

## HOMEWORK # 2 : LÉVY PROCESSES

Due on January 30th

### Exercise 1 *Stable and self-similar Lévy processes*

1. Let  $X$  be a non-zero real random variable. Suppose that  $b, c \in (0, \infty)$  satisfy  $bX \sim cX$ . Show that  $b = c$ . Suppose further that  $b, c \in (0, \infty)$  and  $u, v \in \mathbb{R}$  satisfy  $bX + u \sim cX + v$ . Show that  $b = c$  and  $u = v$ .
2. A real random variable  $X$  is called *stable* if  $\forall n \in \mathbb{N}, \exists b_n > 0, c_n \in \mathbb{R}$  such that  $X'_1 + \dots + X'_n \sim b_n X + c_n$ , where  $X'_1, \dots, X'_n$  are i.i.d. copies of  $X$ . If  $c_n = 0$  then  $X$  is called *strictly stable*. Give an equivalent condition for the stability of a r.v. in terms of its characteristic function. Prove that a stable r.v. is infinitely divisible.
3. A Lévy process  $X = (X_t : t \geq 0)$  is (*strictly*) *stable* if  $X_1$  is a (strictly) stable r.v. Show that A Lévy process  $X$  is stable if and only if all random variables  $X_t$  are stable. Hint : use the previous point.
4. Let  $X = (X_t : t \geq 0)$  be a stochastically continuous process in  $\mathbb{R}$ . The process is called *self-similar* if  $\forall a \geq 0, \exists b = b(a)$  such that  $(X(at) : t \geq 0) \sim b(X_t : t \geq 0)$  (in the sense that both sides have the same finite-dimensional distributions). Prove that if  $X$  is a non-degenerate self-similar process there exists a unique index  $H \geq 0$  of self-similarity such that  $b(a) = a^{H-1}$ . Hint : find a functional equation satisfied by  $b(a)$  and then use the first point, the continuity in probability and the convergence of types result<sup>2</sup> to conclude.
5. Assume that  $X = (X_t : t \geq 0)$  is a self-similar Lévy process. Show that  $X_1$ , hence  $X = (X_t : t \geq 0)$  is a strictly stable process.
6. We show that, conversely, that a strictly stable Lévy process is self-similar. Assume that  $X = (X_t : t \geq 0)$  is a strictly stable Lévy process with Lévy exponent  $\eta$ 
  - (a) For all  $t \geq 0, u \in \mathbb{R}$ ,  $e^{mt\eta(u)} = e^{t\eta(b_m u)}$  for  $m \geq 1$ . Deduce that  $e^{t/m\eta(u)} = e^{t\eta(b_m^{-1} u)}$ , for  $m \geq 1$ , and  $e^{qt\eta(u)} = e^{t\eta(b(q)u)}$  for  $q = n/m \in \mathbb{Q}_+$  and where  $b(q) = b_n/b_m$ .
  - (b) Deduce that  $X_{qt} \sim b(q)X_t$  and then  $X_{at} \sim b(a)X_t$  for all  $t \geq 0$ . Conclude.

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**Remark :** A real-valued random variable  $X$  is stable if and only if there exist  $\sigma > 0, -1 \leq \beta \leq 1$  and  $m \in \mathbb{R}$  such that for all  $u \in \mathbb{R}$  :

- when  $\alpha = 2$ ,  $\varphi_X(u) = \exp \left\{ imu - \frac{1}{2}\sigma^2 u^2 \right\}$  : normal distribution ;
- when  $\alpha \neq 1, 2$ ,  $\varphi_X(u) = \exp \left\{ imu - \sigma^\alpha |u|^\alpha [1 - i\beta \operatorname{sgn}(u) \tan(\pi\alpha/2)] \right\}$  : Lévy distribution for  $\alpha = 1/2$  and  $\beta = 1$  having density  $f_X(x) = \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x-m)^{3/2}} \exp \left( -\frac{\sigma}{2(x-m)} \right)$  ;
- when  $\alpha = 1$ ,  $\varphi_X(u) = \exp \left\{ imu - \sigma |u| [1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log(|u|)] \right\}$  : Cauchy distribution for  $\beta = 0$  having density  $f_X(x) = \frac{\sigma}{\pi[(x-m)^2 + \sigma^2]}$ .

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1. It can be shown that if  $X$  is a non-degenerate self-similar Lévy process then  $H \geq \frac{1}{2}$  (this result is due to Lamperti and is admitted). It is common to call  $\alpha = 1/H \in (0, 2]$  the index of stability and  $X_{nt} \sim n^{1/\alpha} X_t$ .

2. Convergence of types : let  $(Y_n)_{n \geq 1}$ ,  $Y$  and  $Y'$  be random variables such that  $Y$  and  $Y'$  are non-degenerate. Suppose that there are constants  $a_n > 0$  and  $c_n \in \mathbb{R}$  such that  $Y_n \rightarrow Y$  and  $a_n Y_n + c_n \rightarrow Y'$  in distribution. Then the limits  $a = \lim a_n$  and  $c = \lim c_n$  exist and  $Y' \sim aY + c$ .