Homework # 3 : Martingales càdlàg/ Itô-Lévy décomposition

Due on February 22nd, 2022

Exercise 1 *Product of martingales*
Let $(M_t^1)_{t\geq 0}$ and $(M_t^2)_{t\geq 0}$ two centered càdlàg real-valued martingales with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $M_0^1 = M_0^2 = 0$. We assume that

$$\mathbb{E}\big[|M_t^1|^2\big] < \infty \text{ and } \mathbb{E}\big[|V(M^2)_t)|^2\big] < \infty, \; \forall t \geq 0,$$

where $V(M^2)_t$ denotes the variation of M^2 on $[0,t]$.

1. Denote $\pi : 0 = t_0 < t_1 < \ldots < t_k = t$ a partition of the interval $[0,t]$. Show that

$$\mathbb{E}\big[M_t^1 M_t^2\big] = \sum_{j=0}^{k-1} \mathbb{E}\big[(M_{t_{j+1}}^1 - M_{t_j}^1)(M_{t_{j+1}}^2 - M_{t_j}^2)\big].$$

2. Find a sequence of partitions $(\pi_n)_{n\geq 1}$ of the interval $[0,t]$ such that $\lim_{n\to\infty} \text{mesh}(\pi_n) = \lim_{n\to\infty} \max_{0\leq j\leq k(n)-1} |t_{j+1}^{(n)} - t_j^{(n)}| = 0$. We will prove that, with probability 1,

$$(\star) \quad \lim_{n\to\infty} \sum_{j=0}^{k(n)-1} (M_{t_{j+1}^{(n)}}^1 - M_{t_j^{(n)}}^1)(M_{t_{j+1}^{(n)}}^2 - M_{t_j^{(n)}}^2) = \sum_{0<s\leq t} \Delta M_s^1 \Delta M_s^2.$$

Fix $\omega \in \Omega$ and set $D = \{\tau_r : r \geq 1\}$ the common points of discontinuity of $[0,\infty) \ni t \mapsto M_t^1(\omega)$ and $[0,\infty) \ni t \mapsto M_t^2(\omega)$. With this notation the right hand side of $(\star)$ is

$$\sum_{0<s\leq t} \Delta M_s^1 \Delta M_s^2 = \sum_{r\geq 1} \Delta M_{\tau_r}^1 \Delta M_{\tau_r}^2.$$

Prove $(\star)$ by using the following steps (for simplicity we will omit ω).

(a) Denote by $K := 2\big(\sup_{0<s\leq t} |M_s^1| + \sup_{0<s\leq t} |M_s^2|\big) < \infty$. Fix $\varepsilon > 0$ arbitrary and show that there exists $\{\eta_r^{(\varepsilon)} : r \geq 1\}$ such that for each $s \in (0, \eta_r^{(\varepsilon)}]$:

$$\max\Big\{|M_{\tau_r}^1 - M_{\tau_r - s}^1 - \Delta M_{\tau_r}^1|, |M_{\tau_r}^2 - M_{\tau_r - s}^2 - \Delta M_{\tau_r}^2|\Big\} \leq \frac{\varepsilon}{K2^r}.$$

(b) Let $(\pi_n)_{n\geq 1}$ a partition of the interval $[0,t]$ containing exactly $\{\tau_1, \ldots, \tau_{r(n)}\}$ among the points of D and such that $\text{mesh}(\pi_n) \leq \inf_{i\leq r(n)} \eta_i^{(\varepsilon)}$. Show that the quantity

$$\Big|\sum_{j=0}^{k(n)-1} (M_{t_{j+1}^{(n)}}^1 - M_{t_j^{(n)}}^1)(M_{t_{j+1}^{(n)}}^2 - M_{t_j^{(n)}}^2) - \sum_{r\geq 1} \Delta M_{\tau_r}^1 \Delta M_{\tau_r}^2\Big|$$

can be bounded by a sum of the following two quantities

$$A := \Big|\sum_{i=0}^{r(n)} (M_{\tau_i}^1 - M_{t_{p_{i-1}}^{(n)}}^1)(M_{\tau_i}^2 - M_{t_{p_{i-1}}^{(n)}}^2) - \sum_{j=0}^{r(n)} \Delta M_{\tau_i}^1 \Delta M_{\tau_i}^2\Big|$$

and

$$B := \sum_{j=0,\ldots,k(n)-1, j\notin\{p_i, i\leq r(n)\}} (M_{t_{j+1}^{(n)}}^1 - M_{t_j^{(n)}}^1)(M_{t_{j+1}^{(n)}}^2 - M_{t_j^{(n)}}^2),$$

where $\tau_i = t_{p_i}^{(n)}$, $\forall i \leq r(n)$, and thus $t_j^{(n)}$ for $j \notin \{p_1, \ldots, p_{r(n)}\}$ is not in $\{\tau_1, \ldots \tau_{r(n)}\}$.

