On an operator related to the convergence of Uzawa's algorithm for the Stokes equation

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Dedicated to Roland Glowinski on his 60th birthday

1. INTRODUCTION

Let Ω be an open subset of $\mathbf{R}^d$, $d = 2$ or 3. We assume that it is bounded, connected and that its boundary is Lipschitz continuous. We consider the space $L_0^2(\Omega) = \{p \in L^2(\Omega); \int_\Omega p(x)dx = 0\}$ equipped with the inner product $(p,q) = \int_\Omega p(x)q(x)dx$ and the norm $| p | = (p,p)^{1/2}$. In this paper we are concerned with the study of the operator D which is defined in the following way. Given $p \in L_0^2(\Omega)$, we denote by $\vec{u}$ the unique solution of

$$\vec{u} \in (H_0^1(\Omega))^d \quad \text{and} \qquad \Delta\vec{u} = \overrightarrow{\mathrm{grad}}\, p, \qquad \text{in } \Omega, \tag{0.1}$$

and we set

$$Dp = \mathrm{div}\ \vec{u}. \tag{0.2}$$

This operator is related for instance to the convergence of Uzawa's algorithm used for solving the stationary Stokes equation : given $\nu > 0$ and $\vec{f}$ we are looking for a velocity $\vec{v}$ and a pressure q satisfying

$$-\nu\Delta\vec{v} + \overrightarrow{\mathrm{grad}}\, q = \vec{f} \qquad \text{in } \Omega,$$

$$\vec{v} = 0, \qquad \text{on } \partial\Omega,$$

$$\mathrm{div}\ \vec{v} = 0, \qquad \text{in } \Omega.$$

Starting from $q_0 \in L^2_0(\Omega)$, Uzawa's algorithm provides a sequence of approximations $(\vec{v}_n, q_n)$ of $(\vec{v}, q)$ from the induction scheme (Glowinski and al. 1976)

$$-\nu \Delta \vec{v}_{n+1} = \vec{f} - \overrightarrow{\mathrm{grad}}\, q_n \qquad \text{in } \Omega,$$

$$\vec{v}_{n+1} = 0, \qquad \text{on } \partial\Omega,$$

$$q_{n+1} = q_n - \rho_n \nu \ \mathrm{div}\ \vec{v}_{n+1}.$$

We have $q - q_{n+1} = (I - \rho_n D)(q - q_n)$, therefore $q - q_n = (I - \rho_{n-1} D) \cdots (I - \rho_0 D)\,(q - q_0)$. Since D is a self-adjoint operator on $L^2_0(\Omega)$ we get the L^2-estimate

$$|q - q_n| \le \max_{\lambda \in Sp(D)} \Big| \prod_{k=0}^{n-1} (I - \rho_k \lambda) \Big|\ |q - q_0|. \tag{0.3}$$

It is clear from this estimate that a good knowledge of the operator D, and particularly of its spectrum $Sp(D)$, is of interest for analysing the convergence of q_n towards q and for judiciously determining the parameters ρ_k (Crouzeix 1974). A similar analysis will be working for an augmented Lagrangian version of this algorithm (Fortin-Glowinski 1982).

2. BASIC PROPERTIES OF D

First we remark that the restriction of D to the following subspace of $L^2_0(\Omega)$,

$$N = \{p \in L^2_0(\Omega); \exists \theta \text{ with } \Delta\theta = p \text{ in } \Omega \text{ and } \overrightarrow{\mathrm{grad}}\, \theta \in (H^1_0(\Omega))^d\},$$

is the identity operator. Indeed if $p = \Delta\theta \in N$, we have in (0.1) $\vec{u} = \overrightarrow{\mathrm{grad}}\, \theta$, therefore $Dp = \mathrm{div}\ \vec{u} = p$.

