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DISTRIBUTION TAILS FOR SOLUTIONS OF SDE DRIVEN BY AN ASYMMETRIC STABLE LÉVY PROCESS

BY

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Abstract. The behaviour of the tails of the invariant distribution for stochastic differential equations driven by an asymmetric stable Lévy process is obtained. We generalize a result by Samorodnitsky and Grigoriu [8] where the stable driving noise was supposed to be symmetric.

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1. INTRODUCTION

The goal of this paper is to extend a result obtained by Samorodnitsky and Grigoriu [8]. They consider the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}L_t - f(X_t)\mathrm{d}t, \quad X_0 = x, \tag{1.1}$$

where f is a function which is regularly varying at infinity and L is a symmetric Lévy motion and they study the exact rate of decay of the tail probabilities of the random variables X_t, $t > 0$. The proof in [8] is technical and in Remark 3.2, p. 76, the authors conjecture that their main result remains true without the assumption of symmetry of the Lévy process. The present paper (Section 2) contains a proof of this conjecture and we reduce the technical difficulties announced in the cited remark by assuming that the Lévy process is α-stable. More precisely, we assume that X is a solution of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}\ell_t - f(X_t)\mathrm{d}t, \quad X_0 = x, \tag{1.2}$$

where ℓ is an asymmetric α-stable Lévy process with Lévy measure given by

$$\nu(\mathrm{d}z) = |z|^{-1-\alpha}[a_- \mathbb{1}_{\{z<0\}} + a_+ \mathbb{1}_{\{z>0\}}]\mathrm{d}z. \tag{1.3}$$

Here $\alpha \in (0,2) \setminus \{1\}$, $a_+ \neq a_-$ and x is a real number.

Dynamics of integrated processes driven by Lévy noises appears in financial mathematics models or in physics. Moreover, diffusions in heterogeneous materials or prices in finance could be modelled by stochastic differential equations driven by asymmetric Lévy noises (see for instance [9]). In [3] we studied a scaling limit of the position process whose speed satisfies a one-dimensional stochastic differential equation driven by an α-stable Lévy process, multiplied by a small parameter $\varepsilon > 0$, in a potential of the form a power function of exponent $\beta + 1$. More precisely, we considered the stochastic differential equation

$$\mathrm{d}v_t^\varepsilon = \varepsilon \mathrm{d}\ell_t - |v_t^\varepsilon|^\beta \operatorname{sgn}(v_t^\varepsilon)\mathrm{d}t, \quad v_0^\varepsilon = 0, \tag{1.4}$$

and assumed that ℓ is an α-stable Lévy noise. We proved that when the driving noise ℓ is a symmetric stable process and take a natural scaling of the position process $x_t^\varepsilon = \int_0^t v_t^\varepsilon \,\mathrm{d}t$, there is convergence in distribution toward a Brownian motion. One can wonder if this is still true when ℓ is an asymmetric α-stable Lévy noise. To get the limit in distribution as $\varepsilon \to 0$ of the position process one needs to know the exact rate of decay of the tail probabilities for the speed process (see also [2, §4, pp. 70–80]).

Let us end this section by introducing some notations and by stating our results. We will always assume that ℓ is an asymmetric α-stable Lévy process with Lévy measure given by (1.3), with $\alpha \in (0, 2) \setminus \{1\}$, $a_+ \neq a_-$ and $a_+ \neq 0$ and $a_- \neq 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function with $f(0) = 0$ which is regularly varying at $+\infty$ with exponent $\beta > 1$: for all $a > 0$, $\lim_{x\to+\infty} f(ax)/f(x) = a^\beta$. The function f could also be supposed to be regularly varying at $-\infty$ with exponent $\beta_1 > 1$, but one can only assume that for all $x \geqslant 1$, $f(-x) \leqslant -\kappa x^{\beta_1}$ for some constants $\kappa > 0$ and $\beta_1 > 1$ (see also Remark 5 and Step 9 in the proof of Theorem 1 below). Finally, we will assume furthermore that f is a locally Lipschitz function.

Recall that the process X satisfies

$$X_t = x + \ell_t - \int_0^t f(X_s)\,\mathrm{d}s, \qquad t \geqslant 0. \tag{1.5}$$

The existence and uniqueness of a global solution for (1.5) is justified in [8] for a general Lévy driving noise, and it is a consequence of [1, Theorem 6.2.11, p. 376] (see also [2, Proposition 1.2.10, p. 27]). Our main result is the following:

THEOREM 1. *Assume all the previous hypotheses on the function f, and denote, for all $u > 0$,*

$$h(u) := \int_u^{+\infty} \frac{\nu((y, +\infty))}{f(y)}\,\mathrm{d}y. \tag{1.6}$$

Then

$$\lim_{u\to+\infty} \frac{\mathbb{P}_x(X_t > u)}{h(u)} = 1, \tag{1.7}$$

uniformly with respect to $x \in \mathbb{R}$ and $t \geqslant 1$.

