Université de Rennes 1 | Année 2021-2022
Master 2ème Mathématiques | Processus stochastiques à sauts

Homework # 4 : Feller semigroups and martingales

Due on March 10th

Exercise 1 *A basic martingale*
Let $X = (X_t : t \geq 0)$ be a Feller process with generator $(A, \mathcal{D}(A))$ and semigroup $(P_t : t \geq 0)$. Show that for every $f \in \mathcal{D}(A)$, the process

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Af(X_s)ds, \quad t \geq 0, \tag{1}$$

is a martingale for the canonical filtration $\mathcal{F}_t^X = \sigma(X_r : r \leq t)$ and for any $\mathbb{P}_x$, $x \in \mathbb{R}^d$.

Exercise 2 *A martingale transformation and a martingale problem*

1. Let (M_t) be a càdlàg $(\mathcal{F}_t)$-martingale and let $f : [0, \infty) \to \mathbb{R}$ be a continuously differentiable deterministic function. Show that for all $s < t$

$$\mathbb{E}(M_t\, f(t) - M_s\, f(s)|\mathcal{F}_s) = (f(t) - f(s))M_s = \mathbb{E}\Big(\int_s^t M_u\, f'(u)du|\mathcal{F}_s\Big) \tag{2}$$

   and that $M_t\, f(t) - \displaystyle\int_0^t M_u\, f'(u)du$ is also an $(\mathcal{F}_t)$-martingale.
   Hint: to get the second equality in (2) introduce a division $s = u_0 < u_1 < \ldots < u_n = t$ of the interval $[s, t]$ with $u_j = j(t - s)/n$, $j = 0, 1, \ldots, n$ and use the same argument as for the first equality in (2).

2. Let $X = (X_t : t \geq)$ solve the $(L, \mathcal{D})$ martingale problem[1] and let $\phi : [0, \infty) \to \mathbb{R}$ be a bounded continuously differentiable deterministic function. Show that, for any $f \in \mathcal{D}$,

$$S_t := f(X_t)\phi(t) - \int_0^t \big[\phi'(s)\, f(X_s) + \phi(s)\, Lf(X_s)\big]\, ds \quad \text{ is a martingale.}$$

   Hint : use the first part.

3. Application : if $\phi(s) = e^{-\lambda s}$, $\lambda > 0$, what is S ?

Exercise 3 *Resolvent equation*
Let $(P_t : t \geq 0)$ be a Feller semigroup. Prove that its resolvent operator R_λ satisfies the following equation

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu, \quad \text{for all } \lambda, \mu > 0.$$

[1] Let L be a linear operator $L : \mathcal{D} \to B_b(\mathbb{R}^d)$, $\mathcal{D} \subset \mathrm{C}(\mathbb{R}^d)$: we say that a $\mathbb{R}^d$-valued process $X = (X_t : t \geq 0)$ solves the $(L, \mathcal{D})$-martingale problem if processes (1) with A replaced by L are martingales for any $f \in \mathcal{D}$ with respect to its own filtration. Hence any Feller process X solves the $(A, \mathcal{D})$-martingale problem, where A is the generator of X and $\mathcal{D} \subset \mathcal{D}(A)$.