Homework # 1 : Infinitely Divisible distributions

Due on 23rd of January, 2022

Exercise 1 *(basics on Lévy processes)*

a) If X is a Lévy process with characteristics (b, Γ, ν) show that $-X = (-X_t : t \geq 0)$ is also a Lévy process and find its characteristics. Show also that for each $\beta \in \mathbb{R}$ the process $(X_t + \beta t : t \geq 0)$ is a Lévy process and find its characteristics.

b) Show that if X and Y are stochastically continuous processes then so is their sum $X + Y = (X_t + Y_t : t \geq 0)$.

c) Show that the sum of two independent Lévy processes is again a Lévy process.

Exercise 2 *(the set of infinite divisible probability measures)*

a) Show that the weak limit (or limit in distribution) of a sequence of infinitely divisible probability measures (or random variables) on $\mathbb{R}^d$ is itself infinitely divisible.

b) Show that any infinitely divisible probability measure μ (or random variable X) on $\mathbb{R}^d$ is a weak limit (or limit in distribution) of a sequence of compound Poisson distributions (or random variables).

c) Conclude : the set of all infinite divisible probability measures on $\mathbb{R}^d$ coincides with the weak closure of the set of all compound Poisson distributions on $\mathbb{R}^d$.

Exercice (Optional - bonus) *(weak convergence and tightness)*

1. If $P \in \mathcal{M}_1(\mathbb{R}^d)$ is the distribution of a random variable X, then its characteristic function satisfies, for any $y, h \in \mathbb{R}^d$ and any $\delta > 0$,

$$|\varphi_X(u+h) - \varphi_X(u)| \leq \max_{|x| \leq \delta} |e^{ihx} - 1| + 2\mathbb{P}(|X| \geq \delta).$$

2. Assume that a sequence $\{P_n : n \geq 1\}$ of probability measures on $\mathbb{R}^d$ *converges weakly* to a probability measure P i.e. assume that

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} f(x) P_n(dx) = \int_{\mathbb{R}^d} f(x) P(dx),$$

then the family $\{P_n : n \geq 1\}$ is *tight* i.e.

$$\forall \varepsilon > 0, \exists K > 0 \, : \, \forall n, \, \mathbb{P}_n(|x| > K) < \varepsilon,$$

and their characteristic functions φ_n converges uniformly on compact sets. Hint : use the tightness and the first point to show that the family $\{\varphi_n : n \geq 1\}$ is *equicontinuous*, i.e.

$$\forall \varepsilon > 0 \, \exists \delta > 0 : |\varphi_n(u+h) - \varphi_n(u)| < \varepsilon, \, \forall |h| < \delta, \, n \geq 1.$$

3. Show that if a family of probability measures $\{P_n : n \geq 1\}$ is tight, then it is *relatively weakly compact*, i.e. any sequence of this family has a weakly convergent subsequence. Hint : use the idea of the preceding point and Lévy's theorem.