As a consequence we obtain the behaviour of the tail for the invariant probability measure. According to [5, Proposition 0.1, p. 604], and under the assumptions on f, the exponential ergodicity of the solution X of (1.1) is ensured. Moreover its unique invariant probability measure, denoted by $m_{\alpha,\beta}$, satisfies

$$\forall x \in \mathbb{R}, \quad \|\mathbb{P}_x^t - m_{\alpha,\beta}\|_{\mathrm{TV}} = \mathrm{O}(\exp(-Ct)) \quad \text{as } t \to \infty, \tag{1.8}$$

where $\mathbb{P}_x^t$ is the distribution of X_t under $\mathbb{P}_x$ and $\|\cdot\|_{\mathrm{TV}}$ is the total variation norm. In other words, by the definition of the total variation norm,

$$\begin{aligned}\forall x \in \mathbb{R}, \quad & \sup_{u>0} |\mathbb{P}_x(X_t > u) - m_{\alpha,\beta}((u,+\infty))| \\ & \qquad \leqslant \sup_{B\in\mathcal{B}(\mathbb{R})} |\mathbb{P}_x(X_t \in B) - m_{\alpha,\beta}(B)| \leqslant \kappa e^{-Ct}\end{aligned}$$

for some constants κ and C. Therefore, letting $t \to \infty$ in Theorem 1, we get

COROLLARY 2. *Under the assumptions of Theorem* 1*, we have*

$$\lim_{u\to+\infty} \frac{m_{\alpha,\beta}((u,+\infty))}{h(u)} = 1. \tag{1.9}$$

2. PROOF OF THEOREM 1

We split the proof of Theorem 1 into several steps.

STEP 1. For $\sigma > 0$ and for some $c > 0$ to be chosen, we introduce a Lévy process $\ell^{(\sigma)}$ with small jumps prescribed by the Lévy measure

$$\nu^{(\sigma)}(\mathrm{d}z) = |z|^{-1-\alpha}[a_- \mathbb{1}_{\{z<-\sigma\}} + a_+ \mathbb{1}_{\{z>c\sigma\}}]\mathrm{d}z. \tag{2.1}$$

The process $\ell^{(\sigma)}$ has a finite number of jumps in each finite interval of time. Denote by T_j the time when the jth jump occurs (with the convention $T_0 = 0$) and by $W_j^{(\sigma)}$ its size. The random variables $(W_j^{(\sigma)})$ are i.i.d. and, by using the underlying compound Poisson process (see for instance [1, Theorem 2.3.10, p. 93]), the probability density of $W_1^{(\sigma)}$ is given by

(2.2)
$$z \mapsto \frac{1}{\lambda_\sigma}|z|^{-1-\alpha}[a_- \mathbb{1}_{\{z<-\sigma\}} + a_+ \mathbb{1}_{\{z>c\sigma\}}] \quad \text{with} \quad \lambda_\sigma := \frac{\sigma^{-\alpha}}{\alpha}(a_- + a_+ c^{-\alpha}).$$

We will choose the constant c such that, for all y and σ,

$$\mathbb{E}(W_1^{(\sigma)}\mathbb{1}_{\{-y\leqslant W_1^{(\sigma)}\leqslant cy\}})=0.$$

Hence, by (2.2) we find $\frac{1}{\lambda_\sigma}(-a_-+c^{1-\alpha}a_+)(y^{1-\alpha}-\sigma^{1-\alpha})=0$ for all y and σ. We deduce that the only possible value of the constant is

$$c=(a_-/a_+)^{1/(1-\alpha)}. \tag{2.3}$$

Let us point out that, by the definition of $\nu^{(\sigma)}$, for $u>c\sigma>0$,

$$\nu^{(\sigma)}((u,+\infty))=\nu((u,+\infty))=:\rho(u). \tag{2.4}$$

STEP 2. Let us denote

$$X_t^{(\sigma)}=x+\ell_t^{(\sigma)}-\int_0^t f(X_s^{(\sigma)})\,\mathrm{d}s,\qquad t\geqslant 0. \tag{2.5}$$

According to [4, Theorem 19.25, p. 385], $X^{(\sigma)}$ converges in distribution to X as $\sigma\to 0$. To get (1.7) it is enough to prove that there exists σ_0 such that

$$\left|\frac{\mathbb{P}_x(X_t^{(\sigma)}>u)}{h(u)}-1\right|\leqslant \mathrm{o}(1)\quad \text{as } u\to+\infty, \tag{2.6}$$

uniformly in $x\in\mathbb{R}$, $\sigma\leqslant\sigma_0$ and $t\geqslant 1$.

STEP 3. The ordinary differential equation, starting from an arbitrary $x>0$,

$$x(t)=x-\int^t f(x(s))\,\mathrm{d}s,\qquad t\geqslant 0, \tag{2.7}$$

has a unique solution. As in [8, p. 93], we introduce, for all $u>0$,

$$g(u):=\int_u^{+\infty}\frac{1}{f(y)}\,\mathrm{d}y. \tag{2.8}$$

This function is clearly finite, non-negative, continuous and strictly decreasing for large u. Let us fix $1\leqslant s\leqslant t$. It is not difficult to see that the solution of (2.7) is non-increasing and satisfies $g(x(t))=g(x(s))+t-s$. In particular,

$$\forall u>0,\quad \text{if } x(t)>u,\ \text{then}\ g(u)>g(x(t))\geqslant t-s. \tag{2.9}$$

We now recall an important result from [8, Lemma 5.1, p. 94]. Let $A>0$ and denote by y the solution of the deterministic equation (2.7) on each interval of the form (S_{i-1}, S_i) with $0 = S_0 < \cdots < S_n < A$ but with jumps at time S_i of size j_i. More precisely,

$$y'(t) = -f(y(t)) \quad \text{on } (S_{i-1}, S_i), \qquad y(S_i) = y(S_i^-) + j_i, \quad y(0) = x. \tag{2.10}$$

As previously, it is not difficult to see that $g(y(A)) = g(y(S_n)) + A - S_n$ and in particular, for any $u > 0$, if $y(A) > u$, then $A - S_n \leqslant g(u)$. Moreover, one can compare the solution x of (2.7) with y:

$$-\max_{k=1,\dots,n}\Big(\sum_{i=k}^{n} j_i\Big)_- \leqslant y(A) - x(A) \leqslant \max_{k=1,\dots,n}\Big(\sum_{i=k}^{n} j_i\Big)_+.$$

