha}\lambda^\alpha,$$
   where here $\Gamma(\cdot)$ is the Euler's gamma function. Deduce that the stable subordinators with index $\alpha$ and rate $r$ have the Laplace exponent (4) with Lévy measure
   $$\nu(dy) = \frac{\alpha r}{\Gamma(1-\alpha)}\,\frac{1}{y^{1+\alpha}}. \tag{6}$$
   (d) An example : the stable subordinator with index $\frac{1}{2}$ and rate 1 have the probability density
   $$f_{X_t}(s) := \Big(\frac{t}{2\sqrt{\pi}}\Big)s^{-3/2}e^{-t^2/(4s)}, \quad s \geq 0.$$
   Indeed, set
   $$g_t(\lambda) := \mathbb{E}\big(e^{-\lambda X_t}\big) = \int_0^\infty e^{-\lambda s} f_{X_t}(s)ds,$$
   prove that $g_t'(\lambda) = -(t/2\sqrt{\lambda})g_t(\lambda)$, that $g_t(0) = 1$ and deduce that $g_t(\lambda) = e^{-t\lambda^{1/2}}$.

**Exercise 3** *Rising scale : transience and recurrence*
Let $X = (X_t : t \geq 0)$ be a Lévy process with the characteristic exponent $\eta$. Denote by $\mathrm{P}_t$ the associated semigroup, given by
$$\mathrm{P}_t f(x) = \int_{\mathbb{R}} f(x+y)\mathbb{P}(X_t \in dy),$$
with $f$ a non-negative measurable function. Recall that the resolvent $\mathrm{R}_\lambda$ is given, for measurable $f \geq 0$ by
$$\mathrm{R}_\lambda f(x) = \int_0^\infty e^{-\lambda t}\,\mathrm{P}_t f(x)dt = \mathbb{E}_x\Big(\int_0^\infty e^{-\lambda t} f(X_t)dt\Big).$$
1. Let $\tau$ an exponential random time with parameter $\lambda$. Show that $\mathbb{E}_\bullet f(X_\tau) = \lambda \mathrm{R}_\lambda f(\bullet)$.

2. Denote the Fourier transform of a function $g \in \mathrm{L}^1(\mathbb{R})$ by $\mathcal{F}g(u) = \int_{\mathbb{R}} e^{iux} g(x)dx$ (**attention**, not the same definition as in the course). Show that for every $f \in \mathrm{L}^1 \cap \mathrm{L}^\infty$,

$$\mathcal{F}(\mathrm{P}_t f)(u) = \exp\{t\eta(-u)\}\mathcal{F}f(u), \quad \text{and} \quad \mathcal{F}(\mathrm{R}_\lambda f)(u) = \Big(\frac{1}{\lambda - \eta(-u)}\Big)\mathcal{F}f(u), \tag{7}$$

where $t \geq 0$ and $\lambda > 0$. Moreover if A denotes the generator of $\mathrm{P}_t$ and $D$ its domain, show that if $f \in D$ and $\mathrm{A}f \in \mathrm{L}^1$, then $\mathcal{F}(\mathrm{A}f)(u) = \eta(-u)\mathcal{F}f(u)$.

3. Let us introduce a familiy of measures called the potential measures $\{U(x,\cdot) : x \in \mathbb{R}\}$ given, for $B \in \mathcal{B}(\mathbb{R})$, by

$$U(x,B) = \int_0^\infty \mathbb{P}_x(X_t \in B)dt = \mathbb{E}_x\Big(\int_0^\infty \mathbf{1}_{\{X_t \in B\}}dt\Big) \in [0,\infty].$$

If $T_B = \inf\{t \geq 0 : X_t \in B\}$ denotes the first entrance time into $B$, show that

$$U(x,B) = \mathbb{E}_x\Big(\int_{T_B}^\infty \mathbf{1}_{\{X_t \in B\}}dt\Big) = \int_{\overline{B}} U(y,B)\mathbb{P}_x(X_{T_B} \in dy), \tag{8}$$

where $\overline{B}$ is the closure of $B$.

4. We say that the process $X$ is *transient* if the for every compact set $K$, $U(x,K) < \infty$, $x \in \mathbb{R}$, or equivalent if $U(0,K) < \infty$ since $U(x,K) = U(0, K - \{x\})$. Here we denoted by $B - B' = \{x - x' : x \in B, x' \in B'\}$. We say that a process is *recurrent* if $U(0,B) = \infty$ for every open ball $B$ centered in 0. We want to prove that the process is either transient or recurrent.

