Homework # 5 : Martingales, integrals and Itô formula

Due on March 6th

Exercise 1 *Martingale problem*
Let $L : D \to \mathrm{B}(\mathbb{R}^d)$ be a linear operator with $D \subset \mathrm{C}_b(\mathbb{R}^d)$. Assume that $X = (X_t : t \geq 0)$ solve the (L, D)-martingale problem with initial distribution δ_x, for each $x \in \mathbb{R}^d$. Show that the operator L is dissipative, i.e. $\|(\lambda I - L)f\| \geq \lambda \|f\|$ for some (all) $\lambda > 0$ and $f \in D$. Recall that a consequence of Ex. 2 in HW#4 is that

$$e^{-\lambda t} f(X_t) - f(X_0) - \int_0^t e^{-\lambda s}\big(\lambda f(X_s) - Lf(X_s)\big)ds, \quad t \geq 0,$$

is a martingale. Use it to express $f(x)$ as an expectation and deduce that for each x, $|f(x)| \leq \lambda^{-1}\|\lambda f - Lf\|$.

Exercise 2 *Wiener-Lévy integral*
Let $X = (X_t, t \geq 0)$ be a Lévy process taking values in $\mathbb{R}^d$ and let $f \in \mathrm{L}^2(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$. We consider the Wiener–Lévy integral $Y = (Y_t, t \geq 0)$ where for each $t \geq 0$, $Y_t = \int_0^t f(s)dX_s$.

1. For each $0 \leq s < t < \infty$, $Y_t - Y_s$ is independent of $\mathcal{F}_s = \sigma(X_r : r \leq s)$.
2. Assume that the Lévy measure of the Lévy process X satisfies $\int_{|z|\geq 1} |z|\nu(dz) < \infty$. Show that $Y = (Y_t, t \geq 0)$ is stochastically continuous.

Exercise 3 *Itô's type formula*
Let $P(t) = \int_{|z|\geq 1} zN(t, dz)$, $t \geq 0$ be a compound Poisson process, where N is a PRM associated to a Lévy process with Lévy measure $\int_{|z|\geq 1} z^2\nu(dz) < \infty$. Define the Poisson stochastic integral by

$$\int_0^t \int_{|z|\geq 1} K(s, z)N(ds, dz) := \sum_{0\leq s\leq t} K(s, \Delta P(s))\mathbb{1}_{\{|\Delta P(s)|\geq 1\}},$$

where K is a predictable real process that means that for each $s \in [0, t]$, $(z, \omega) \to K(s, z, \omega)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_s$ -measurable and for each $z \in \mathbb{R}$, $\omega \in \Omega$ the mapping $s \mapsto K(s, z, \omega)$ is left-continuous.

1. Set, for $t \geq 0$,
$$Y(t) = Y(0) + \int_0^t \int_{|z|\geq 1} K(s, z)N(ds, dz).$$
Show that for each $f \in \mathrm{C}(\mathbb{R})$ and each $t \geq 0$ with probability 1, we have
$$f(Y(t)) - f(Y(0)) = \int_0^t \int_{|z|\geq 1} \big[f(Y(s-) + K(s, z)) - f(Y(s-))\big]N(ds, dz).$$
One can uses the jump times of $P(t)$, $T_0 = 0$, $T_n = \inf\{t > T_{n-1} : |\Delta P(t)| \geq 1\}$.
2. Set, for $t \geq 0$,
$$Z(t) = Z(0) + \int_0^t G(s)ds + \int_0^t F(s)dB_s + \int_0^t \int_{|z|\geq 1} K(s, z)N(ds, dz)$$

$$= Z(0) + Z_c(t) + \int_0^t \int_{|z|\geq 1} K(s,z)N(ds,dz),$$

where G is a predictable real process such that $\int_0^t G(s)ds < \infty$ and where F is a predictable real process such that $\mathbb{P}(\int_0^t |F(s)|^2 ds < \infty) = 1$. Show that for each $f \in rmC^2(\mathbb{R})$ and each $t \geq 0$, with probability 1, we have

$$f(Z(t)) - f(Z(0)) = \int_0^t f'(Z(s-))ds + \frac{1}{2}\int_0^t f''(Z(s-))d[Z_c, Z_c](s)$$

$$+ \int_0^t \int_{|z|\geq 1} \big[f(Y(s-) + K(s,z)) - f(Y(s-))\big] N(ds,dz).$$