We introduce now the orthogonal subspace

$$M = \{p \in L^2_0(\Omega);\ (p, q) = 0 \text{ for all } q \in N\} = N^{\perp}.$$

The subset N is closed, thus we have $L^2_0(\Omega) = M \oplus N$. Since D is self-adjoint on $L^2_0(\Omega)$, it maps M into M. As $D_{|N}$ is completely known, the remaining part of this paper will be mainly concerned with the study of the restriction $D_{|M}$ of D to M.

The space M

We need a better knowledge of the space M. We first remark that it follows of its definition that $M \subset H = \{p \in L^2_0(\Omega)\ ;\ \Delta p = 0\}$ and that $M = H$ when the boundary of Ω is connected (indeed we have $H \cap N = \{0\}$ in this case). We equip now the set $(H^1_0(\Omega))^d$ with the inner product and the associated norm

$$(\vec{v}, \vec{w})_{1,\Omega} = \sum_{i,j=1}^{d} (\frac{\partial v_i}{\partial x_j}, \frac{\partial w_i}{\partial x_j}) = (\mathrm{div}\ \vec{v}, \mathrm{div}\ \vec{w}) + (\overrightarrow{\mathrm{curl}}\, \vec{v}, \overrightarrow{\mathrm{curl}}\, \vec{w}), \qquad |\,\vec{v}\,|_{1,\Omega} = (\vec{v}, \vec{v})^{1/2}_{1,\Omega}.$$

We introduce the sets

$$V_1 = \{\vec{v} \in (H_0^1(\Omega))^d \ ; \ \text{div } \vec{v} = 0\}, \qquad V_2 = \{\vec{v} \in (H_0^1(\Omega))^d \ ; \ \overrightarrow{\text{curl}}\, \vec{v} = 0\},$$

$$W = \{\vec{v} \in (H_0^1(\Omega))^d \ ; \ (\vec{v}, \vec{w})_{1,\Omega} = 0, \forall \vec{w} \in V_1 \text{ and } \forall \vec{w} \in V_2\}.$$

These subsets are orthogonal with respect to the inner product in $(H_0^1(\Omega))^d$ and we have $(H_0^1(\Omega))^d = V_1 \oplus V_2 \oplus W$.

We admit for a while the following lemma

Lemma 1 *The operator "div " is an isomorphism from W onto M. The three quantities $|\,\vec{v}\,|_{1,\Omega}$, $|\text{div } \vec{v}|$ and $|\overrightarrow{\text{curl}}\, \vec{v}|$ define equivalent norms on W.*

Now we can summarize some basic properties of the operator D.

Theorem 2 *The operator D is self-adjoint positive definite on $L_0^2(\Omega)$ and there exist two real numbers $0 < \alpha < \beta < 1$ such that*

a) $\forall p \in L_0^2(\Omega), \quad \alpha |\,p\,|^2 \leq (Dp, p) \leq |\,p\,|^2,$

b) $\forall p \in M, \quad \alpha |\,p\,| \leq |Dp| \leq \beta |\,p\,|,$

c) $Sp(D) \subset [\alpha, \beta] \cup \{1\}$ *and we have* $\beta = 1 - \alpha$ *when* $d = 2$.

Proof.

a) Using (0.1) and (0.2) we deduce

$$|Dp|^2 \leq |\text{div } \vec{u}|^2 + |\overrightarrow{\text{curl}}\, \vec{u}|^2 = |\,\vec{u}\,|_{1,\Omega}^2 = (p, \text{div } \vec{u}) = (p, Dp).$$

Therefore from the Cauchy-Schwarz inequality, we have $|Dp| \leq |\,p\,|$, which proves the right part of the inequality a). We also have $(Dp, p) = |\,\vec{u}\,|_{1,\Omega}^2 = \|\overrightarrow{\text{grad}}\, p\|_{H^{-1}(\Omega)}^2 \geq \alpha |\,p\,|^2$, since $|\,p\,|$ and $\|\overrightarrow{\text{grad}}\, p\|_{H^{-1}(\Omega)}$ are equivalent norms on $L_0^2(\Omega)$ (Nečas 1965).