For $a > 0$, we set $N(a) = \sup\{i \leqslant n : j_i + \cdots + j_n > a\}$ ($= 0$ if the set is empty). Therefore $\max_{N(a)+1\leqslant k\leqslant n}(\sum_{i=k}^{n} j_i) \leqslant a$. Let $t \in [S_{N(a)}, A]$ be such that $y(t) \leqslant b$. Then the solution of (2.7) starting at t from $y(t)$ satisfies $x(A) \leqslant b$, since $x(\cdot)$ is a non-increasing function. We deduce that in this case

$$y(A) \leqslant x(A) + \max_{N(a)+1\leqslant k\leqslant n}\Big(\sum_{i=k}^{n} j_i\Big) \leqslant a + b,$$

in other words,

$$\text{for } t \in [S_{N(a)}, A] \text{ with } y(t) \leqslant b, \quad \text{we have} \quad y(A) \leqslant a + b. \tag{2.11}$$

STEP 4. For $t \geqslant 1$, denote by $N_t^{(\sigma)}$ the number of jumps of $\ell^{(\sigma)}$ in $[0,t]$ and define, for all $a < 0$ and $b > 0$,

$$M_1^{(\sigma)}(a,b) := \sup\{j \leqslant N_t^{(\sigma)} : W_j^{(\sigma)} \notin [a,b]\}, \qquad \text{and } := 0 \text{ if the set is empty.} \tag{2.12}$$

To simplify notations we will denote by $\tau_1 := T_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}$ the time of the jump with index $M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)$. We can write

$$\begin{aligned}\mathbb{P}_x(X_t^{(\sigma)} > u) = {} & \mathbb{P}_x(X_t^{(\sigma)} > u,\, \tau_1 < t - g(\delta u)) \\ & + \mathbb{P}_x(X_t^{(\sigma)} > u,\, \tau_1 \in [t - g(\delta u), t]) =: p_1(u) + p_2(u).\end{aligned} \tag{2.13}$$

Let us fix $s \leqslant t$ and for $\varepsilon, \gamma, \delta, u > 0$, introduce the event

$$A_{\varepsilon,\gamma,\delta,u,s} := \left\{ \sup_{\substack{1\leqslant i\leqslant N_t^{(\sigma)} \\ s-g(\delta u)\leqslant T_i\leqslant s}} \sum_{i\leqslant j\leqslant N_t^{(\sigma)}} W_j^{(\sigma)} \mathbb{1}_{\{-\varepsilon u\leqslant W_j^{(\sigma)}\leqslant c\varepsilon u\}} \geqslant \gamma u \right\}. \tag{2.14}$$

We can state the following lemma:

LEMMA 3. *If $(1 \vee c)\varepsilon \leqslant \gamma/4$ then there exist $u_0(\varepsilon,\gamma,\delta)$, σ_0 and a positive constant $C(\varepsilon,\gamma)$ such that, for all $u \geqslant u_0(\varepsilon,\gamma,\delta)$ and $\sigma \leqslant \sigma_0$,*

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s}) \leqslant C(\varepsilon,\gamma) g(\delta u)\rho(u)^{\gamma/(4\varepsilon(1\vee c))}. \tag{2.15}$$

REMARK 4. The constants in (2.15) do not depend on t.

We postpone the proof of Lemma 3 and we proceed with the proof of our main result.

STEP 5. First, we study the term p_1 in (2.13). We can write

$$p_1(u) \leqslant \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) + \mathbb{P}_x\big(A^c_{\varepsilon,\gamma,\delta,u,t} \cap \{X^{(\sigma)}_t > u,\ \tau_1 < t - g(\delta u)\}\big). \tag{2.16}$$

By a similar reasoning to that for (2.11) (see also (2.9)), we get

$$X^{(\sigma)}_t \leqslant \delta u + \gamma u \quad \text{on the event} \quad A^c_{\varepsilon,\gamma,\delta,u,t} \cap \{\tau_1 < t - g(\delta u)\}.$$

If $\delta + \gamma \leqslant 1$, the second term on the right hand side of (2.16) is equal to 0. Furthermore, assuming that $(1 \vee c)\varepsilon \leqslant \gamma/4$, using Lemma 3, we see that there exist $u_0(\varepsilon,\gamma,\delta)$ and σ_0 such that, for all $u \geqslant u_0(\varepsilon,\gamma,\delta)$ and $\sigma \leqslant \sigma_0$,

$$p_1(u) \leqslant \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) \leqslant C(\varepsilon,\gamma) g(\delta u)\rho(u)^{\gamma/(4\varepsilon(1\vee c))}. \tag{2.17}$$

We now analyse the term p_2 in (2.13). Let us introduce, for all $a < 0$ and $b > 0$,

$$M^{(\sigma)}_2(a,b) := \sup\{j < M^{(\sigma)}_1(a,b) : W^{(\sigma)}_j \notin [a,b]\}, \tag{2.18}$$

and again to simplify we write $\tau_2 := T_{M^{(\sigma)}_2(-\varepsilon u, c\varepsilon u)}$ for the time of the jump with index $M^{(\sigma)}_2(-\varepsilon u, c\varepsilon u)$. We can write

$$\begin{aligned} p_2(u) &= \mathbb{P}_x\big(X^{(\sigma)}_t > u,\ \tau_1 \in [t - g(\delta u), t]\big) \\ &\leqslant \mathbb{P}\big(t - \tau_1 \leqslant g(\delta u),\ \tau_1 - \tau_2 \leqslant g(\delta u)\big) \\ &\quad + \mathbb{P}_x\big(X^{(\sigma)}_t > u,\ t - \tau_1 \leqslant g(\delta u),\ \tau_1 - \tau_2 > g(\delta u)\big) \\ &=: p_{21}(u) + p_{22}(u). \end{aligned} \tag{2.19}$$

