Inequalities related to the numerical range for a sectorial or a parabolic domain

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Abstract

In [2] we have proved some inequalities for holomorphic functions of a linear operator, when this operator has its numerical range contained in a conic domain. Here we present direct proofs in the particular cases of sector and parabola.

1 Introduction.

The numerical range (or field of values) of a square matrix $A \in \mathbb{C}^{d,d}$ is the set

$$W(A) := \{\langle Av, v\rangle \,;\, v \in \mathbb{C}^d,\ \|v\| = 1\},$$

where $\langle .,.\rangle$ denotes the usual inner product on the euclidean space $\mathbb{C}^d$ and $\|.\|$ the corresponding norm. In this paper we are concerned with the following problem: given a convex open set $\Omega \subset \mathbb{C}$, find upper bounds for the smallest constant $C(\Omega)$ depending only on Ω such that for any matrix $A \in \mathbb{C}^{d,d}$ with $\overline{W(A)} \subset \Omega$ and for any rational function r there holds

$$\|r(A)\| \leq C(\Omega) \sup_{z\in\Omega} \|r(z)\|. \tag{1}$$

In [2] some new bounds for $C(\Omega)$ have been given in the case where the boundary of Ω is a (branch of a) conic curve. By a limiting argument this induces the bound

$$C(S_\alpha) \leq 2\,\frac{\pi-\alpha}{\pi} + \mu(\alpha), \quad \mu(\alpha) := \frac{\sin 2\alpha}{\pi}\int_0^\infty \frac{dy}{y^2\cos\alpha - 2\,y\,\cos 2\alpha + \cos\alpha}, \tag{2}$$

if S_α is a sector of angle $2\alpha \in (0,\pi)$, and the bound $C(P) \leq 2 + 2/\sqrt{3} < 3.16$ for a parabola. These bounds improved previous ones given for instance in [3] , [1] and [4].

This paper does not contain new result with respect to [2], but the proofs given here are direct, and simpler in the sector case.

2 Some useful lemmas

We consider two matrix-valued functions M and S satisfying the following assumptions

$$(H)\quad \left\{\begin{array}{lll} M \text{ and } S \in C^0([a,+\infty)\,;\mathbb{C}^{d,d}) & & \\ \forall x > a, \quad S(x) = S^*(x), & \operatorname{Re} M(x) \geq S(x), & S(x) \text{ is positive definite,} \\ \text{the integral } \int_a^\infty S^{-1}(x)\,dx & \text{is convergent.} & \end{array}\right.$$

We use the notation $\operatorname{Re} M = \frac{1}{2}(M+M^*)$ for the selfadjoint part of M, and $\operatorname{Im} M = \frac{1}{2i}(M-M^*)$.

Lemma 1. *We assume that the functions M and S satisfy the assumptions (H) and that g is a continuous function such that $|g(x)| \le 1$ for all $x \in [a,\infty)$. Then the matrix $\int_a^\infty g(x)\,(M^*(x))^{-1}\,dx$ is well defined and we have*

$$\Big\| \int_a^\infty g(x)\,(M^*(x))^{-1}\,dx \Big\| \le \Big\| \int_a^\infty (S(x))^{-1}\,dx \Big\|.$$

Proof. It is sufficient to give the proof when $S = \operatorname{Re} M$. Then we introduce $L = S^{-1/2}$ and $D = L\,(\operatorname{Im} M)\,L$. Since the matrix $D(x)$ is selfadjoint, the matrix $I - iD(x)$ is invertible and $\|(I - iD(x))^{-1}\| \le 1$. We have $M^* = L^{-1}(I - iD)L^{-1}$, thus

$$|g(x)\langle (M^*(x))^{-1}u, v\rangle| = |\langle (I - iD(x))^{-1}L(x)u, L(x)v\rangle| \le \|L(x)u\|\,\|L(x)v\|, \quad \forall\, u, v \in \mathbb{C}^d.$$

This yields

$$\begin{aligned} \int_a^\infty |g(x)\langle (M^*(x))^{-1}u, v\rangle|\,dx &\le \Big(\int_a^\infty \|L(x)u\|^2 dx\Big)^{1/2}\Big(\int_a^\infty \|L(x)v\|^2 dx\Big)^{1/2} \\ &\le \Big\| \int_a^\infty (S(x))^{-1}\,dx \Big\|\,\|u\|\,\|v\|. \end{aligned} \tag{3}$$