b) The left inequality follows from the part a). Since D is self-adjoint, the optimal value for β is

$$\beta = \max_{p \in M} \frac{(Dp, Dp)}{(Dp, p)}$$

From the definition (0.1), (0.2), $Dp = \text{div } \vec{u}$ and $(\vec{u}, \vec{w})_{1,\Omega} = (p, \text{div } \vec{w})$. It follows that $\vec{u} \in V_1^{\perp}$; moreover if $\vec{w} \in V_2$ we can write $\vec{w} = \overrightarrow{\text{grad}}\, \theta$ and then $(\vec{u}, \vec{w})_{1,\Omega} = (p, \Delta\theta) = 0$ since $p \in M$. Therefore $\vec{u} \in W$ and we get from the lemma

$$\beta = \max_{\vec{u} \in W} \frac{|\text{div } \vec{u}|^2}{|\,\vec{u}\,|_{1,\Omega}^2} = 1 - \min_{\vec{u} \in W} \frac{|\overrightarrow{\text{curl}}\, \vec{u}|^2}{|\,\vec{u}\,|_{1,\Omega}^2} < 1.$$

c) It remains to prove that in two dimensions we have $\beta = 1 - \alpha$. From the previous part, the optimal value of α is

$$\alpha = \min_{p \in M} \frac{(Dp, Dp)}{(Dp, p)} = \min_{\vec{u} \in W} \frac{|\text{div } \vec{u}|^2}{|\,\vec{u}\,|_{1,\Omega}^2}.$$

But it is easy to see that if $\vec{v} = (v_1, v_2) \in W$ then $\vec{w} = (-v_2, v_1)$ satisfies $\text{div } \vec{v} = \text{curl } \vec{w}$, $\text{curl } \vec{v} = -\text{div } \vec{w}$ and $\vec{w} \in W$, therefore we have

$$\min_{\vec{u} \in W} \frac{|\text{div } \vec{u}|^2}{|\,\vec{u}\,|_{1,\Omega}^2} = \min_{\vec{u} \in W} \frac{|\text{curl } \vec{u}|^2}{|\,\vec{u}\,|_{1,\Omega}^2}.$$

We turn now to the proof of the lemma

Proof of the lemma

If $\vec{v} \in W$ and $\overrightarrow{\mathrm{grad}}\,\theta \in (H_0^1(\Omega))^d$ then $(\mathrm{div}\ \vec{v}, \Delta\theta) = (\vec{v}, \overrightarrow{\mathrm{grad}}\,\theta)_{1,\Omega} = 0$, since $\overrightarrow{\mathrm{grad}}\,\theta \in V_2$; therefore "div" maps W into M. If $q \in M$ then there exists $p \in M$ such that $q = Dp = \mathrm{div}\ \vec{u}$ (see the part a) of the theorem) and we have noticed that $\vec{u} \in W$. Therefore the linear operator "div", which is bijective and continuous from W onto M, is an isomorphism ; consequently $|\,\vec{v}\,|_{1,\Omega}$ and $|\mathrm{div}\ \vec{v}|$ are equivalent norms on W. If $d = 2$ a similar proof gives that "curl" is an isomorphism from W onto M and therefore the norms $|\vec{v}|_{1,\Omega}$ and $|\mathrm{curl}\ \vec{v}|$ are equivalent on W.