STEP 6. First, we estimate p_{21}. Since $N^{(\sigma)}_{g(\delta u)}$ has the same distribution as the number of jumps of $\ell^{(\sigma)}$ in $[t - g(\delta u), t]$, we get

$$\mathbb{P}(\tau_1 \leqslant t - g(\delta u)) = \mathbb{P}\big(\forall j \in \{1,\dots,N^{(\sigma)}_{g(\delta u)}\},\ -\varepsilon u \leqslant W^{(\sigma)}_j \leqslant c\varepsilon u\big).$$

By using the fact that $N^{(\sigma)}_{g(\delta u)}$ is a Poisson distributed random variable of parameter $\lambda_\sigma g(\delta u)$ and is independent of the $W_i^{(\sigma)}$, we deduce that

$$\begin{aligned}\mathbb{P}(\tau_1 \leqslant t - g(\delta u)) &= e^{-\lambda_\sigma g(\delta u)} \sum_{n=0}^{+\infty} \frac{(\lambda_\sigma g(\delta u))^n}{n!} \mathbb{P}(-\varepsilon u \leqslant W_1^{(\sigma)} \leqslant c\varepsilon u)^n \\ &= \exp\bigl\{-\lambda_\sigma g(\delta u)\bigl(1 - \mathbb{P}(-\varepsilon u \leqslant W_1^{(\sigma)} \leqslant c\varepsilon u)\bigr)\bigr\} \\ &= \exp\{-\lambda_\sigma g(\delta u)\mathbb{P}(W_1^{(\sigma)} \notin [-\varepsilon u, c\varepsilon u])\}.\end{aligned}$$

Since

$$\mathbb{P}(W_1^{(\sigma)} \notin [-\varepsilon u, c\varepsilon u]) = \frac{c^{1-\alpha} + c^{-\alpha}}{\lambda_\sigma} \rho(\varepsilon u),$$

we get

$$\mathbb{P}(\tau_1 \leqslant t - g(\delta u)) = e^{-(c^{1-\alpha} + c^{-\alpha}) g(\delta u)\rho(\varepsilon u)}.$$

Since $t - \tau_1$ and $\tau_1 - \tau_2$ are independent and have the same distribution, we obtain

$$\begin{aligned}p_{21}(u) &= \mathbb{P}\bigl(t - \tau_1 \leqslant g(\delta u),\, \tau_1 - \tau_2 \leqslant g(\delta u)\bigr) \\ &= (1 - e^{-(c^{1-\alpha}+c^{-\alpha})g(\delta u)\rho(\varepsilon u)})^2 \leqslant (c^{1-\alpha} + c^{-\alpha})^2 \rho(\varepsilon u)^2 g(\delta u)^2.\end{aligned} \tag{2.20}$$

To estimate p_{22}, we fix η that will be chosen later. We can write

$$\begin{aligned}p_{22}(u) &\leqslant \mathbb{P}_x\bigl(X_t^{(\sigma)} > u,\, t - \tau_1 \leqslant g(\delta u),\, X_{\tau_1 -}^{(\sigma)} \leqslant \eta u\bigr) \\ &+ \mathbb{P}_x\bigl(t - \tau_1 \leqslant g(\delta u),\, X_{\tau_1 -}^{(\sigma)} > \eta u,\, \tau_1 - \tau_2 > g(\delta u)\bigr) =: p_{221}(u) + p_{222}(u).\end{aligned} \tag{2.21}$$

STEP 7. We begin with the study of p_{221}. We have

$$\begin{aligned}p_{221}(u) &\leqslant \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) \\ &\quad + \mathbb{P}_x\bigl(A^c_{\varepsilon,\gamma,\delta,u,t} \cap \{X_t^{(\sigma)} > u,\, t - \tau_1 \leqslant g(\delta u),\, X_{\tau_1 -}^{(\sigma)} \leqslant \eta u\}\bigr) \\ &=: \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) + p_{\mathrm{main}}(u).\end{aligned} \tag{2.22}$$

By using Lemma 3, for all $u \geqslant u_0(\varepsilon, \gamma, \delta)$ and $\sigma \leqslant \sigma_0$,

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,t}) \leqslant C(\varepsilon, \gamma) g(\delta u) \rho(u)^{\gamma/(4\varepsilon(1\vee c))}. \tag{2.23}$$

Let $\overline{x}_t$ be the deterministic solution of (2.7) with initial value $X_{\tau_1 -}^{(\sigma)} + W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}$. Then $g(\overline{x}_t) = g(X_{\tau_1 -}^{(\sigma)} + W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}) + t - \tau_1$. Moreover, for all $u \geqslant u_0$, on the event

$$A^c_{\varepsilon,\gamma,\delta,u,t} \cap \{X_t^{(\sigma)} > u,\, t - \tau_1 \leqslant g(\delta u),\, X_{\tau_1 -}^{(\sigma)} \leqslant \eta u\}, \tag{2.24}$$

we find, since g is decreasing, $g(\overline{x}_t) \leqslant g(\eta u + W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}) + t - \tau_1$. By using (2.11), for all $u \geqslant u_0$, on the same event (2.24) we get

$$u < X_t^{(\sigma)} < \overline{x}_t + \gamma u, \quad \text{hence} \quad (1-\gamma)u < \overline{x}_t.$$

Therefore, for all $u \geqslant u_0$, on the event (2.24), the magnitude $W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}$ of the jump at time τ_1 should satisfy

$$t - \tau_1 + g(\eta u + W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}) \leqslant g((1-\gamma)u).$$