We have used that

$$\begin{aligned} \int_a^\infty \|L(x)u\|^2 dx &= \int_a^\infty \langle (S(x))^{-1}u, u\rangle dx = \Big\langle \textstyle\int_a^\infty (S(x))^{-1} dx\, u, u \Big\rangle \\ &\le \Big\| \int_a^\infty (S(x))^{-1}\,dx \Big\|\,\|u\|^2. \end{aligned}$$

The lemma easily follows from (3). □

Le us consider now the sector S_θ with $\theta \in (0, \pi/2)$, defined by

$$S_\theta := \{z \in \mathbb{C}\,;\, z \ne 0,\ |\arg(z)| < \theta\}.$$

We will use frequently the following lemma.

Lemma 2. *The condition $W(M) \subset S_\theta$ implies* $\operatorname{Re} M^{-1} \ge \cos^2\theta\ (\operatorname{Re} M)^{-1}$.

Proof. We first note that the assumption implies that M is invertible and $B := \operatorname{Re} M$ is positive definite. We set $D = B^{-1/2}(\operatorname{Im} M)\,B^{-1/2}$; then D is selfadjoint and $M = B^{1/2}(I + iD)B^{1/2}$. The condition $W(M) \subset S_\theta$ yields

$$\langle (I + iD)B^{1/2}u, B^{1/2}u\rangle \in S_\theta, \qquad \forall\, u \in \mathbb{C}^d, u \ne 0.$$

Setting $v = B^{1/2}u$ we deduce $\|Dv\| \le \tan\theta\ \|v\|$, $\forall v \in \mathbb{C}^d$, and thus $\|D\| \le \tan\theta$. We have

$$\operatorname{Re}(I + iD)^{-1} \ge \inf_{\lambda \in Sp(D)} \operatorname{Re} \frac{1}{1 + i\lambda} = \operatorname{Re} \frac{1}{1 + i\tan\theta} = \cos^2\theta.$$

Finally we deduce $\operatorname{Re} A^{-1} = \operatorname{Re} B^{-1/2}(I + iD)^{-1}B^{-1/2} \ge \cos^2\theta\ B^{-1}$. □

3 The sector case

Since the constant $C(S_\alpha)$ only depends on the angle of the sector, we can assume in this section that $S_\alpha = \{z \in \mathbb{C}\,; z \neq 0 \text{ and } |\arg z| < \alpha\}$, $\alpha \in (0, \pi/2)$.

Theorem 3. *We have the estimate*

$$C(S_\alpha) \le 2\,\frac{\pi-\alpha}{\pi} + \frac{\sin 2\alpha}{\pi}\int_0^\infty \frac{dy}{y^2 - 2\,y\,\cos 2\alpha + \cos^2\alpha}.$$

Proof. Let us consider a rational function r bounded by 1 in S_α and satisfying $r(\infty)=0$. Let us consider also a square matrix $A \in \mathbb{C}^{d,d}$ such that $W(A) \subset S_\alpha$. It suffices to show that

$$\|r(A)\| \le 2\,\frac{\pi-\alpha}{\pi} + \frac{\sin 2\alpha}{\pi}\int_0^\infty \frac{dy}{y^2 - 2\,y\,\cos 2\alpha + \cos^2\alpha}.$$

For that we denote by σ the generic point on the counterclockwise oriented boundary ∂S_α and by s its curvilinear abscissa. Then we deduce from the Cauchy formula

$$r(A) = \frac{1}{2\pi i}\int_{\partial S_\alpha} r(\sigma)\,(\sigma I - A)^{-1} d\sigma,$$

that we have

$$r(A) = \int_{\partial S_\alpha} r(\sigma)\,\mu(\sigma, A)\,ds + \tilde r(A^*),$$

with

$$\begin{aligned}
\mu(\sigma, A) &= \frac{1}{2\pi}\big(\nu(\sigma - A)^{-1} + \bar\nu(\bar\sigma - A^*)^{-1}\big), \quad \nu = \frac{1}{i}\frac{d\sigma}{ds},\\
\tilde r(\bar z) &= \frac{1}{2\pi i}\int_{\partial S_\alpha} r(\sigma)\,\frac{d\bar\sigma}{\bar\sigma - \bar z}.
\end{aligned}$$