We turn now to the case $d = 3$ and assume first that the complementary set Ω^c of Ω is simply connected. We introduce the space $Q = \{\vec{q} \in (L_0^2(\mathbf{R}^3))^3\ ;\ \vec{q} = 0 \text{ in } \Omega^c, \mathrm{div}\ \vec{q} = 0 \text{ in } \mathbf{R}^3\}$ which is closed in $(L^2(\mathbf{R}^3))^3$. It is sufficient to prove that "$\overrightarrow{\mathrm{curl}}$" is an isomorphism from $V_1 \oplus W$ onto Q in order to get the equivalence between the norms $|\,\vec{v}\,|_{1,\Omega}$ and $|\overrightarrow{\mathrm{curl}}\,\vec{v}|$ in $V_1 \oplus W$, and a fortiori in W. Since this operator is clearly continuous and injective from $V_1 \oplus W$ into Q, it is sufficient to prove that it is surjective from $V_1 \oplus W$ onto Q, or equivalently from $(H_0^1(\Omega))^3$ onto Q.

Let us consider $\vec{q} \in Q$; then there exists $\vec{u} \in (H^1(\mathbf{R}^3))^3$ such that $\mathrm{div}\ \vec{u} = 0$ and $\overrightarrow{\mathrm{curl}}\,\vec{u} = \vec{q}$ in $\mathbf{R}^3$ (see for instance Girault-Raviart 1986 page 45). Since $\overrightarrow{\mathrm{curl}}\,\vec{u} = 0$ in the simply connected set Ω^c, there exists $\rho \in H^2_{loc}(\Omega^c)$ such that $\vec{u} = \overrightarrow{\mathrm{grad}}\,\rho$ in Ω^c. We still denote by $\rho \in H^2_{loc}(\mathbf{R}^3)$ an extension of ρ to all the space $\mathbf{R}^3$ and we set $\vec{v} = \vec{u} - \overrightarrow{\mathrm{grad}}\,\rho$. We now have $\vec{v} = 0$ in Ω^c and $\vec{v} \in (H^1_{loc}(\mathbf{R}^3)^3$, thus $\vec{v} \in (H_0^1(\Omega))^3$; moreover $\overrightarrow{\mathrm{curl}}\,\vec{v} = \overrightarrow{\mathrm{curl}}\,\vec{u} = \vec{q}$ in Ω, which proves the surjectivity.

When the set Ω^c is not simply connected, the previous proof still works if we restrict the space Q to the functions which satisfy $\int_\omega \vec{q}.\overrightarrow{dS} = 0$ for all smooth 2-manifold ω with border γ satisfying $\gamma \subset \Omega^c$.

3. STUDY IN THREE DIMENSION

The following theorem shows that the operator $D - \frac{1}{2}I$ has regularizing properties as soon as the boundary of Ω is smooth.

Theorem 3 *Assume that the boundary $\partial\Omega$ is of class C^3, then $D - \frac{1}{2}I$ maps M continuously into $H^1(\Omega)$. Furthermore if $\partial\Omega$ is of class C^{k+3} then $D - \frac{1}{2}I$ maps $M \cap H^k(\Omega)$ into $H^{k+1}(\Omega)$.*

Proof

From the smoothness assumption on $\partial\Omega$ we can find a function $\xi \in C^3(\bar{\Omega})$ such that $\xi = 0$ on $\partial\Omega$ and $\overrightarrow{\mathrm{grad}}\,\xi = \vec{n}$ (the unitary normal) on $\partial\Omega$. We now consider $p \in H^1(\Omega)$ such that $\Delta p = 0$ in Ω; we define $\vec{u} \in (H^2(\Omega) \cap H_0^1(\Omega))^3$ by (1) and we set $w = \vec{u} \cdot \overrightarrow{\mathrm{grad}}\,\xi - \frac{1}{2}\,p\xi$. Clearly we have $w \in H_0^1(\Omega)$ and

$$\Delta w = 2\ \overrightarrow{\mathrm{grad}}\,\vec{u} \cdot \overrightarrow{\mathrm{grad}}\,\overrightarrow{\mathrm{grad}}\,\xi + \vec{u} \cdot \overrightarrow{\mathrm{grad}}\,\Delta\xi - \frac{1}{2}\ p\Delta\xi \in L^2(\Omega).$$

Therefore $w \in H^2(\Omega)$ and $\|\,w\,\|_{H^2(\Omega)} \leq C_1|\Delta w| \leq C_2|\,p\,|$.

