Numerical range, holomorphic calculus and applications

Michel CROUZEIX

version du 21 juillet 2005

Abstract

This paper is devoted to the presentation of two elementary inequalities concerning polynomial functions of one matrix. After having extended them to more general functions and more general operators we give the proofs. Then we detail some applications to different areas of mathematics.

Mathematical subject classification : 47A12 ; 47A25

Keywords : functional calculus, numerical range, spectral sets.

1 Introduction

Let $A \in \mathbb{C}^{d,d}$ be a square matrix with complex entries ; its numerical range $W(A)$ is defined by

$$W(A) = \{\langle Af, f\rangle\,;\, f \in \mathbb{C}^d,\ \langle f, f\rangle = 1\},$$

where $\langle f, g\rangle = \sum_{j=1}^{d} f_j \overline{g}_j$ denotes the inner product in $\mathbb{C}^d$ and $\|f\| = \langle f, f\rangle^{1/2}$ the associated norm. Our first result is the following

Theorem 1. *There exists a constant* $\mathcal{Q} \le 30.46$ *such that the inequality*

$$\|p(A\| \le \mathcal{Q} \sup_{z \in W(A)} |p(z)|, \tag{1}$$

holds for all matrices $A \in \mathbb{C}^{d,d}$*, for all polynomials* $p : \mathbb{C} \to \mathbb{C}$ *and for all values of* $d \in \mathbb{N}^*$*.*

By definition $\mathcal{Q}$ will be the best constant such that the previous inequality holds. The proof of this result will be the object of Sections 4, 5 and 6.

Remark. For $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ *it is easily seen that its numerical range is the closed unit disk* $W(A) = \overline{D}$*. Taking* $p(z) = z$ *we get* $\|p(A)\| = \|A\| = 2$ *and* $\sup_{z \in W(A)} |p(z)| = 1$*. This shows that necessarily* $\mathcal{Q} \ge 2$*.*

Up to now I have failed to prove my conjecture $\mathcal{Q} = 2$*.*

The second theorem is the tensorial form of the previous result. If $P : \mathbb{C} \to \mathbb{C}^{m,n}$ is a polynomial with $m \times n$ matrix values $P(z) = \big(p_{k\ell}(z)\big)$, we define $P(A) \in \mathbb{C}^{dm,dn}$ by :

$$\text{if } f = \begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ f_n \end{pmatrix}, g = \begin{pmatrix} g_1 \\ \cdot \\ \cdot \\ g_m \end{pmatrix} \quad \text{are block-vectors with } \ f_\ell \in \mathbb{C}^d,\ g_k \in \mathbb{C}^d, \quad \textit{then}$$

$$g = P(A)\, f \qquad \text{means} \quad g_k = \sum_{\ell=1}^{n} p_{k\ell}(A)\, f_\ell,\ k = 1, \dots, m.$$

Theorem 2. *There exists a constant* $\mathcal{Q}_{cb} \leq 30.46$ *such that the inequality*

$$\|P(A\| \leq \mathcal{Q}_{cb} \sup_{z \in W(A)} \|P(z)\|, \tag{2}$$

holds for all matrices $A \in \mathbb{C}^{d,d}$*, for all polynomials* $P : \mathbb{C} \to \mathbb{C}^{m,n}$ *and for all values of* $d, m, n \in \mathbb{N}^*$*.*

We clearly have $2 \leq \mathcal{Q} \leq \mathcal{Q}_{cb}$. We have also the conjectures that $\mathcal{Q} = \mathcal{Q}_{cb}$ and $\mathcal{Q}_{cb} = 2$.

Remark. In the language of operator theory, Theorem 1 means that the map $p \mapsto p(A)$*, acting from the algebra of polynomials equipped with the norm* $\sup_{z \in W(A)} |p(z)|$ *into the algebra* $\mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)$*, is bounded with the constant* $\mathcal{Q}$*. Theorem 2 means that this map is "completely " bounded with constant* $\mathcal{Q}_{cb}$*, the adverb "completely" points out that the constant* $\mathcal{Q}_{cb}$*can be chosen independently of* m *and* n*.*

.

Remark. Let us consider now a matrix B*, the unit disk* D *and the map* $p \mapsto p(B)$*, from the algebra of polynomials equipped with the uniform norm* $\sup_{z \in D} |p(z)|$ *into the algebra* $\mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)$*. A famous conjecture due to Halmos was "bounded implies completely bounded", but it has been disproved by Gilles Pisier. Here we are not in the same situation ; we do not consider all matrices* B *corresponding to a bounded map but, for each matrix* A *given, only the matrices with* $W(B) = W(A)$*.*

The organisation of the paper is the following

We will not give the proof of Theorem 2, leaving to the reader the care to adapt the one of Theorem 1. The tools which are used are Cauchy-Schwarz inequality in Theorem 6 and Lemmas 14, 16, 17, von Neumann inequality in Theorem 8 and Lemma 11 ; it is well known that these inequalities are also valid in completely bounded form.

2 Some remarks on the numerical range

To the square matrix $A \in \mathbb{C}^{d,d}$ we associate the two self-adjoint matrices $B = \frac{1}{2}(A + A^*)$ and $C = \frac{1}{2i}(A - A^*)$; so we have $A = B + iC$. Recall that the numerical range is defined by

$$W(A) := \{\tfrac{\langle Af, f\rangle}{\langle f, f\rangle} \,;\, f \in \mathbb{C}^d, f \neq 0\} = \{\langle Af, f\rangle \,;\, f \in \mathbb{C}^d, \langle f, f\rangle = 1\}.$$

The following properties are easily verified :

- $W(A)$ is compact and contains the spectrum $\sigma(A)$ of the matrix A,
- $W(\lambda A + \mu I) = \lambda W(A) + \mu$,
- if U is a unitary matrix, then $W(U^*AU) = W(A)$,
- if $\lambda \notin W(A)$ then $\|(\lambda - A)^{-1}\| \leq \frac{1}{d(\lambda, W(A))}$.

Now we give a simple proof of the result due to Toeplitz and Hausdorff

Lemma 3. *The numerical range is convex.*

Proof. Let $\lambda = \frac{\langle Af,f\rangle}{\langle f,f\rangle}$ and $\mu = \frac{\langle Ag,g\rangle}{\langle g,g\rangle}$ be two distinct points of $W(A)$. Replacing if needed g by $e^{i\theta}g$, we can assume that

$$\frac{\langle (A-\lambda)f,g\rangle}{\mu-\lambda} - \frac{\langle (A^*-\bar\lambda)f,g\rangle}{\bar\mu-\bar\lambda} \in \mathbb{R}, \qquad \text{thus}$$
$$\frac{\langle (A-\lambda)f,g\rangle + \langle (A-\lambda)g,f\rangle}{\mu-\lambda} = \frac{\langle (A-\lambda)f,g\rangle}{\mu-\lambda} - \frac{\langle (A^*-\bar\lambda)f,g\rangle}{\bar\mu-\bar\lambda} + 2\,\mathrm{Re}\,\frac{\langle (A-\lambda)g,f\rangle}{\mu-\lambda} \in \mathbb{R}.$$

Then, for $t \in \mathbb{R}$, we set

$$\begin{aligned}\varphi(t) &:= \frac{\langle A(f+tg),f+tg\rangle - \lambda\langle f+tg,f+tg\rangle}{(\mu-\lambda)\langle f+tg,f+tg\rangle}\\ &= t\,\frac{\langle (A-\lambda)f,g\rangle + \langle (A-\lambda)g,f\rangle}{(\mu-\lambda)\langle f+tg,f+tg\rangle} + t^2\,\frac{\langle g,g\rangle}{\langle f+tg,f+tg\rangle} \in \mathbb{R}.\end{aligned}$$

Clearly the rational function φ is continue on $\overline{\mathbb{R}}$, and takes real values. One has $\varphi(0)=0$, $\varphi(\infty)=1$, thus $(0,1)\subset\varphi(\mathbb{R})$. Noticing that

$$(1-\varphi(t))\lambda + \varphi(t)\mu = (\lambda-\mu)\varphi(t) + \lambda = \frac{\langle A(f+tg),f+tg\rangle}{\langle f+tg,f+tg\rangle} \in W(A),$$

we deduce that the segment joining λ to μ is contained in $W(A)$. □

It is also easy to prove the following properties :

- if A is a normal matrix, $W(A)$ is the convex hull $\mathrm{co}(\sigma(A))$ of $\sigma(A)$,
- If $\mathbb{C}^d = E_1 \oplus \cdots \oplus E_n$ is a orthogonal sum of subspaces E_j which are invariant by A, then

  $$W(A) = \mathrm{co}(W(A_1),\dots,W(A_n)), \quad \text{(convex hull of the union of } W(A_j)\text{ ,}$$

  where A_j denotes any matrix which represents A (restricted to E_j) w.r.t. an orthonormal basis of E_j.

Remark on the algebraic character of the boundary $\partial W(A)$ of the numerical range.

Let $\sigma_0 = x_0+iy_0 = \langle Ag,g\rangle$, with $\|g\|=1$, be a point of this boundary. There exists a half-plane $\Pi = \{(x,y)\,; ux+vy+w \ge 0\}$ with $\sigma_0 \in \partial\Pi$ (thus $ux_0+vy_0+w=0$) which contains the numerical range : $\Pi \supset W(A)$. We set

$$G(f) := u\ \mathrm{Re}\langle Af,f\rangle + v\ \mathrm{Im}\langle Af,f\rangle + w\,\langle f,f\rangle.$$

Then clearly $G(g)=0$ and, since $\langle Af,f\rangle/\langle f,f\rangle \in \Pi$, $G(f)\ge 0$ for all $f\in\mathbb{C}^d$. Recall that $A = B+i\,C$, we have

$$G(g+h) = G(g) + 2\ \mathrm{Re}\,\big(u\langle Bg,h\rangle + v\langle Cg,h\rangle + w\langle g,h\rangle\big) + O(\|h\|^2),$$

which implies $\mathrm{Re}\langle (uB+vC+wI)g,h\rangle = 0$, $\forall\, h\in\mathbb{C}^d$, *and then the equality $(uB+vC+wI)g=0$. Since $g\ne 0$, we deduce*

$$P(u,v,w) := \det(uB+vC+wI) = 0. \tag{3}$$

Thus all the tangent lines (and even all the “ droites d'appui“) to the convex subset $W(A)$ satisfy $P(u,v,w)=0$ (furthermore, if $P(u,v,w)=0$, the straight line with cartesian equation $ux+vy+w=0$ has at least a point belonging to the numerical range).

It follows that the numerical range $W(A)$ is included in the algebraic curve $\mathcal{C}$ with tangential equation $P(u,v,w)=0$. The homogeneous polynomial P having the degree d, the curve $\mathcal{C}$ is said of class d. Note also that, if $\lambda=x+iy$ is an eigenvalue of A, then $P(1,i,-\lambda)=det(A-\lambda I)=0$; there exists a tangent to the curve $\mathcal{C}$ with slope i (i.e. an isotropic tangent) passing through the point λ. In short $\mathcal{C}$ is an algebraic curve of class d whose the foci are the eigenvalues of A ; from the Plücker formulas the degree of this curve is at most $d(d-1)$ [17].

Now if H is a complex Hilbert space, with inner product $\langle .,. \rangle$ and associated norm $\|.\|$, and if $A \in \mathcal{L}(H,H)$ is a bounded linear operator on H, we define similarly the numerical range of A by

$$W(A) := \{\langle Af,f\rangle \,;\, f\in H, \langle f,f\rangle = 1\}.$$

We obtain in the same way than before that $W(A)$ is convex ; it is not necessarily closed (if H is infinite dimensional) but now we have for the spectrum $\sigma(A)\subset \overline{W(A)}$. Indeed, if $\lambda\notin\overline{W(A)}$, we have

$$\big|\langle(\lambda-A)v,v\rangle\big| \geq \mathrm{d}(\lambda,W(A))\|v\|^2, \qquad \forall v\in H,$$

which shows (Lax-Milgram lemma) that λ belongs to the resolvant and provides the estimate $\|(\lambda-A)^{-1}\|\leq 1/\mathrm{d}(\lambda,W(A))$.

For the applications it is important to consider the case where A is a closed linear operator unbounded on H. We assume that $A\in\mathcal{L}(D(A),H)$ has its domain $D(A)\subset H$ dense in H ; we then define the numerical range of A by

$$W(A) := \{\langle Af,f\rangle \,;\, f\in D(A), \langle f,f\rangle = 1\}.$$

We also assume that the spectrum satisfies $\sigma(A)\subset\overline{W(A)}$. Then it is easily seen that (same proof that previously) $W(A)$ is convex and, if $\lambda\notin\overline{W(A)}$, the resolvant estimate $\|(\lambda-A)^{-1}\|\leq 1/\mathrm{d}(\lambda,W(A))$ still holds.

Finally we describe a framework which is frequently met in the variational study of partial differential equations. We assume that we have two complex Hilbert spaces $V\subset H$, V dense in H with a continuous imbedding, and a sesquilinear form $a(.,.)$ which satisfies for some constants $\alpha>0$ and M

$$\operatorname{Re} a(v,v)\geq \alpha\|v\|_V^2, \qquad \text{and} \qquad |a(v,v)| \leq M\|v\|_V^2, \qquad \forall v\in V. \tag{4}$$

Then we define the operator $A\in\mathcal{L}(D(A),H)$ by

$$D(A)=\{v\in V\,;\exists K_v \ s.t.\ |a(v,w)|\leq K_v\|w\|_H, \forall\, w\in H\} \quad \text{and} \quad \langle Av,w\rangle = a(v,w).$$

It is an easy consequence of the Lax-Milgram lemma that A is an isomorphism from $D(A)$ onto H and that $D(A)$ is dense in H and in V. If we define

$$W_V(A) := \{a(v,v)\,; v\in V, \|v\|_H = 1\},$$

we have

$$W(A)\subset W_V(A), \quad \text{and} \quad \overline{W(A)} = \overline{W_V(A)}.$$

Furthermore, using one more time the Lax-Milgram lemma, we see that $\sigma(A)\subset\overline{W(A)}$.

Remark. Note that the assumptions (4) on the sesquilinear form give some informations on the numerical range :

$$W(A)\subset \overline{S}_\theta := \{z\in\mathbb{C}\,; z=0 \text{ or } |\arg z|\leq\theta\}, \quad \text{where } \ \theta := \arccos\tfrac{\alpha}{M}.$$

3 Extension to a more general context

Using the fact that the constants are independent of d, we will deduce that Theorems 1 and 2 are still valid for any bounded linear operator $A \in \mathcal{L}(H)$ on any Hilbert space H. We now consider the algebra $\mathbb{C}[z]$ of polynomials p from $W(A)$ into $\mathbb{C}$, provided with the norm $\|p\|_{\infty,A} = \sup_{z\in W(A)} |p(z)|$, and, for a convex subset E of $\mathbb{C}$, we introduce the algebra $\mathcal{H}_b(E)$ of continuous and bounded functions in $\overline{E}$ which are holomorphic in the interior of E. The following statement shows the existence of a functional calculus based on the numerical range.

Theorem 4. *Let H be a Hilbert space. For any bounded linear operator $A \in \mathcal{L}(H)$ the homomorphism $p \mapsto p(A)$ from the algebra $\mathbb{C}[z]$, with norm $\|\,.\,\|_{\infty,A}$, into the algebra $\mathcal{L}(H)$, is bounded with constant $\mathcal{Q}$. It admits a unique bounded extension from $\mathcal{H}_b(W(A))$ into $\mathcal{L}(H)$. This extension is bounded with constant $\mathcal{Q}$ and completely bounded with constant $\mathcal{Q}_{cb}$.*

Proof. We assume that (1) is valid if $H = \mathbb{C}^d$, for all integers d. This implies that (1) holds for any finite dimensional Hilbert space. Let now H be infinite-dimensional and consider an operator $A \in \mathcal{L}(H)$ and a polynomial p of degree n. For $u \in H$ given, we introduce the Krylov space $\mathcal{K}_n = \operatorname{Span}\{u, Au, \dots, A^n u\} \subset H$, denote by Π_n the orthogonal projection onto $\mathcal{K}_n$, and set $A_n := \Pi_n A|_{\mathcal{K}_n}$. We clearly have $p(A)\,u = p(A_n)\,u$ and $W(A_n) \subset W(A)$, and A_n acts on the $n{+}1$ dimensional space $\mathcal{K}_n$. We then deduce that

$$\|p(A)\,u\| = \|p(A_n)\,u\| \le \mathcal{Q} \sup_{z\in W(A_n)} |p(z)|\ \|u\| \le \mathcal{Q} \sup_{z\in W(A)} |p(z)|\ \|u\|,$$

which implies $\|p(A)\| \le \mathcal{Q} \sup\limits_{z\in W(A)} |p(z)|$. The same proof works also for polynomials with matrix values, thus the inequality (2) is also valid.

The Mergelyan theorem states that $\mathbb{C}[z]$ is dense in $\mathcal{H}_b(W(A))$, which completes the proof. □

For many applications it is useful to consider unbounded operators. Assume now that $A \in \mathcal{L}(D(A), H)$ is a closed linear operator with domain $D(A)$ densely and continuously embedded in H. Its numerical range is then defined by $W(A) := \{\langle A\,v, v\rangle\,;\, v \in \Sigma_H \cap D(A)\}$. We assume that the spectrum $\sigma(A)$ is included in $\overline{W(A)}$. The numerical range is still convex but unbounded and we hence cannot apply polynomials. We therefore instead consider the algebra $\mathbb{C}_b(z)$ of rational functions which are bounded on $W(A)$, and provide it with the norm $\|r\|_{\infty,A} = \sup_{z\in W(A)} \|r(z)\|$. We then have the following statement similar to Theorem 4

Theorem 5. *For any closed linear unbounded operator A such that $\sigma(A) \subset \overline{W(A)}$, the homomorphism $r \mapsto r(A)$ from the algebra $\mathbb{C}_b(z)$, with norm $\|\,.\,\|_{\infty,A}$, into the algebra $\mathcal{L}(H)$, is bounded with constant $\mathcal{Q}$. It admits a unique bounded extension from the algebra $\mathcal{H}_b(W(A))$ into $\mathcal{L}(H)$. This extension is bounded with constant $\mathcal{Q}$ and completely bounded with constant $\mathcal{Q}_{cb}$.*

Proof. We consider now a closed linear unbounded operator A such that $\sigma(A) \subset \overline{W(A)}$ and a rational function r bounded on $\overline{W(A)}$. It is clear that $r(A)$ is well defined and belongs to $\mathcal{L}(H)$ (write r in simple partial fraction form). In the case $W(A) = \mathbb{C}$, the space $\mathcal{H}_b(W(A))$ is reduced to constant functions and Theorem 2 follows immediately (but has no interest). In the other cases, changing, if needed, A to $\alpha + e^{i\theta}A$, $\alpha \in \mathbb{C}$, $\theta \in \mathbb{R}$, we have only two possibilities to consider :

a) The numerical range is a strip : $\overline{W(A)} = \{z \in \mathbb{C}\,;\, 0 \le \operatorname{Re} z \le a\}$, $a > 0$.

