Homework # 3 : Poisson random measures / Itô-Lévy décomposition

Due on February 6th

Exercise 1 *Functional of Poisson Random measure*
Let Φ be a Poisson Random Measure (PRM) with intesity measure μ on $S \in \mathcal{B}(\mathbb{R}^d)$ and let $f : S \to \mathbb{R}$ be a Borel measurable function. We set $X := \int_S f(z)\Phi(dz)$. We want to prove that X is a.s. absolutely convergent if and only if $\int_S \big(1 \wedge |f(z)|\big)\mu(dz) < \infty$ and then to compute $\mathbb{E}\big(e^{iuX}\big)$.

1. Assume first that f is a simple function $f(z) = \sum_{j=1}^p f_j \mathbb{1}_{A_j}(z)$ with f_j positive constants, $A_1, \ldots, A_p$ disjoint subsets in S such that $\mu(A_1 \cup \ldots \cup A_p) < \infty$. Show that $X < \infty$ a.s. and compute $\mathbb{E}\big(e^{-vX}\big)$ for $v > 0$.
2. Assume now that f is a positive measurable function. Show that, for any $v > 0$,
$$(\star) \qquad \mathbb{E}\big(e^{-vX}\big) = \exp\Big\{ -\int_S \big(1 - e^{-v f(z)}\big)\mu(dz)\Big\}.$$
Deduce that $X < \infty$ a.s. if and only if, for any $v > 0$, $\int_S \big(1 - e^{-v f(z)}\big)\mu(dz) < \infty$. [1]
3. For a general measurable function f write $f = f^+ - f^-$ and introduce two random measures $\Phi_+ = \Phi\big(\,\cdot \cap \{z : f(z) \geq 0\}\big)$ and $\Phi_- = \Phi\big(\,\cdot \cap \{z : f(z) < 0\}\big)$. Show that $\Phi_\pm$ are independent PRM and $X = X_+ - X_-$. Conclude.
4. Under the same condition, show that, for any $u \in \mathbb{R}$, $\mathbb{E}\big(e^{iuX}\big) = \exp\big\{-\int_S \big(1 - e^{iuf(z)}\big)\mu(dz)\big\}$.
Hint : follow the same steps as precedingly (for instance, show that if f is positive, $(\star)$ could be extended analytically with v replaced by $v - iu$ and conclude for this case). [2]

Exercise 2 *Interlacing*
Let $Y = (Y(t) : t \geq 0)$ be a Lévy process with jumps bounded by 1, but may have jumps of arbitrarily small size, i.e. that there exists no $a \in (0,1)$ such that $\nu\big((-a,a)\big) = 0$, where ν is the Lévy measure of Y. Assume that its Lévy-Itô decomposition is $Y(t) = bt + B_\Gamma(t) + \int_{|z|<1} z\widetilde{N}(t,dz)$, $t \geq 0$. Define the sequence $(\varepsilon_n)_{n\geq 1}$ given by $\varepsilon_n = \sup\{y \geq 0 : \int_{0<|z|<y} z^2\nu(dz) \leq 1/8^n\}$ and introduce a sequence of Lévy processes having the size of each jump bounded below by ε_n and above by 1, as follows : $Y_n(t) = bt + B_A(t) + \int_{\varepsilon_n \leq |z| < 1} z\widetilde{N}(t,dz)$, $t \geq 0$.

1. Show that Y_n can be written as the sum of a Brownian motion with drift C_n and of a compound Poisson process with jumps ΔY. [3] What is the expression of the drift ?
2. Verify that the sequence $(\varepsilon_n)_n$ is decreasing and converges to 0.
3. Fix $T > 0$. Show that for each $n \geq 1$ the process $Y_{n+1} - Y_n$ is an L^2-martingale, satisfying $\mathbb{E}\big(\sup_{0\leq t\leq T} |Y_{n+1}(t) - Y_n(t)|^2\big) \leq 4T/8^n$ and $\mathbb{P}\big(\liminf_{n\to\infty} \big\{\sup_{0\leq t\leq T} |Y_{n+1}(t) - Y_n(t)| < 1/2^n\big\}\big) = 1$.
4. Deduce that the sequence $\{Y_n(t)\}_{n\geq 1}$ is almost surely uniformly Cauchy on compact intervals. Conclude that Y_n tends to Y uniformly on compact intervals of $[0, +\infty)$. [4]

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1. <u>Bonus :</u> Show that the condition in Ex. 1.2 is equivalent with $\int_S \big(1 \wedge f(z)\big)\mu(dz) < \infty$ (f is still positive).
2. <u>Bonus :</u> Show that $\mathbb{E}(X) = \int_S f(z)\mu(dz)$, if $\int_S |f(z)|\mu(dz) < \infty$ and $\mathbb{E}\big(X^2\big) = \int_S f(z)^2\mu(dz) + \int_S f(z)\mu(dz))^2$, if $\int_S f(z)^2\mu(dz) < \infty$.
3. Thus the process Y_n can be built by interlacing. <u>Bonus :</u> write the interlaced expression of Y_n.
4. If X is an arbitrary Lévy process, then by the Lévy–Itô decomposition, $X(t) = Y(t) + \int_{|z|\geq 1} zN(t,dz)$, $t \geq 0$, so its paths can be obtained by a further interlacing of Y by a compound Posson process with jumps of size bigger than 1.