Hence, since g is positive and decreasing, we get

$$t - \tau_1 \leqslant g((1-\gamma)u), \qquad W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)} \geqslant g^{-1}\big(g((1-\gamma)u) - (t-\tau_1) - \eta u\big).$$

Now assume that $(1 \vee c)\varepsilon + \gamma + \eta < 1$. For all $s \in (0, g((1-\gamma)u))$,

$$\begin{aligned}
\mathbb{P}\big(W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)} &\geqslant g^{-1}(g((1-\gamma)u) - s - \eta u)\big) \\
&= \mathbb{P}\big(W_1^{(\sigma)} \geqslant g^{-1}\big(g((1-\gamma)u) - s - \eta u\big) \mid W_1^{(\sigma)} \notin [-\varepsilon u, c\varepsilon u]\big) \\
&= \frac{\mathbb{P}(W_1^{(\sigma)} \geqslant g^{-1}(g((1-\gamma)u) - s - \eta u))}{\mathbb{P}(W_1^{(\sigma)} \notin [-\varepsilon u, c\varepsilon u])} \\
&= \frac{\rho(g^{-1}(g((1-\gamma)u) - s - \eta u))}{(c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u)}.
\end{aligned}$$

Since $t - \tau_1$ and $W^{(\sigma)}_{M_1^{(\sigma)}(-\varepsilon u, c\varepsilon u)}$ are independent and the distribution of $t - \tau_1$ is exponential with parameter $(c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u)$, we obtain

$$\begin{aligned}
p_{\text{main}}(u) &= \mathbb{P}_x\big(A^c_{\varepsilon,\gamma,\delta,u,t} \cap \big\{X_t^{(\sigma)} > u,\, t - \tau_1 \leqslant g(\delta u),\, X^{(\sigma)}_{\tau_1 -} \leqslant \eta u\big\}\big) \\
&\leqslant \int_0^{g((1-\gamma)u)} e^{-(c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u)s} \rho\big(g^{-1}(g((1-\gamma)u) - s) - \eta u\big)\,\mathrm{d}s \\
&\leqslant \int_0^{g((1-\gamma)u)} \rho\big(g^{-1}(g((1-\gamma)u) - s) - \eta u\big)\,\mathrm{d}s.
\end{aligned}$$

The change of variable $y = g^{-1}(g((1-\gamma)u) - s)$ yields

$$\begin{aligned}
p_{\text{main}}(u) &\leqslant \int_{(1-\gamma)u}^{+\infty} \frac{\rho(y - \eta u)}{f(y)}\,\mathrm{d}y \leqslant \int_{(1-\gamma)u}^{+\infty} \frac{\rho(y(1 - \eta/(1-\gamma)))}{f(y)}\,\mathrm{d}y \\
&= \left(1 - \frac{\eta}{1-\gamma}\right)^{-\alpha} \int_{(1-\gamma)u}^{+\infty} \frac{\rho(y)}{f(y)}\,\mathrm{d}y = \left(1 - \frac{\eta}{1-\gamma}\right)^{-\alpha} h((1-\gamma)u).
\end{aligned} \tag{2.25}$$

Putting together (2.22), (2.23) and (2.25), we deduce, for all $u \geqslant u_0(\varepsilon,\gamma,\delta)$ and $\sigma \leqslant \sigma_0$,

$$p_{221}(u) \leqslant \left(1-\frac{\eta}{1-\gamma}\right)^{-\alpha} h((1-\gamma)u) + C(\varepsilon,\gamma) g(\delta u)\rho(u)^{\gamma/(4\varepsilon(1\vee c))}. \tag{2.26}$$

It remains to estimate p_{222}. Since $\tau_1-\tau_2$ and $t-\tau_1$ are independent, we can split

$$p_{222}(u) = \mathbb{P}(t-\tau_1 \leqslant g(\delta u)) \cdot \mathbb{P}_x\big(X^{(\sigma)}_{\tau_1-} > \eta u,\, \tau_1-\tau_2 > g(\delta u)\big).$$

We can write

$$\mathbb{P}_x\big(X^{(\sigma)}_{\tau_1-} > \eta u,\, \tau_1-\tau_2 > g(\delta u)\big) \leqslant \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,\tau_1}) + \mathbb{P}_x\big(A^c_{\varepsilon,\gamma,\delta,u,\tau_1} \cap \{X^{(\sigma)}_{\tau_1-} > \eta u,\, \tau_1-\tau_2 > g(\delta u)\}\big).$$

By choosing γ, δ and ε small enough, we can assume that $\delta+\gamma<\eta$. By employing the same argument used to estimate p_1, we deduce

$$\mathbb{P}_x\big(A^c_{\varepsilon,\gamma,\delta,u,\tau_1} \cap \{X^{(\sigma)}_{\tau_1-} > \eta u,\, \tau_1-\tau_2 > g(\delta u)\}\big) = 0.$$

We use again Lemma 3 and the exponential distribution of $t-\tau_1$ with parameter $(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)$ to find that, for all $u \geqslant u_0(\varepsilon,\delta,\gamma)$ and $\sigma \leqslant \sigma_0$,

$$p_{222}(u) \leqslant C(\varepsilon,\delta,\gamma,\eta)\rho(u)^{(1+\gamma/(4(1\vee c)\varepsilon))} g(u)^2. \tag{2.27}$$