We then set $A_\varepsilon = (1{+}\varepsilon A^*)^{-1} A (1{+}\varepsilon A)^{-1}$, for $\varepsilon > 0$.

b) $\{x\,;x>0\}\subset W(A)\subset\{z\in\mathbb{C}\,;\operatorname{Re}z\ge 0\}$.
We then set $A_\varepsilon=A(1+\varepsilon A)^{-1}$, for $\varepsilon>0$.

In both cases it is easy to verify that A_ε is a bounded operator and that $W(A_\varepsilon)\subset W(A)$. Let r be a rational function bounded in $W(A)$, clearly $r\in\mathcal{H}_b(W(A_\varepsilon))$. From Theorem 2 we have

$$\|r(A_\varepsilon)\|\le\mathcal{Q}\sup_{z\in W(A_\varepsilon)}|r(z)|\le\mathcal{Q}\sup_{z\in W(A)}|r(z)|. \tag{5}$$

We now note that $\lim_{\varepsilon\to 0}r(A_\varepsilon)=r(A)$, strongly on H. Indeed, it suffices to verify that

$$\lim_{\varepsilon\to 0}(\lambda-A_\varepsilon)^{-1}u=(\lambda-A)^{-1}u\quad\text{ for all }\lambda\notin\overline{W(A)}\text{ and all }u\in D(A).$$

We consider the case of a strip, the other situation being simpler. We have by a simple calculation

$$\begin{aligned}(\lambda-A_\varepsilon)^{-1}u-(\lambda-A)^{-1}u=&-\varepsilon(\lambda-A_\varepsilon)^{-1}(1+\varepsilon A^*)^{-1}(A+A^*)(1+\varepsilon A)^{-1}(\lambda-A)^{-1}Au\\&-\varepsilon(\lambda-A_\varepsilon)^{-1}(1+\varepsilon A^*)^{-1}(\varepsilon A^*)(1+\varepsilon A)^{-1}A(\lambda-A)^{-1}Au.\end{aligned}$$

The convergence to 0 of the second member follows from the bounds

$$\|A+A^*\|\le 2\,a,\quad\|(1+\varepsilon A^*)^{-1}\|\le 1,\quad\|(1+\varepsilon A^*)^{-1}(\varepsilon A^*)\|\le 1,$$
$$\|(\lambda-A_\varepsilon)^{-1}\|\le\frac{1}{d(\lambda,W(A))},\quad\|(\lambda-A)^{-1}\|\le\frac{1}{d(\lambda,W(A))},$$
$$\|A\,(\lambda-A)^{-1}\|\le\frac{|\lambda|}{d(\lambda,W(A))}.$$

Taking the limit as $\varepsilon\to 0$ in (5), we obtain that the map $r\mapsto r(A)$ is bounded with constant $\mathcal{Q}$. The space $\mathbb{C}_b(z)$ is dense in the subset $\mathcal{H}_{b,0}(W(A))$ of functions of $\mathcal{H}_b(W(A))$ which have a limit at ∞. This therefore allows a first extension of the map to this subspace.

Let us consider now $f\in\mathcal{H}_b(W(A))$ and $\lambda\notin\overline{W(A)}$. With $g_\lambda(z)=(\lambda-z)^{-1}f(z)$ we have $g_\lambda\in\mathcal{H}_{b,0}(W(A))$, and from the previous extension, $g_\lambda(A)\in\mathcal{L}(H)$ is well defined. If, in addition, $f\in\mathcal{H}_{b,0}(W(A))$, then $f(A)\in\mathcal{L}(H)$ and $g_\lambda(A)=(\lambda-A)^{-1}f(A)$. This shows that $g_\lambda(A)\in\mathcal{L}(H,D(A))$ and $f(A)=(\lambda-A)g_\lambda(A)$.

We return to the general case $f\in\mathcal{H}_b(W(A))$. Recall that we have assumed $W(A)\subset\{z\in\mathbb{C}\,;\operatorname{Re}z\ge 0\}$. From the inequality, with $\varepsilon>0$,

$$\|A(1+\varepsilon A)^{-1}g_\lambda(A)\|\le\mathcal{Q}\sup_{z\in W(A)}\Big|\frac{z}{1+\varepsilon z}\,\frac{f(z)}{\lambda-z}\Big|\le\mathcal{Q}\,\frac{|\lambda|}{d(\lambda,W(A))}\,\|f\|_{\infty,A},$$

we deduce, by taking the limit as $\varepsilon\to 0$, that $\|A\,g_\lambda(A)\|\le\mathcal{Q}\frac{|\lambda|}{d(\lambda,W(A))}\,\|f\|_{\infty,A}$ and $g_\lambda(A)\in\mathcal{L}(H,D(A))$. We can therefore define $f(A)\in\mathcal{L}(H)$ by $f(A)=(\lambda-A)g_\lambda(A)$. Passing to the limit in the inequality

$$\|(1+\varepsilon A)^{-1}f(A)\|\le\mathcal{Q}\sup_{z\in W(A)}\Big|\frac{\lambda-z}{1+\varepsilon z}\,\frac{f(z)}{\lambda-z}\Big|\le\mathcal{Q}\,\|f\|_{\infty,A},$$

we deduce $\|f(A)\|\le\mathcal{Q}\,\|f\|_{\infty,A}$. It is easily seen that our definition does not depend on the particular choice of λ.

The same arguments work for the complete bound.

$\square$

.

4 Constants associated with a convex domain

Let $\Omega \neq \emptyset$ be an open convex subset of $\mathbb{C}$. We introduce the following constants

$$C(\Omega) := \sup\{\|r(A)\|; W(A) \subset \Omega, |r(z)| \leq 1, \forall z \in \overline{\Omega}\}, \tag{6}$$

$$C_{cb}(\Omega) := \sup\{\|R(A)\|; W(A) \subset \Omega, \|R(z)\| \leq 1, \forall z \in \overline{\Omega}\}. \tag{7}$$

In (6) the supremum is taken over all matrices $A \in \mathbb{C}^{d,d}$, all values of d and all rational functions r satisfying the prescribed constraints. In the second definition the rational functions R take their values in $\mathbb{C}^{m,n}$, the supremum is taken also over all m and n. Note that, in the case of a bounded convex domain Ω, we can replace rational functions by polynomial functions without changing the values of the constants. Remark also that these constants are only depending on the shape of Ω, more precisely $C(\Omega) = C(f(\Omega))$ and $C_{cb}(\Omega) = C_{cb}(f(\Omega))$ if $f(z) = \lambda z + \mu$ or if $f(z) = \lambda \bar{z} + \mu$, with $0 \neq \lambda,\ \mu \in \mathbb{C}$..

The oldest result is a classical inequality of von Neumann [16] which states that $C(\Pi) = C_{cb}(\Pi) = 1$ if Π is a half plane. My research on this subject has started after the reading of the fundamental work of Bernard and François Delyon [8] where they have shown that $C(\Omega) < \infty$ for all bounded convex domains. In [5] we have proved that $C_{cb}(S) \leq 2+2/\sqrt{3}$ if S is a strip or a convex sector ; in [2] some other constants are estimated for particular domains, in particular for the disk case D it is shown that $C(D) = C_{cb}(D) = 2$. An estimate $C_{cb}(\mathcal{P}) \leq 4.75$ where $\mathcal{P}$ is the interior of a parabola is given in [6].

Let us consider now a bounded convex domain Ω and a square matrix $A \in \mathbb{C}^{d,d}$ satisfying $W(A) \subset \Omega$. We denote by $\sigma \in \partial\Omega$ a generic boundary point and ν the outward unit normal in σ. To this point and the matrix A we associate the selfadjoint matrix

$$\mu(\sigma, A) := \frac{1}{2\pi}\big(\nu(\sigma - A)^{-1} + \bar{\nu}(\bar{\sigma} - A^*)^{-1}\big).$$

The following remark is of crucial importance

$$W(A) \subset \Omega \quad \Longleftrightarrow \quad \forall \sigma \in \partial\Omega, \quad \mu(\sigma, A) > 0. \tag{8}$$

Indeed we have $2\pi\mu(\sigma, A) = (\sigma - A)^{-1}\big(\bar{\nu}(\sigma - A) + \nu(\bar{\sigma} - A^*)\big)(\bar{\sigma} - A^*)^{-1}$, and the median term $\bar{\nu}(\sigma - A) + \nu(\bar{\sigma} - A^*)$ is positive definite if and only if the numerical range $W(A)$ is included in the open half-plane, tangent in σ to $\partial\Omega$, which contains Ω.

To a rational function r bounded in Ω, we associate a function $\tilde{r}$ and a matrix $\tilde{r}(A^*)$ defined by

$$\tilde{r}(\bar{z}) = \frac{1}{2\pi i}\int_{\partial\Omega} r(\sigma)\,\frac{d\bar{\sigma}}{\bar{\sigma} - \bar{z}}, \text{ if } z \in \Omega, \qquad \tilde{r}(A^*) = \frac{1}{2\pi i}\int_{\partial\Omega} r(\sigma)\,(\bar{\sigma} - A^*)^{-1} d\bar{\sigma}. \tag{9}$$

Let s be the arclength abscissa on $\partial\Omega$ (which will be counterclockwise oriented) ; we have $d\sigma = -i\nu\, ds$. Using the Cauchy formula $r(A) = (2\pi i)^{-1}\int_{\partial\Omega} r(\sigma)\,(\sigma - A)^{-1}\,d\sigma$ we deduce

$$r(A) = \int_{\partial\Omega} r(\sigma)\,\mu(\sigma, A)\,ds + \tilde{r}(A^*).$$

Remark. In general $\tilde{r}$ *is not a rational function but an antimorphic function in* $z \in \Omega$*. Taking the limit as* $\zeta \in \Omega$ *tends to* $z \in \partial\Omega$ *in the relation*

$$\tilde{r}(\bar{\zeta}) = r(\zeta) - \int_{\partial\Omega} r(\sigma)\,\frac{d\arg(\sigma - \zeta)}{\pi}, \quad \big(= r(\zeta) - \int_{\partial\Omega} r(\sigma)\,\mu(\sigma, \zeta)\,ds\big)$$

We deduce that $\tilde r$ admits a continuous extension to $\overline{\Omega}$ defined by

$$\tilde r(\bar z) = -\int_{\partial\Omega} r(\sigma)\,\frac{d\arg(\sigma - z)}{\pi} = -\int_{\partial\Omega} r(\sigma)\,\mu(\sigma, z)\,ds, \qquad \text{if } z \in \partial\Omega.$$

We remark also that, from the maximum principle, the positivity of the measure $d\arg(\sigma - z)$ and the relation $\int_{\partial\Omega} d\arg(\sigma - z) = \pi$, we have

$$\sup_{z\in\Omega} |\tilde r(\bar z)| = \sup_{z\in\partial\Omega} |\tilde r(\bar z)| \le \sup_{z\in\partial\Omega} |r(z)|. \tag{10}$$

The following theorem allows to replace the problem of the research of a bound for $\|r(A)\|$ by that of a bound for $\|\tilde r(A^*)\|$.

Theorem 6. *Let Ω be a bounded convex domain of $\mathbb{C}$, A a square matrix with $W(A) \subset \Omega$ and r a rational function with complex values satisfying $|r(z)| \le 1$ for all in Ω . Then (with $\tilde r(A^*)$ defined by* (9)*)*

$$\|r(A)\| \le 2 + \|\tilde r(A^*)\|.$$

Proof. We have seen in (8) that $\mu(\sigma, A) > 0$ therefore we get, if $u, v \in \mathbb{C}^d$ satisfy $\|u\| = \|v\| = 1$,

$$\begin{aligned} \big|\langle (r(A) - r(A^*))\, u, v\rangle\big| &\le \int_{\partial\Omega} |\langle \mu(\sigma, A)\, u, v\rangle|\, ds \\ &\le \tfrac{1}{2}\int_{\partial\Omega} \langle \mu(\sigma, A)\, u, u\rangle\, ds + \tfrac{1}{2}\int_{\partial\Omega} \langle \mu(\sigma, A)\, v, v\rangle\, ds = 2. \end{aligned}$$

The last equality comes from $\int_{\partial\Omega} \mu(\sigma, A)\, ds = 2\, I$, which holds from the Cauchy formula. We deduce

$$\|r(A) - r(A^*)\| \le 2.$$

□

Remark. It can appear quite surprizing to look for a bound of $\|\tilde r(A^*)\|$ in place of a bound directly on $\|r(A)\|$, but it appears that the first bound seems easier to obtain. An instructive example (which has been a guide for our investigations) is given by the case where Ω is the unit disk. Then $\bar\sigma = \sigma^{-1}$ and a simple calculation gives $\tilde r(A^*) = -r(0)\, I$. Thus, in the case of a disk D we obtain in a simple way $C(D) \le 3$.

Note that this result is not optimal ; a nice result due to Catalin Badea [2] states that $C(D) = C_{cb}(D) = 2$, but the corresponding proof is very specific to the disk case. Note also that Theorem 6 cannot be used for proving my conjecture $\mathcal{Q} = 2$.

It is natural now to introduce the following constants

$$\begin{aligned} D(\Omega) &:= \sup\{\|\tilde r(A^*)\|; W(A) \subset \Omega, |r(z)| \le 1, \forall z \in \overline{\Omega}\}, \\ D_{cb}(\Omega) &:= \sup\{\|\tilde R(A^*)\|; W(A) \subset \Omega, \|R(z)\| \le 1, \forall z \in \overline{\Omega}\}. \end{aligned}$$

The previous theorem implies that $C(\Omega) \le D(\Omega) + 2$; similarly $C_{cb}(\Omega) \le D_{cb}(\Omega) + 2$.

Conjecture. I have also the conjecture $D(\Omega) = D_{cb}(\Omega) = 1$.

It follows from Theorem 1 that we have $C(\Omega) \le 30.46$ for all convex domains Ω. Conversely, if we prove $D(\Omega) \le 28.46$, and consequently $C(\Omega) \le 30.46$, for any bounded convex domain which has a smooth boundary with a strictly positive curvature in each of its point, then we deduce Theorem 1 by using the next lemma

Lemma 7. *Let A be a matrix with complex entries. There exists a sequence of convex domains $\Omega_n \supset W(A)$ such that $\Omega_n \to W(A)$, with respect to the Hausdorff distance, and satisfying the constraints : the boundary $\partial\Omega_n$ is analytic and in each point of $\partial\Omega_n$ the curvature is stricltly positive.*

Proof. In the case where the interior of $W(A)$ is empty, this is easily realised with a sequence of ellipses. Otherwise, we can assume that the origin O is interior to $W(A)$. Then let a be the Riemann conformal mapping from the (open) unit disk D onto the interior of $W(A)$. We set $D_n = (1-1/n)D$; with an appropriate choice of $\varepsilon_n \to 0$, $\Omega_n = (1+\varepsilon_n)a(D_n)$ fullfills the requirements. Indeed classical calculations give that, for $z = (1-1/n)e^{i\theta}$ the curvature $c(z)$ at the point $a(z) \in \partial a(D_n)$ is given by

$$c(z) = \frac{1}{|z\, a'(z)|}\,\mathrm{Re}\,\big(1 + \frac{a''(z)z}{a'(z)}\big),$$

The strict positivity of $|z\, a'(z)|\, c(z)$ follows from the maximum principle. ☐

From this lemma, starting with $\|p(A)\| \le C(\Omega_n)\sup_{z\in\Omega_n}|p(z)|$, we deduce that

$$\|p(A)\| \le \liminf_{n\to\infty} C(\Omega_n) \sup_{z\in W(A)} |p(z)|,$$

Therefore $\mathcal{Q} \le \liminf_{n\to\infty} C(\Omega_n)$.

In order to prove Theorem 1 we will estimate the constants $D(\Omega)$ first in the case of smooth thick domains, then in the more tricky case of thin domains.

5 The thick domain case

Let $\Omega \neq \emptyset$ be a bounded convex domain of the complex plane. In order to avoid technical difficulties without interest we will assume that its boundary is analytic and that in each point of $\partial\Omega$ the curvature is strictly positive. We define the rate of flatness τ_{Ω} by

$$\tau_{\Omega} := \min_{\omega\in\Omega} \frac{\max\{|\sigma-\omega|\,;\sigma\in\partial\Omega\}}{\min\{|\sigma-\omega|\,;\sigma\in\partial\Omega\}}. \tag{11}$$

We will say that the domain is thick if $\tau_{\Omega} \le 1001$, otherwise it will be considered as thin.

Theorem 8. *Let $\Omega \neq \emptyset$ be a bounded convex domain of the complex plane. We set*

$$\gamma := \arccos\tfrac{1}{\tau_\Omega}, \quad g(t) := (\tfrac{1}{2} + \tfrac{|\arctan t|}{\pi})\sqrt{1+t^2} \quad \textit{and} \quad G(\gamma) := \int_0^\gamma g(\tan\varphi)\,d\varphi.$$

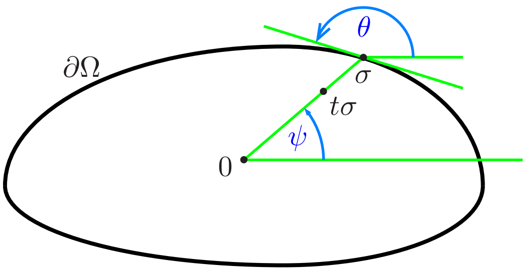
Then we have

$$D(\Omega) \le 4\,G(\gamma) + 2\,G(\pi-2\gamma), \quad \textit{if} \ \ \tau_\Omega \ge 2, \qquad D(\Omega) \le 6\,G(\tfrac{\pi}{3}), \quad \textit{otherwise}.$$

In particular $D(\Omega) \le 28.46$, *if* $\tau_\Omega \le 1001$.