   (a) Suppose that $\exists \varepsilon > 0$ such that $U(0,B) < \infty$, where $B = (-\varepsilon, \varepsilon)$, and let $B' = \big[-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\big]$. Justify the following relations

   $$U(x,B') \leq \sup_{y \in B'} U(y,B') = \sup_{y \in B'} U(0, B' - \{y\}) \leq U(0, B' - B') \leq U(0,B) < \infty.$$

   Use (8) for the first inequality.

   (b) Deduce that for every $y \in \mathbb{R}$, $U(x, \{y\} + B') < \infty$ and then $U(x,K) < \infty$ for any compact $K$.

5. Test of transience : the Lévy process $X$ is transient iff for some $r > 0$ small enough

$$\limsup_{\lambda \downarrow 0} \int_{-r}^{r} \mathrm{Re}\Big(\frac{1}{\lambda - \eta(u)}\Big)du < \infty. \tag{9}$$

   (a) For $r > 0$ arbitrary small consider $f = \mathbf{1}_{[-r,r]} \star \mathbf{1}_{[-r,r]}$ (the convolution). Clearly it can be (proved) seen that $0 \leq f \leq 2r\mathbf{1}_{[-2r,2r]}$ is continuous non-negative with support $[-2r, 2r]$ and also that

   $$\mathcal{F}f(u) = \begin{cases} [(2/u)\sin(ru)]^2 & \text{if } u \neq 0 \\ 4r^2 & \text{otherwise} \end{cases}$$

   is a bounded continuous non-negative function. Show that for $\lambda > 0$,

   $$\mathrm{R}_\lambda f(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \Big[\frac{2}{u}\sin(ru)\Big]^2 \mathrm{Re}\Big(\frac{1}{\lambda - \eta(-u)}\Big)du. \tag{10}$$

   Use Fourier inversion, (7) and the fact that $\mathrm{R}_\lambda f(0)$ is a real number.

(b) Deduce that $2rU(0,[-2r,2r]) \geq \limsup_{\lambda\downarrow 0} \mathrm{R}_\lambda f(0)$, and latter quantity is infinite whenever

$$\limsup_{\lambda\downarrow 0} \int_{-r}^{r} \mathrm{Re}\Big(\frac{1}{\lambda-\eta(-u)}\Big)du = \infty$$

and then $X$ is recurrent.

(c) Conversely, assume that for $r>0$,

$$\limsup_{\lambda\downarrow 0} \int_{-2r}^{2r} \mathrm{Re}\Big(\frac{1}{\lambda-\eta(-u)}\Big)du < \infty.$$

Set

$$g(x) = f(u) = \begin{cases} [(2/x)\sin(rx)]^2 & \text{if } x \neq 0 \\ 4r^2 & \text{otherwise} \end{cases}$$

having its Fourier transform $\mathcal{F}g(u) = 2\pi \mathbf{1}_{[-r,r]} \star \mathbf{1}_{[-r,r]}$. Deduce un expression of $\mathrm{R}_\lambda g(0)$. One can use the same argument as in the previous point.

(d) Prove that $U(0,[-\frac{\pi}{3r},\frac{\pi}{3r}]) < \infty$. For instance one can (prove and) use that $g(x) \geq r^2$, when $|x| \leq \frac{\pi}{3r}$. Conclude that $X$ is transient.

**Exercise 4** *Final step : pathwise uniqueness*
Let $X = (X_t : t \geq))$ be a one-dimensional symmetric stable with index $\alpha \in (1,2)$ having its Lévy measure given by $\nu(dz) = |z|^{-1-\alpha}$ on $\mathbb{R}^*$ and its generator, for $f$ a $\mathrm{C}^2$−function

$$Lf(x) = \int_{\mathbb{R}^*} [f(x+z) - f(x) - \mathbf{1}_{\{|z|\leq 1\}} z f'(x)]|z|^{-1-\alpha}dz.$$