STEP 8. Finally, summarizing the inequalities (2.17), (2.20), (2.26) and (2.27), for ε, γ, δ and η such that $\delta+\gamma<\eta<1$, $(1\vee c)\varepsilon < \gamma/4$ and $(1\vee c)\varepsilon+\gamma+\eta<1$, there exist $u_0(\varepsilon,\gamma,\delta,\eta)$ and σ_0 such that, for all $u \geqslant u_0(\varepsilon,\gamma,\delta,\eta)$ and $\sigma \leqslant \sigma_0$,

$$\mathbb{P}_x(X^{(\sigma)}_t > u) \leqslant \left(1-\frac{\eta}{1-\gamma}\right)^{-\alpha} h((1-\gamma)u) + (c^{1-\alpha}+c^{-\alpha})^2\rho(\varepsilon u)^2 g(\delta u)^2 + C(\varepsilon,\gamma,\delta,\eta) g(u)\rho(u)^{\gamma/(4(1\vee c)\varepsilon)}.$$

Since h is regularly varying at infinity with exponent $1-\alpha-\beta$, while g is regularly varying at infinity with exponent $1-\beta$ and $\rho(u)$ is regularly varying at infinity with exponent $-\alpha$, choosing ε, γ, δ and η small enough we find that for all $\xi>0$, there exists $u_0(\xi)$ such that, for all $u \geqslant u_0(\xi)$, all $x\in\mathbb{R}$ and all $t \geqslant 1$,

$$\frac{\mathbb{P}_x(X^{(\sigma)}_t > u)}{h(u)} \leqslant 1+\xi,$$

hence we have established the upper bound of the main result.

REMARK 5. If instead of regular variation at infinity of f, we only assume $f(x) \geqslant \hat{f}(x)$ for all $x \geqslant A$ for some function $\hat{f}$ which is regularly varying at infinity with exponent greater than 1, we would still have the upper bound: for all $\xi > 0$, there exists $u_0(\xi)$ such that, for all $u \geqslant u_0(\xi)$, all $x \in \mathbb{R}$ and all $t \geqslant 1$,

$$\frac{\mathbb{P}_x(X_t^{(\sigma)} > u)}{\hat{h}(u)} \leqslant 1 + \xi \quad \text{with} \quad \hat{h}(u) = \int_u^{+\infty} \frac{\nu((y, +\infty))}{\hat{f}(y)} \, \mathrm{d}y.$$

STEP 9. We proceed to the proof of the lower bound. For all $\varepsilon, \delta, \eta < 1$ we get, by the strong Markov property and (2.11),

$$\begin{aligned} \mathbb{P}_x(X_t^{(\sigma)} > u) &\geqslant \mathbb{P}_x\big(X_t^{(\sigma)} > u,\, \tau_1 \geqslant t - g(u(1+\delta)),\, X_{\tau_1-}^{(\sigma)} \geqslant -\eta u\big) \\ &\geqslant \int_0^{g(u(1+\delta))} (c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u) e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s} \, \mathbb{P}_x(X_{(t-s)-}^{(\sigma)} \geqslant -\eta u) \\ &\qquad \times \int_{c\varepsilon u}^{+\infty} \mathbb{P}_{y-\eta u}(X_s^{(\varepsilon u)} > u) \frac{\nu(\mathrm{d}y)}{(c^{1-\alpha} + c^{-\alpha})\rho(\varepsilon u)} \, \mathrm{d}s. \end{aligned}$$

Observe that $X^{(\sigma)}$ has, under $\mathbb{P}_x$, the same distribution as $-X^{(\sigma)}$ under the distribution $\mathbb{P}_{-x}$, but with a drift $\hat{f}(x) = -f(-x)$ and an asymmetric driving noise where the coefficients a_+, a_- in the expressions of its Lévy measure are inverted. By using the hypothesis on f and Remark 5, we find that for all $u \geqslant u_0$, all $\sigma \leqslant \sigma_0$, all $x \in \mathbb{R}$ and all $s < g(u(1+\delta))$,

$$\mathbb{P}_x(X_{(t-s)-}^{(\sigma)} \geqslant -\eta u) \geqslant 1 - r(u),$$

where r is a function converging to zero as $u \to +\infty$. In what follows, the function r can change from line to line. Observe that, according to (2.11), much as for p_1, if

$$y \geqslant \eta u + g^{-1}(g(u(1+\delta)) - s) \tag{2.28}$$

then, under the distribution $\mathbb{P}_{y-\eta u}$, the event $\{X_s^{(\varepsilon u)} > u\}$ contains, up to an event of probability zero, the event $A^c_{\varepsilon,\delta,1+\delta,u,t}$. Hence, for all s and y satisfying (2.28), we get

$$\mathbb{P}_{y-\eta u}(X_s^{(\varepsilon u)} > u) \geqslant 1 - \mathbb{P}_x(A_{\varepsilon,\delta,1+\delta,u,t}).$$

Therefore, by using Lemma 3, for all $\sigma \leqslant \sigma_0$ and $u \geqslant u_0(\varepsilon, \delta)$,

$$\mathbb{P}_{y-\eta u}(X_s^{(\varepsilon u)} > u) \geqslant 1 - r(u)$$

for all s and y satisfying (2.28), as long as ε is small relative to δ. So, for all

$\varepsilon, \delta, \eta < 1$ such that ε is small relative to δ, for all $\sigma \leqslant \sigma_0$ and all $u \geqslant u_0(\varepsilon, \delta)$,

$$\begin{aligned}
\mathbb{P}_x(X_t^{(\sigma)} > u) \geqslant & \int_0^{g(u(1+\delta))} e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s} \mathbb{P}_x(X_{(t-s)-}^{(\sigma)} \geqslant -\eta u) \\
& \times \int_{\eta u + g^{-1}(g(u(1+\delta))-s)}^{+\infty} \mathbb{P}_{y-\eta u}(X_s^{(\varepsilon u)} > u)\, \nu(\mathrm{d}y)\, \mathrm{d}s \\
\geqslant & (1-r(u))^2 \int_0^{g(u(1+\delta))} e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u)s} \rho\big(\eta u + g^{-1}(g(u(1+\delta))-s)\big)\, \mathrm{d}s \\
\geqslant & (1-r(u))^2 e^{-(c^{1-\alpha}+c^{-\alpha})\rho(\varepsilon u) g(u(1+\delta))} \int_{u(1+\delta)}^{+\infty} \frac{\rho(\eta u + y)}{f(y)}\, \mathrm{d}y \\
\geqslant & (1-r(u))^3 \int_{u(1+\delta)}^{+\infty} \frac{\rho(y(1+\eta/(1+\delta)))}{f(y)}\, \mathrm{d}y \\
= & (1-r(u))^3 \left(1 + \frac{\eta}{1+\delta}\right)^{-\alpha} h(u(1+\delta)).
\end{aligned}$$