Proof. Without lost of generality we can assume that the minimum in the definition of τ_Ω is reached when $\omega = 0$ is the origin of the complex plane, that $\max_{\sigma\in\partial\Omega} = 1$ and $\min_{\sigma\in\partial\Omega} = \cos\gamma$. We will assume also $\tau_\Omega \ge 2$. Thus we consider a matrix A with numerical range $W(A) \subset \Omega$ and a rational function r satisfying $|r(z)| \le 1$ in Ω. Now we introduce the angles θ and ψ such that (recall that s is the curvilinear abscissa)

$e^{i\theta} = \frac{d\sigma}{ds}$
$\psi = \arg \sigma$



Taking into account the conditions $\max_{\sigma\in\partial\Omega} = 1$ and $\min_{\sigma\in\partial\Omega} = \cos\gamma$ we can assume that the determinations of θ and ψ are choosed such that $0 < \frac{\pi}{2} - \gamma \leq \theta - \psi \leq \frac{\pi}{2} + \gamma$. From our assumptions on the smoothness of the boundary and the strict positivity of the curvature, σ can be considered as a C^∞ function of θ. Then we introduce

$$\vec{v}(\theta, t) = \frac{1}{2\pi i} \frac{r(t\sigma(\theta))}{(t-1)\sigma(\theta) + e^{2i\theta}(\bar{\sigma}(\theta)-\bar{z})} \begin{pmatrix} -\sigma(\theta) \\ \\ t\sigma'(\theta) \end{pmatrix}.$$

We have clearly for the second component $v_2(\theta, 0) = 0$, $v_2(\theta, 1) = \frac{1}{2\pi i}\frac{r(\sigma)}{\bar{\sigma}-\bar{z}} e^{-2i\theta}\sigma'(\theta)$, and also $d\bar{\sigma} = e^{-2i\theta} d\sigma$. Therefore using the Green formula we have, if $z \in \Omega$,

$$\begin{aligned} \tilde{r}(\bar{z}) &= \frac{1}{2\pi i}\int_{\partial\Omega} r(\sigma)\,\frac{d\bar{\sigma}}{\bar{\sigma}-\bar{z}} = \int_0^{2\pi} \big(v_2(\theta,1) - v_2(\theta,0)\big)\,d\theta \\ &= \int_0^{2\pi}\int_0^1 \operatorname{div}\vec{v}(\theta,t)\,dt\,d\theta. \end{aligned}$$

Setting

$$Jr(\theta, \bar{z}) := \int_0^1 \operatorname{div}\vec{v}(\theta,t)\,dt = e^{2i\theta}\,\frac{\sigma(\theta)(\bar{\sigma}(\theta)-\bar{z}))}{\pi} \int_0^1 \frac{r(t\sigma(\theta))}{\big((t-1)\sigma(\theta) + e^{2i\theta}(\bar{\sigma}(\theta)-\bar{z})\big)^2}\,dt,$$

we deduce

$$\tilde{r}(\bar{z}) = \int_0^{2\pi} Jr(\theta,\bar{z})\,d\theta, \qquad \text{and therefore} \quad \tilde{r}(A^*) = \int_0^{2\pi} Jr(\theta, A^*)\,d\theta.$$

We consider now the open half-plane Π_σ tangent in σ to $\partial\Omega$, which contains Ω. We remark that $Jr(\theta,\bar{z})$ is antiholomorphic for $z \in \Pi_\sigma$, bounded and continuous in $\overline{\Pi}_\sigma$, and that $W(A) \subset \Pi_\sigma$. Using a von Neumann inequality and then the maximum principle, we obtain

$$\begin{aligned} \|Jr(\theta, A^*)\| &\leq \sup_{z\in\Pi_\sigma} |Jr(\theta,\bar{z})| = \sup_{x\in\mathbb{R}} |Jr(\theta, \bar{\sigma} + xe^{-i\theta})| \\ &\leq \sup_{x\in\mathbb{R}} \int_0^1 \frac{|\sigma|\,|x|}{\pi\,|(t-1)\sigma - xe^{i\theta}|^2}\,dt \\ &\leq \max_{\varepsilon=\pm1} \int_0^\infty \frac{d\tau}{\pi\,|\tau e^{i\psi} - \varepsilon e^{i\theta}|^2} \\ &= \frac{\max(\theta-\psi, \pi-\theta+\psi)}{\pi\,\sin(\theta-\psi)} = g(\tan(\tfrac{\pi}{2}-\theta+\psi)). \end{aligned}$$

(We have used the change of variables $\tau = (t-1)|\sigma|/|x|$). We deduce the estimate

$$\|\tilde{r}(A^*)\| \leq \int_0^{2\pi} g(\tan(\tfrac{\pi}{2}-\theta+\psi))\,d\theta = \int_0^{2\pi} g(\tan(\tfrac{\pi}{2}-\theta+\psi))\,d\psi.$$

If we consider $\rho(\psi) := |\sigma|$ as a function of ψ, we have $\tan(\theta-\psi) = \rho(\psi)/\rho'(\psi)$. We have obtain the bound

$$D(\Omega) \leq \int_0^{2\pi} g\big(\tfrac{\rho'(\psi)}{\rho(\psi)}\big)\, d\psi.$$

Note that the function g is even and convex on $\mathbb{R}$. It can be shown [7] that, with the constraints $\cos\gamma \leq \rho(\psi) \leq 1$, the right member is bounded by $4\int_0^{\gamma} g((\tan(\varphi))\,d\varphi + 2\int_0^{\pi-2\gamma} g(\tan(\varphi))\,d\varphi$. $\square$

6 The thin domain case

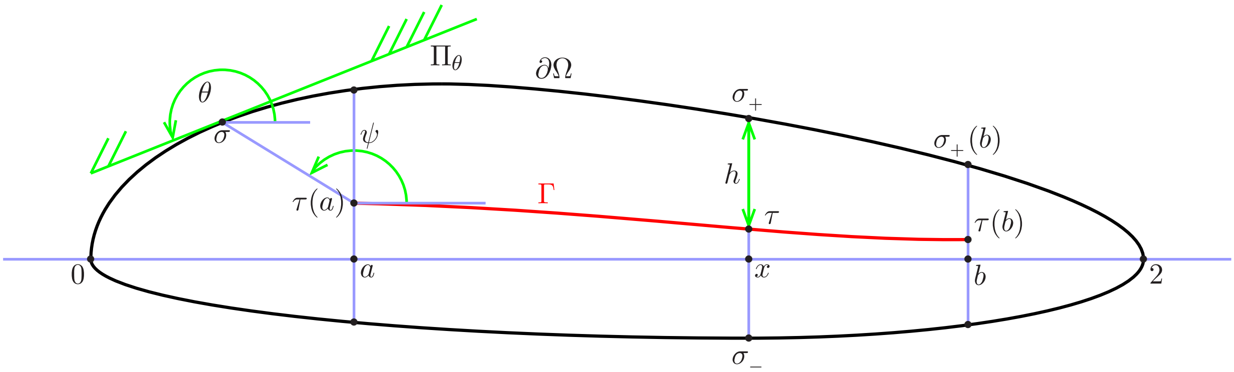
We assume that the flatness of the convex domain Ω satisfies $\tau_\Omega \geq 1001$ and that its diameter is equal to 2. After a change of origin and a rotation of the axes if needed, we can assume that there exist two functions $\eta_\pm \in C^0([0,2]) \cap C^\infty(]0,2[)$ such that

$$\begin{aligned} 0 \in \partial\Omega, \qquad 2 = \text{ diameter } (\Omega) \in \partial\Omega, \\ \Omega = \{x+i\,y\,; 0 < x < 2,\ \eta_-(x) < y < \eta_+(x)\}. \end{aligned}$$

We set $h(x) := \frac{1}{2}(\eta_+(x) - \eta_-(x))$.

We set $\alpha = h(1)$; for $\beta > \alpha$ given, there exist unique a, b such that $0 < a < 1 < b < 2$ and $\frac{h(a)}{a} = \frac{h(b)}{2-b} = \beta$. We introduce the median curve

$$\Gamma = \{z = \tau(x)\,; a \leq x \leq b,\ \tau(x) = x + \frac{i}{2}\,(\eta_+(x) + \eta_-(x))\}.$$



We have not respected the scale in order to make the figure readable.

As previously σ denotes a generic point of arclength s on the counterclockwise oriented boundary $\partial\Omega$, $\frac{d\sigma}{ds} = e^{i\theta}$. Due to the strict convexity, the point σ is a C^∞ function of θ. We define some functions of the real variable $x \in (0,2)$

$$\begin{aligned} \sigma_+(x) = x + i\,\eta_+(x), \qquad \sigma_-(x) = x + i\,\eta_-(x), \\ \theta_+ = \arg(\frac{d\sigma_+}{ds}), \qquad \theta_- = \arg(\frac{d\sigma_-}{ds}). \end{aligned}$$

Note that σ_+ and σ_- belongs to $\partial\Omega$ and that $\sigma_+ > 0$ and $\sigma_- < 0$.

We use also the notations

$$\begin{aligned} \xi(\theta,t) &= t\sigma(\theta) + (1-t)\,\tau(a) && \text{if} \quad \operatorname{Re}\sigma(\theta) \leq a, \\ &= t\sigma(\theta) + (1-t)\,\tau(\operatorname{Re}\sigma(\theta)) && \text{if} \quad \operatorname{Re}\sigma(\theta) \in [a,b], \\ &= t\sigma(\theta) + (1-t)\,\tau(b) && \text{if} \quad \operatorname{Re}\sigma(\theta) \geq b, \end{aligned}$$

$$\begin{aligned}
Jr(\theta,\bar z) &= \frac{1}{\pi}\int_0^1 r(\xi(\theta,t))\,\frac{\partial\xi}{\partial t}\,\frac{e^{2i\theta}(\bar\sigma(\theta)-\bar z)}{(\xi(\theta,t)-\sigma(\theta)+e^{2i\theta}(\bar\sigma(\theta)-\bar z))^2}\,dt,\\
G_+r(\bar z) &= \frac{1}{2\pi i}\int_a^b \frac{r(\tau)\,e^{-2i\theta_+}}{\bar\tau-\bar z-i\,(1+e^{-2i\theta_+})\,h}\,\frac{d\tau}{dx}\,dx,\\
G_-r(\bar z) &= \frac{1}{2\pi i}\int_a^b \frac{r(\tau)\,e^{-2i\theta_-}}{\bar\tau-\bar z+i\,(1+e^{-2i\theta_-})\,h}\,\frac{d\tau}{dx}\,dx.
\end{aligned}$$

The estimation of the constant $D(\Omega)$ is based on the following lemma.

Lemma 9. *For any rational function r bounded in Ω and for any $z\in\Omega$, we have the representation formula*

$$\tilde r(\bar z)=\int_0^{2\pi} Jr(\theta,\bar z)\,d\theta - G_+r(\bar z)+G_-r(\bar z). \tag{12}$$

Furthermore if A is a square matrix with $W(A)\subset\Omega$, we have

$$\tilde r(A^*)=\int_0^{2\pi} Jr(\theta,A^*)\,d\theta - G_+r(A^*)+G_-r(A^*). \tag{13}$$

Proof. We first note that $Jr(\theta,\bar z)$ is well defined, continuous, analytic in $\bar z$, for all $\theta\in\mathbb{R}$ and all $z\in\overline{\Omega}$. We set

$$\vec v(\theta,t)=\frac{1}{2\pi i}\,\frac{r(\xi(\theta,t))}{\xi(\theta,t)-\sigma+e^{2i\theta}(\bar\sigma-\bar z)}\begin{pmatrix}-\frac{\partial\xi}{\partial t}\\[2ex] \frac{\partial\xi}{\partial\theta}\end{pmatrix}.$$

Using that $\frac{\partial}{\partial\theta}\big(-\sigma+e^{2i\theta}(\bar\sigma-\bar z)\big)=2\,i\,e^{2i\theta}(\bar\sigma-\bar z)$, we get

$$\operatorname{div}\vec v=\frac{1}{\pi}\,\frac{\partial\xi}{\partial t}\,r(\xi(\theta,t))\,\frac{e^{2i\theta}(\bar\sigma-\bar z)}{(\xi(\theta,t)-\sigma+e^{2i\theta}(\bar\sigma-\bar z))^2}.$$

Then, using the Green formula,

$$\begin{aligned}
\int_0^{2\pi} Jr(\theta,z)\,d\theta &= \int_0^{2\pi}\int_0^1 \operatorname{div}\vec v\;dt\,d\theta\\
&= \int_0^{2\pi} v_2(\theta,1)\,d\theta-\int_{\pi/2}^{5\pi/2} v_2(\theta,0)\,d\theta.
\end{aligned}$$

We clearly have

$$\int_0^{2\pi} v_2(\theta,1)\,d\theta=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{r(\sigma)}{e^{2i\theta}(\bar\sigma-\bar z)}\,d\sigma=\frac{1}{2\pi i}\int_{\partial\Omega} r(\sigma)\,\frac{d\bar\sigma}{\bar\sigma-\bar z},$$

$$\int_{\pi/2}^{3\pi/2} v_2(\theta,0)\,d\theta \;=\; \frac{1}{2\pi i}\int_{\theta_+(b)}^{\theta_+(a)}\frac{r(\tau)}{\tau-\sigma+e^{2i\theta}(\bar\sigma-\bar z)}\,\frac{d\tau}{d\theta}\,d\theta,$$

thus, using that $\overline{\sigma}_+-\bar\tau=\tau-\sigma_+=-i\,h$,

$$\int_{\pi/2}^{3\pi/2} v_2(\theta,0)\,d\theta=\frac{1}{2\pi i}\int_b^a\frac{r(\tau)\,e^{-2i\theta_+}}{\bar\tau-\bar z-i\,(1+e^{-2i\theta_+})\,h}\,d\tau=-G_+r(\bar z).$$

Similarly

$$\int_{3\pi/2}^{5\pi/2} v_2(\theta,0)\,d\theta=G_-r(\bar z),\qquad\text{which shows (12).}$$

The representation (13) follows immediately. □

Now we write $A = B + i\,C$, with $B = \frac{1}{2}(A + A^*)$, $C = \frac{1}{2i}(A - A^*)$ self-adjoint.

Lemma 10. *We have the estimate*

$$D(\Omega) \le D_1(\alpha,\beta) + D_2(\alpha,\beta) + 2\,D_3(\alpha,\beta),$$

where

$$\begin{aligned}
D_1(\alpha,\beta) &= \sup\big\{\|\int_0^{2\pi} Jr(\theta,A^*)\,d\theta\| \ ; W(A)\subset\Omega,\ |r(z)|\le 1 \ in\ \Omega\big\},\\
D_2(\alpha,\beta) &:= \sup_{\Omega,\lambda,r}\big\{|G_+r(\lambda) - G_-r(\lambda)|\,;\lambda\in(0,2)\ |r(z)|\le 1\ in\ \Omega\big\},\\
D_3(\alpha,\beta) &= \sup\big\{\|G_+r(A^*) - G_+r(B)\|\,; W(A)\subset\Omega,\ |r(z)|\le 1\ in\ \Omega\big\}.
\end{aligned}$$

In these definitions the suprema are understood for all square matrices $A \in \mathbb{C}^{d,d}$, for all rational functions r (satisfying the mentionned constraints), and all domains Ω with $h(1) = \alpha$ and diameter $(0,2)$.

Proof. We deduce from (13) that

$$\begin{aligned}
\tilde r(A^*) &= \int_0^{2\pi} Jr(\theta,A^*)\,d\theta - (G_+r(B)) - G_-r(B))\\
&\qquad -(G_+r(A^*) - G_+r(B)) + (G_-r(A^*) - G_-r(B)).
\end{aligned}$$

From the spectral theory for the self adjoint matrix B we have, noticing that $\sigma(B)\subset(0,2)$,

$$\|G_+r(B)) - G_-r(B)\| \le \sup_{\lambda\in(0,2)} |G_+r(\lambda)) - G_-r(\lambda)| \le D_2(\alpha,\beta).$$

We have clearly $\|G_+r(A^*) - G_+r(B)\| \le D_3(\alpha,\beta)$; changing Ω in its conjugate and A in A^*, we obtain also $\|G_-r(A^*) - G_-r(B)\| \le D_3(\alpha,\beta)$. ☐

We first begin with an estimation of $D_1(\alpha,\beta)$.

Lemma 11. *We have (using the notations of Theorem 8)*

$$D_1(\alpha,\beta) \le 4\,G(\gamma') + 2\,G(\pi - 2\gamma')$$

$$\textit{where}\quad \gamma' = \arccos\frac{\beta}{\sqrt{1+4\beta^2}\sqrt{1+(\beta{+}2\alpha)^2}}.$$

For the values $\alpha \le \frac{1}{1000}$ and $\beta = \frac{1}{10}$ that gives $D_1(\alpha,\beta) \le 10.074$.

Proof. In the same way as in the previous section we obtain the estimate

$$\|Jr(\theta,A^*)\| \le \sup_{z\in\Pi_\sigma} |Jr(\theta,\bar z)| \le g(\tfrac{\pi}{2}{-}\theta{+}\psi),$$

where

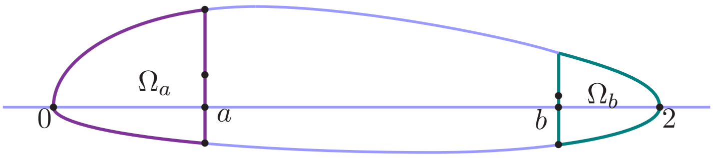
$$\begin{aligned}
\psi &= \arg(\sigma - \tau(a)) \text{ if } \operatorname{Re}\sigma \le a, \quad \psi = \arg(\sigma - \bar\sigma) \text{ if } a < \operatorname{Re}\sigma < b,\\
\psi &= \arg(\sigma - \tau(b)) \text{ if } \operatorname{Re}\sigma \ge b.
\end{aligned}$$

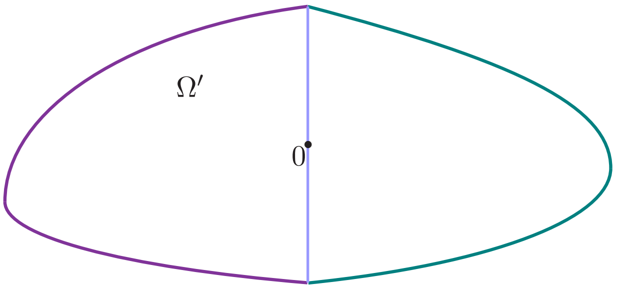
The determination of ψ is chosen in such a way that $\theta - \psi \in (0,\pi)$; Π_σ denotes the half-plane tangent in σ to Ω which contains Ω. Then we obtain

$$\int_0^{2\pi} \|Jr(\theta,A^*)\|\,d\theta \le \int_0^{2\pi} g(\tan(\tfrac{\pi}{2}{-}\theta{+}\psi))\,d\theta = \int_0^{2\pi} g(\tan(\tfrac{\pi}{2}{-}\theta{+}\psi))\,d\psi.$$

We introduce now the sets Ω_a, Ω_b, Ω' defined by

$$\Omega_a := \{x+iy \in \Omega\,;\, x \leq a\}, \qquad \Omega_b := \{x+iy \in \Omega\,;\, x \geq b\},$$
$$\Omega' := \{x+iy\,;\, h(a)(x+iy)+\tau(a) \in \Omega_a\} \cup \{x+iy\,;\, h(b)(x+iy)+\tau(b) \in \Omega_b\}.$$