1. Set $X_t^n = \sum_{s\leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s|\leq n\}}$. Show that $X^n$ is a Lévy process and give its Lévy measure, then show that $X^n$ is a square integrable martingale.
2. Let $H_t$ be a bounded predictable process. Show that $Z_t^n := \int_0^t H_s dX_s^n$ is a square integrable martingale.
3. Set $U_t^n = X_t - X_t^n$. Show that
$$\mathbb{E}\Big|\int_0^t H_s dU_s^n\Big| \leq C\mathbb{E}\Big(\sum_{s\leq t} |\Delta X_s| \mathbf{1}_{\{|\Delta X_s|>n\}}\Big),$$
for some constant $C$. Show that the right hand side of the latter inequality is finite and tends to 0, as $n \to \infty$.
4. Deduce that the process $Z_t = \int_0^t H_s dX_s$ is a martingale.
5. Let $f$ be a $\mathrm{C}_b^2$ function (with bounded first and second derivatives) and set $K(s,z) := f(Z_{s-} + H_s\, z) - f(Z_{s-}) - f'(Z_{s-})H_s\, z$. Justify each of following equalities
$$f(Z_t) = f(Z_0) + \int_0^t f'(Z_{s-})dZ_s + \sum_{s\leq t}[f(Z_s) - f(Z_{s-}) - f'(Z_{s-})\Delta Z_s]$$
$$= f(Z_0) + \int_0^t f'(Z_{s-})dZ_s + \int_0^t \int K(s,z)N(ds,dz) = f(Z_0) + M_t + \int_0^t \int K(s,z)ds\nu(dz),$$
where $M_t = \displaystyle\int_0^t f'(Z_{s-})dZ_s + \int_0^t \int K(s,z)\widetilde{N}(ds,dz)$. Here $N$ is the PRM associated to $X$ with intensity measure $dt\,\nu(dz)$.

6. Prove that for each $m$, $V_t^m = \int_0^t \int_{|z|\le m} K(s,z)\widetilde{N}(ds,dz)$ is a martingale and that $M_t$ is the limit of martingales $\int_0^t f'(Z_{s-})dZ_s + V_t^m$.

   One can use the fact that for each $k > m$, $V_t^k - V_t^m$ is a martingale and show that

$$\mathbb{E}\int_0^t \int_{m<|z|} |K(s,z)|\big(N(ds,dz) + ds\nu(dz) \le C' m^{1-\alpha},$$

   for some constant $C'$ not depending on $m$.

7. Show that, if $H_s \neq 0$, we have

$$\int_0^t \int K(s,z)ds\nu(dz) = \int_0^t |H_s|^\alpha Lf(Z_{s-}).$$

   One should come back to the expression of $K$, perform the change of variable $w = H_s z$ and recall the expression of $L$. Conclude that even if $H_s = 0$,

$$f(Z_t) = f(Z_0) + M_t + \int_0^t |H_s|^\alpha Lf(Z_{s-})ds. \tag{11}$$

8. (Please read also Bonus Exercice) We study the uniqueness of the following SDE

$$dY_t = F(Y_{t-})dX_t, \quad Y_0 = y_0, \tag{12}$$

   where $F$ is supposed bounded continuous such that

$$|F(x) - F(y)| \le \rho(|x-y|), \quad \forall x, y \in \mathbb{R}, \tag{13}$$

   with $\rho : [0,\infty) \to \mathbb{R}$ a non-decreasing continuous function, $\rho(0) = 0$. More precisely we try to prove that if

$$\int_{0+} \frac{1}{\rho(x)^\alpha} dx = \infty, \tag{14}$$

   then the solution of the SDE (12) is pathwise unique.

   (a) Let $Y^1$ and $Y^2$ be any two solutions of (12) set $Z_t = Y_t^1 - Y_t^2$ and $H_t = F(Y_{t-}^1) - F(Y_{t-}^2)$. Deduce that $Z_t = \int_0^t H_s dX_s$.

   (b) Define $A_t = \int_0^t |H_s|^\alpha ds$. Justify that $A_t < \infty$ (use the fact that $F$ is bounded).

   (c) Denote by $p_t(x,y)$ the transition density for $X_t$, that is the density of $\mathbb{P}_x(X_t \in dy)$. Fix $\lambda > 0$, let $r_\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(x,0)dt$ and $R_\lambda f(x) = \int f(y) r_\lambda(x-y)dy$. Recognize $R_\lambda$? It can be proved that $r_\lambda(x)$ is bounded continuous in $x$, and $r_\lambda(x) < r_\lambda(0)$, if $x \neq 0$(admitted).

   (d) Let $a_n \downarrow 0$ such that $\int_{a_{n+1}}^{a_n} \rho(x)^{-\alpha} dx = n$. For each $n$ one can choose $h_n$ a non-negative $\mathrm{C}^2$−function with the support in $[a_{n+1}, a_n]$, whose integral is 1 and with $h_n(x) \le 2/(n\rho(x)^\alpha)$ (admitted). Set $f_n = R_\lambda h_n(x)$. Show that $f_n$ is $\mathrm{C}^2$.

   One can interchanges differentiation and integration and uses translation invariance.

   (e) Show that $Lf_n = LR_\lambda h_n = \lambda R_\lambda h_n - h_n = \lambda f_n - h_n$.

