Université de Rennes 1 Année 2021-2022
Master 2ème année Math Fonda Processus stochastiques à sauts

Homework # 2 : Stable and self-similar processes

Due on February 6th

Exercise 1 *Stable and self-similar Lévy processes*

1. Let X be a non-zero real random variable. Suppose that $b, c \in (0, \infty)$ satisfy $bX \sim cX$. Show that $b = c$. Suppose further that $b, c \in (0, \infty)$ and $u, v \in \mathbb{R}$ satisfy $bX + u \sim cX + v$. Show that $b = c$ and $u = v$.

2. A real random variable X is called *stable* if $\forall n \in \mathbb{N}$, $\exists b_n > 0$, $c_n \in \mathbb{R}$ such that $X'_1 + \ldots + X'_n \sim b_n X + c_n$, where $X'_1, \ldots, X'_n$ are i.i.d. copies of X. If $c_n = 0$ then X is called *strictly stable*.

   Give an equivalent condition for the stability of a r.v. in terms of its characteristic function. Prove that a stable r.v. is infinitely divisible.

3. A Lévy process $X = (X_t : t \geq 0)$ is *(strictly) stable* if X_1 is a (strictly) stable r.v.

   Show that A Lévy process X is stable if and only if all random variables X_t are stable. Hint : use the previous point.

4. Let $X = (X_t : t \geq 0)$ be a stochastically continuous process in $\mathbb{R}$. The process is called *self-similar* if $\forall a \geq 0$, $\exists b = b(a)$ such that $(X(at) : t \geq 0) \sim b\,(X_t : t \geq 0)$ (in the sense that both sides have the same finite-dimensional distributions).

   Prove that if X is a non-degenerate self-similar process there exists a unique index $H \geq 0$ of self-similarity such that $b(a) = a^H$ [1].

   Hint: find a functional equation satisfied by $b(a)$ and then use the first point, the continuity in probability and the convergence of types result[2] to conclude.

5. Assume that $X = (X_t : t \geq 0)$ is a self-similar Lévy process. Show that X_1, hence $X = (X_t : t \geq 0)$ is a strictly stable process.

6. Conversely we will show that a strictly stable Lévy process is self-similar. Assume that $X = (X_t : t \geq 0)$ is a strictly stable Lévy process with Lévy exponent η:

   (a) For all $t \geq 0, u \in \mathbb{R}$, $e^{m\,t\,\eta(u)} = e^{t\,\eta(b_m\,u)}$ for $m \geq 1$. Deduce that $e^{t/m\,\eta(u)} = e^{t\,\eta(b_m^{-1}\,u)}$, for $m \geq 1$, and $e^{q\,t\,\eta(u)} = e^{t\,\eta(b(q)\,u)}$ for $q = n/m \in \mathbb{Q}_+$ and where $b(q) = b_n/b_m$.

   (b) Deduce that $X_{qt} \sim b(q)\,X_t$ and then $X_{at} \sim b(a)\,X_t$ for all $t \geq 0$. Conclude.

Remark : A real-valued random variable X is stable if and only if there exist $\sigma > 0$, $-1 \leq \beta \leq 1$ and $m \in \mathbb{R}$ such that for all $u \in \mathbb{R}$:

- when $\alpha = 2$, $\varphi_X(u) = \exp\left\{imu - \frac{1}{2}\sigma^2 u^2\right\}$: normal distribution;
- when $\alpha \neq 1, 2$, $\varphi_X(u) = \exp\left\{imu - \sigma^\alpha |u|^\alpha [1 - i\beta \mathrm{sgn}(u) \tan(\pi\alpha/2)]\right\}$: Lévy distribution for $\alpha = 1/2$ and $\beta = 1$ having density $f_X(x) = (\frac{\sigma}{2\pi})^{1/2} \frac{1}{(x-m)^{3/2}} \exp\left(-\frac{\sigma}{2(x-m)}\right)$;
- when $\alpha = 1$, $\varphi_X(u) = \exp\left\{imu - \sigma|u|[1 + i\beta\frac{2}{\pi}\mathrm{sgn}(u)\log(|u|)]\right\}$: Cauchy distribution for $\beta = 0$ having density $f_X(x) = \frac{\sigma}{\pi[(x-m)^2 + \sigma^2]}$.

[1] It can be shown that if X is a non-degenerate self-similar Lévy process then $H \geq \frac{1}{2}$ (this result is due to Lamperti and is accepted). It is common to call $\alpha = 1/H \in (0, 2]$ the index of stability and $X_{nt} \sim n^{1/\alpha} X_t$.

[2] Convergence of types : let $(Y_n)_{n\geq 1}$, Y and Y' be random variables such that Y and Y' are non-degenerate. Suppose that there are constants $a_n > 0$ and $c_n \in \mathbb{R}$ such that $Y_n \to Y$ and $a_n Y_n + c_n \to Y'$ in distribution. Then the limits $a = \lim a_n$ and $c = \lim c_n$ exist and $Y' \sim aY + c$.