It is easily verify that Ω' is convex and that the quantity $\int_0^{2\pi} g(\tan(\frac{\pi}{2}-\theta+\psi))\,d\psi$ for Ω is the same that this quantity defined in Theorem 8 for the thick domain Ω'. Therefore the present lemma holds with the constant $\gamma' = \arccos\frac{1}{\tau_{\Omega'}}$, where $\tau_{\Omega'}$ is the flatness of Ω'. We have

$$\tau_{\Omega'} \leq \frac{\max\limits_{\sigma\in\partial\Omega'} |\sigma'|}{\min\limits_{\sigma\in\partial\Omega'} |\sigma'|} = \frac{\max(\sup\limits_{\sigma\in\partial\Omega\cap\partial\Omega_a} \frac{|\sigma-\tau(a)|}{h(a)}, \sup\limits_{\sigma\in\partial\Omega\cap\partial\Omega_b} \frac{|\sigma-\tau(b)|}{h(b)})}{\min(\inf\limits_{\sigma\in\partial\Omega\cap\partial\Omega_a} \frac{|\sigma-\tau(a)|}{h(a)}, \inf\limits_{\sigma\in\partial\Omega\cap\partial\Omega_b} \frac{|\sigma-\tau(b)|}{h(b)})}.$$

We deduce from the convexity that Ω_a contains the triangle with vertices $\sigma_-(a)$,0, $\sigma_+(a)$. Therefore the $\inf_{\sigma\in\partial\Omega\cap\partial\Omega_a} |\sigma-\tau(a)|$ is less than the inf for σ belonging to the broken line $\sigma_-(a)$,0, $\sigma_+(a)$. The worst case corresponds to $\eta_-(a)=0$, which gives,

$$\inf_{\sigma\in\partial\Omega\cap\partial\Omega_a} \frac{|\sigma-\tau(a)|}{h(a)} \geq \frac{1}{\sqrt{4\,\beta^2+1}}, \quad \text{and similarly} \quad \inf_{\sigma\in\partial\Omega\cap\partial\Omega_b} \frac{|\sigma-\tau(b)|}{h(b)} \geq \frac{1}{\sqrt{4\,\beta^2+1}}.$$

We have also from the convexity on the boundary $\partial\Omega$

$$\frac{\operatorname{Im}\sigma_+}{\operatorname{Re}\sigma_+ - 2} \leq \tan\theta_+ \leq \frac{\operatorname{Im}\sigma_+}{\operatorname{Re}\sigma_+}, \qquad \frac{\operatorname{Im}\sigma_-}{\operatorname{Re}\sigma_-} \leq \tan\theta_- \leq \frac{\operatorname{Im}\sigma_-}{\operatorname{Re}\sigma_- - 2}, \tag{14}$$

$$|\operatorname{Im}\sigma| \leq 2\,h(\operatorname{Re}\sigma) \qquad \text{and} \qquad \forall x\in(0,2), \quad h(x) \leq \alpha\,\max(x, 2-x).$$

Using (14) at the points $\sigma_\pm(a)$, we also deduce that the points $\sigma \in \partial\Omega\cap\partial\Omega_a$ satisfy

$$\begin{aligned} 0 &\leq \operatorname{Re}(\tau(a)-\sigma) \leq a \quad \text{and} \\ |\operatorname{Im}(\tau(a)-\sigma)| &\leq h(a) + (a-\operatorname{Re}\sigma)\,\frac{|\operatorname{Im}\sigma(a)|}{2-a} \leq \beta\,a\,(1+\frac{2\,a}{2-a}). \end{aligned}$$

which shows

$$\sup\{|\sigma-\tau(a)|\,;\,\sigma\in\partial\Omega\cap\partial\Omega_a\} \leq a\,\sqrt{1+\beta^2\left(\frac{2+a}{2-a}\right)^2}.$$

Note that $\beta\, a = h(a) \leq \alpha\,(2-a)$ thus $a \leq \frac{2\,\alpha}{\alpha+\beta}$, therefore

$$\sup\{\frac{|\sigma - \tau(a)|}{h(a)}\,; \sigma \in \partial\Omega \cap \partial\Omega_a\} \leq \frac{\sqrt{1+(2\alpha+\beta)^2}}{\beta}.$$

The same estimate holds for $\sup\{\frac{|\sigma-\tau(b)|}{h(b)}\,; \sigma \in \partial\Omega \cap \partial\Omega_b\}$, therefore

$$\tau_{\Omega'} \leq \frac{\sqrt{1+4\beta^2}\,\sqrt{1+(\beta+2\alpha)^2}}{\beta}.$$

$\square$

Recall that $\tau_\Omega \geq 1001$; the assumption $\alpha \leq \frac{1}{1000}$ is then justified by the following lemma

Lemma 12. *If the convex domain* Ω *satisfies* $\tau_\Omega > 7.5$, *then we have* $\alpha \leq \dfrac{1}{\tau_\Omega - 1}$.

Proof. In the same way as in the previous lemma (but with $\beta = \alpha$) we have

$$\inf_{\sigma\in\partial\Omega} \frac{|\sigma - \tau(1)|}{h(1)} \geq \frac{1}{\sqrt{1+4\,\alpha^2}}, \quad \text{and} \quad \sup_{\sigma\in\partial\Omega} \frac{|\sigma - \tau(1)|}{h(1)} \leq \frac{\sqrt{1+9\,\alpha^2}}{\alpha}.$$

Therefore

$$\tau_\Omega \leq \frac{\sqrt{1+9\,\alpha^2}\sqrt{1+4\,\alpha^2}}{\alpha} \leq \frac{1+6.5\,\alpha^2}{\alpha} \quad \text{thus} \quad q(\alpha) := 6.5\,\alpha^2 - \alpha\,\tau + 1 \geq 0.$$

It is clear that $\alpha < 1$; since $q(1) < 0$, α is less than the smallest root of q. Since $q(1/(\tau-1)) = (7.5-\tau)/(\tau-1)^{-2} < 0$, we deduce $\alpha \leq 1/(\tau-1)$. $\square$

Lemma 13. *We assume* $\alpha < \beta \leq \frac{1}{4}$. *Then we have the estimate*

$$D_2(\alpha,\beta) \quad \leq \quad \frac{4}{\pi\delta_1}\,(\frac{\pi}{2} + \arctan\tfrac{2\beta}{\delta_1}) + \frac{4\beta}{1-\delta_2}(\sqrt{3} + \frac{2\beta}{\pi}),$$

with

$$\delta_1 = \sqrt{3 - \frac{52\beta^2 + 16\beta^4}{1+4\beta^2}}, \qquad \delta_2 = \frac{8\beta^2}{1+4\beta^2} + \sqrt{\frac{4\beta^2}{3} + (\frac{8\beta^2}{1+4\beta^2})^2}.$$

. *For the values* $\alpha \leq \frac{1}{1000}$ *and* $\beta = \frac{1}{10}$ *that gives* $D_2(\alpha,\beta) \leq 2.215$.

Proof. We have

$$\begin{aligned} G_+r(\lambda) - G_-r(\lambda) \quad &= \quad \frac{1}{2\pi i}\int_a^b r(\tau)\,R_0(x,\lambda)\,\tfrac{d\tau}{dx}\,dx, \\ \text{where} \quad R_0(x,\lambda) \quad &= \quad \frac{e^{-2i\theta_+}}{\bar{\tau} - \lambda - i\,(1+e^{-2i\theta_+})\,h} - \frac{e^{-2i\theta_-}}{\bar{\tau} - \lambda + i\,(1+e^{-2i\theta_-})\,h} \end{aligned}$$

Using that $|r(\tau)| \leq 1$ and $|\frac{d\tau}{dx}| = \sqrt{1+\gamma'(x)^2}$, we deduce

$$2\,\pi\,|G_+r(\lambda) - G_-r(\lambda)| \leq \int_a^b |R_0(x,\lambda)|\sqrt{1+\gamma'(x)^2}\,dx.$$

Now we introduce (recall that $\tau(x)=x+i\,\gamma(x)$)

$$\begin{aligned}
D_0(x,\lambda) &:= (\bar\tau-\lambda-i\,h\,(1+e^{-2i\theta_+}))(\bar\tau-\lambda+i\,h\,(1+e^{-2i\theta_-})),\\
R_1(x,\lambda) &:= \frac{(x-\lambda)(e^{-2i\theta_+}-e^{-2i\theta_-})+ih(1+e^{-2i\theta_+})(1+e^{-2i\theta_-})}{D_0(x,\lambda)},\\
R_2(x,\lambda) &:= -i\frac{\gamma(x)\,(e^{-2i\theta_+}-e^{-2i\theta_-})+\,h(x)\,(1-e^{-2i\theta_+}e^{-2i\theta_-})}{D_0(x,\lambda)}\\
&= 2\,e^{-i(\theta_++\theta_-)}\frac{-\gamma(x)\,\sin(\theta_+-\theta_-)+\,h(x)\,\sin(\theta_++\theta_-)}{D_0(x,\lambda)},
\end{aligned}$$

and we note that

$$R_0=R_1+R_2.$$

Therefore we have

$$\begin{aligned}
|G_+r(\lambda)-G_-r(\lambda)| &\le T_1(\lambda)+T_2(\lambda),\\
\text{where}\quad T_i(\lambda) &= \frac{1}{2\pi}\int_a^b |R_i(x,\lambda)|\,\sqrt{1+\gamma'(x)^2}\,dx,\quad i=1,2.
\end{aligned}$$

a) *Estimation of* T_1. We first begin by a lower bound for $|D_0|$. We have

$$\begin{aligned}
\operatorname{Re}D_0 &= (x-\lambda)^2-\gamma^2-(x-\lambda)\,h\,(\sin 2\theta_+-\sin 2\theta_-)\\
&\quad +h^2(1+\cos 2\theta_++\cos 2\theta_-+\cos 2(\theta_++\theta_-))-\gamma\,h\,(\cos 2\theta_+-\cos 2\theta_-).
\end{aligned}$$

Using that $|\gamma|\le h$ we obtain

$$\begin{aligned}
|D_0|\ge \operatorname{Re}D_0 \ge (x-\lambda)^2-|x-\lambda)|\,h\,|\sin 2\theta_+-\sin 2\theta_-|\\
+h^2\big(\cos 2(\theta_++\theta_-)+2\,\min(\cos 2\theta_+,\cos 2\theta_-)\big).
\end{aligned}$$

From the convexity of Ω we have

$$-\frac{h+\gamma}{2-x}\le \tan\theta_+\le\frac{h+\gamma}{x}\quad \text{et}\quad -\frac{h-\gamma}{2-x}\le -\tan\theta_-\le\frac{h-\gamma}{x}.$$

This implies, for $x\in[a,b]$,

$$-\frac{2h}{2-x}\le\tan\theta_+-\tan\theta_-\le\frac{2\,h}{x},\quad \text{thus}\quad |\tan\theta_+-\tan\theta_-|\le 2\,\beta.$$

We deduce

$$|\sin 2\theta_+-\sin 2\theta_-|=2\frac{|(\tan\theta_+-\tan\theta_-)(1+\tan\theta_+\,\tan\theta_-)|}{(1+\tan^2\theta_+)(1+\tan^2\theta_-)}\le 2|\tan\theta_+-\tan\theta_-|\le 4\,\beta.$$

Assume for instance that $|\sin\theta_+|\ge|\sin\theta_-|$. Then, using that $|\tan\theta_+|\le 2\beta$,

$$\begin{aligned}
\cos 2(\theta_++\theta_-)+2\,\min(\cos 2\theta_+,\cos 2\theta_-) &= 3-2\sin^2(\theta_++\theta_-)-4\sin^2\theta_+\ge 3-12\sin^2\theta_+\\
&\ge 3-\frac{12\,\tan^2\theta_+}{1+\tan^2\theta_+}\ge 3\big(1-\frac{16\beta^2}{1+4\beta^2}\big).
\end{aligned}$$

We obtain in this way the lower bound

$$|D_0|\ge(x-\lambda)^2-4\,\beta\,|x-\lambda)|\,h+3\,h^2\,\big(1-\frac{16\beta^2}{1+4\beta^2}\big). \tag{15}$$

Note that the right member is always positive since it is assumed that $\beta \le 1/4$. Now we remark that

$$2\,h' = \tan\theta_+ - \tan\theta_- = \frac{1}{i}\Big(\frac{1-e^{-2i\theta_+}}{1+e^{-2i\theta_+}} - \frac{1-e^{-2i\theta_-}}{1+e^{-2i\theta_-}}\Big) = -\frac{2}{i}\,\frac{e^{-2i\theta_+}-e^{-2i\theta_-}}{(1+e^{-2i\theta_+})(1+e^{-2i\theta_-})}.$$

Therefore

$$|R_1(x,\lambda)| = |(1+e^{-2i\theta_+})(1+e^{-2i\theta_-})|\,\frac{|(x-\lambda)h'-h|}{|D_0|} = 4\,\cos\theta_+\,\cos\theta_-\,\frac{|(x-\lambda)h'-h|}{|D_0|}.$$

We remark now that

$$\begin{aligned}
\gamma'(x) &= \frac{1}{2}(\tan\theta_+ - \tan\theta_-) = \frac{1}{2}\,\frac{\sin(\theta_+ + \theta_-)}{\cos\theta_+\,\cos\theta_-}\\
1+\gamma'^2 &= \frac{4\cos^2\theta_+\,\cos^2\theta_- + \sin^2(\theta_+ + \theta_-)}{4\cos^2\theta_+\,\cos^2\theta_-}\\
&\le \frac{4\cos^2\theta_+\,\cos^2\theta_- + 2(\sin^2\theta_+\cos^2\theta_- + \sin^2\theta_-\cos^2\theta_+)}{4\cos^2\theta_+\,\cos^2\theta_-}\\
&\le \frac{2\cos^2\theta_- + \cos^2\theta_+}{4\cos^2\theta_+\,\cos^2\theta_-} \le \frac{1}{\cos^2\theta_+\,\cos^2\theta_-}.
\end{aligned}$$

This shows that $(\cos\theta_+\,\cos\theta_-)\sqrt{1+\gamma'(x)^2} \le 1$ and then

$$\begin{aligned}
T_1(\lambda) &= \frac{1}{2\pi}\int_a^b |R_1(x,\lambda)|\,\sqrt{1+\gamma'(x)^2}\,dx \le \frac{2}{\pi}\int_a^b \frac{|(x-\lambda)h'-h|}{|D_0|}\,dx\\
&\le \frac{2}{\pi}\int_a^b \frac{|(x-\lambda)h'-h|}{(x-\lambda)^2 - 4\,\beta\,|x-\lambda|\,h + 3\,h^2\left(1-\frac{16\beta^2}{1+4\beta^2}\right)}\,dx.
\end{aligned}$$

Now we set $u(x) = \frac{x-\lambda}{h(x)}$ and note that u is an increasing function of x. The previous inequality becomes

$$\begin{aligned}
T_1(\lambda) &\le \frac{2}{\pi}\int_a^b \frac{u'(x)}{u(x)^2 - 4\,\beta\,|u(x)| + 3(1-\frac{16\beta^2}{1+4\beta^2})}\,dx\\
&\le \frac{4}{\pi}\int_0^\infty \frac{1}{(u-2\beta)^2+\delta^2}\,du = \frac{4}{\pi\delta_1}\,\big(\frac{\pi}{2} + \arctan\tfrac{2\beta}{\delta_1}\big)\\
&\qquad\qquad\text{with } \delta_1 = \sqrt{3 - \frac{52\beta^2+16\beta^4}{1+4\beta^2}}.
\end{aligned}$$

b) *Estimation of* T_2. We have

$$|R_2(x,\lambda)| \;:=\; 2\,\cos\theta_+\,\cos\theta_-\frac{|-\gamma(x)\,(\tan\theta_+ - \tan\theta_-) +\, h(x)\,(\tan\theta_+ + \tan\theta_-)|}{|D_0(x,\lambda)|},$$

We deduce from $\tan\theta_+ = h' + \gamma'$ and $\tan\theta_- = -h' + \gamma'$ that

$$|-\gamma(x)\,(\tan\theta_+ - \tan\theta_-) +\, h(x)\,(\tan\theta_+ + \tan\theta_-)| = 2|\gamma' h - h'\gamma|,$$

and then

$$T_2(\lambda) = \frac{1}{2\pi}\int_a^b |R_2(x,\lambda)|\,\sqrt{1+\gamma'(x)^2}\,dx \le \frac{2}{\pi}\int_a^b \frac{|\gamma' h - h'\gamma|}{|D_0(x,\lambda)|}\,dx.$$

Now if we set

$$\delta_2 = \frac{8\beta^2}{1+4\beta^2} + \sqrt{\frac{4\beta^2}{3} + \big(\frac{8\beta^2}{1+4\beta^2}\big)^2},$$

we deduce from (15) that

$$\begin{aligned} -(1-\delta_2)\big((x-\lambda)^2 + 3h^2\big) + |D_0(x,\lambda)| \ &\geq \ \delta_2(x-\lambda)^2 - 4\beta|x-\lambda|\,h + 3\,h^2\big(\delta_2 - \tfrac{16\beta^2}{1+4\beta^2}\big) \\ &= \delta_2\,\big(|x-\lambda| - 2\beta h/\delta_2\big)^2 \geq 0. \end{aligned}$$

This shows that

$$\frac{1}{|D_0(x,\lambda)|} \leq \frac{1}{(1-\delta_2)((x-\lambda)^2 + 3h^2)},$$

and thus

$$\begin{aligned} T_2(\lambda) \ &\leq \ \frac{2}{\pi(1-\delta_2)} \int_a^b \frac{|\gamma' h - h'\gamma|}{(x-\lambda)^2 + 3h^2}\, dx \\ &\leq \ \frac{2}{\pi(1-\delta_2)} (S_1(\lambda) + S_2(\lambda)), \end{aligned} \tag{16}$$

where

$$S_1(\lambda) := \int_a^\lambda \frac{|\gamma' h - h'\gamma|}{(x-\lambda)^2 + 3h^2}\, dx, \quad S_2(\lambda) := \int_\lambda^b \frac{|\gamma' h - h'\gamma|}{(x-\lambda)^2 + 3h^2}\, dx.$$

From now we assume that $\lambda \in [0,1]$; by a symmetry argument it will be clear that our estimate would be still valid if $\lambda \in [1,2]$. We remark that, if $\lambda \in [0,a]$, then $T_2(\lambda) \leq T_2(a)$. Thus we have just to consider $\lambda \in [a,1]$.