The condition $W(A) \subset S_\alpha$ implies that the selfadjoint matrix $\mu(\sigma, A)$ is positive definite for $\sigma \in \partial S_\alpha$. Thus we have the estimate

$$\Big\|\int_{\partial S_\alpha} r(\sigma)\,\mu(\sigma, A)\,ds\Big\| \le \Big\|\int_{\partial S_\alpha} \mu(\sigma, A)\,ds\Big\| = 2\,\frac{\pi - \alpha}{\pi}.$$

Therefore it suffices to show that

$$\|\tilde r(A^*)\| \le \mu(\alpha), \quad \text{with } \mu(\alpha) := \frac{\sin 2\alpha}{\pi}\int_0^\infty \frac{dy}{y^2 - 2\,y\,\cos 2\alpha + \cos^2\alpha}.$$

We split the boundary in $\Gamma_+ := \{\sigma \in \partial S_\alpha\,; \operatorname{Im}\sigma > 0\}$ and $\Gamma_- := \{\sigma \in \partial S_\alpha\,; \operatorname{Im}\sigma < 0\}$. Note that, on $\Gamma_\pm$, we have $\bar\sigma = \sigma\, e^{\mp 2i\alpha}$, thus

$$\tilde r(\bar z) = \frac{1}{2\pi i}\int_{\Gamma_+} \frac{r(\sigma)}{\sigma - e^{2i\alpha}\bar z}\,d\sigma + \frac{1}{2\pi i}\int_{\Gamma_-} \frac{r(\sigma)}{\sigma - e^{-2i\alpha}\bar z}\,d\sigma.$$

Now we remark that, in the sector S_α, the integrands are holomorphic functions of σ and have a $O(\sigma^{-2})$ behaviour at ∞. Using the Cauchy theorem, we can move the integrals on $\Gamma_\pm$ in integrals on the half-real axis. This gives

$$\int_{\Gamma_+} \frac{r(\sigma)}{\sigma - e^{2i\alpha}\bar z}\,d\sigma = \int_\infty^0 \frac{r(x)}{x - e^{2i\alpha}\bar z}\,dx, \quad \text{and} \quad \int_{\Gamma_-} \frac{r(\sigma)}{\sigma - e^{-2i\alpha}\bar z}\,d\sigma = \int_0^\infty \frac{r(x)}{x - e^{-2i\alpha}\bar z}\,dx.$$

We deduce

$$\tilde r(\bar z) = \frac{1}{2\pi i}\int_0^\infty r(x)\Big(\frac{1}{x - e^{-2i\alpha}\bar z} - \frac{1}{x - e^{2i\alpha}\bar z}\Big)\,dx = -\frac{\sin 2\alpha}{\pi}\int_0^\infty \frac{r(x)}{x^2\bar z^{-1} - 2\,x\,\cos 2\alpha + \bar z}\,dx.$$

Therefore

$$\tilde r(A^*) = -\frac{\sin 2\alpha}{\pi}\int_0^\infty r(x)\big(M^*(x)\big)^{-1}\,dx, \quad \text{with} \quad M(x) := x^2A^{-1} - 2\,x\,\cos 2\alpha + A.$$

Now we set $S(x) := x^2\cos^2\alpha\, B^{-1} - 2\,x\,\cos 2\alpha + B$, with $B = \operatorname{Re} A$. Clearly we have $S(x) = S^*(x)$ and

$$S(x) \geq \min_{\lambda\in Sp(B)}\big(\frac{x^2\cos^2\alpha}{\lambda} - 2\,x\,\cos 2\alpha + \lambda\big) \geq 2\,x(\cos\alpha - \cos 2\alpha) = 4\,x\,\sin\tfrac{\alpha}{2}\,\sin\tfrac{3\alpha}{2} > 0.$$

Furthermore, from Lemma 2, we have $\operatorname{Re} M(x) \geq S(x)$. Using then Lemma 1, we get

$$\|\tilde r(A^*)\| \leq \frac{\sin 2\alpha}{\pi}\Big\|\int_0^\infty \big(S(x)\big)^{-1}\,dx\Big\|.$$

We note now that, for all $\lambda > 0$, by setting $y = 1/(\lambda x)$, we have

$$\varphi(\lambda) := \frac{\sin 2\alpha}{\pi}\int_0^\infty \frac{dx}{\frac{x^2\cos^2\alpha}{\lambda} - 2\,x\,\cos 2\alpha + \lambda} = \mu(\alpha).$$