We now consider $q = (D - \frac{1}{2}I)p = \text{div}\,\vec{u} - \frac{1}{2}p$ which belongs to $H^1(\Omega)$. We remark that $\Delta q = 0$ in Ω and $q = \overrightarrow{\text{grad}}\, w \cdot \overrightarrow{\text{grad}}\, \xi$ on $\partial\Omega$, therefore we have $|\, q\,|_{1,\Omega} \leq |\overrightarrow{\text{grad}}\, w \cdot \overrightarrow{\text{grad}}\, \xi|_{1,\Omega} \leq C_3 \|\, w\,\|_{H^2(\Omega)} \leq C_2 C_3 |\, p\,|$. We have proved for all harmonic $p \in H^1(\Omega)$ that the inequality $|(D - \frac{1}{2}I)p|_{1,\Omega} \leq C(\Omega)|\, p\,|$ holds; by a density argument this is still valid for all harmonic $p \in L^2(\Omega)$, and a fortiori for $p \in M$.

If the boundary $\partial\Omega$ is of class C^{k+3} we can choose $\xi \in C^{k+3}(\bar{\Omega})$. Then for $p \in H^k(\Omega) \cap M$ we get successively $\vec{u} \in H^{k+1}(\Omega)$, $w \in H^{k+2}(\Omega)$ and $q \in H^{k+1}(\Omega)$.

Remark

The assumption on the regularity of $\partial\Omega$ may be weakened but the increase of order 1 with respect to the smoothness of p is optimal. Indeed, in the case where Ω is the unit ball, taking $\xi(x) = (x_1^2 + x_2^2 + x_3^2 - 1)/2$, then $\Delta w = 2\text{div}\,\vec{u} - 3p/2 = 2q - p/2$. If $p \in H^s(\Omega)$ and $\notin H^{s+\varepsilon}(\Omega)$, we have $w \in H^{s+2}(\Omega)$ and $\notin H^{s+2+\varepsilon}(\Omega)$. Since $\Delta^2 w = 0$ in Ω and $w = 0$ on $\partial\Omega$, we deduce $\frac{\partial w}{\partial n} \in H^{s+1/2}(\Omega)$ and $\notin H^{s+1/2+\varepsilon}(\Omega)$. Using $\Delta q = 0$ and $q = \frac{\partial w}{\partial n}$ on $\partial\Omega$ we get finally $q \in H^{s+1}(\Omega)$ and $\notin H^{s+1+\varepsilon}(\Omega)$.

From this theorem we deduce the following corollary which is similar to the results of Mikhlin (1973) concerning another operator related to elasticity. The approach in this work is different, it can be easily transposed to our case and it is certainly more general, but we think that our analysis which is more elementary is more precise on some aspects and therefore worthwhile. See also the work of Grubb and Geymonat (1977).

Corollary 4 *Assume that the boundary of Ω is of class C^{k+3}, $k \geq 0$. Then the operator $D - \frac{1}{2}I$ is compact from M into M. The values 1/2 and 1 belong to the spectrum of $D : L_0^2(\Omega) \to L_0^2(\Omega)$. The other points of the spectrum consist in a countable set of eigenvalues which have finite multiplicities. The corresponding eigenfunctions belong to the space $H^{k+1}(\Omega)$.*

In his article Mikhlin mentions the old papers (1898) of the brothers Eugène and François Cosserat. Transposed to our operator their results state

If Ω is an ellipsoid the eigenfunctions of $D_{|M}$ are harmonic polynomials.