We begin by an estimate of S_1. For that we use that $|\gamma(x)| \leq h(x) \leq h(\lambda)\frac{2-x}{2-\lambda} \leq 2h(\lambda)$. From the convexity of Ω we have, for $x \in (a,\lambda)$, $|h'(x) \pm \gamma'(x)| \leq 2\beta$, thus $|\gamma' h - h'\gamma| \leq 4\beta h(\lambda)$. We also use the estimate $h(x) \geq h(\lambda)x/\lambda$. That gives

$$\begin{aligned} S_1(\lambda) \ &\leq \ 4\beta h(\lambda) \int_a^\lambda \frac{dx}{(x-\lambda)^2 + 3\,h(\lambda)^2 x^2/\lambda^2} = \frac{4\beta}{\sqrt{3}} \arctan \frac{(1+3h(\lambda)^2/\lambda^2)x - \lambda}{\sqrt{3}h(\lambda)}\Big]_a^\lambda \\ &\leq \ \frac{4\beta\pi}{\sqrt{3}}. \end{aligned}$$

It remains to estimate S_2. We now use

$$h'(x)\gamma(x) - h(x)\gamma'(x) = h'(x)(\gamma(x)-\gamma(\lambda)) - (h(x)-h(\lambda))\gamma'(x) + h'(x)\gamma(\lambda) - h(\lambda)\gamma'(x),$$

therefore

$$|h'(x)\gamma(x) - h(x)\gamma'(x)| \leq |h'(x)(\gamma(x)-\gamma(\lambda)) - (h(x)-h(\lambda))\gamma'(x)| + 2\,\beta\, h(\lambda).$$

We also remark that

$$h'(x)(\gamma(x)-\gamma(\lambda)) - (h(x)-h(\lambda))\gamma'(x) = \int_\lambda^x h''(t)(\gamma(t)-\gamma(\lambda)) - (h(t)-h(\lambda))\gamma''(t)\, dt,$$

which yields

$$\begin{aligned} |h'(x)(\gamma(x)-\gamma(\lambda)) - (h(x)-h(\lambda))\gamma'(x)| \ &\leq \ \int_\lambda^x |h''(t)(\gamma(t)-\gamma(\lambda)) - (h(t)-h(\lambda))\gamma''(t)|\, dt \\ &\leq \ \beta \int_\lambda^x (|h''(t)| + |\gamma''(t)|)(t-\lambda)\, dt. \end{aligned}$$

We note that $h(x) \geq h(\lambda)\frac{2-x}{2-\lambda}$, for $x \in (\lambda, b)$, and $|\gamma''| \leq |h''|$, thus

$$\begin{aligned} S_2(\lambda) &\leq 2\,\beta \int_\lambda^b \int_\lambda^x |h''(t)|(t-\lambda)\frac{1}{(x-\lambda)^2}\,dt\,dx \\ &\qquad +2\,\beta h(\lambda)\int_\lambda^b \frac{dx}{(x-\lambda)^2 + 3\,h(\lambda)^2(2-x)^2/(2-\lambda)^2} \\ &\leq 2\,\beta \int_\lambda^b \int_t^b |h''(t)|(t-\lambda)\frac{1}{(x-\lambda)^2}\,dx\,dt + \frac{2\beta\pi}{\sqrt{3}} \end{aligned}$$

Finally we have obtained

$$\begin{aligned} S_2(\lambda) &\leq 2\,\beta \int_\lambda^b |h''(t)|\,dt + \frac{2\beta\pi}{\sqrt{3}} \\ &\leq 2\beta(h'(\lambda) - h'(b)) + \frac{2\beta\pi}{\sqrt{3}} \leq 4\beta^2 + \frac{2\beta\pi}{\sqrt{3}}. \end{aligned}$$

Taking this estimate and the estimate of S_1 in (16) we obtain

$$T_2(\lambda) \leq \frac{4\beta}{1-\delta_2}(\sqrt{3} + \frac{2\beta}{\pi}).$$

□

Before estimating the constant $D_3(\alpha, \beta)$ we need some technical lemmas

Lemma 14. *We set*

$$\Gamma(\lambda, x) := h'(x)\,(\lambda - x) + h(x) \quad \textit{and} \quad \gamma(x) = \operatorname{Im}\tau(x), \qquad \textit{for} \;\; x \in (0,2).$$

The condition $W(A) \subset \overline{\Omega}$ is equivalent to : $\quad 0 \leq B \leq 2$ *and,* $\forall\, x \in (0,2)$,

$$\forall\, u, v \in H, \quad |\langle (C - \gamma'(x)\,(B-x) - \gamma(x))u, v\rangle| \leq \|(\Gamma(B,x))^{1/2}u\|\;\|\Gamma(B,x))^{1/2}v\|.$$

This condition implies

$$\forall\, u, v \in H, \quad |\langle C\,u, v\rangle| \leq 2\,\|(\Gamma(B,x))^{1/2}u\|\;\|\Gamma(B,x))^{1/2}v\|.$$

Proof. a) We first remark that $X + iY \in \overline{\Omega}$ is equivalent to

$$0 \leq X \leq 2 \quad \text{and,} \quad \forall\, x \in (0,2), \quad |Y - \gamma'(x)\,(X-x) - \gamma(x)| \leq \Gamma(X,x).$$

Indeed, $Y - \gamma'(X-x) - \gamma(x) = \pm\Gamma(X,x)$ are the equations of the tangents to $\partial\Omega$ in the points σ_+ and σ_-. Therefore the condition $W(A) \subset \overline{\Omega}$ means $0 \leq B \leq 2$ and

$$\forall v \in H,\ \forall\, x \in (0,2), \qquad |\langle (C - \gamma'(x)(B-x) - \gamma(x))v, v\rangle| \leq \langle \Gamma(B,x)\,v, v\rangle.$$

Thus $\Gamma(B,x) \geq 0$ and the lemma follows from the generalized Cauchy-Schwarz inequality.

b) We note that, if $X \in (0,2)$, then

$$\begin{aligned} \Gamma(X,x) - \gamma'(x)\,(X-x) - \gamma(x) &= -\eta'_-(x)\,(X-x) - \eta_-(x) > 0, \\ \Gamma(X,x) + \gamma'(x)\,(X-x) + \gamma(x) &= +\eta'_+(x)\,(X-x) + \eta_+(x) > 0. \end{aligned}$$

Therefore $X + iY \in \overline{\Omega}$ implies $|Y| \leq 2\,\Gamma(X,x)$. We conclude as in part a). □

Lemma 15. *We define*

$$K_1(\lambda) := \Big(\int_a^b \frac{\Gamma(\lambda, x)}{|x-\lambda-i\,\eta_+-i\,h\,e^{-2i\theta_+}|^2}\,dx \Big)^{1/2}.$$

Then we have, with $\varepsilon = \frac{4\,\alpha}{1+4\alpha^2}$ *and* $\eta = \frac{4\,\beta}{1+4\beta^2}$*,*

$$\begin{aligned}\sup_{\lambda\in[0,2]} (K_1(\lambda))^2 \;&\le\; \frac{1}{\sqrt{1-\varepsilon^2}}\big(\frac{\pi}{2} + \arctan\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\big)\\ &\qquad + \Big(\int_0^{4\beta} + \int_{1/\beta}^{\infty} \Big) \frac{u^2+4}{u^4-3\,u^2+4}\,du + \int_{4\beta}^{1/\beta} \frac{du}{u^2+1-\eta\,u}.\end{aligned}$$

Proof. We only look at the case $\lambda \le 1$, the other situation being symmetric. In order to lighten the notations we will omit the subscript $+$ in the proof. We have

$$\begin{aligned}|x-\lambda-i\,\eta-i\,h\,e^{-2i\theta}|^2 \;&=\; (x-\lambda-h\,\sin 2\theta)^2 + (\eta + h\,\cos 2\theta)^2\\ &\ge\; (x-\lambda-h\,\sin 2\theta)^2 + h^2\cos^2 2\theta\\ &=\; (x-\lambda)^2 + h^2 - 2\,h\,\sin 2\theta\,(x-\lambda).\end{aligned}$$

Setting $u(x) = \dfrac{x-\lambda}{h(x)}$ and $v(x) = \sin 2\theta$, we deduce

$$(K_1(\lambda))^2 \;\le\; \int_a^b \frac{h'(\lambda-x) + h}{(x-\lambda)^2 + h^2 - 2\,h\,v\,(x-\lambda)}\,dx = \int_a^b \frac{u'}{u^2+1-2\,u\,v}\,dx.$$

a) First case $a \le \lambda \le 1$. We note that for $x \in (a,b)$ we have

$$-2\,\beta \le -2\,\frac{h(x)}{2-x} \le \tan\theta(x) \le 2\,\frac{h(x)}{x} \le 2\,\beta,$$

therefore $\quad -\dfrac{4\frac{h(x)}{c-x}}{1+4\frac{h(x)^2}{(2-x)^2}} \le v(x) = \sin 2\theta(x) \le \dfrac{4\frac{h(x)}{x}}{1+4\frac{h(x)^2}{x^2}} \le \dfrac{4\,\beta}{1+4\beta^2}.$

- *If* $a < x < \lambda$. Then $u(x) \le 0$, $\frac{h(x)}{2-x} \le \alpha$ and $v(x) \ge -\frac{4\,\alpha}{1+4\alpha^2}$. We deduce, with $\varepsilon = \frac{4\,\alpha}{1+4\alpha^2}$,

$$\int_a^\lambda \frac{u'}{u^2+1-2\,u\,v}\,dx \le \int_{-\infty}^0 \frac{du}{u^2+1+2\,\varepsilon\,u} \le \frac{1}{\sqrt{1-\varepsilon^2}}\big(\frac{\pi}{2} + \arctan\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\big).$$

- *If* $\lambda < x$. Then $u(x) \ge 0$; in any case we have $v(x) \le \eta := \frac{4\,\beta}{1+4\beta^2}$. But we have also $\frac{h(x)}{x} \le \frac{h(x)}{x-\lambda} = \frac{1}{u}$. Thus

$$\text{if } u < 4\,\beta \text{ or } u < \tfrac{1}{\beta}, \qquad v(x) = \sin 2\theta \le \frac{4\,u}{u^2+4}.$$

We deduce

$$\begin{aligned}\int_\lambda^b \frac{u'}{u^2+1-2\,u\,v}\,dx \;&\le\; \Big(\int_0^{4\beta} + \int_{1/\beta}^{\infty} \Big) \frac{du}{u^2+1-\frac{8\,u^2}{u^2+4}} + \int_{4\beta}^{1/\beta} \frac{du}{u^2+1-\eta\,u}\\ &\le\; \Big(\int_0^{4\beta} + \int_{1/\beta}^{\infty} \Big) \frac{u^2+4}{u^4-3\,u^2+4}\,du + \int_{4\beta}^{1/\beta} \frac{du}{u^2+1-\eta\,u}.\end{aligned}$$

a) Second case $0 \le \lambda < a$. The proof is the same except that the first term is missing since there is no x such that $a < x < \lambda$. $\square$

We introduce now the operator, for $t \in [0,1]$, $x \in [a,b]$

$$\begin{aligned} \Psi(t,x) &= (\Gamma(B,x))^{1/2}(\bar{\tau} - i\,(1+e^{-2i\theta_+})\,h - B + i\,t\,C)^{-1} \\ &= (\Gamma(B,x))^{1/2}(x - i\eta_+ - i\,h\,e^{-2i\theta_+} - B + i\,t\,C)^{-1}. \end{aligned}$$

Lemma 16. *We have for all $u \in H$*

$$\begin{aligned} \|\Psi(0,.)\,u\|_{L^2(a,b)} &\le \sup_{\lambda\in(0,2)} K_1(\lambda)\,\|u\|, \\ \|\Psi(1,.)\,u\|_{L^2(a,b)} &\le \exp(2\sqrt{1+\beta^2})\,\|\Psi(0,.)\,u\|_{L^2(a,b)}. \end{aligned}$$

Proof. Since, for $t=0$, C does not appear in Ψ, we deduce from spectral theory

$$\|\Psi(0,.)\,u\|_{L^2(a,b)} \le \sup_{\lambda\in[0,2]} K_1(\lambda)\,\|u\|.$$

We will bound now $\partial_t \Psi(t,x)\,u$

$$\begin{aligned} \partial_t\Psi(t,x) &= -i\,\Gamma(B,x)^{1/2}(x - i\eta - i\,h\,e^{-2i\theta} - B + i\,t\,C)^{-1}C(x - i\eta - i\,h\,e^{-2i\theta} - B + i\,t\,C)^{-1} \\ &= -i\,\Psi(t,x)\,C\,\Gamma(B,x)^{-1/2}\,\Psi(t,x). \end{aligned}$$

Using Lemma 14 we get

$$\begin{aligned} \|\partial_t\Psi(t,x)\,u\| &= \sup_{\|v\|\le 1} \left|\langle \partial_t\Psi(t,x)\,u, v\rangle\right| \\ &\le \sup_{\|v\|\le 1} \left|\langle C\,\Gamma(B,x)^{-1/2}\Psi(t,x)u, \Psi(t,x)^* v\rangle\right| \\ &\le 2\,\|\Psi(t,x)\,u\|\,K_2(t), \end{aligned} \tag{17}$$

with

$$\begin{aligned} K_2(t) &= \sup_{\|v\|\le 1} \|\Gamma(B,x)^{1/2}\Psi(t,x)^* v\| \\ &= \sup_{\|v\|\le 1} \|\Gamma(B,x)^{1/2}(x + i\eta + i\,h\,e^{2i\theta} - B - i\,t\,C)^{-1}\Gamma(B,x)^{1/2} v\|. \end{aligned}$$

Let us consider

$$w := \Psi(t,x)^* v = (x + i\eta + i\,h\,e^{2i\theta} - B - i\,t\,C)^{-1}\Gamma(B,x)^{1/2} v$$

and

$$\Delta := \langle (x + i\eta + i\,h\,e^{2i\theta} - B - i\,t\,C)\,w, w\rangle = \langle v, \Gamma(B,x)^{1/2} w\rangle.$$

We deduce from Cauchy-Schwarz inequality

$$|\Delta| \le \|\Gamma(B,x)^{1/2} w\|\;\|v\|. \tag{18}$$

We have also

$$|\Delta|^2 = \langle (x - B - \sin 2\theta\,h)\,w, w\rangle^2 + \langle (\eta + \cos 2\theta\,h - t\,C)\,w, w\rangle^2 = |\Delta_1|^2 + |\Delta_2|^2.$$

Using that $\langle C\,w, w\rangle \le \langle (\eta + \eta'(B - x))w, w\rangle$ and $\eta' = \tan\theta$, we deduce

$$\Delta_2 = \langle (\eta + \cos 2\theta\,h - t\,C)\,w, w\rangle \ge \langle (\cos 2\theta\,h - \tan\theta\,(B - x))\,w, w\rangle.$$

We introduce

$$X_1 := \langle (B-x)\, w, w\rangle, \quad \text{and} \quad X_2 := h\,\langle w, w\rangle,$$

then we have $X_2 \geq 0$ and

$$|\Delta_1| = |X_1 + \sin 2\theta\, X_2|, \quad \Delta_2 \geq \cos 2\theta\, X_2 - \tan\theta\, X_1, \quad \|\Gamma(B,x)^{1/2} w\|^2 = h'\, X_1 + X_2.$$

We consider the two possible cases

1st case: $\cos 2\theta\, X_2 - \tan\theta\, X_1 \geq 0$. Then we have

$$\begin{aligned} |\Delta|^2 &\geq |X_1 + \sin 2\theta\, X_2|^2 + (\cos 2\theta\, X_2 - \tan\theta\, X_1)^2 \\ &\geq (1+\tan^2\theta) X_1^2 + 2\tan\theta\, X_1\, X_2 + X_2^2 \\ &\geq (1 + (\tan\theta - h')^2) X_1^2 + 2(\tan\theta - h')\, X_1\, (X_2 + h' X_1) + (X_2 + h' X_1)^2. \end{aligned}$$

Therefore

$$(1 + (\tan\theta - h')^2)|\Delta|^2 \geq \big((1 + (\tan\theta - h')^2) X_1 + (\tan\theta - h')\,(X_2 + h' X_1)\big)^2 + (X_2 + h' X_1)^2,$$

which shows that

$$\|\Gamma(B,x)^{1/2} w\|^2 \leq \sqrt{1 + (\tan\theta - h')^2}|\Delta|.$$

Remark that $|\tan\theta - h'| = \frac{1}{2}|\tan\theta_+ - \tan\theta_-| \leq \beta$, joint with (18) we deduce

$$\|\Gamma(B,x)^{1/2} w\|^2 \leq \sqrt{1+\beta^2}\, \|\Gamma(B,x)^{1/2} w\| \ \|v\|,$$

and then

$$\|\Gamma(B,x)^{1/2} w\| \leq \sqrt{1+\beta^2}\, \|v\|.$$

2nd case: $\tan\theta\, X_1 > \cos 2\theta\, X_2$. Then we have

$$\begin{aligned} \frac{\|\Gamma(B,x)^{1/2} w\|^2}{\Delta_1} &= \frac{|h' X_1 + X_2|}{|X_1 + \sin 2\theta\, X_2|} = \frac{|h' X_1 \tan\theta + X_2 \tan\theta|}{|X_1 \tan\theta + 2\, \sin^2\theta\, X_2|} \\ &\leq \max(|h'|, \frac{|h' \cos 2\theta + \tan\theta|}{|cos 2\theta + 2\sin^2\theta|}) = \max(|h'|, |h' \cos 2\theta + \tan\theta|) \leq 1. \end{aligned}$$

This shows $\|\Gamma(B,x)^{1/2} w\| \leq |\Delta|$ and consequently

$$\|\Gamma(B,x)^{1/2} w\| \leq \|v\|.$$

Going back to K_2 we deduce

$$K_2(t) = \sup_{\|v\|\leq 1} \|\Gamma(B,x)^{1/2} \Psi(t,x)^* v\| = \sup_{\|v\|\leq 1} \|\Gamma(B,x)^{1/2} w\| \leq \sqrt{1+\beta^2},$$

and from (17)

$$\|\partial_t \Psi(t,x)\, u\| \leq 2\, \sqrt{1+\beta^2}\, \|\Psi(t,x)\, u\|.$$

By integration of this differential inequality we deduce

$$\|\Psi(1,x)\, u\| \leq \exp(2\, \sqrt{1+\beta^2})\, \|\Psi(0,x)\, u\|.$$

□

Lemma 17. *We have the estimate*

$$D_3(\alpha,\beta) \le \frac{\sqrt{1+\beta^2}}{\pi}\exp(2\sqrt{1+\beta^2}) \sup_{\lambda\in(0,2)} K_1(\lambda)^2.$$

Therefore $D_3(\alpha,\beta) \le 8.085$ *if* $\alpha \le \frac{1}{1000}$ *and* $\beta = \frac{1}{10}$.