We conclude that, for all $\xi > 0$, choosing η, ε and δ small enough, there exist $u_0(\xi)$ and $\sigma_0(\xi)$ such that

$$\frac{\mathbb{P}_x(X_t^{(\sigma)} > u)}{\hat{h}(u)} \geqslant 1 - \xi$$

for all $u \geqslant u_0(\xi)$, all $\sigma \leqslant \sigma_0(\xi)$, all $x \in \mathbb{R}$ and $t \geqslant 1$. ∎

Proof of Lemma 3. Recall that we denoted $\rho(u) = \nu((u, +\infty))$ and

$$\lambda_\sigma = \frac{\sigma^{-\alpha}}{\alpha}(a_- + a_+ c^{-\alpha}).$$

Set $q := \frac{a_-}{a_- + a_+ c^{-\alpha}}$. For all ε, u and σ, 0 is a quantile of order q for the random variable $W_1^{(\sigma)} \mathbb{1}_{\{W_1^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}}$ since, by using (2.2),

$$\begin{aligned}
\mathbb{P}(W_1^{(\sigma)} \mathbb{1}_{\{W_1^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} < 0) &= \mathbb{P}(W_1^{(\sigma)} \in [-\varepsilon u, -\sigma]) \\
&= \frac{1}{\lambda_\sigma \alpha}(a_- \sigma^{-\alpha} - a_-(\varepsilon u)^{-\alpha}) = \frac{q}{\sigma^{-\alpha}}(\sigma^{-\alpha} - (\varepsilon u)^{-\alpha}) \leqslant q,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(W_1^{(\sigma)} \mathbb{1}_{\{W_1^{(\sigma)} \in [-\varepsilon u, c\varepsilon u]\}} \leqslant 0) &= \mathbb{P}(W_1^{(\sigma)} \leqslant -\sigma) + \mathbb{P}(W_1^{(\sigma)} > c\varepsilon u) \\
&= \frac{1}{\lambda_\sigma \alpha}(a_- \sigma^{-\alpha} + a_+ c^{-\alpha}(\varepsilon u)^{-\alpha}) \geqslant \frac{a_- \sigma^{-\alpha}}{\lambda_\sigma \alpha} = q.
\end{aligned}$$

Recall that $N^{(\sigma)}_{g(\delta u)}$ has the same distribution as the number of jumps of $\ell^{(\sigma)}$ in $[s-g(\delta u),s]$. By using [6, Theorem 2.1, p. 50], we get

$$\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s}) \leqslant \frac{1}{q}\mathbb{P}\Big(\sum_{i=1}^{N^{(\sigma)}_{g(\delta u)}} W_i^{(\sigma)}\mathbb{1}_{\{W_i^{(\sigma)}\in[-\varepsilon u,c\varepsilon u]\}} \geqslant \gamma u\Big).$$

Again we use the fact that $N^{(\sigma)}_{g(\delta u)}$ is a Poisson distributed random variable of parameter $\lambda_\sigma g(\delta u)$ and is independent of the $W_i^{(\sigma)}$. By conditioning, we obtain

$$\begin{aligned}\mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s}) &\leqslant \frac{1}{q}\exp(-\lambda_\sigma g(\delta u))\\ &\quad\times\sum_{n\geqslant 1}\frac{(\lambda_\sigma g(\delta u))^n}{n!}\mathbb{P}\Big(\sum_{i=1}^{n} W_i^{(\sigma)}\mathbb{1}_{\{W_i^{(\sigma)}\in[-\varepsilon u,c\varepsilon u]\}} \geqslant \gamma u\Big).\end{aligned}\tag{2.29}$$

Recalling that $W_i^{(\sigma)}\mathbb{1}_{\{W_i^{(\sigma)}\in[-\varepsilon u,c\varepsilon u]\}}$ are i.i.d. random variables with expectation 0, bounded by $(1\vee c)\varepsilon u$, we can use [7, Theorem 1, p. 201] to get

$$\begin{aligned}&\mathbb{P}\Big(\sum_{i=1}^{n} W_i^{(\sigma)}\mathbb{1}_{\{W_i^{(\sigma)}\in[-\varepsilon u,c\varepsilon u]\}} \geqslant \gamma u\Big)\\ &\qquad\leqslant \exp\Big[-\frac{\gamma}{2\varepsilon(1\vee c)}\operatorname{arcsinh}\Big(\frac{\gamma u^2\varepsilon(1\vee c)}{n\,\mathbb{V}\mathrm{ar}(W_1^{(\sigma)}\mathbb{1}_{\{W_1^{(\sigma)}\in[-\varepsilon u,c\varepsilon u]\}})}\Big)\Big].\end{aligned}$$