This shows, by using the spectral theory for the selfadjoint matrix B,

$$\frac{\sin 2\alpha}{\pi}\int_0^\infty (S(x))^{-1}dx = \varphi(B) = \mu(\alpha)\,I.$$

Thus we have obtained the bound

$$\|r(A)\| \leq 2\frac{\pi-\alpha}{\pi} + \mu(\alpha),$$

for rational functions bounded by 1 in S_α and satisfying $r(\infty) = 0$. Now if r still is bounded by 1 in the sector but $r(\infty) \neq 0$, we introduce $r_\varepsilon(z) = (1+\varepsilon z)^{-1}r(z)$, $\varepsilon > 0$. Then r_ε also is bounded by 1 in the sector and $r_\varepsilon(\infty) = 0$. Thus we have the bound $\|r_\varepsilon(A)\| \leq 2\frac{\pi-\alpha}{\pi} + \mu(\alpha)$, which yields the bound for $\|r(A)\|$ with ε tending to 0. $\square$

4 Case of a parabola

Since all the parabolas are similar, $C(P)$ is the same for all parabolas P. Here we consider

$$P := \{x+i\,y\,;x > y^2, y \in \mathbb{R}\}.$$

Theorem 4. *We have the estimate $C(P) \leq 2 + 2/\sqrt{3}$.*

Proof. As in the previous section we consider a matrix $A \in \mathbb{C}^{d,d}$ such that $W(A) \subset P$ and a rational function such that $|r(z)| \leq 1$ for all $z \in P$ and $r(\infty) = 0$. In order to show that $C(P) \leq 2 + 2/\sqrt{3}$ it suffices to show that $\|r(A)\| < 2 + 2/\sqrt{3}$ for all such A and r. With the previous notations, we have

$$r(A) = \int_{\partial P} r(\sigma)\,\mu(\sigma, A)\,ds + \tilde r(A^*), \quad \text{with} \quad \tilde r(A^*) = \int_{\partial P} r(\sigma)\,(\bar\sigma - A^*)^{-1}d\bar\sigma.$$

We know that $\big\|\int_{\partial P} r(\sigma)\,\mu(\sigma, A)\,ds\big\| \leq 2$, therefore it suffices to show that $\|\tilde r(A^*)\| \leq 2/\sqrt{3}$.

Recall that the boundary ∂P is counterclockwise oriented. On this boundary we have the relation $2(\sigma+\bar\sigma) + (\sigma-\bar\sigma)^2 = 0$, thus $\bar\sigma = \sigma - 1 + \sqrt{1-4\sigma}$, here the notation $\sqrt{1-4\sigma}$ denotes the continuous determination of the square root off the cut $\Gamma = \{x \in \mathbb{R}\,; x > \frac{1}{4}\}$ which takes the value 1 when $\sigma = 0$. Note also that, if $\sigma \in P$, then $\sigma - 1 + \sqrt{1-4\sigma} \notin \overline{P}$. So, for $z \in P$,

$$\tilde{r}(\bar{z}) = \frac{1}{2\pi i}\int_{\partial P} \frac{r(\sigma)}{\bar{\sigma} - \bar{z}}\, d\bar{\sigma} = \frac{1}{2\pi i}\int_{\partial P} \frac{r(\sigma)}{\sigma - 1 + \sqrt{1-4\sigma} - \bar{z}}\Big(1 - \frac{2}{\sqrt{1-4\sigma}}\Big)\, d\sigma.$$

Using the holomorphy (in σ) of the integrand in $P \setminus \Gamma$, we can replace the path ∂P by the path $\Gamma_+ \cup \Gamma_-$, where $\Gamma_\pm = \lim_{\varepsilon \to 0_+} \Gamma + i\varepsilon$. We also note that $\sqrt{1-4\sigma}$ tends to $\mp i\sqrt{4x-1}$ as $\sigma \in P$ tends to $x \in \Gamma_\pm$. Then we get

$$\begin{aligned}\tilde{r}(\bar{z}) &= \frac{1}{2\pi i}\int_{\Gamma_+} \frac{r(x)}{x - 1 - i\sqrt{4x-1} - \bar{z}}\Big(1 - \frac{2i}{\sqrt{4x-1}}\Big)\, dx \\ &\quad + \frac{1}{2\pi i}\int_{\Gamma_-} \frac{r(x)}{x - 1 + i\sqrt{4x-1} - \bar{z}}\Big(1 + \frac{2i}{\sqrt{4x-1}}\Big)\, dx.\end{aligned}$$