Proof. We start from

$$2\pi\|Gr(A^*) - Gr(B)\| =$$

$$\sup_{\|u\|\le 1, \|v\|\le 1} \Big| \int_a^b r(\tau) e^{-2i\theta} \Big\langle \big((\bar\tau - ih(1+e^{-2i\theta}) - A^*)^{-1}$$

$$- (\bar\tau - ih(1+e^{-2i\theta}) - B)^{-1} \big) u, v \Big\rangle \, d\tau \Big|$$

$$\le \sup_{\|u\|\le 1, \|v\|\le 1} \int_a^b \Big| \big\langle C\left((\bar\tau - ih(1+e^{-2i\theta}) - A^*)^{-1} u, (\tau + ih(1+e^{2i\theta}) - B)^{-1} v \right\rangle \Big| \, |d\tau|.$$

We remark that $|d\tau| = \sqrt{1+\gamma'(x)^2}\, dx \le \sqrt{1+\beta^2}\, dx$. Using Lemma 14 we deduce

$$2\pi\|Gr(A^*) - Gr(B)\| \le 2\sqrt{1+\beta^2}, \quad \sup_{\|u\|\le 1, \|v\|\le 1} \int_a^b \|\Psi(1,x)\, u\| \, \|\Psi(0,x)^*\, v\| \, dx.$$

Using Lemma 16 we get

$$2\pi\|Gr(A^*) - Gr(B)\| \le 2\sqrt{1+\beta^2} \exp(2\sqrt{1+\beta^2}) \sup_{\lambda\in(0,2)} K_1(\lambda)^2.$$

□

Now we summarize the results obtained in this section. We make the choice $\beta = \frac{1}{10}$ and we assume $\tau \ge 1001$ then we have from Lemma 13

$$D(\Omega) \le D_1(\alpha,\beta) + D_2(\alpha,\beta) + 2\, D_3(\alpha,\beta) \le 10.074 + 2.215 + 2 \times 8.085 = 28.459.$$

Joint with the result of the previous section we have proved Theorem 1.

7 Domains which are symmetric with respect to the real axis

Note that if the matrix $A \in \mathbb{R}^{d,d}$ then its numerical range $W(A)$ is symmetric with respect to the real axis. This is also true if $A \in \mathcal{L}(D(A), H)$ corresponds to the complexification of an operator defined on a real Hilbert space, which is a very common situation. Then it is natural to consider only domains which are also symmetric w.r.t. the real axis ; that allows simplifications and better bounds than in the previous sections, except may be if this domain is too thin in the vertical direction.

More precisely we assume in this section that Ω is bounded convex and symmetric with respect to the real axis and that 0 and 2 are the extremal point of Ω on this axis. As in the previous section we set $\alpha = h(1)$. Here we will consider that Ω is thick if $\frac{1}{7} \le \alpha \le 7$, and thin otherwise. Using Theorem 8 we deduce in the thick situation

$$C(\Omega) \le 11.18, \qquad \text{if } \frac{1}{7} \le \alpha \le 7.$$

Now we look at the case where Ω is thin in the horizontal direction $0 < \alpha < \frac{1}{7}$. We use the same notations that in Section 6, but now we choose the value $\beta = .49$. We have from (13)

$$\tilde{r}(A^*) = \int_0^{2\pi} Jr(\theta, A^*)\, d\theta + G_+r(B) - G_+r(A^*) + G_-r(A^*) \\ -G_-r(B) - (G_+r(B) - G_-r(B)).$$

Using the notations of Lemma 10 and noticing that the spectrum of the symmetric matrix B is included in $[0, 2]$, we have

$$C(\Omega) \le 2 + D(\Omega) \le 2 + D_1(\alpha, \beta) + D_2(\alpha, \beta) + 2\, D_3(\alpha, \beta). \tag{19}$$

We can also use the symmetry for improving the bound on the quantities $D_1(\alpha, \beta)$ and $D_3(\alpha, \beta)$. For instance Lemma 11 may be replaced by

Lemma 18. *We have*

$$D_1(\alpha, \beta) \le 4 \int_0^{\gamma''} g(\tan\varphi)\, d\varphi + 2 \int_0^{\pi - 2\gamma''} g(\tan\varphi)\, d\varphi,$$

$$\text{where} \quad \gamma'' = \arccos \frac{\beta}{\sqrt{1+\beta^2}\sqrt{1+\beta^2+2\alpha\beta+\alpha^2}}.$$

Therefore $D_1(\alpha, \beta) \le 5.806$ *if* $\alpha \le \frac{1}{7}$ *and* $\beta = .49$.

Proof. The proof is the same that for Lemma 11 except that we have a better estimate for the flatness $\tau_{\Omega'}$. Indeed with the notations of this proof we have now $\tau(a) = a$ and

$$\inf_{\sigma \in \partial\Omega \cap \partial\Omega_a} \frac{|\sigma - \tau(a)|}{h(a)} \ge \frac{1}{\sqrt{\beta^2+1}}, \quad \text{and similarly} \quad \inf_{\sigma \in \partial\Omega \cap \partial\Omega_b} \frac{|\sigma - \tau(b)|}{h(b)} \ge \frac{1}{\sqrt{\beta^2+1}}.$$

Noticing that $\tan\theta_+(a) \ge -\alpha$ we deduce that the points $\sigma \in \partial\Omega \cap \partial\Omega_a$ satisfy

$$0 \le \mathrm{Re}(\tau(a) - \sigma) \le a \quad \text{and}$$
$$|\,\mathrm{Im}(\tau(a) - \sigma)| = |\,\mathrm{Im}\,\sigma| \le h(a) + a\alpha = a(\alpha + \beta),$$

which shows

$$\sup\{\frac{|\sigma - \tau(a)|}{h(a)}\,; \sigma \in \partial\Omega \cap \partial\Omega_a\} \le \frac{\sqrt{1 + (\alpha+\beta)^2}}{\beta}.$$

The same estimate holds for $\sup\{\frac{|\sigma - \tau(b)|}{h(b)}\,; \sigma \in \partial\Omega \cap \partial\Omega_b\}$, therefore

$$\tau_{\Omega'} \le \frac{\sqrt{1+\beta^2}\,\sqrt{1+\beta^2+2\alpha\beta+\alpha^2}}{\beta}.$$

□

Lemma 19. *We have the estimate*

$$D_2(\alpha, \beta) \le \frac{2}{\pi\sqrt{1-\beta^2}}\,(\frac{\pi}{2} + \arctan\frac{\beta}{\sqrt{1-\beta^2}}).$$

Therefore $D_2(\alpha, \beta) \le 1.53$ *if* $\beta = .49$.

Proof. We set $\theta = \theta_+ = 2\pi - \theta_-$ and $u_\lambda(x) := \frac{x-\lambda}{h(x)}$, then

$$\begin{aligned}2\pi\,|G_+r(\lambda) - G_-r(\lambda)| &\le \Big|\int_a^b r(x)\big(\frac{e^{-2i\theta}}{x-\lambda-2i\,h\cos\theta e^{-i\theta}} - \frac{e^{2i\theta}}{x-\lambda+2i\,h\cos\theta e^{i\theta}}\big)\,dx\Big| \\ &\le \int_a^b \frac{4|(x-\lambda)\sin\theta\cos\theta - h\cos^2\theta|}{|x-\lambda|^2 + 4\,h^2\cos^2\theta - 4\,h\,\sin\theta\cos\theta(x-\lambda)}\,dx \\ &\le \int_a^b \frac{4\,u_\lambda'(x)}{u_\lambda^2(x)(1+\tan^2\theta) + 4 - 4\,u_\lambda(x)\tan\theta}\,dx,\end{aligned}$$

Note that u_λ is an increasing function in x and that, for $x \in [a,b]$, $|\tan\theta| \le \beta$. Therefore

$$\begin{aligned}2\pi\,|G_+r(\lambda) - G_-r(\lambda)| &\le \int_{-\infty}^{\infty} \frac{4}{u^2 + 4 - 4\beta\,|u|}\,du \\ &= 2\,\frac{4}{2\sqrt{1-\beta^2}}\,(\frac{\pi}{2} + \arctan\frac{2\beta}{2\sqrt{1-\beta^2}}).\end{aligned}$$

□

Lemma 17 can be modified now in

Lemma 20. *In the symmetric case we have the estimate*

$$\begin{aligned}D_3(\alpha,\beta) &\le \frac{e}{2\pi}\sup_{\lambda\in(0,2)} K_1(\lambda)^2, \\ &\le \frac{e}{2\pi}\big(\sqrt{1+\beta^2}\,\frac{\pi}{4} + \frac{1+\beta^2}{2}(\frac{\pi}{2} + \arctan\beta)\big).\end{aligned}$$

Therefore $D_3(\alpha,\beta) \le 0.922$ *if* $\beta = .49$.

Proof. Since $\tau(x) = x$ (i.e. $\gamma(x) = 0$) is real we deduce from Lemma 14 that the condition $W(A) \subset \overline{\Omega}$ is equivalent to : $0 \le B \le 2$ and, $\forall\, x \in (0,2)$,

$$\forall\, u, v \in H, \quad |\langle C\,u, v\rangle| \le \|(\Gamma(B,x))^{1/2}u\|\;\|\Gamma(B,x))^{1/2}v\|.$$

Using this in Lemma 16 we obtain now

$$\|\Psi(1,.)\,u\|_{L^2(a,b)} \le \exp(\sup_{t\in[0,1]} K_2(t))\,\|\Psi(0,.)\,u\|_{L^2(a,b)}.$$

Therefore the result will follow if we show that $K_2(t)) \le 1$ for $t \in [0,1]$.

For that we use the same proof that for Lemma 16. We define in the same way

$$X_1 := \langle (B-x)\,w, w\rangle, \quad \text{and} \quad X_2 := h\,\langle w, w\rangle,$$

we have

$$|\Delta_1| = |X_1 + \sin 2\theta\,X_2|, \quad \Delta_2 \ge \cos 2\theta\,X_2 - \tan\theta\,X_1,$$

but now $\|\Gamma(B,x)^{1/2}w\|^2 = \tan\theta\,X_1 + X_2$. We have seen in the proof of Lemma 16 that $K_2(t)) \le 1$ if $\cos 2\theta\,X_2 - \tan\theta\,X_1 \le 0$, it remains to consider the other case: $\cos 2\theta\,X_2 - \tan\theta\,X_1 > 0$. Then we have

$$\begin{aligned}|\Delta|^2 &\ge |X_1 + \sin 2\theta\,X_2|^2 + (\cos 2\theta\,X_2 - \tan\theta\,X_1)^2 \\ &\ge (1+\tan^2\theta)X_1^2 + 2\tan\theta\,X_1\,X_2 + X_2^2 \ge \|\Gamma(B,x)^{1/2}w\|^4,\end{aligned}$$

Therefore, using(18) we get

$$\|\Gamma(B,x)^{1/2}w\| \leq \|v \qquad \text{which induces } K_2(t)) \leq 1.$$

Recall that

$$(K_1(\lambda))^2 = (\int_a^b \frac{\Gamma(\lambda,x)}{|x-\lambda-2\,i\,h\,\cos\theta\,e^{-i\theta}|^2}\,dx.$$

We have

$$|x-\lambda-2\,i\,h\,\cos\theta\,e^{-i\theta}|^2 = (x-\lambda)^2 + 4\,h^2\cos^2\theta - 2\,(x-\lambda)\,h\,\sin 2\theta.$$

Therefore, setting $u(x) = \dfrac{x-\lambda}{h(x)}$, we deduce

$$\begin{aligned}
(K_1(\lambda))^2 &= \int_a^b \frac{h'(\lambda-x)+h}{(x-\lambda)^2 + 4\,h^2\cos^2\theta - 2\,(x-\lambda)\,h\,\sin 2\theta}\,dx \\
&= \int_a^b \frac{u'}{u^2 + 4\cos^2\theta - 2\,u\,\sin 2\theta}\,dx \\
&\leq \int_{-\infty}^0 \frac{du}{u^2 + \frac{4}{1+\beta^2}} + \int_0^\infty \frac{du}{u^2 + \frac{4}{1+\beta^2} - \frac{4\beta}{1+\beta^2}\,u} = \sqrt{1+\beta^2}\,\frac{\pi}{4} + \frac{1+\beta^2}{2}(\frac{\pi}{2} + \arctan\beta).
\end{aligned}$$

We have used $0 \leq \sin 2\theta \leq \frac{2\beta}{1+\beta^2}$ and $\cos^2\theta \geq \frac{1}{1+\beta^2}$ which follow from $|\tan\theta| \leq \beta$.

□

Coming back to (19) we deduce in the horizontal thin symmetric case

$$C(\Omega) \leq 2 + 5.806 + 2 \times 0.922 + 1.53 = 11.18\ .$$

Coupling with the begining of this section, we deduce that the estimate $C(\Omega) \leq 11.18$ holds if the symmetric domain satisfies $\alpha = h(1) \leq 7$.

8 Unbounded symmetric domains

We assume now that the domain $\Omega \neq \mathbb{C}$ is convex, unbounded and symmetric with respect to some axis. If Ω is a strip, it is known that $C(\Omega) \leq 2 + 2/\sqrt{3}$, see [5]. Otherwise, after a rotation and a translation if needed, we can assume that $\mathbb{R}_*^+ \subset \Omega \subset \{z\,;\mathrm{Re}\,z > 0\}$, and that Ω is symmetric with respect to the real axis. We remark that

$$\begin{aligned}
C(\Omega) &:= \sup\{\|r(A)\|; W(A) \subset \Omega, |r(z)| \leq 1, \forall z \in \overline{\Omega}\} \\
&= \sup\{\|r(A)\|; W(A) \subset \Omega, |r(z)| \leq 1, \forall z \in \overline{\Omega}, \text{ and } r(\infty) = 0\}.
\end{aligned}$$

Indeed we remark that if the rational function r satisfies $|r(z)| \leq 1$, $\forall z \in \overline{\Omega}$; then for $\varepsilon > 0$, setting $r_\varepsilon(z) := r(z)/(1+\varepsilon z)$ we have $|r_\varepsilon(z)| \leq 1$, $\forall z \in \overline{\Omega}$, $r_\varepsilon(\infty) = 0$ and $\lim_{\varepsilon\to 0} r_\varepsilon(A) = r(A)$.

From the assumptions made on Ω there exists a larger angle $\gamma \in [0, \frac{\pi}{2}]$ such that the closed sector $S_\gamma := \{z \in \mathbb{C}\,; z = 0 \text{ or } |\arg z| \leq \gamma\}$ satisfies $S_\gamma \subset \overline{\Omega}$. The angle 2γ will be called the aperture or Ω; we will consider that Ω is thin if $\gamma < \frac{\pi}{50}$ otherwise we will say that it is thick.

The thick case can be treated by Theorem 4.1 in [2], which provides the estimate

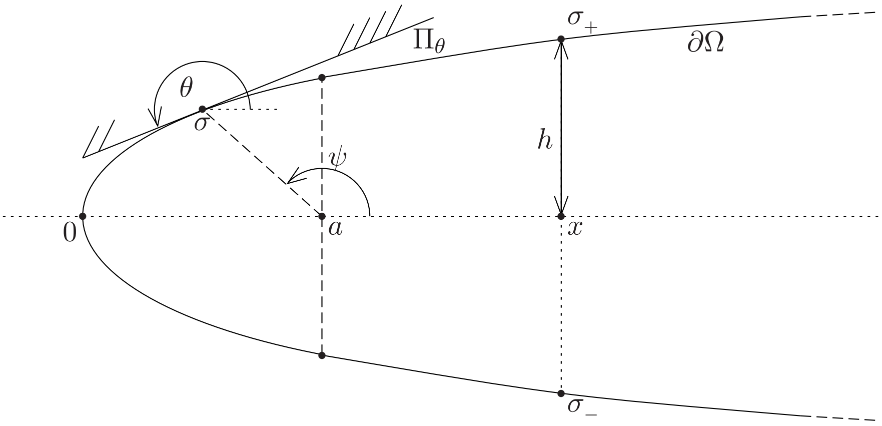
$$C(\Omega) \leq 1 + \frac{2}{\pi}\int_\gamma^{\pi/2} \frac{\pi - x + \sin x}{\sin x}\,dx.$$

With $\gamma \geq \frac{\pi}{50}$ that implies $C(\Omega) \leq 7.76$.

We turn now to the thin case $\gamma < \frac{\pi}{50}$. There exists a function $h \in C^0([0,\infty) \cap C^\infty(]0,\infty[)$ such that

$$\Omega = \{x{+}i\,y\,; x>0, -h(x)<y<h(x)\}, \quad h(0)=0 \quad \text{and} \quad \lim_{x\to\infty} \frac{h(x)}{x} = \gamma.$$

Let us fix $\beta = 0.4$. From the aperture asssumption there exists a unique a such that $\frac{h(a)}{a} = \beta$. The figure of Section 6 becomes now



We use also the notations

$$\begin{aligned}
\xi(\theta,t) &= t\sigma(\theta)+(1{-}t)\,a \qquad\qquad \text{if} \quad \operatorname{Re}\sigma(\theta)\leq a,\\
&= t\sigma(\theta)+(1{-}t)\,\operatorname{Re}\sigma(\theta) \quad \text{if} \quad \operatorname{Re}\sigma(\theta)\geq a,\\
Jr(\theta,\bar z) &= \frac{1}{\pi}\int_0^1 r(\xi(\theta,t))\,\frac{\partial\xi}{\partial t}\,\frac{e^{2i\theta}(\bar\sigma(\theta)-\bar z)}{(\xi(\theta,t)-\sigma(\theta)+e^{2i\theta}(\bar\sigma(\theta){-}\bar z))^2}\,dt,\\
G_+r(\bar z) &= \frac{1}{2\pi i}\int_a^\infty \frac{r(x)\,e^{-2i\theta_+}}{x-\bar z-2\,i\,h(x)\,\cos\theta_+ e^{-i\theta_+}}\,dx,\\
G_-r(\bar z) &= \frac{1}{2\pi i}\int_a^\infty \frac{r(x)\,e^{-2i\theta_-}}{x-\bar z+2\,i\,h(x)\,\cos\theta_- e^{-i\theta_-}}\,dx.
\end{aligned}$$

Note that we assume that the rational function r satisfies $|r(z)|\leq 1$ in Ω and $r(\infty)=0$. This last condition insures that the integrals are well defined.

Therefore we have in a similar way as previously

$$\begin{aligned}
r(A) &= \int_{\partial\Omega} r(\sigma)\,\mu(\sigma,A)\,ds+\tilde r(A^*),\\
\tilde r(A^*) &= \int_{\pi+\gamma}^{2\pi-\gamma} Jr(\theta,A^*)\,d\theta + G_+r(B)-G_+r(A^*)+G_-r(A^*)-G_-r(B)\\
&\qquad -(G_+r(B)-G_-r(B)).
\end{aligned}$$

Consequently

$$\begin{aligned}
C(\Omega) &\leq 2+D_1(\beta)+D_2(\beta)+2\,D_3(\beta), \quad \text{with} \qquad (20)\\
D_1(\beta) &= \sup\Big\{\|\int_{\pi+\gamma}^{2\pi-\gamma} Jr(\theta,A^*)\,d\theta\|\,; W(A)\subset\Omega,\ |r(z)|\leq 1 \text{ in } \Omega, r(\infty)=0\Big\},\\
D_2(\beta) &= \sup\big\{|G_+r(\lambda)-G_-r(\lambda)|\,;\lambda\in[0,\infty)\ |r(z)|\leq 1 \text{ in } \Omega, r(\infty)=0\big\},\\
D_3(\beta) &= \sup\big\{\|G_+r(A^*)-G_+r(B)\|\,; W(A)\subset\Omega,\ |r(z)|\leq 1 \text{ in } \Omega, r(\infty)=0\big\}.
\end{aligned}$$

Lemma 21. *We have*

$$D_1(\beta) \leq 2\int_0^{\gamma'''} g(\tan\varphi)\,d\varphi + \int_0^{\pi-2\gamma''} g(\tan\varphi)\,d\varphi,$$

$$\textit{where}\quad \gamma''' = \arccos\frac{\beta}{(1+\beta^2)}.$$

Therefore $D_1(\beta) \leq 3.99$ *for* $\beta = 0.4$.