We can give a simple proof of that. First we note that the harmonic polynomials are dense in M and secondly we remark that, if $\mathbf{P}_m$ denotes the set of polynomials of degree $\leq m$, then $D\mathbf{P}_m \subset \mathbf{P}_m$ so that $\mathbf{P}_m \cap M$ is an invariant set for D. Indeed $p \in \mathbf{P}_m \Rightarrow \Delta\vec{u} = \overrightarrow{\text{grad}}\, p \in (\mathbf{P}_{m-1})^3 \Rightarrow \vec{u} \in (\mathbf{P}_{m+1})^3$ (since Ω is an ellipsoid) $\Rightarrow Dp = \text{div}\,\vec{u} \in \mathbf{P}_m$.

In the case where Ω is a ball we have a full description of the eigen-elements of $D_{|M}$.

If Ω is a ball the eigenvalues of D are $\lambda_m = \frac{m}{2m+1}$; $m = 1, 2, \ldots$; the multiplicity of λ_m is $2m+1$ and the corresponding eigenspace is the set of harmonic polynomials which are homogeneous of degree p.

Indeed we remark that $\vec{x} \cdot \overrightarrow{\text{grad}}\, q = kq$ when q is homogeneous of degree k. Therefore if p is harmonic and homogeneous of degree m, assuming that Ω is the unit ball in $\mathbf{R}^d$, we have $\vec{u} = \Delta^{-1}(\overrightarrow{\text{grad}}\, p) = \frac{1}{4m+2d-4}(x_1^2 + \cdots + x_d^2 - 1)\, \overrightarrow{\text{grad}}\, p$. Then $Dp = \text{div}\,\vec{u} = \frac{1}{2m+d-2}\,\vec{x} \cdot \overrightarrow{\text{grad}}\, p = \frac{m}{2m+d-2}\, p$, and here $d = 3$.

In this case of a threedimensional ball we can give explicitly the values of the constants which appear in the theorem 2; they are $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{2}$.

4. THE TWO-DIMENSIONAL CASE

We can improve the previous analysis in two dimensions. We first remark that the case of a twodimensional ball is very simple since

$$\textit{If } \Omega \textit{ is a disk then } D_{|M} = \tfrac{1}{2}\, I_{|M}.$$

Therefore in this case the spectrum of D contains only two points $Sp(D) = \{\frac{1}{2}\} \cup \{1\}$. Making the choice $\rho_0 = 1$, $\rho_1 = 2$ in Uzawa's algorithm we get in (0.3), $q - q_2 = 0$. Then we obtain the convergence after only two iterations.

In order to improve the previous result for more general domains we will use the following lemma

Lemma 5 *Assume that the boundary of Ω is of class C^{m+3}, $m \geq 0$. Then there exists a function $\vec{a} \in (C^{m+2}(\bar{\Omega}) \cap H_0^1(\Omega))^2$ such that* curl $\vec{a} = 0$ *in Ω and* div $\vec{a} - 1/2 \in (C^{m+1}(\bar{\Omega}) \cap H_0^{m+1}(\Omega))$. *Furthermore if the boundary is analytic, $\vec{a}$ can be chosen such that $\vec{a} \in (C^{\infty}(\bar{\Omega}) \cap H_0^1(\Omega))^2$,* curl $\vec{a} = 0$ *in Ω and* div $\vec{a} = 1/2$ *in a neighborhood of $\partial\Omega$.*

Proof

It is sufficient to choose $\vec{a} = \overrightarrow{\mathrm{grad}}\,\theta + (x_1, x_2)/4$ where the function $\theta \in C^{m+3}(\bar{\Omega})$ and satisfies on the boundary $\partial\Omega$

$$\theta = -\frac{x_1^2 + x_2^2}{8}, \quad \frac{\partial\theta}{\partial n} = -\frac{\vec{x} \cdot \vec{n}}{4}, \quad \Delta\theta = 0, \quad \frac{\partial\Delta\theta}{\partial n} = 0, \quad \cdots \quad , \quad \frac{\partial^m \Delta\theta}{\partial n^m} = 0.$$

Such a function θ exists from usual trace theorems.