Furthermore, we can estimate

$$\begin{aligned}\mathbb{V}\mathrm{ar}(W_1^{(\sigma)}\mathbb{1}_{\{W_1^{(\sigma)}\in[-\varepsilon u,c\varepsilon u]\}}) &= \mathbb{E}((W_1^{(\sigma)})^2\mathbb{1}_{\{W_1^{(\sigma)}\in[-\varepsilon u,c\varepsilon u]\}})\\ &= \frac{1}{\lambda_\sigma}\Big(\int_{-\varepsilon u}^{-\sigma} a_-|z|^{1-\alpha}\,\mathrm{d}z + \int_{c\sigma}^{c\varepsilon u} a_+ z^{1-\alpha}\,\mathrm{d}z\Big) \leqslant \frac{\alpha(c^{1-\alpha}+c^{2-\alpha})}{\lambda_\sigma(2-\alpha)}\varepsilon^{2-\alpha}u^2\rho(u).\end{aligned}$$

Setting $\hat{C} := \frac{(1\vee c)(2-\alpha)}{\alpha(c^{1-\alpha}+c^{2-\alpha})}$, we can write

$$\begin{aligned}&\mathbb{P}\Big(\sum_{i=1}^{n} W_i^{(\sigma)}\mathbb{1}_{\{W_i^{(\sigma)}\in[-\varepsilon u,c\varepsilon u]\}} \geqslant \gamma u\Big)\\ &\qquad\qquad\leqslant \exp\Big[-\frac{\gamma}{2\varepsilon(1\vee c)}\operatorname{arcsinh}\Big(\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{n\rho(u)}\Big)\Big].\end{aligned}$$

Since $\operatorname{arcsinh}(x)\sim\log(x)$ as $x\to+\infty$, there exists $a>0$ such that for all $x\geqslant a$,

$\operatorname{arcsinh}(x) \geqslant \frac{1}{2}\log(x)$. Therefore, if $n \leqslant \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\,\rho(u)}$, we get

$$\mathbb{P}_x\Big(\sum_{i=1}^n W_i^{(\sigma)}\mathbb{1}_{\{W_i^{(\sigma)}\in[-\varepsilon u, c\varepsilon u]\}} \geqslant \gamma u\Big)$$
$$\leqslant \exp\Big[-\frac{\gamma}{4\varepsilon(1\vee c)}\log\Big(\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{n\rho(u)}\Big)\Big] = \Big(\frac{n\rho(u)}{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}\Big)^{\gamma/(4\varepsilon(1\vee c))}.$$

By inserting this result in (2.29), we obtain

$$\text{(2.30)}\qquad \mathbb{P}_x(A_{\varepsilon,\gamma,\delta,u,s}) \leqslant \frac{1}{q}\Big(\frac{\rho(u)}{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}\Big)^{\gamma/(4\varepsilon(1\vee c))}\mathbb{E}((N_{g(\delta u)}^{(\sigma)})^{\gamma/(4\varepsilon(1\vee c))})$$
$$+\frac{1}{q}\mathbb{P}\Big(N_{g(\delta u)}^{(\sigma)} > \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\Big).$$

It is not difficult to see that if ξ is a Poisson distributed random variable, then for all $p \geqslant 1$ there exists C_p such that

$$\mathbb{E}\xi^p \leqslant C_p(\mathbb{E}\xi + (\mathbb{E}\xi)^p).$$

Since $(1\vee c)\varepsilon \leqslant \gamma/4$, we can apply this result to $N_{g(\delta u)}^{(\sigma)}$ to deduce

$$\mathbb{E}((N_{g(\delta u)}^{(\sigma)})^{\gamma/(4\varepsilon(1\vee c))}) \leqslant C'_{\varepsilon,\gamma}\big(\lambda_\sigma g(\delta u) + (\lambda_\sigma g(\delta u))^{\gamma/(4\varepsilon(1\vee c))}\big).$$

We estimate the first term on the right hand side of (2.30): there exists $C(\varepsilon,\gamma)$ such that

$$\text{(2.31)}\qquad \frac{1}{q}\Big(\frac{\rho(u)}{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}\Big)^{\gamma/(4\varepsilon(1\vee c))}\mathbb{E}((N_{g(\delta u)}^{(\sigma)})^{\gamma/(4\varepsilon(1\vee c))})$$
$$\leqslant C(\varepsilon,\gamma)g(\delta u)\rho(u)^{\gamma/(4\varepsilon(1\vee c))}.$$

To study the second term on the right hand side of (2.30), we set

$$\vartheta := \log\Big(\frac{\varepsilon^{\alpha-1}\gamma}{g(\delta u)\rho(u)}\Big).$$

There exists $u_0(\varepsilon,\gamma,\delta)$ such that for all $u \geqslant u_0(\varepsilon,\gamma,\delta)$, ϑ is strictly positive. We get, for all $u \geqslant u_0(\varepsilon,\gamma,\delta)$,

$$\mathbb{P}\Big(N_{g(\delta u)}^{(\sigma)} > \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\Big) = \mathbb{P}\Big(e^{\vartheta N_{g(\delta u)}^{(\sigma)}} > \exp\Big(\vartheta\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\Big)\Big)$$
$$\leqslant \exp\Big((e^\vartheta - 1)\lambda_\sigma g(\delta u) - \vartheta\frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\Big),$$

by using Markov's inequality. By choosing $C(\varepsilon,\gamma)$ and $u_0(\varepsilon,\gamma,\delta)$ large enough, we obtain, using the expression of ϑ,

$$\mathbb{P}\left(N_{g(\delta u)}^{(\sigma)} > \frac{\hat{C}\varepsilon^{\alpha-1}\gamma\lambda_\sigma}{a\rho(u)}\right) \leqslant C(\varepsilon,\gamma)\,(g(\delta u)\rho(u))^{C(\varepsilon,\gamma)\lambda_\sigma/\rho(u)}\,. \tag{2.32}$$

Inserting (2.31) and (2.32) in (2.30), we get (2.15). ■

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