Proof. We have

$$\begin{aligned}\|\int_{\pi+\gamma}^{2\pi-\gamma} Jr(\theta, A^*)\,d\theta\| &\leq \int_{\pi+\gamma}^{2\pi-\gamma} g(\tan(\tfrac{\pi}{2}-\theta+\psi))\,d\theta \\ &\leq \int_{\pi/2}^{3\pi/2} g(\tan(\tfrac{\pi}{2}-\theta+\psi))\,d\psi + \int_{\partial\Omega} g(\tan(\tfrac{\pi}{2}-\theta+\psi))\,d(\theta-\psi).\end{aligned}$$

Note that in the last integral $\theta-\psi$ runs from $\frac{\pi}{2}+\gamma$ to $\frac{\pi}{2}-\gamma$, therefore this integral is ≤ 0 and

$$\|\int_{\pi+\gamma}^{2\pi-\gamma} Jr(\theta, A^*)\,d\theta\| \leq \int_{\pi/2}^{3\pi/2} g(\tan(\tfrac{\pi}{2}-\theta+\psi))\,d\psi.$$

We introduce now the sets Ω''' defined by symmetrisation

$$\Omega''' := \{z = x+iy\,;\, x\in(0,a] \text{ and } z\in\Omega,\ \text{ or }\quad x\in[a,2a) \text{ and } 2a-z\in\Omega\}.$$

It is easily verify that Ω''' is convex and that the quantity $\int_{\pi/2}^{3\pi/2} g(\tan(\frac{\pi}{2}-\theta+\psi))\,d\psi$ for Ω is half of the similar quantity defined in Theorem 8 for the thick domain Ω'''. Therefore the present lemma holds with the constant $\gamma''' = \arccos\frac{1}{\tau_{\Omega'''}}$, where $\tau_{\Omega'''}$ is the flatness of Ω'''. We have

$$\tau_{\Omega'''} \leq \frac{\max\limits_{\sigma\in\partial\Omega'''}|\sigma'|}{\min\limits_{\sigma\in\partial\Omega'''}|\sigma|} = \frac{\sup\left\{\dfrac{|\sigma-a|}{h(a)}\,;\sigma\in\partial\Omega, \operatorname{Re}\sigma\leq a\right\}}{\inf\left\{\dfrac{|\sigma-a|}{h(a)}\,;\sigma\in\partial\Omega, \operatorname{Re}\sigma\leq a\right\}}.$$

As in Lemma 18 we have

$$\inf_{\sigma\in\partial\Omega\,\operatorname{Re}\sigma\leq a}\frac{|\sigma-a|}{h(a)} \geq \frac{1}{\sqrt{1+\beta^2}}.$$

Using that $0 \leq \tan\theta_+(a) = -\tan\theta_-(a)$ we obtain easily

$$|\sigma-a| \leq a\sqrt{1+\beta^2},$$

which shows

$$\tau_{\Omega'''} \leq \frac{1+\beta^2}{\beta}.$$

□

Note that the estimate given by Lemma 20 is still valid

$$D_3(\beta) \leq \frac{e}{2\pi}\left(\sqrt{1+\beta^2}\,\frac{\pi}{4} + \frac{1+\beta^2}{2}(\frac{\pi}{2}+\arctan\beta)\right).$$

Therefore $D_3(\beta) \leq 0.86$ if $\beta = 0.4$.

Lemma 22. *We have the estimate*

$$D_2(\beta) \leq \frac{1}{2} + \frac{1}{\pi\sqrt{1-\beta^2}}\,(\frac{\pi}{2} + \arctan\frac{\beta}{\sqrt{1-\beta^2}}).$$

Therefore $D_2(\beta) \leq 1.19$ *if* $\beta = 0.4$.

Proof. We set $\theta = \theta_+ = 2\pi - \theta_-$ and $u_\lambda(x) := \frac{x-\lambda}{h(x)}$, then

$$\begin{aligned} 2\pi\,|G_+r(\lambda) - G_-r(\lambda)| &\leq \Big|\int_a^\infty r(x)(\frac{e^{-2i\theta}}{x-\lambda-2i\,h\cos\theta e^{-i\theta}} - \frac{e^{2i\theta}}{x-\lambda+2i\,h\cos\theta e^{i\theta}})\,dx\Big| \\ &\leq \int_a^\infty \frac{4|(x-\lambda)\sin\theta\cos\theta - h\cos^2\theta|}{|x-\lambda|^2 + 4\,h^2\cos^2\theta - 4\,h\,\sin\theta\cos\theta(x-\lambda)}\,dx \\ &\leq \int_a^\infty \frac{4\,u_\lambda'(x)}{u_\lambda^2(x)(1+\tan^2\theta) + 4 - 4\,u_\lambda(x)\tan\theta}\,dx, \end{aligned}$$

Note that u_λ is an increasing function in x and that, for $x \in [a,\infty)]$, $0 < \tan\theta \leq \beta$. Therefore

$$\begin{aligned} 2\pi\,|G_+r(\lambda) - G_-r(\lambda)| \leq \int_{-\infty}^0 \frac{4}{u^2+4}\,du + \int_0^\infty \frac{4}{u^2+4-4\,\beta\,u}\,du \\ = \pi + \frac{4}{2\sqrt{1-\beta^2}}\,(\frac{\pi}{2} + \arctan\frac{2\beta}{2\sqrt{1-\beta^2}}). \end{aligned}$$

□

Coming back to (20) we have obtain

$$C(\Omega) \leq 2 + 3.99 + 1.19 + 2 \times 0.86 = 7.9.$$

From the begining of this section, we deduce that the estimate $C(\Omega) \leq 7.9$ holds for all unbounded symmetric domains.

9 Application to error analysis of Krylov methods

In this section we want to show that our results are well suited for the study of Krylov subspace methods. We consider such a method, for instance look-ahead BiCG, QMR, FOM or GMRES, in order to compute an approximation of the solution for a linear system

$$Ax = b,$$

where $A \in \mathbb{C}^{d,d}$ is a non singular complex matrix of (large) dimension and $b \in \mathbb{C}^d$ a given vector. The iterative method provides a sequence of approximations $x^{(n)}$ of the solution x, where $x^{(n)} \in \text{Span}(b, Ab, \ldots, A^{n-1}b)$. This means that we can write the residual vector $r^{(n)} := b - Ax^{(n)}$ on the form $r^{(n)} = q_n(A)b$ for some polynomial function q_n of degree $\leq n$. Then it can be shown [12] that the errror satisfies

$$\|x - x^{(n)}\| \leq \|A^{-1}P_n\|\ \min\big\{\|p_n(A)b\|\,; p_n(0) = 1,\ \deg(p_n) \leq n\big\},$$

where P_n is some projection operator and p_n runs among polynomial functions of degree $\leq n$ satisfying $p_n(0) = 1$. We deduce from Theorem 1

$$\begin{aligned} \|x - x^{(n)}\| &\leq \mathcal{Q}\,\delta_n(A)\,\|A^{-1}P_n\|\,\|b\|, \qquad \text{where} \\ \delta_n(A) &:= \min\big\{\|p_n\|_{L^\infty(W(A))}\,; p_n(0) = 1,\ \deg(p_n) \leq n\big\}. \end{aligned}$$

We assume now that $0 \notin W(A)$; then let ϕ be a conformal mapping from the exterior of $W(A)$ onto the exterior of the unit disk which sends ∞ to ∞. We consider the Faber polynomial ϕ_n of degree $\leq n$ defined by $\phi_n(z) = (\phi(z))^n + \mathrm{O}(1/z)$, as $z \to \infty$, and choose $p_n(z) = \phi_n(z)/\phi_n(0)$. From a theorem of Kövary and Pommerenke we know that

$$|\phi_n(z) - (\phi(z))^n| \leq 1, \qquad \text{if } \; z \notin W(A).$$

Thus $|\phi_n(z)| \leq 2$ on $\partial W(A)$ and $|\phi_n(0)| \geq |\phi(0)|^n - 1$, which shows that $\delta_n(A) \leq \frac{2}{|\phi(0)|^n - 1}$. Therefore we have the estimate (recall that $|\phi(0)| > 1$)

$$\|x - x^{(n)}\| \leq \frac{2\,\mathcal{Q}}{|\phi(0)|^n - 1}\, \|A^{-1}P_n\|\, \|b\|.$$

This estimate is not the best possible. In particular it is possible to improve the previous bound on $\delta_n(A)$, see [3]. If we set

$$r(A) = \max_{z \in W(A)} |z|, \quad \beta = \arccos \frac{d(0, W(A))}{r(A)}, \quad \gamma = 2 \sin \frac{\beta}{4 - 2\beta/\pi},$$

($r(A)$ is called the numerical radius of A), it is shown in [3]

$$\delta_n(A) \leq \min\{2 + \gamma, \frac{2}{1 - \gamma^{n+1}}\}\, \gamma^n.$$

Remark. Our method does not work if $0 \in W(A)$ *since then* $\delta_n(A) = 1$. *However, if* 0 *does not belong to the convex hull of the spectrum* $\sigma(A)$, *then it is possible to find an invertible matrix* H *such that* $0 \notin W(H^{-1}AH)$. *It is easy to show that* $\delta_n(A) \leq \delta_n(H^{-1}AH)\, \|H\|\, \|H^{-1}\|$, *which allows to use the previous results.*

In this section we have in fact consider an approximation of A^{-1} by a polynomial $q_n(A)$. This can be generalized for other functions and we think that our results could be used for improving some details in the analysis of the nice papers [9] or [11].

9.1 New result

The previous study has just been nicely improved by Bernhard Beckermann [4]. He has noticed that, for the Faber polynomials, one has, $\|\phi_n(A)\| \leq 2$. Consequently we obtain $\|p_n(A)\| \leq 2/|\phi_n(0)|$ with $p_n(.) = \phi_n(.)/\phi_n(0)$ and then

$$\|x - x^{(n)}\| \leq \frac{2\,\|A^{-1}P_n\|\, \|b\|}{|\phi(0)|^n - 1}.$$

The proof is simple. We write, with $z \in \Omega$, (Ω = interior of $W(A)$),

$$\int_{\partial\Omega} \phi(\sigma)^n\, \mu(\sigma, z)\, ds = \frac{1}{2\pi i} \int \phi_n(\sigma) \frac{d\sigma}{\sigma - z} + \frac{1}{2\pi i} \int (\phi(\sigma)^n - \phi_n(\sigma)) \frac{d\sigma}{\sigma - z} - \frac{1}{2\pi i} \int \frac{1}{\bar{\phi}(\bar{\sigma})^n} \frac{d\bar{\sigma}}{\bar{\sigma} - \bar{z}}.$$

The second term is null since $\phi(\sigma)^n - \phi_n(\sigma)$ is analytic in σ in the complementary set of Ω and vanishes at ∞. Similarly the third term is null since $1/(\bar{\phi}(\bar{\sigma}))$ is analytic in $\bar{\sigma}$ and vanishes at ∞. Therefore we have

$$\int_{\partial\Omega} \phi(\sigma)^n\, \mu(\sigma, z)\, ds = \frac{1}{2\pi i} \int \phi_n(\sigma) \frac{d\sigma}{\sigma - z} = \phi_n(z).$$

Arguing as in Theorem 6, we deduce

$$\phi_n(A) = \int_{\partial\Omega} \phi(\sigma)^n\, \mu(\sigma, A)\, ds, \qquad \text{and thus} \quad \|\phi_n(A)\| \leq 2 \sup_{\sigma \in \partial\Omega} |\phi^n(\sigma)| = 2.$$

10 An application in probability : proof of the Burkholder conjecture

Theorem 23. *Let* $T = P_1 \dots P_k$ *be a finite product of conditional expectations with respect to the* σ*-fields* $\mathcal{F}_1, \dots, \mathcal{F}_k$ *Then, for all functions* f *in* L^2,

$$\lim_{n\to\infty} T^n f = E[f \big| \mathcal{F}_1 \cap \dots \cap \mathcal{F}_k], \qquad \textit{almost surely.}$$

This statement was known as the Burkholder conjecture and has been proved by Bernard and François Delyon in the nice paper [8] which is at the origin of the present work. We essentially follow their proof.

Lemma 24. *The operators* T *and* $S := I - T$ *satisfy*

$$\|T - 2^{-k}\| = \|S - r_k\| \le r_k, \quad \textit{and} \quad W(S) \subset \overline{S}_{\theta_k} \tag{21}$$

$$\textit{where} \quad r_j = 1 - 1/2^j, \quad \theta_1 = 0,\ \theta_{j+1} = \arctan \frac{2\,\tan\theta_j + \sqrt{r_j(2-r_j)}}{2 - r_j}, \quad j = 1, \dots$$

$$\textit{and} \quad \overline{S}_{\theta_k} := \{z\,;\, z = 0 \quad \textit{or}\ |\arg z| \le \theta_k\}.$$

Proof. Note that $P_j = P_j^*$ is an orthogonal projector and T a contraction in L^2. The relations are clearly satisfied if $k = 1$. Assume that they are satisfied for T and consider TP where P is a orthogonal projector. Then we have

$$\begin{aligned} TP - 2^{-(k+1)} &= (T - 2^{-k})P + 2^{-k}(P - 2^{-1}), \quad \text{therefore} \\ \|TP - 2^{-(k+1)}\| &\le 1 - 2^{-k} + 2^{-k}2^{-1} = 1 - 2^{-(k+1)}, \end{aligned}$$

which shows by induction the left estimate of (21).

Now we consider $z = \langle (I-TP)u, u\rangle \in W(I-TP)$; it remains to show that $z \in \overline{S}_{\theta_{k+1}}$. For this we set $v = (I-P)^{1/2}u$, $w = Pu$. Then noticing that

$$I - TP = I - P + P(I-T)P + (I-P)(I-T)P,$$

we deduce

$$z = \|v\|^2 + \langle Sw, w\rangle + \zeta, \quad \text{with} \quad \zeta = \langle Sw, (I-P)^{1/2}v\rangle, \quad \text{thus} \quad |\zeta| \le \|Sw\|\,\|v\|.$$

Note that

$$\|Sw\|^2 = \|(S-r_k)w\|^2 - r_k^2\|w\|^2 + 2\,r_k\ \mathrm{Re}\langle Sw, w\rangle \le 2\,r_k\ \mathrm{Re}\langle Sw, w\rangle.$$

We introduce the notations

$$x = \|v\|, \quad y = \sqrt{\mathrm{Re}\langle Sw, w\rangle}, \quad \varphi = \arg\zeta, \quad \text{thus} \quad |\zeta| \le \sqrt{2\,r_k}\,xy.$$

From the induction hypothesis we have $\langle Sw, w\rangle \in S_{\theta_k}$, thus

$$\mathrm{Re}\, z \ge x^2 + y^2 - \sqrt{2r_k}\,xy\,|\cos\varphi| > 0, \quad |\,\mathrm{Im}\, z| \le y^2\tan\theta_k + \sqrt{2r_k}\,xy\,|\sin\varphi|.$$

We deduce

$$\begin{aligned} \frac{|\,\mathrm{Im}\, z|}{\mathrm{Re}\, z} &\le \frac{y^2\tan\theta_k}{x^2 + y^2 - \sqrt{2r_k}\,xy} + \frac{\sqrt{2r_k}\,xy\,|\sin\varphi|}{x^2 + y^2 - \sqrt{2r_k}\,xy\,|\cos\varphi|} \\ &\le \frac{\tan\theta_k}{1 - r_k/2} + \frac{\sqrt{2r_k}\,|\sin\varphi|}{2 - \sqrt{2r_k}\,|\cos\varphi|} \\ &\le \frac{2\tan\theta_k}{2 - r_k} + \frac{\sqrt{2r_k}\,\sqrt{1 - r_k/2}}{2 - r_k} = \tan\theta_{k+1}. \end{aligned}$$

This shows that $z \in S_{\theta_{k+1}}$. ☐

Proof of Theorem 23. It is known that

$$T^n f \to E[f \big| \mathcal{F}_1 \cap \cdots \cap \mathcal{F}_k] \text{ in } L^2, \quad \text{and from Chacon's theorem,}$$

$$A^n f := \frac{1}{n}\sum_{k=0}^{n-1} T^k f \to E[f \big| \mathcal{F}_1 \cap \cdots \cap \mathcal{F}_k], \qquad \text{a.s.}$$

Therefore it is sufficient to show that $(A_n - T^n)f \to 0$, (a.s.).