b) If the boundary is analytic, it can be defined by an equation $\bar{z} = f(z)$ where $f(z) = r(x_1, x_2) + i\ s(x_1, x_2)$ is an holomorphic function of $z = x_1 + ix_2$, defined in a neighborhood of $\partial\Omega$. We then set $\vec{a} = (x_1 - r(x_1, x_2), x_2 + s(x_1, x_2))/4$ so that div $\vec{a} = 1/2$ and curl $\vec{a} = 0$ in this neighborhood. Using this last property and $\vec{a} = 0$ on $\partial\Omega$, we can find φ such that $\vec{a} = \overrightarrow{\mathrm{grad}}\,\varphi$ in the neighborhood; then we extend φ to a function $\tilde{\varphi} \in C^{\infty}(\bar{\Omega})$ and finally we take $\vec{a} = \overrightarrow{\mathrm{grad}}\,\tilde{\varphi}$.

We recall also a classical result

Lemma 6 *Assume that p is harmonic, belongs to $L_0^2(\Omega)$ and that $\psi \in C^m(\bar{\Omega}) \cap H_0^m(\Omega)$. Then we have $\psi\partial^j p \in H_0^{m-|j|}(\Omega)$ for all partial derivatives $\partial^j p$ of order $|j| \leq m$.*

We can now state the theorem

Theorem 7 *Assume that the boundary of Ω is of class C^{m+3}, $m \geq 0$. Then the operator $D - \frac{1}{2}I$ maps M into $H^{m+1}(\Omega)$; furthermore, if the boundary is analytic, then, for all $p \in M$, $Dp - \frac{1}{2}\ p$ is analytic in $\bar{\Omega}$.*

Proof

Let us consider $p \in M$. Since $p \in \text{div } W$ we have $(\overrightarrow{\text{grad}}\, p, \vec{v}) = 0$, $\forall \vec{v} \in (\mathcal{D}(\Omega))^2$ with curl $\vec{v} = 0$ in Ω. Therefore there exists $q \in L_0^2(\Omega)$ such that $\overrightarrow{\text{curl}}\, q = \overrightarrow{\text{grad}}\, p$. We have also $\Delta q = \text{curl } \overrightarrow{\text{curl}}\, q = 0$. With $\vec{a}$ given by Lemma 5 and $\vec{u}$ by (0.1), we set

$$\vec{h} = p\ \vec{a} + q(a_2, -a_1) - \vec{u}$$

We have $\vec{h} \in (H_0^1(\Omega))^2$; using the previous lemmata and noticing that $\Delta a_j \in C^m(\bar{\Omega}) \cap H_0^m(\Omega)$

$$\Delta \vec{h} = p\Delta\vec{a} + q(\Delta a_2, -\Delta a_1) + 2(\text{div } \vec{a} - \frac{1}{2})\overrightarrow{\text{grad}}\, p \in H^m(\Omega).$$

Therefore $\vec{h} \in (H^{m+2}(\Omega) \cap H_0^1(\Omega))^2$. We remark now that

$$Dp - \frac{1}{2}\ p = \text{div } \vec{u} - \frac{1}{2}\ p = (\text{div } \vec{a} - \frac{1}{2})p - \text{div } \vec{h},$$

which proves that $Dp - \frac{1}{2}\ p \in H^{m+1}(\Omega)$.

If $\partial\Omega$ is analytic we have furthermore $Dp - \frac{1}{2}\ p = -\text{div } \vec{h}$ and $\Delta\vec{h} = 0$ in a neighborhood of $\partial\Omega$. Together with $\vec{h} = 0$ on $\partial\Omega$ that implies the analyticity up to the boundary.

We complete this paper by showing that we loose the regularizing properties of $D - \frac{1}{2}\ I$ if the domain is no more smooth. More precisely

If the boundary of Ω is piecewise smooth with corners, then the operator $D - \frac{1}{2}\ I$ is not compact from M into M.