(c) Denote $A(\eta^{(\varepsilon)}) := \sum_{r=1}^{\infty}\Big[(M^1_{\tau_r}-M^1_{\tau_r-\eta_r^{(\varepsilon)}})(M^2_{\tau_r}-M^2_{\tau_r-\eta_r^{(\varepsilon)}})-\Delta M^1_{\tau_r}\Delta M^2_{\tau_r}\Big]$. By noting that

$$\begin{aligned}&\Big|(M^1_{\tau_r}-M^1_{\tau_r-\eta_r^{(\varepsilon)}})(M^2_{\tau_r}-M^2_{\tau_r-\eta_r})-\Delta M^1_{\tau_r}\Delta M^2_{\tau_r}\Big|\\ &\qquad\le \Big|M^1_{\tau_r}-M^1_{\tau_r-\eta_r^{(\varepsilon)}}-\Delta M^1_{\tau_r}\Big|\,\Big|M^2_{\tau_r}-M^2_{\tau_r-\eta_r^{(\varepsilon)}}\Big|+\Big|M^2_{\tau_r}-M^2_{\tau_r-\eta_r^{(\varepsilon)}}-\Delta M^2_{\tau_r}\Big|\,\big|\Delta M^1_{\tau_r}\big|,\end{aligned}$$

show that,

$$|A(\eta^{(\varepsilon)})|\le 2\big(\sup_{0<s\le t}|M^1_s|+\sup_{0<s\le t}|M^2_s|\big)\sum_{r=1}^{\infty}\frac{\varepsilon}{K2^r}<\varepsilon.$$

Conclude that $A\le\varepsilon$ as $\mathrm{mesh}(\pi_n)\le\inf_{i\le r(n)}\eta_k^{(\varepsilon)}$.

(d) Show that

$$B\le\max_{0\le j\le k(n)-1,j\notin\{p_i,i\le r(n)\}}|M^1_{t^{(n)}_{j+1}}-M^1_{t^{(n)}_j}|\,V_{\pi_n}(M^2),\quad\text{where}\quad V_{\pi_n}(M^2)=\sum_{j=0}^{k(n)-1}|M^2_{t^{(n)}_{j+1}}-M^2_{t^{(n)}_j}|.$$

Prove that,

$$\lim_{n\to\infty}\max_{1\le j\le k(n),,j\notin\{p_i,i\le r(n)\}}|M^1_{t^{(n)}_j}-M^1_{t^{(n)}_{j-1}}|=0.$$

Hint* : suppose that there exists $\delta>0$ and $(n_\ell)\uparrow$ s.t. exists $i_\ell\in\{1,\dots,k(n_\ell)\}\setminus\{p_1,\dots,p_{n_\ell}\}$ with $\big|M^1(t^{(n_\ell)}_{i_\ell})-M^1(t^{(n_\ell)}_{i_\ell-1})\big|>\delta$ for all ℓ, and reveal a contradiction.

3. Deduce that $\mathbb{E}\big[M^1_tM^2_t\big]=\mathbb{E}\sum_{0<s\le t}\Delta M^1_s\Delta M^2_s$, by noting that $\mathbb{E}\Big[\Big(\sup_{0<s\le t}|M^1_s|\Big)V(M^2)_t\Big]<\infty$, and the fact that

$$\Big|\sum_{j=0}^{k(n)-1}\Big((M^1_{t^{(n)}_{j+1}}-M^1_{t^{(n)}_j})\Big(M^2_{t^{(n)}_{j+1}}-M^2_{t^{(n)}_j})\Big)\Big|\le 2\Big(\sup_{0<s\le t}|M^1_s|\Big)V(M^2)_t.$$

4. Let B be a Borel set bounded away from 0 and let $g\in\mathrm{L}^2(B,\mu_X)$ where μ_X is the Lévy measure associated to a Lévy process X. Show that $M_t=\int_B g(z)\widetilde{N}_X(t,dz)$ satisfies

$$\mathbb{E}\big[|M_t|^2\big]<\infty\ \text{ and }\ \mathbb{E}\big[|V(M)_t)|^2\big]<\infty,\ \forall t\ge0.$$

Exercise 2 *Interlacing*
Let $Y=(Y(t):t\ge0)$ be a Lévy process with jumps bounded by 1, but may have jumps of arbitrarily small size, i.e. that there exists no $a\in(0,1)$ such that $\nu\big((-a,a)\big)=0$, where ν is the Lévy measure of Y. Assume that its Lévy-Itô decomposition is $Y(t)=bt+B_\Gamma(t)+\int_{|z|<1}z\widetilde{N}(t,dz)$, $t\ge0$. Define the sequence $(\varepsilon_n)_{n\ge1}$ given by $\varepsilon_n=\sup\{y\ge0:\int_{0<|z|<y}z^2\nu(dz)\le 1/8^n\}$ and introduce a sequence of Lévy processes having the size of each jump bounded below by ε_n and above by 1, as follows: $Y_n(t)=bt+B_A(t)+\int_{\varepsilon_n\le|z|<1}z\widetilde{N}(t,dz)$, $t\ge0$.

1. Show that Y_n can be written as the sum of a Brownian motion with drift C_n and of a compound Poisson process with jumps ΔY.[1] What is the expression of the drift?
2. Verify that the sequence $(\varepsilon_n)_n$ is decreasing and converges to 0.
3. Fix $T>0$. Show that for each $n\ge1$ the process $Y_{n+1}-Y_n$ is an L^2-martingale, satisfying $\mathbb{E}\big(\sup_{0\le t\le T}|Y_{n+1}(t)-Y_n(t)|^2\big)\le 4T/8^n$ and $\mathbb{P}\big(\liminf_{n\to\infty}\big\{\sup_{0\le t\le T}|Y_{n+1}(t)-Y_n(t)|<1/2^n\big\}\big)=1$.
4. Deduce that the sequence $\{Y_n(t)\}_{n\ge1}$ is almost surely uniformly Cauchy on compact intervals. Conclude that Y_n tends to Y uniformly on compact intervals of $[0,+\infty)$.[2]

[1] Thus the process Y_n can be built by interlacing. Bonus : write the interlaced expression of Y_n.
[2] If X is an arbitrary Lévy process, then by the Lévy–Itô decomposition, $X(t)=Y(t)+\int_{|z|\ge1}zN(t,dz)$, $t\ge0$, so its paths can be obtained by a further interlacing of Y by a compound Posson process with jumps of size bigger than 1.