Due to the counterclockwized orientation of ∂P x runs from $+\infty$ to $1/4$ on Γ_+ and from $1/4$ to $+\infty$ on Γ_-. Setting $x = y^2 + 1/4$ we obtain

$$\begin{aligned}\tilde{r}(\bar{z}) &= \frac{1}{\pi i}\int_0^\infty r(y^2 + \tfrac{1}{4})\Big(\frac{y+i}{y^2 - 3/4 + 2iy - \bar{z}} - \frac{y-i}{y^2 - 3/4 - 2iy - \bar{z}}\Big)\, dy \\ &= \frac{-2}{\pi}\int_0^\infty r(y^2 + \tfrac{1}{4})\,\frac{y^2 + 3/4 + \bar{z}}{(y^2 - 3/4 - \bar{z})^2 + 4y^2}\, dy \\ &= \frac{-2}{\pi}\int_0^\infty r(y^2 + \tfrac{1}{4})\,\frac{1}{(y^2 + 3/4 + \bar{z}) - 4y^2 + 4y^2(1+y^2)(y^2 + 3/4 + \bar{z})^{-1}}\, dy.\end{aligned}$$

Therefore we have

$$\tilde{r}(A^*) = -\int_0^\infty r(y^2 + \tfrac{1}{4})\,\big(M^*(y)\big)^{-1}\, dy,$$

with

$$M(y) := \frac{\pi}{2}\Big((A + y^2 + 3/4) - 4y^2 + 4y^2(1+y^2)(A + y^2 + 3/4)^{-1}\Big).$$

Remark. *The straight lines with equations $y = \pm\frac{1}{2t}(x + t^2)$ are tangent to the parabola in the points $t^2 \pm it$. Thus, setting $\theta(t) := \arctan(1/2t)$, we have $P \subset -t^2 + S_{\theta(t)}$ and the condition $W(A) \subset P$ implies $W(A + t^2) \subset S_{\theta(t)}$. Therefore, setting $B := \operatorname{Re} A$ and applying Lemma 2, we obtain*

$$\operatorname{Re}(A + t^2)^{-1} \geq \cos^2\theta(t)\ (B + t^2)^{-1} = \frac{4t^2}{4t^2 + 1}\ (B + t^2)^{-1}, \quad \forall t > 0. \qquad (4)$$

We now define

$$S(y) := \frac{\pi}{2}\Big((B + y^2 + 3/4) - 4y^2 + y^2(4y^2 + 3)(B + y^2 + 3/4)^{-1}\Big).$$

Using (4) with $t = \sqrt{y^2 + 3/4}$ it follows

$$\operatorname{Re} M(y) \geq S(y) = S^*(y).$$

In order to study more precisely the matrix $S(.)$ we introduce the function $\varphi(., \lambda)$

$$\varphi(y, \lambda) := \frac{\pi}{2}\big((\lambda + y^2 + 3/4) - 4y^2 + y^2(4y^2 + 3)(\lambda + y^2 + 3/4)^{-1}\big),$$

that we consider for $\lambda > 0$ and $y \geq 0$. Simple calculations give

$$\begin{aligned}\frac{1}{\varphi(y,\lambda)} &= \frac{2}{\pi}\,\frac{y^2+3/4+\lambda}{(y^2-3/4-\lambda)^2+3y^2}\\ &= \frac{1}{\pi}\Big(\frac{1}{(y-\sqrt{\lambda})^2+3/4}+\frac{1}{(y+\sqrt{\lambda})^2+3/4}\Big).\end{aligned}$$

This implies that $\varphi(y,\lambda) > 0$ for all $y, \lambda > 0$, and therefore $S(y) = \varphi(y,B)$ is positive definite. Furthermore we have

$$\int_0^\infty \frac{1}{\varphi(y,\lambda)}\,dy = \frac{1}{\pi}\Big(\int_{-\sqrt{\lambda}}^\infty \frac{dv}{v^2+3/4}+\int_{\sqrt{\lambda}}^\infty \frac{dv}{v^2+3/4}\Big) = \frac{2}{\sqrt{3}},$$

which shows that

$$\int_0^\infty (S(y))^{-1}dy = 2/\sqrt{3}$$

and, by using Lemma 1, provides the estimate $\|\tilde{r}(A^*)\| \leq 2+2/\sqrt{3}$.

$\square$

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