We have

$$\begin{aligned}
A_n - T^n &= \frac{1}{n}(I + 2T + \cdots + nT^{n-1})(I - T), \\
\sup_{n \geq n_0} \big|(A_n - T^n)f\big| &= \sup_{n \geq n_0} \frac{1}{n}\big|(I + 2T + \cdots + nT^{n-1})(I - T)f\big| \\
&\leq \sup_{n \geq n_0} \frac{1}{n}(1 + 2 + \cdots + n)^{1/2} \Big(\sum_{j=1}^{n} j\big(T^{j-1}(I - T)f\big)^2\Big)^{1/2} \\
&\leq \Big(\sum_{j=1}^{\infty} j\big(T^{j-1}(I - T)f\big)^2\Big)^{1/2}.
\end{aligned}$$

$$\| \sup_{n \geq n_0} \big|(A_n - T^n)f\big| \| \leq \Big(\sum_{j=1}^{\infty} j\|T^{j-1}(I - T)f\|^2\Big)^{1/2}.$$

We deduce from theorem 4 used with $R(z) = (1 - z)(1, \sqrt{2}z, \ldots, \sqrt{n}z^{n-1}) \in \mathbb{C}^n$ that

$$\begin{aligned}
\Big(\sum_{j=1}^{n} j\|T^{j-1}(I - T)f\|^2\Big)^{1/2} &\leq \mathcal{Q}_{cb} \sup_{z \in W(T)} \Big(\sum_{j=1}^{n} j|z^{j-1}(1 - z)|^2\Big)^{1/2} \|f\| \\
&\leq \mathcal{Q}_{cb} \sup_{z \in W(T)} \frac{|1 - z|}{1 - |z|^2}\|f\| \leq C_k\|f\|.
\end{aligned}$$

Indeed we have seen in the previous lemma that $W(T) \subset B(2^{-k}, r_k) \cap (1 - S_{\theta_k})$. Therefore, for all $f \in L^2$ and all n_0,

$$\| \sup_{n \geq n_0} \big|(A_n - T^n)f\big| \| \leq C_k\|f\|.$$

Similarly starting from $(A_n - T^n)(I - T) = \frac{1}{n}(I + T + \cdots + T^{n-1})(I - T)$, we deduce

$$\begin{aligned}
\sup_{n \geq n_0} \big|(A_n - T^n)(I - T)f\big| &\leq \sup_{n \geq n_0} \frac{1}{n}(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n})^{1/2} \Big(\sum_{j=1}^{n} j\big(T^{j-1}(I - T)f\big)^2\Big)^{1/2} \\
&\leq \frac{\sqrt{\log(2n_0 + 1)}}{n_0} \Big(\sum_{j=1}^{\infty} j\big(T^{j-1}(I - T)f\big)^2\Big)^{1/2},
\end{aligned}$$

which gives

$$\| \sup_{n \geq n_0} \big|(A_n - T^n)(I - T))f\big| \| \leq C_k \frac{\sqrt{\log(2n_0 + 1)}}{n_0} \|f\|.$$

Now we remark that, since T is a contraction, $\mathrm{Ker}\,(I-T)=\mathrm{Ker}\,(I-T^*)$. Consequently the range of $I-T$ is dense into the orthogonal of $\mathrm{Ker}\,(I-T)$ and, for all $f\in L^2$ and for all $\varepsilon>0$ we can write

$$f=(I-T)g+h+s,\quad \text{with}\quad g,h\in L^2, \|h\|\le\varepsilon, s\in \mathrm{Ker}\,(I-T).$$

We have $(A_n-T^n)s=0$, thus

$$\|\sup_{n\ge n_0}\big|(A_n-T^n)f\big|\,\|\le C_k\,\big(\frac{\sqrt{\log(2n_0+1)}}{n_0}\,\|g\|+\varepsilon).$$

Since ε si arbitrary we deduce that $(A_n-T^n)f$ almost surely converges to 0. □

11 Application to second-order time derivative problems

We assume here that we have two complex Hilbert spaces $V\subset H$, V dense in H with a continuous imbedding, and a sesquilinear form $a(.,.)$ which satisfies for some constants $0<\alpha\le M$, N, $\lambda\in\mathbb{R}$ and for all $v\in V$,

$$\alpha\|v\|_V^2\le \mathrm{Re}\,a(v,v)+\lambda\|v\|_H^2\le M\|v\|_V^2,\qquad\text{and}\qquad |\,\mathrm{Im}\,a(v,v)|\;\le N\|v\|_V\,\|v\|_H. \tag{22}$$

Then we define the operator $A\in\mathcal{L}(D(A),H)$ by

$$D(A)=\{v\in V\,;\exists K_v\ s.t.\ |a(v,w)|\le K_v\|w\|_H, \forall\, w\in H\}\quad\text{and}\quad \langle Av,w\rangle=a(v,w).$$

The Lax-Milgram lemma shows that $A+x\,I$ is an isomorphism from $D(A)$ onto H for all $x\ge\lambda$; this implies that $D(A)$ is dense in V and in H.

We define the parabolic domain $\mathcal{P}_\omega$ by

$$\mathcal{P}_\omega=\{z=x+i\,y\,;x>-\omega^2+\frac{y^2}{4\,\omega^2}\}=\{\zeta^2\,;\zeta\in\mathbb{C},\,|\,\mathrm{Im}\,\zeta|<\omega\}.$$

We set $\omega=\max(\frac{N}{2\sqrt{\alpha}},\lambda)$ if $\lambda>0$, $\omega=\frac{N}{2\sqrt{\alpha}}$ otherwise. Then it is easy to see that $a(v,v)\in\overline{\mathcal{P}_\omega}$ if $\|v\|_H=1$. That means that $W(A)\subset\overline{\mathcal{P}_\omega}$ and implies that $\sigma(A)\subset\overline{\mathcal{P}_\omega}$.

Recall that from [6] we have $C(\mathcal{P}_\omega)\le 4.75$. We consider now the function

$$f_t(z):=\cos(t\,z^{1/2}),\quad t\in\mathbb{R},\ z\in\mathbb{C}.$$

Note that, since the function cos is even, f_t is holomorphic for all $z\in\mathbb{C}$ and does not depend on the choice of determination of the square root. Furthermore we have $|f_t(z)|\le e^{\omega|t|}$, for all $z\in\mathcal{P}_\omega$. From Section 3

$$C(t):=f_t(A)\in\mathcal{L}(H)\quad\text{is well defined for } t\in\mathbb{R}\text{ and}\quad \|C(t)\|\le C(\mathcal{P}_\omega)\,e^{\omega|t|}.$$

From the relations $f_0=1$ and $f_{t+h}+f_{t-h}=2f_tf_h$ we deduce

$$C(0)=I\quad\text{and}\quad\forall\,t,h\in\mathbb{R},\ C(t+h)+C(t-h)=2\,C(t)\,C(h). \tag{23}$$

Furthermore

$$\forall\,u_0\in H,\quad\text{the function } u(t):=C(t)u_0\quad\text{satisfies}\ \ u\in C^0(\mathbb{R};H). \tag{24}$$

By definition, a function $C(.)$ with values in $\mathcal{L}(H)$ which satisfies the conditions (23) and (24) is called a cosine function on the Hilbert space H.

Proof of (24). We set

$$s_{t,h}(z) := \frac{f_{t+h}(z) - f_t(z)}{h(2\omega^2 + z)} = -\frac{z^{1/2}}{2\omega^2+z} \int_0^1 \sin((t+sh)z^{1/2})\,ds.$$

It is clear that $s_{t,h}$ is analytic in a neighborhood of $\mathcal{P}_\omega$ and we have the estimate

$$|s_{t+h}(z)| \le \frac{e^{\omega T}}{\omega}, \quad \forall\, z \in \mathcal{P}_\omega,\ \forall\, |t| \le T \text{ and } |t+h| \le T.$$

From the relation $f_{t+h}(z) - f_t(z) = h\, s_{t,h}(z)(2\omega^2+z)$ we deduce, if $u_0 \in D(A)$,

$$u(t+h) - u(t) = h\, s_{t,h}(A)(2\omega^2 u_0 + Au_0),$$

$$\text{thus } \|u(t+h) - u(t)\| \le 4.75\, h\, \frac{e^{\omega T}}{\omega}\ \|2\omega^2 u_0 + Au_0\|.$$

That shows the continuity of u if $u_0 \in D(A)$. If we only have $u_0 \in H$, then there exists a sequence of $u_{n0} \in D(A)$ which tends to u_0 in H. We set $u_n(t) = C(t)u_{n0}$; for $|t| \le T$ we have $\|u_n(t) - u(t)\| \le 4.75\, e^{\omega T} \|u_{n0} - u_0\|$, which shows that the sequence of continuous functions $u_n(.)$ uniformly converges towards $u(.)$ on $[-T, T]$, and whence $u(.)$ is continuous. □

From the previous part it is easy to see that, if $u_0 \in D(A)$, then, for all $t \in \mathbb{R}$, $u(t) \in D(A)$ and $A\, u(t) = C(t)A\, u_0 \in C^0(\mathbb{R}; H)$. More generally we have $u(t) \in D((\lambda{+}A)^s)$ and $(\lambda{+}A)^s u(t) = C(t)(\lambda{+}A)^s u_0 \in C^0(\mathbb{R}; H)$ as soon as $u_0 \in D((\lambda{+}A)^s)$. That may be considered as a smoothness result with respect to the spaces. With respect to the time we have

$$\begin{aligned} &\forall\, u_0 \in D(A) \quad \text{the function } u(t) := C(t)u_0 \quad \text{satisfies } \ u \in C^2(\mathbb{R}; H) \cap C^0(\mathbb{R}; D(A)) \\ &\qquad \text{and} \qquad u''(t) + A\, u(t) = 0, \qquad u(0) = u_0,\ u'(0) = 0. \end{aligned} \tag{25}$$

Proof of (25). It is sufficient to prove the relation (note that $u(t) = u(-t)$)

$$u(t) = u_0 - \int_0^t (t-s)A\, u(s)\, ds, \quad \forall\, t > 0.$$

For that we set, with fixed $t > 0$, $n \in \mathbb{N}$, $h = t/N$ and $s_n = n\, h$,

$$\Sigma_h(z) = \Big(f_t(z) - 1 + h \sum_{j=1}^{N} (t-s_n) f_{s_n}(z) z\Big) / (2\omega^2 + z).$$

It is clear that Σ_h is analytic and bounded in a neighborhood or $\mathcal{P}_\omega$. Thus we have

$$u(t) - u_0 + h \sum_{j=1}^{N} (t-s_n)A\, u(s_n) = \Sigma_h(A)(2\,\omega^2 u_0 + A\, u_0)$$

$$\begin{aligned} \text{and} \quad \|\Sigma_h(A)\| \ &\le\ 4.75 \sup_{z \in \mathcal{P}_\omega} |\Sigma_h(z)| \\ &\le\ 4.75\, h \sup_{z \in \mathcal{P}_\omega} \int_0^t \Big| \frac{|\cos s\sqrt{z} + (t-s)\sqrt{z} \sin t\sqrt{z}|}{|2\omega^2 + z|} \Big| ds \\ &\le\ 4.75\, h\, \frac{(t + t^2\omega)\, e^{\omega t}}{\omega^2}. \end{aligned}$$

(We have used a well-known error estimate for the rectangular rule). We obtain the result by considering the limit as $h \to 0$. □

In order to treat an initial data $u'(0) \neq 0$, it is useful to introduce

$$g_t(z) = \int_0^t \cos(s\sqrt{z})\, ds, \quad t \in \mathbb{R},\ z \in \mathbb{C}, \quad \text{i.e.} \quad g_t(z) = \frac{\sin t\sqrt{z}}{\sqrt{z}}, \quad \text{if } z \neq 0.$$

This function is analytic for $z \in \mathbb{C}$ and we have, using the maximum principle,

$$\forall\, z \in \mathcal{P}_\omega, \quad |g_t(z)| \leq \sup_{z \in \partial\mathcal{P}_\omega} \frac{|\sin t\sqrt{z}|}{\sqrt{|z|}} = \sup_{y \in \mathbb{R}} \frac{|\sinh t(\omega{+}iy)}{|\omega{+}iy|} \leq \frac{e^{|t|\omega}}{\omega}.$$

Therefore we can define

$$S(t) := g_t(A) \in \mathcal{L}(H), \quad \text{and we have} \quad \|S(t)\| \leq 4.75\, \frac{e^{|t|\omega}}{\omega}.$$

Then we have, $\forall\, u_0 \in D(A)$ and $\forall\, v_0 \in V$,

$$\text{the function } u(t) := C(t)u_0 + S(t)v_0 \in C^2(\mathbb{R}; H) \cap C^0(\mathbb{R}; D(A)) \quad \text{and satisfies}$$
$$u''(t) + A\,u(t) = 0, \qquad u(0) = u_0,\ u'(0) = v_0. \tag{26}$$

Proof of (26). We set $v(t) = S(t)\, v_0 = \int_0^t C(s) v_0\, ds$. It is sufficient to prove that $v \in C^2(\mathbb{R}; H) \cap C^0(\mathbb{R}; D(A))$ and $v''(t) + A\, v(t) = 0$.

This is a simple consequence of the previous part if $v_0 \in D(A)$. If we only have $v_0 \in V$ we will use that $(2\,\omega{+}A)^{-1/2}$ is an isomorphism from H onto V and from V onto $D(A)$; this can be deduced from the assumptions (22) on the sesquilinear form, see [14]. Thus there exists $w_0 = (2\,\omega{+}A)^{-1/2} v_0 \in D(A)$. Setting $h_t(z) = \frac{\sin t\sqrt{z}}{\sqrt{z}}\sqrt{2\omega + z}$, $T(t) := h_t(A)$ is well defined and we have $S(t) = T(t)(2\,\omega{+}A)^{-1/2}$. This shows that $v(t) = T(t)\, w_0$. Then we obtain the result by mimicking the proofs of (24) and (25), but with f_t replaced by h_t.

□

Keeping the notations of this proof, we deduce from the definition of g_t that $v'(t) = C(t)\, v_0 = C(t)(2\,\omega{+}A)^{-1/2} w_0 = (2\,\omega{+}A)^{-1/2} C(t)\, w_0$. This shows that $v \in C^1(\mathbb{R}; V)$.

Similarly we have

$$(C(t)\, u_0)' = -\int_0^t A\, C(s)\, u_0\, ds = -A\, S(t)\, u_0 = -(2\,\omega{+}A)^{-1/2} A\, T(t)\, u_0$$

Since we have $u_0 \in D(A)$, then $T(.)u_0 \in C^0(\mathbb{R}; D(A))$ and then $C(.)\, u_0 \in C^1(\mathbb{R}, V)$. Alltogether we have shown that $u \in C^1(\mathbb{R}; V)$.

In order to give a more general statement we introduce some spaces H_a^s, for real $s \geq 0$

$$H_a^s := (2\,\omega{+}A)^{-s/2} H \quad \text{i.e. } H^s \text{ is the image of } H \text{ by the operator } (2\,\omega{+}A)^{-s/2}.$$

Note that, if k is an integer $H_a^{2k} = D(A^k)$ and $H_a^{1/2} = V$.

Theorem 25. *Assume that the sesquilinear form satisfies the assumptions (22) and that $u_0 \in D(A)$, $v_0 \in V$. Then the problem*

Find $u \in C^2(\mathbb{R}; H) \cap C^0(\mathbb{R}; D(A))$ such that
$u''(t) + A\, u(t) = 0, \qquad u(0) = u_0,\ u'(0) = v_0,$
has one and only one solution which is given by $u(t) := C(t)u_0 + S(t)v_0$. Furthermore,
if $u_0 \in H_a^{s+2}$ and $v_0 \in H_a^{s+1}$, then $u \in C^k(\mathbb{R}; H_a^{s+2-k})$, for each integer $k \leq s + 2$.

Proof. a) *Existence and smoothness.* For $k = 0, 1, 2$, this follows from (25), (26) and from the relations

$$C(t)(2\,\omega{+}A)^{-s/2} = (2\,\omega{+}A)^{-s/2}C(t) \quad \text{and} \quad S(t)(2\,\omega{+}A)^{-s/2} = (2\,\omega{+}A)^{-s/2}S(t).$$

For greater values of k, we use that $Au = -u''$...

b) *Uniqueness.* It is sufficient to prove that, if $u \in C^2(\mathbb{R}; H) \cap C^0(\mathbb{R}; D(A))$ satisfies

$$u''(t) + A\,u(t) = 0, \qquad u(0) = 0,\ u'(0) = 0, \qquad \text{then } u = 0.$$

Replacing if needed u by $(\lambda{+}A)^{-1}u$ we can assume that $u \in C^2(\mathbb{R}; D(A)) \cap C^0(\mathbb{R}; D(A^2))$. Then we set

$$v(t,s) = C(s)u(t) - S(s)u'(t), \qquad w(t,s) = C(s)u'(t) + S(s)A\,u(t).$$

We deduce from the equality $C(t)^2 + S(t)^2A = Id_{|_{D(A)}}$ (which is valid on $D(A)$) that

$$u(t) = C(t)\,v(t,t) - S(t)\,w(t,t).$$

We have

$$\frac{\partial v}{\partial t}(t,s) = C(s)u'(t) + S(s)Au(t), \qquad \frac{\partial v}{\partial s}(t,s) = -C(s)u'(t) - S(s)A\,u(t).$$

Thus

$$\frac{d}{dt}v(t,t) = \frac{\partial v}{\partial t}(t,t) + \frac{\partial v}{\partial s}(t,t) = 0.$$

Joint together $v(0,0) = 0$ that shows that $v(t,t) = 0$ for all t. Similarly we have $w(t,t) = 0$ for all t and thus $u(t) = C(t)\,v(t,t) - S(t)\,w(t,t) = 0$. ☐

Remark. We have seen that the problem (26) is well posed. This fact is equivalent to say that $C(.)$ is a cosine function on the Hilbert space H with generator $-A$. A deep result due to Markus Haase [10] says that, if this situation occurs, then there exists an equivalent scalar product $\langle ., . \rangle_0$ and a real ω such that $\langle Av, v \rangle_0 \in \mathcal{P}_\omega$ for all $v \in D(A)$. Using this new scalar product and setting $a(u,v) = \langle Au, v \rangle_0$, the assumptions (22) can be verified for some appropriate constants. That shows that (in the hilbertian situation) our framework is general.

Remark. Note that the problem (26) can be well posed in a Banach framework. But there is not plenty of interesting examples. For instance it is known that the Laplacian generates a cosine function in $L^p(\mathbb{R}^d)$ if, and only if, either $p = 2$ or $d = 1$.

References

[1] W. Arveson : Subalgebras of C^*-algebras. II *Acta Math.* **128** (1972), 271–308.

[2] C. Badea, M. Crouzeix, B. Delyon, Convex domains and K-spectral sets, preprint

[3] B. Beckerman, S.A. Goreinov, E.E. Tyrtyshnikov, Some remarks on the Elman estimate for GMRES, preprint.

[4] B. Beckerman, Image numérique, GMRES et polynômes de Faber, CRAS, to appear.

[5] M. Crouzeix, B. Delyon, Some estimates for analytic functions of strip or sectorial operators, *Arch. Math.*, **81** (2003), 553-566.

[6] M. Crouzeix, Operators with numerical range in a parabola, *Arch. Math.*, **82** (2004), 517-527.

[7] M. Crouzeix, Une inégalité sur les ensembles convexes du plan, preprint.

[8] B. & F. Delyon, Generalization of Von Neumann's spectral sets and integral representation of operators, *Bull. Soc. Math. France*, **1**, (1999), 25–42.

[9] E. Gallopoulos, Y. Saad, Efficient solution of parabolic equations by Krylov approximation methods, *SIAM J. Sci. Comput.*, **13**,(1992), 1236–1264.

[10] M. Haase, *The Functional Calculus for Sectorial Operators*, book manuscript, available at www.mathematik.uni-ulm.de/m5/haase

[11] M. Hochbruck, Ch. Lubich, On Krylov subspace approximations to the matrix exponential operator, *SIAM J. Numer. Anal.*, **34**, n° 5, (1997), 1911–1925.

[12] M. Hochbruck, Ch. Lubich, Error analysis of Krylov methods in a nutshell, *SIAM J. Sci. Comput.*, **19**, n° 2, (1998), 695–701.

[13] T. Kövari, Ch Pommerenke, On Faber polynomials and Faber expansions, *Math. Z.*, **99**, (1967), 193–206.

[14] A. McIntosh, On representing closed accretive sesquilinear forms as $(A^{1/2}u, A^{*1/2}v)$. In Non linear partial differential equations and their applications. Collège de France Seminar, vol. 3, (Paris 1980-1981), (1982), 252–267.

[15] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Univ. Press, 2002.

[16] J. von Neumann, Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes, *Math. Nachrichten* **4** (1951) 258–281.

[17] J. Walker, *Algebraic curves*, Princeton Univ. Press, 1951.