Proof

For the sake of simplicity we give the proof in the case where $\partial\Omega$ is smooth, except at the origin, contains at the origin the straight corner $C = \{z = x_1 + ix_2 = re^{\pm i\theta}; 0 \le r \le \rho_0\}$, $(\theta \neq \pi/2)$, and is symmetric with respect to the axis "$x_2 = 0$". We set

$$\alpha_n = -1 + 1/n, \qquad p_n = \Im m\ z^{\alpha_n}, \qquad q_n = \Re e\ z^{\alpha_n}.$$

We define also $\vec{u}_n$ by $\Delta\vec{u}_n = \overrightarrow{\text{grad}}\, p_n$ and $\vec{u}_n \in (H_0^1(\Omega))^2$, and we introduce

$$h_{n1} = u_{n1} - \frac{x_1}{2}p_n + \frac{\cos\theta \sin\alpha_n\theta}{2\sin(\alpha_n+1)\theta}\ \Im m\ z^{\alpha_n+1},$$

$$h_{n2} = u_{n2} - \frac{x_1}{2}q_n + \frac{\cos\theta \cos\alpha_n\theta}{2\cos(\alpha_n+1)\theta}\ \Re e\ z^{\alpha_n+1}.$$

We clearly have $\Delta\vec{h}_n = 0$, in Ω and $\vec{h}_n = 0$, on C, furthermore $\vec{h}_n$ is uniformly (w.r.t. n) smooth in $\partial\Omega \setminus C$. We deduce (Grisvard (1985) $|\vec{h}_n|_{1,\Omega} \le C$ (constant independent of n) and $\vec{h}_n \in H^{3-2\theta/\pi-\varepsilon}(\Omega)$, for all $\varepsilon > 0$.

Now we remark that

$$\text{div } \vec{h}_n = \text{div } \vec{u}_n - \frac{1}{2}\ p_n - (\alpha_n+1)\frac{\sin 2\theta}{2\sin 2(\alpha_n+1)\theta}\Im m\ z^{\alpha_n},$$

therefore
$$Dp_n - \frac{1}{2}\ p_n = (\alpha_n + 1)\frac{\sin 2\theta}{2\sin 2(\alpha_n+1)\theta}\ p_n + \text{div}\ \vec{h}_n. \tag{0.4}$$

Since $p_n \notin H^{1/n}$ (except for $n=1$) and div $\vec{h}_n \in H^{2-2\theta/\pi-\varepsilon}(\Omega)$ this formula shows that $Dp_n - \frac{1}{2}\ p_n$ is no more regular than p_n if $n > \frac{\pi}{2(\pi-\theta)}$.

We normalize now with respect to the L^2-norm by setting $\gamma_n = |p_n|^{-1}$ and $\tilde{p}_n = \gamma_n p_n$. We have $|\tilde{p}_n| = 1$ and, as $n \to \infty$, $\gamma_n \to 0$ and $\tilde{p}_n \rightharpoonup 0$ weakly in L^2. If $D - \frac{1}{2}\ I$ were compact from M into M, we would have $|D\tilde{p}_n - \frac{1}{2}\ \tilde{p}_n| \to 0$. But we deduce from (0.4)
$$D\tilde{p}_n - \frac{1}{2}\ \tilde{p}_n = (\alpha_n + 1)\frac{\sin 2\theta}{2\sin 2(\alpha_n+1)\theta}\ \tilde{p}_n + \gamma_n\ \text{div}\ \vec{h}_n.$$

Therefore $|D\tilde{p}_n - \frac{1}{2}\ \tilde{p}_n| \to \frac{\sin 2\theta}{4\theta} \neq 0$, which is in contradiction with the compactness of $D - \frac{1}{2}\ I$.

We remark also that we have $|D\tilde{p}_n - (\frac{1}{2} + \frac{\sin 2\theta}{4\theta})\tilde{p}_n| \to 0$, which implies that the value $\frac{1}{2} + \frac{\sin 2\theta}{4\theta}$ belongs to the spectrum of D.

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