   (f) Use Itô's product formula and (11) to deduce

$$\mathbb{E}\big(e^{-\lambda A_t} f_n(Z_t)\big) - f_n(0) = \mathbb{E}\int_0^t e^{-\lambda A_s} |H_s|^\alpha Lf_n(Z_{s-})ds - \mathbb{E}\int_0^t e^{-\lambda A_s} \lambda |H_s|^\alpha f_n(Z_{s-})ds.$$

(g) Conclude from the preceding two points that

$$f_n(0) - \mathbb{E}\big(e^{-\lambda A_t} f_n(Z_t)\big) = \mathbb{E}\int_0^t e^{-\lambda A_s}|H_s|^\alpha h_n(Z_{s-})ds.$$

Show, by using the properties of $h_n$ and the fact that $|H_s| \le \rho(Z_{s-})|$, that the right hand side of the latter equality is less that $2t/n$ hence tends to 0, as $n\to\infty$.

(h) Justify that $h_n(y)dy \to \delta_0$ weakly, as $n\to\infty$ and that $f_n(x)\to r_\lambda(x)$ as $n\to\infty$. Show that

$$r_\lambda(0) - \mathbb{E}\big(e^{-\lambda A_t} r_\lambda(Z_t)\big) = 0.$$

(i) Conclude that $\mathbb{P}(Z_t = 0) = 1$ for each $t$ hence $Z$ is identically 0.

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**Exercice (Bonus)** *Last word : another approach to pathwise uniqueness*
Consider the same SDE (12) as in 8 of the Exercise 4 and make the same assumptions on $F$ and $\rho$. Recall that $N$ is the PRM associated to $X$ with intensity measure $dt\nu(dz)$.

1. Explain why the equation (12) can be written as

$$Y_t = y_0 + \int_0^t \int_{|z|\le 1} F(Y_{s-})z\widetilde{N}(ds,dz) + \int_0^t\int_{|z|>1} F(Y_{s-})zN(ds,dz). \tag{15}$$

2. Consider again the sequence $a_n \downarrow 0$ such that $\int_{a_{n+1}}^{a_n}\rho(x)^{-\alpha}dx = n$ and also $h_n$ non-negative $C^2$ even functions with the support in $[a_{n+1},a_n]$, whose integral is 1 and with $h_n(x)\le 2/(n\rho(x)^\alpha)$. Set $u(x) = |x|^{\alpha-1}$ and $u_n = u\star h_n$. Justify that $u_n(s)\to u(x)$, as $n\to\infty$.
3. In this question we prove that $Lu_n = cu_n$ with $c$ a constant independent of $n$. Set $u^\epsilon(x) = u(x)e^{-\epsilon|x|}$ and $u_n^\epsilon = u^\epsilon \star h_n$. The functions $u_n^\epsilon$ belongs to $\mathcal{S}(\mathbb{R})$.
   (a) Use Exercise 3 point 2, with same notations, and show $\mathcal{F}(Lu_n^\epsilon)(\xi) = c(\alpha)|\xi|^\alpha\mathcal{F}u_n^\epsilon(\xi)$.
   (b) Show that
   $$\mathcal{F}u_n^\epsilon(\xi) = c'(\alpha)\big[(\epsilon - i\xi)^{-\alpha} + (\epsilon - i\xi)^{-\alpha}\big]\mathcal{F}h_n(\xi).$$
   (c) For $\xi\neq 0$ compute $\lim\limits_{\epsilon\to 0}\big[(\epsilon - i\xi)^{-\alpha} + (\epsilon-\xi)^{-\alpha}\big]$ and deduce $Lu_n = \lim_{\epsilon\to 0} Lu_n^\epsilon = c\,u_n$.
4. Denote again $Y^1$ and $Y^2$ two solutions of (12) and set $Z_t = Y_t^1 - Y_t^2$. Show that

$$u_n(Z_t) - u_n(0) = \int_0^t |F(Y_s^1) - F(Y_s^2)|^\alpha Lu_n(Z_s)ds$$

$$+\int_0^t\int\Big[u_n\big(Z_{s-} + (F(Y_s^1) - F(Y_s^2))z\big) - u_n(Z_{s-})\Big]\widetilde{N}(ds,dz).$$

5. Show that $|F(x) - F(y)|^\alpha Lu_n(x,y) \le c\rho(|x-y|)^\alpha h_n(x-y)\le c/n$.
6. Set, for $k\ge 1$, $T_k = \inf\{t>0 : |Z_t| > k\}$. Deduce that

$$\mathbb{E}\Big[u_n(Z_{t\wedge T_k})\Big] \le u_n(0) + \mathbb{E}\Big[\int_0^{t\wedge T_k}\frac{c}{n}ds\Big].$$

Conclude that $\mathbb{E}\big[|Z_{t\wedge T_k}|^{\alpha-1}\big] = 0$ and then $Z_t = 0$ a.s.